Recurrence Extraction for Functional Programs Through Call-by-Push-Value

Alex Kavvos
Department of Computer Science, Aarhus University

j.w.w. Edward Morehouse, Daniel Licata, and Norman Danner

47th ACM SIGPLAN Symposium on Principles of Programming Languages (POPL 2020)

arXiv:1911.04588

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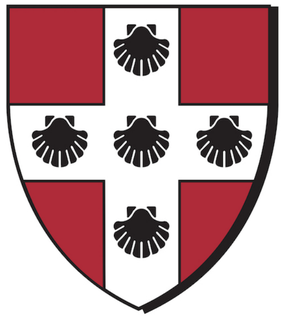

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Recurrence Extraction for FP

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&\qquad\qquad\qquad\qquad\qquad\qquad \mid \text{cons}(z, zs) \mapsto \text{let } q = \mathtt{divide}(\text{cons}(y, ys)) \text{ in} \\
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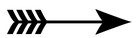

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$$T(n) = 7 + (c + d)n + 2T(n/2).$$

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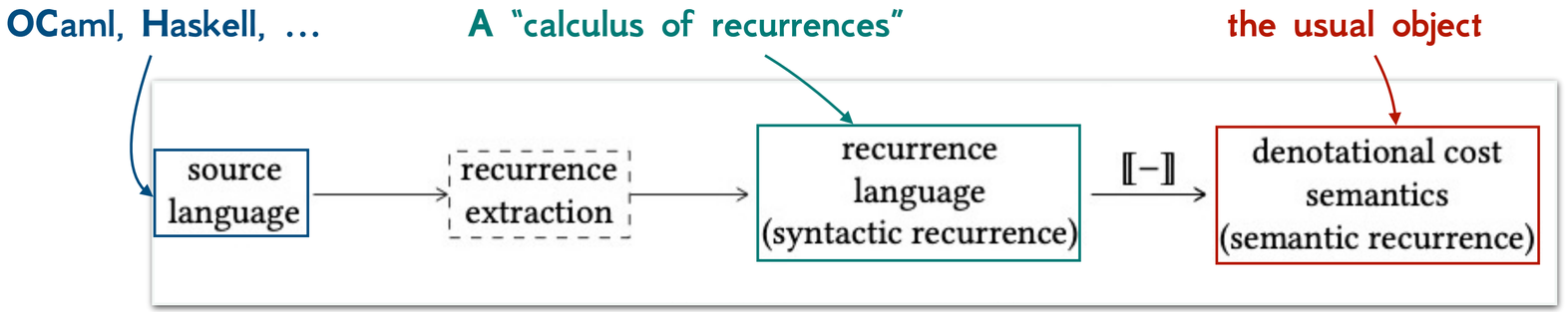

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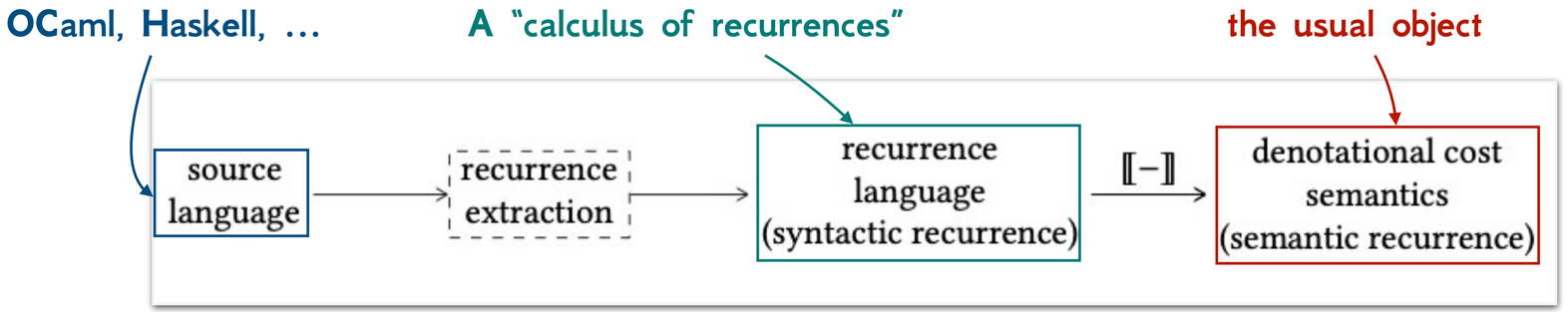

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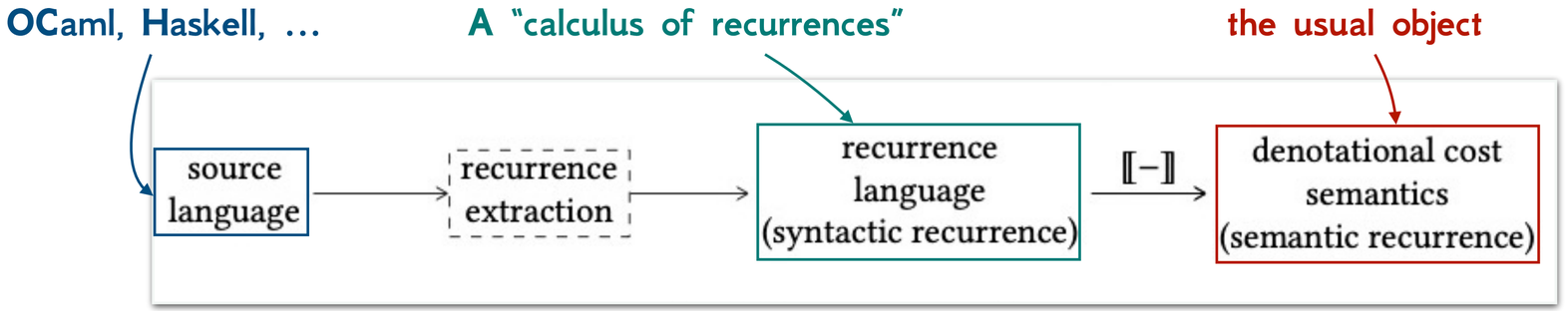

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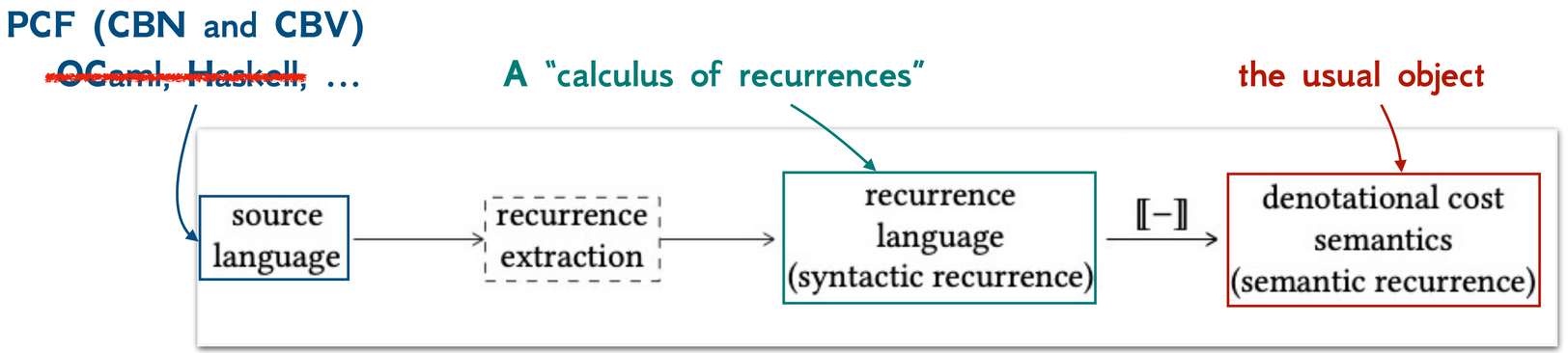

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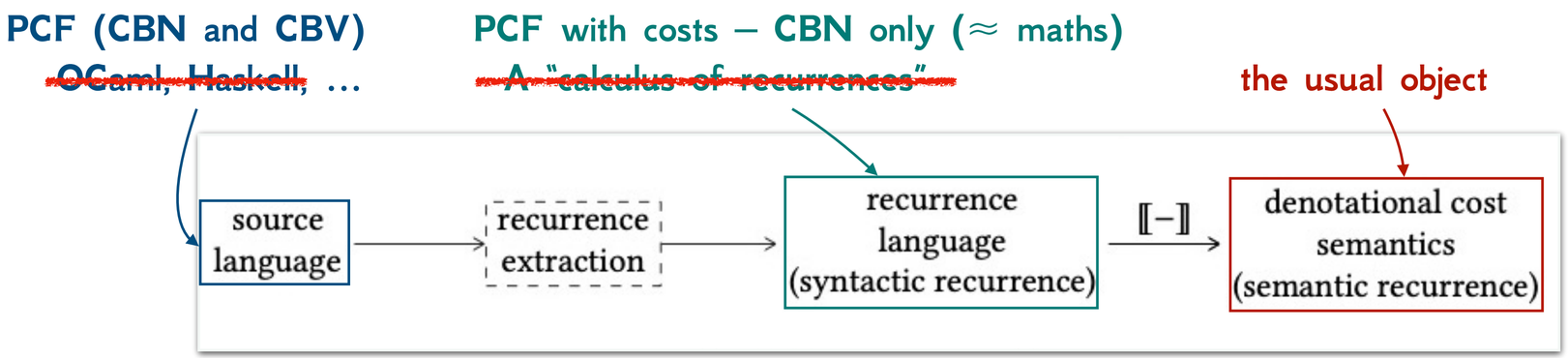

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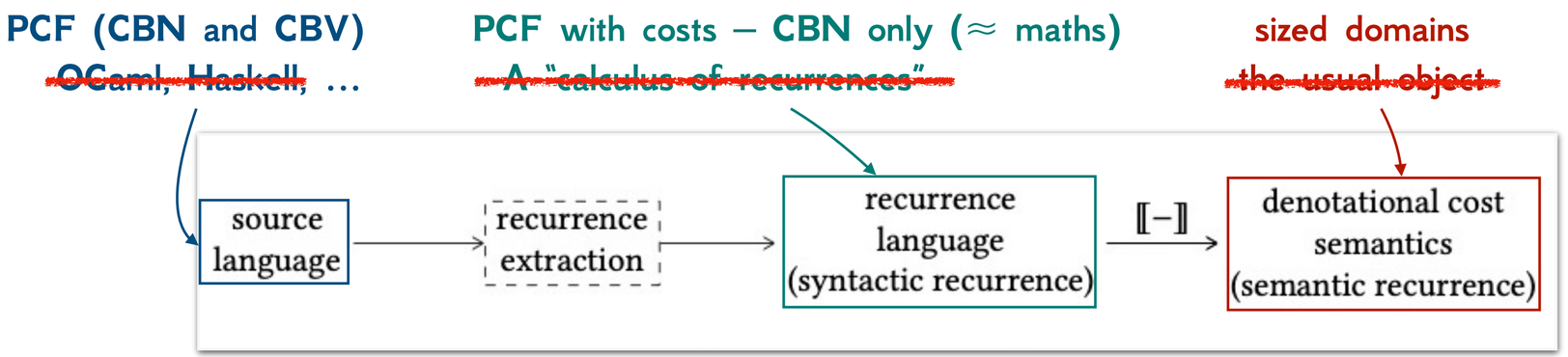

PCF

= simply-typed λ-calculus + fixpoints (CBN & CBV)

Constants

$$\frac{n \in \mathbb{N}}{\Gamma \vdash \underline{n} : \mathsf{nat}} \qquad \frac{\Gamma \vdash M, N : \mathsf{nat} \quad \mathbf{op} \in \{+, *, -, \div, \mathrm{mod}\}}{\Gamma \vdash M \ \mathbf{op}\ N : \mathsf{nat}}$$

Function types

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.\ M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash M\, N : B}$$

CBN recursion

$$\frac{\Gamma, x : A \vdash M : A}{\Gamma \vdash \mathsf{fix}\ x.\ M : A}$$

Big-step rules

$$\frac{M \downarrow^{a} \underline{m} \quad N \downarrow^{b} \underline{n}}{M \ \mathbf{op}\ N \downarrow^{a+b} \underline{m \ \mathbf{op}\ n}}$$

$$\frac{M \downarrow^{m} \lambda x.\ P \quad P[N/x] \downarrow^{n} V}{M N \downarrow^{m+n+1} V}$$

$$\frac{M[\mathsf{fix}\ x.\ M/x] \downarrow^{n} V}{\mathsf{fix}\ x.\ M \downarrow^{n+1} V}$$

PCFc

= simply-typed λ-calculus + fixpoints + costs (CBN only)

Constants

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CBN recursion

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Costs

$$\frac{\widehat{n} \in \{\mathbf{0}, \mathbf{1}\}}{\Gamma \vdash \widehat{n} : \mathbb{C}} \qquad \frac{\Gamma \vdash M : \mathbb{C} \quad \Gamma \vdash N : \mathbb{C}}{\Gamma \vdash M \boxplus N : \mathbb{C}}$$

The Size Preorder

$$\Gamma \vdash M \leqslant N : A$$

"considered as recurrences, M is less than or equal to N at type A"

Extraction for CBV

In CBV:
- **Variables** carry **values**, which are fully evaluated: **only use-cost, no direct cost**
- **All types are observable**, e.g. evaluating a term at A -> B **has direct cost**

1. Translate each PCF type to **two PCFc types:**

Complexity of A
cost of evaluation + potential

Potential of A
size of output value, future use-cost

$$||A|| \stackrel{\text{def}}{=} \mathbb{C} \times \langle\!\langle A \rangle\!\rangle$$

$$\langle\!\langle \mathsf{nat} \rangle\!\rangle \stackrel{\text{def}}{=} \mathsf{nat}$$

$$\langle\!\langle A_1 \times A_2 \rangle\!\rangle \stackrel{\text{def}}{=} \langle\!\langle A_1 \rangle\!\rangle \times \langle\!\langle A_2 \rangle\!\rangle$$

$$\langle\!\langle A \to B \rangle\!\rangle \stackrel{\text{def}}{=} \langle\!\langle A \rangle\!\rangle \to ||B||$$

2. Extract a complexity from each term by induction on syntax.

Given **potentials** for the variables

this **recurrence**

gives the **complexity** of M

Theorem: $\Gamma \vdash M : A \Longrightarrow \langle\!\langle \Gamma \rangle\!\rangle \vdash ||M|| : ||A||$

Extraction for CBN

In CBN:

- **Variables** carry **thunks**, to be evaluated later.
- **Only base types are observable**, e.g. evaluating a term at A -> B has no cost but evaluating a nat might **trigger a cascade of thunk evaluations**.

1. Translate each PCF type to a **PCFc cost algebra**, i.e. a **type &** a **cost action** on

$$||\mathsf{nat}|| \stackrel{\text{def}}{=} \mathbb{C} \times \mathsf{nat}$$

$$||A_1 \times A_2|| \stackrel{\text{def}}{=} ||A_1|| \times ||A_2||$$

$$||A \to B|| \stackrel{\text{def}}{=} ||A|| \to ||B||$$

$$\left(||A||, \alpha_A(c, x)\right)$$

$$c : \mathbb{C}, x : ||A|| \vdash \alpha_A(c, x) : ||A||$$

Algebras are defined by induction on types; they **push costs down to observable types.**

2. Extract a complexity from each term by induction on syntax.

Given **complexities** for the inputs — this **recurrence** — gives the **complexity** of M

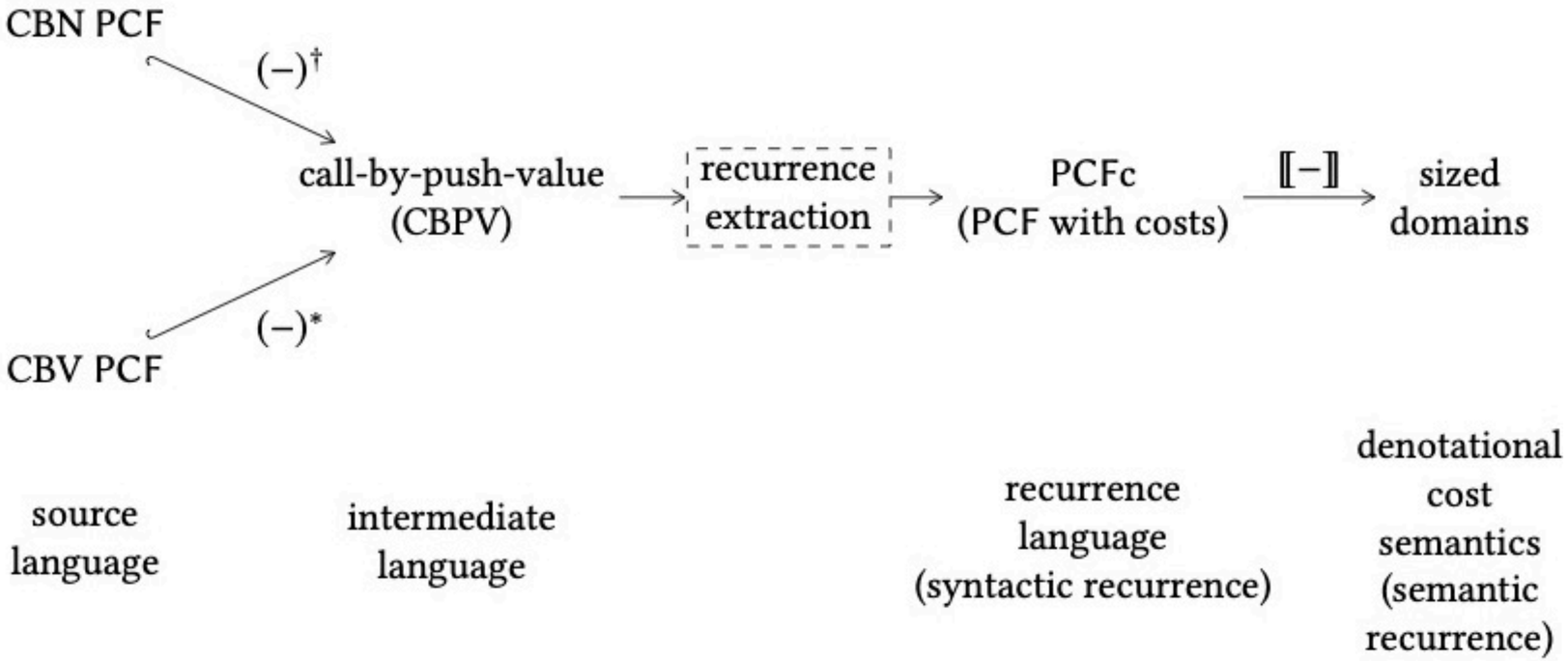
Theorem: $\Gamma \vdash M : A \Longrightarrow ||\Gamma|| \vdash ||M|| : ||A||$

Proving this correct is nontrivial.

- Some difficult issues:
  - Handling **nontermination** is nontrivial.
  - Translating **CBV to CBN.** Continuations?
  - Function types -> need **logical relation** to state correctness. The straightforward definition fails due to fixpoints.
- How did we even come up with the extraction in the first place?

Two for the price of one: Call-by-Push-Value

- CBPV features **modalities** that control the evaluation of terms.
- We can embed both CBV and CBN in it.



Call-by-Push-Value (CBPV)

"A value is, a computation does." — Paul Blain Levy

- Two kinds of types:
  - Terms of **value types** are**...** values.
  - Terms of **computation type** can be **effectful**. Equip with one effect: **charging a unit cost**.
- Extraction: mix'n'match of CBV & CBN styles
  - **Value types -> potentials**
  - **Computation types -> PCFc cost algebras**

thunks

$$A ::= \mathsf{nat} \mid A_1 \times A_2 \mid U\underline{B}$$

$$\underline{B} ::= FA \mid \underline{B}_1 \,\&\, \underline{B}_2 \mid A \to \underline{B}$$

returners

$$\frac{\Gamma \vdash_c M : \underline{B}}{\Gamma \vdash_c \mathsf{charge}.\, M : \underline{B}}$$

$$\frac{M \Downarrow^n T}{\mathsf{charge}.\, M \Downarrow^{n+1} T}$$

Proving correctness

$$\widetilde{n} \lesssim^{\mathsf{val}}_{\mathsf{nat}} E \stackrel{\mathsf{def}}{\equiv} \underline{n} \leqslant E$$

$$(V_1, V_2) \lesssim^{\mathsf{val}}_{A_1 \times A_2} E \stackrel{\mathsf{def}}{\equiv} \begin{cases} V_1 \lesssim^{\mathsf{val}}_{A_1} \pi_1(E) \\ V_2 \lesssim^{\mathsf{val}}_{A_2} \pi_2(E) \end{cases}$$

$$\mathsf{thunk}\ M \lesssim^{\mathsf{val}}_{U\underline{B}} E \stackrel{\mathsf{def}}{\equiv} M \lesssim^{\mathsf{c}}_{\underline{B}} E$$

$$M \lesssim^{\mathsf{c}}_{FA} E \stackrel{\mathsf{def}}{\equiv} E_c{\downarrow} \implies \exists n, V. \begin{cases} M \Downarrow^n \mathsf{return}\ V \\ \widehat{n} \leqslant E_c \\ V \lesssim^{\mathsf{val}}_{A} E_p \end{cases}$$

$$M \lesssim^{\mathsf{c}}_{A \to \underline{B}} E \stackrel{\mathsf{def}}{\equiv} \forall (N \lesssim^{\mathsf{val}}_{A} X).\ M\,N \lesssim^{\mathsf{c}}_{\underline{B}} E\,X$$

$$M \lesssim^{\mathsf{c}}_{\underline{B}_1 \& \underline{B}_2} E \stackrel{\mathsf{def}}{\equiv} \begin{cases} \pi_1(M) \lesssim^{\mathsf{c}}_{\underline{B}_1} \pi_1(E) \\ \pi_2(M) \lesssim^{\mathsf{c}}_{\underline{B}_2} \pi_2(E) \end{cases}$$

Theorem (Bounding Theorem):

$$\cdot \vdash V : A \implies V \lesssim^{\mathsf{val}}_{A} \langle\!\langle V \rangle\!\rangle$$

$$\cdot \vdash M : \underline{B} \implies M \lesssim^{\mathsf{c}}_{\underline{B}} \|M\|$$

Proving correctness

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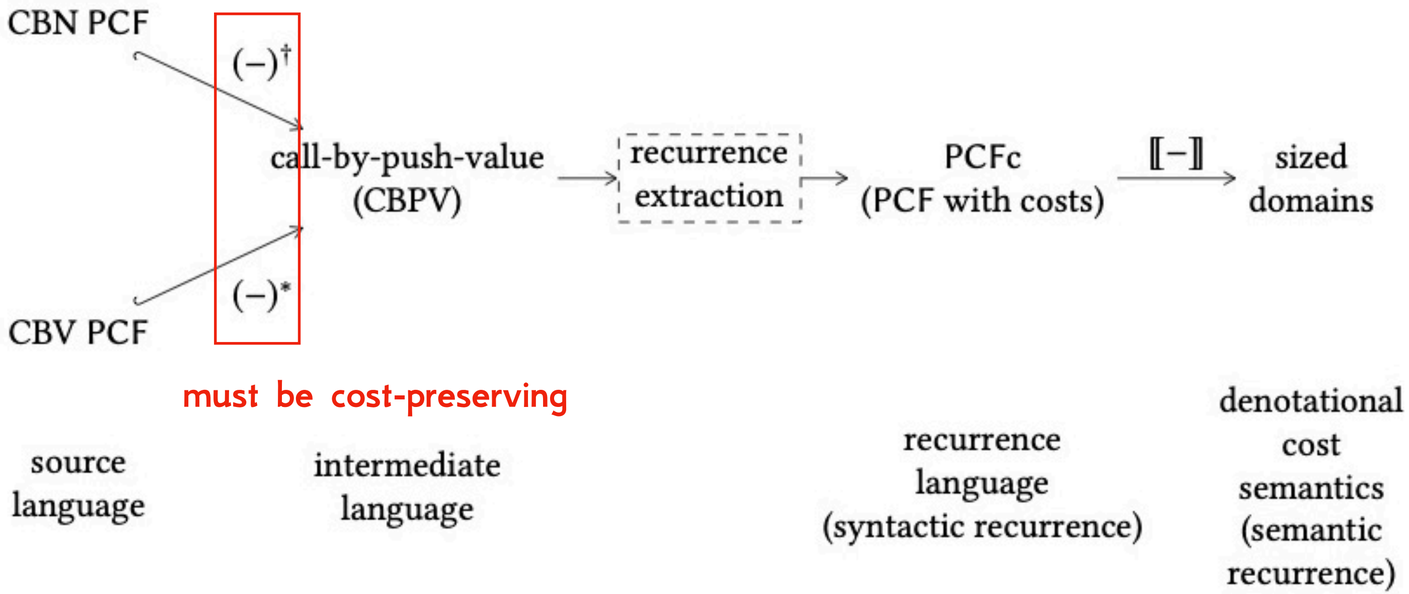
Proving the original theorems for CBN and CBV

- Use the **Levy embedding** of CBN and CBV into CBPV.
- Sprinkle occurrences of "**charge. (-)**" wherever cost should be incurred.
- Prove that the embedding is cost-preserving (a bisimulation-like result).



Denotational semantics of PCFc

- A **sized domain** consists of a set D and
  - An **information order**, i.e. a pointed ω-cpo $(D, \sqsubseteq, \bot)$
  - A **size preorder** $(D, \leqslant, \mathbf{0}, \vee)$

such that

least element

chosen least upper bounds

> **Why preorder?** E.g. the lists [1, 2, 3] and [4, 5, 6] are "size-equal" but not identical.

  - The chosen lubs are continuous w.r.t. the information order.
  - **A better-defined bound is a smaller bound**: $x \sqsubseteq y \Longrightarrow y \leqslant x$
  - **A recursively defined upper bound is an upper bound**:
    for a chain $(x_i)_{i \in \omega}$ we have $(\forall i.\ z \leqslant x_i) \Longrightarrow z \leqslant \sqcup_{i \in \omega} x_i$

Conclusions

- Recurrence extraction for **both CBN and CBV with recursion**: obtained in a uniform way, and **shown to be correct**.
- The extracted recurrences are the expected ones.
- CBPV illuminates the basic concepts of **cost**, **potential** and **complexity:**
  - **cost** is an effect
  - **values** have **potential** (use-cost)
  - **computations** have **complexity** (direct, or deferred cost)
- This is the beginning of a theory of **higher-order recurrences**.
- The analysis can easily be extended to **CBV inductive types**.
- **Recursive types** are harder—but we'll get there!