Type Theory and Homotopy
III. Equivalences, Univalence, and Quotients

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Previously

Intensional identity types seem to support the following view:

- Types are **spaces** (up to homotopy).
- Terms are **points**.
- Elements of the identity type are **paths**.
- Everything given in a **synthetic** manner, not analytic.

This discovery is independently due to

- Awodey and Warren [Math. Proc. Camb. Philos. Soc. 2009]
- Vladimir Voevodsky (1966–2017) [Stanford lecture 2006]

Interpretation of TT into **simplicial sets**: Kapulkin and Lumsdaine, with thanks to Voevodsky [J. Eur. Math. Soc. 2018].

I. Homotopical structure of types

Homotopy Levels

Types are spaces; they have **higher-dimensional structure**.

Yet, some types do not. Let A be a type.

$$\begin{array}{rr} \textit{contractible} & \mathsf{isContr}(A) \stackrel{\text{def}}{=} (c : A) \times ((x : A) \to \mathsf{Id}_A(c, x)) \\ \text{proposition} & \mathsf{isProp}(A) \stackrel{\text{def}}{=} (x, y : A) \to \mathsf{Id}_A(x, y) \\ \text{set} & \mathsf{isSet}(A) \stackrel{\text{def}}{=} (x, y : A) \to (p, q : \mathsf{Id}_A(x, y)) \to \mathsf{Id}(p, q) \\ & \vdots \end{array}$$

In general, we define

$$\text{is-(-2)-type}(A) \stackrel{\text{def}}{=} \mathsf{isContr}(A)$$

$$\text{is-(n+1)-type}(A) \stackrel{\text{def}}{=} (x, y : A) \to \mathsf{isContr}(\mathsf{Id}_A(x, y))$$

Then

$$\text{is-(-1)-type}(A) \simeq \mathsf{isProp}(A) \qquad \text{is-0-type}(A) \simeq \mathsf{isSet}(A)$$

Propositions and Sets

Here is an unusual result:

Theorem (Hedberg, J. Func. Prog 1998)

Let A be a type. If ***identity is decidable****, i.e. if we have*

$$d : (x, y : A) \to \mathrm{Id}_A(x, y) + \neg\mathrm{Id}_A(x, y)$$

then A ***is a set****, i.e. we have a proof of isSet(A).*

Corollary

Nat *is a set.*

In some sense, all the maths we have done so far is **0-dimensional**!

II. Universes

Identity types are not good enough

Theorem (Jan Smith, J. Symb. Log. 1988)

The type correspondong to Peano's fourth axiom, i.e.

$$n : \mathsf{Nat} \vdash \mathsf{Id}_{\mathsf{Nat}}(0, \mathsf{succ}(n)) \to \mathbf{0} \ \mathsf{type}$$

*is **not** inhabited in MLTT with $\to$, $\times$, and identity types.*

Proof: construct a model of MLTT where types are subsingleton sets.

To prove Peano 4, we intuitively want to

1. construct a type family $n : \mathsf{Nat} \vdash B(n)$ type where

$$\begin{array}{rl} \vdash B(\mathsf{zero}) \ \mathsf{type} & \text{is inhabited} \\ n : \mathsf{Nat} \vdash B(\mathsf{succ}(n)) \ \mathsf{type} & \text{is empty, i.e. } \equiv \mathbf{0} \end{array}$$

2. assuming $n : \mathsf{Nat} \vdash P : \mathsf{Id}_{\mathsf{Nat}}(0, \mathsf{succ}(n))$ and $\vdash M : B(\mathsf{zero})$, obtain $n : \mathsf{Nat} \vdash \mathsf{transp}(P)(M) : B(\mathsf{succ}(n)) \equiv \mathbf{0}$

We cannot perform Step 1 because types are not terms.

Universes à la Russell

We introduce the **universe**, a **type of all (small) types**.

$$\frac{}{\Gamma \vdash \mathsf{U}\ \mathsf{type}} \qquad \frac{\Gamma \vdash A : \mathsf{U}}{\Gamma \vdash A\ \mathsf{type}}$$

plus one rule for each type constructor, e.g.

$$\frac{\Gamma \vdash A : \mathsf{U} \qquad \Gamma, x : A \vdash B : \mathsf{U}}{\Gamma \vdash (x : A) \to B : \mathsf{U}}$$

Caution. We must **avoid** the following to avoid paradoxes:

$$\frac{}{\Gamma \vdash \mathsf{U} : \mathsf{U}}$$

Types may then be constructed as terms of U (e.g. by induction).

If $A : \mathsf{U}$ then we say that A is a **small** type.

Homotopy equivalence

Definition

Two topological spaces X and Y are **homotopy-equivalent** if there are continuous functions $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f \sim 1_X \qquad\qquad f \circ g \sim 1_Y$$

where 1_X and 1_Y are the identity functions on X and Y.

We can model this synthetically in MLTT.

Type-theoretic Equivalences

Definition (Voevodsky)

We say that $f : A \to B$ is an **equivalence** just if

$$\mathsf{isEquiv}(f) \stackrel{\text{def}}{=} (y : B) \to \mathsf{isContr}((x : A) \times \mathsf{Id}_B(f(x), y))$$

This is a homotopically well-behaved notion of **isomorphism**.

For $A, B : \mathsf{U}$ define the type of **(type-theoretic) equivalences**

$$A \simeq B \stackrel{\text{def}}{=} (f : A \to B) \times \mathsf{isEquiv}(f)$$

We can use equivalences to **decompose** identity types.

E.g. for any $A, B : \mathsf{U}$ and $p, q : A \times B$:

$$\mathsf{Id}_{A \times B}(p, q) \simeq \mathsf{Id}_A(\mathsf{pr}_1(p), \mathsf{pr}_1(q)) \times \mathsf{Id}_B(\mathsf{pr}_2(p), \mathsf{pr}_2(q))$$

This can be done for most type formers of MLTT.

Univalence

Question:

What is an identity between types?

Voevodsky proposed adding the **univalence axiom** to MLTT:

$$\mathsf{ua} : (A, B : \mathsf{U}) \to (A \simeq B) \simeq \mathsf{Id}_{\mathsf{U}}(A, B)$$

This spoils the computational character of MLTT, but is a revolution:

isomorphic/equivalent types are identical

This principle is often used informally in maths ('abuse of notation'). E.g. the Cauchy reals and the Dedekind reals are "the same."

Its soundness is validated by the simplicial model of type theory. (The identity type elimination rule remains valid!)

III. Quotients

Quotients

It is in general difficult to form **quotients** in MLTT.

Quotient types:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma, x : A, y : A \vdash R \text{ type}}{\Gamma \vdash A/R \text{ type}} \qquad \frac{\Gamma \vdash M : A \qquad \Gamma \vdash A/R \text{ type}_\ell}{\Gamma \vdash [M] : A/R}$$

$$\frac{\Gamma \vdash M, N : A \qquad \Gamma, x : A, y : A \vdash R \text{ type} \qquad \Gamma \vdash P : R[M, N/x, y]}{\Gamma \vdash \mathsf{Qax}(P) : \mathsf{Id}_{A/R}([M], [N])}$$

and so on... but such types are not necessarily **effective**:

$$\frac{\Gamma, x : A, y : A \vdash R(x, y) \text{ type} \qquad \Gamma \vdash P : \mathsf{Id}_{A/R}([M], [N])}{\Gamma \vdash ???(M, N, P) : R(M, N)}$$

Worse:

Theorem (Maietti 1999)

If quotient types are effective and UIP holds then $A + \neg A$ for small A.

Higher Inductive Types (HITs)

Idea:

When building a type, also specify some paths.

For example, to build a type Int of integers we may postulate:

- for each M : Nat a **positive integer** $\mathsf{pos}(M) : \mathsf{Int}$
- for each M : Nat a **negative integer** $\mathsf{neg}(M) : \mathsf{Int}$
- an identity $\mathsf{pnZero} : \mathsf{Id}_{\mathsf{Int}}(\mathsf{pos}(\mathsf{zero}), \mathsf{int}(\mathsf{zero}))$

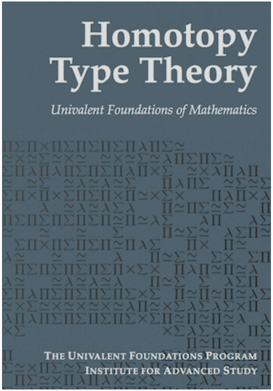
This can be used to specify homotopical spaces synthetically.
E.g. a circle can be specified by postulating

- a base point $\mathsf{base} : \mathbb{S}^1$
- a path $\mathsf{loop} : \mathsf{Id}_{\mathbb{S}^1}(\mathsf{base}, \mathsf{base})$

This leads to **synthetic homotopy theory**. E.g. there is a machine-checked proof that $\pi_4(\mathbb{S}^3) \simeq \mathbb{Z}_2$.

Homotopy Type Theory (HoTT)

The results of Awodey/Warren/Voevodsky led to a flurry of results.
This culminated in a Special Year at the IAS in Princeton:



$$\text{HoTT} \stackrel{\text{def}}{=} \text{MLTT} + \text{univalence axiom} + \text{some HITs}$$

IV. Further directions

50 Years of MLTT

Achievements:

- A number of well-behaved type theories…
- …with well-understood semantics.
- One industrial-strength proof assistant: **Coq**.
  Many machine-checked proofs! Greatest hits:
  - Four color theorem [Gonthier 2008]
  - Feit-Thompson odd order theorem [Gonthier et al. 2013]
  - CompCert, a verified C compiler [Leroy et al. 2005–2018]
  - Iris, for verifying concurrent programs [Jung et al. 2018]
- Many 'experimental' proof assistants: Agda, Lean, Arend, …
  Projects to keep an eye on:
  - Kevin Buzzard's Xena Project (in Lean) at Imperial
  - the CMU Hoskinson Center for Formal Mathematics
  - Tim Gowers' project on automated theorem proving (not TT)
- A deep connection between homotopy theory and MLTT.

Where to go from here

Read the HoTT book!

Many directions of work. To name a few:

- **Synthetic homotopy theory**. Better, possibly computational, calculations of homotopy groups of spheres and other spaces.
- **New formalizations of mathematics**. Constructive, machine-checked proofs of known and new results from mathematics.
  Is there some secret higher-dimensional content?
- **Improved or new type theories**. Either adding more power, or improving the computational behaviour of HoTT.
  - cubical type theories
  - modal type theories
  - metatheory, in particular objective metatheory

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