Type Theory and Homotopy
II. Identity

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Equality

Recall that we could define

$$\vdash \mathsf{add} = \lambda x.\, \lambda y.\, \mathsf{rec}(x; y; n, c.\, \mathsf{succ}(n)) : \mathsf{Nat} \to \mathsf{Nat} \to \mathsf{Nat}$$

and compute that

$$y : \mathsf{Nat} \vdash \mathsf{add}(\mathsf{zero})(y) \equiv y : \mathsf{Nat}$$

It is **not** the case that

$$x : \mathsf{Nat} \vdash \mathsf{add}(x)(\mathsf{zero}) \equiv x : \mathsf{Nat}$$

$\equiv$ only allows unfolding of definitions, **not** non-trivial theorems.

For that we need to introduce the **identity type**.

I. Identity Types

Intensional Identity Types

form. $$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash N : A}{\Gamma \vdash \mathrm{Id}_A(M, N) \text{ type}}$$

intro. $$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathrm{refl}(M) : \mathrm{Id}_A(M, M)}$$

elim. $$\frac{\begin{array}{c}\Gamma, x : A, y : A, p : \mathrm{Id}_A(x, y) \vdash B \text{ type} \\ \Gamma \vdash P : \mathrm{Id}_A(M, N) \qquad \Gamma, z : A \vdash Q : B[z, z, \mathrm{refl}(z)/x, y, p]\end{array}}{\Gamma \vdash \mathrm{J}_{[x,y,p.B]}(P; z.\, Q) : B[M, N, P/x, y, p]}$$

comp. $$\frac{\begin{array}{c}\Gamma, x : A, y : A, p : \mathrm{Id}_A(x, y) \vdash B \text{ type} \\ \Gamma, z : A \vdash Q : B[z, z, \mathrm{refl}(z)/x, y, p]\end{array}}{\Gamma \vdash \mathrm{J}(\mathrm{refl}(M); z.\, Q) \equiv Q[M/z] : B[M, M, \mathrm{refl}(M)/x, y, p]}$$

Because of the **type conversion** and **congruence** rules we always have

$$\frac{\Gamma \vdash M \equiv N : A}{\Gamma \vdash \mathrm{refl}(M) : \mathrm{Id}_A(M, N)}$$

Some examples (I)

- Let $\vdash A$ type and $x : A \vdash P(x)$ type. We have:

  $$x, y : A, p : \mathsf{Id}_A(x, y) \vdash \mathsf{transp}(p) \equiv \mathsf{J}(p; z.\, \lambda w.\, w) : B(x) \to B(y)$$

  Informally:

  Let $x, y : A$ and $p : \mathsf{Id}_A(x, y)$. We want to construct a term of type $B(x) \to B(y)$. By elimination we may assume that $x \equiv y$, so it suffices to give a term $B(x) \to B(x)$. Take the identity function.

- Let $x : A \vdash f(x) : B$. Then $x, y : A \vdash \mathsf{Id}_B(f(x), f(y))$ type. We have

  $$x, y : A, p : \mathsf{Id}_A(x, y) \vdash \mathsf{ap}_f(p) \equiv \mathsf{J}(p; x.\, \mathsf{refl}(f(x))) : \mathsf{Id}_B(f(x), f(y))$$

  Informally:

  Let $x, y : A$ and $p : \mathsf{Id}_A(x, y)$. We want to show $\mathsf{Id}_B(f(x), f(y))$. By elimination we may assume that $x \equiv y$, so it suffices to construct a term of type $\mathsf{Id}_B(f(x), f(x))$. Take $\mathsf{refl}(f(x))$.

Some examples (II)

Here is an informal proof that there is a term of type

$$x : \mathsf{Nat} \vdash \mathsf{Id}_{\mathsf{Nat}}(\mathsf{add}(x)(\mathsf{zero}), x) \ \mathsf{type}$$

We proceed by induction on $x : \mathsf{Nat}$.

- If $x \equiv \mathsf{zero} : \mathsf{Nat}$, then $\mathsf{add}(x)(\mathsf{zero}) \equiv \mathsf{add}(\mathsf{zero})(\mathsf{zero}) \equiv \mathsf{zero}$. Hence it suffices to construct $\mathsf{refl}(\mathsf{zero}) : \mathsf{Id}_{\mathsf{Nat}}(\mathsf{zero}, \mathsf{zero})$.
- If $x \equiv \mathsf{succ}(y) : \mathsf{Nat}$ for some $y : \mathsf{Nat}$, then

  $$\mathsf{add}(x)(\mathsf{zero}) \equiv \mathsf{add}(\mathsf{succ}(y))(\mathsf{zero}) \equiv \mathsf{succ}(\mathsf{add}(y)(\mathsf{zero}))$$

  By the IH we have $p : \mathsf{Id}_{\mathsf{Nat}}(\mathsf{add}(y)(\mathsf{zero}), y)$. Hence

  $$\mathsf{ap}_{\mathsf{succ}(-)}(p) : \mathsf{Id}_{\mathsf{Nat}}(\underbrace{\mathsf{succ}(\mathsf{add}(y)(\mathsf{zero}))}_{\equiv \mathsf{add}(x)(\mathsf{zero})}, \underbrace{\mathsf{succ}(y)}_{\equiv x})$$

  So we have shown the inductive step.

Metatheory

Theorem

The following rule is admissible.

$$\frac{\vdash P : \mathrm{Id}_A(M, N)}{\vdash M \equiv N : A}$$

Any two propositionally equal terms in an **empty context** are also definitionally equal. (Hence the name 'intensional.')

This did not apply to our previous proof because $x : \mathrm{Nat}$ was free.

Theorem

There is a set-theoretic model of MLTT with Π, Σ, *Id,* Nat, *and* $+$ *types.*

Extensional Identity Types

One might argue that $x : \mathsf{Nat} \vdash \ldots : \mathsf{Id}_{\mathsf{Nat}}(\mathsf{add}(x)(\mathsf{zero}), x)$ should be promoted to a definitional equality

$$x : \mathsf{Nat} \vdash \mathsf{add}(x)(\mathsf{zero}) \equiv x : \mathsf{Nat}$$

Add **equality reflection** rule:

$$\frac{\Gamma \vdash P : \mathsf{Id}_A(M, N)}{\Gamma \vdash M \equiv N : A}$$

We then say we have **extensional identity types**. But then

- normalization is no longer decidable, and hence
- **type checking is no longer decidable**

So we are stuck with the 'bureaucracy' of intensional identity types.

But this is a fine type theory for computing by hand.

II. Homotopy

Identity types are very mysterious

Let $\vdash M, N : A$. Construct $\vdash \mathsf{Id}_A(M, N)$ type.

Now suppose $\vdash P, Q : \mathsf{Id}_A(M, N)$.

What is the meaning of the following type?

$$\vdash \mathsf{Id}_{\mathsf{Id}_A(M,N)}(P, Q) \text{ type}$$

Should the following **Uniqueness of Identity Proofs (UIP)** principle be inhabited for any type $\Gamma \vdash A$ type?

$$\vdash (x, y : A) \to (p, q : \mathsf{Id}_A(x, y)) \to \mathsf{Id}_{\mathsf{Id}_A(x,y)}(p, q) \text{ type} \qquad \text{(UIP)}$$

It's certainly true in the set-theoretic model!

Theorem (Hofmann-Streicher, 1998)

There is a model of MLTT in which the above principle of ***uniqueness of identity proofs*** *(UIP) is* ***not*** *true.*

Groupoids

Definition

A **groupoid** $\mathcal{G}$ consists of

- a set of **objects** $\mathrm{ob}(\mathcal{G})$
- for $x, y \in \mathrm{ob}(\mathcal{G})$ a set of **isomorphisms** $\mathrm{Hom}(x, y)$
  We write $f : x \xrightarrow{\cong} y$ if $f \in \mathrm{Hom}(x, y)$.
- for each $x \in \mathrm{ob}(\mathcal{G})$ an **identity** $1_x \in \mathrm{Hom}(x, x)$
- for isos $f : x \xrightarrow{\cong} y$ and $g : y \xrightarrow{\cong} z$ a **composite**

$$g \circ f : x \xrightarrow{\cong} z$$

- for each iso $f : x \xrightarrow{\cong} y$ and **inverse iso** $f^{-1} : y \xrightarrow{\cong} x$, such that

$$f^{-1} \circ f = 1_x : x \xrightarrow{\cong} x \qquad f \circ f^{-1} = 1_y : y \xrightarrow{\cong} y$$

A **one-object** groupoid is …

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A **one-object** groupoid is ... a group!

If $|\mathrm{Hom}(x, y)| \leq 1$ a groupoid is ...

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A **one-object** groupoid is ... a group!

If $|\mathrm{Hom}(x, y)| \leq 1$ a groupoid is ... an equivalence relation!

The Hofmann-Streicher groupoid model of type theory

Hofmann and Streicher interpreted MLTT as follows:

- $\vdash A$ type is interpreted by a groupoid $[\![A]\!]$.
- A type family/dependent type $x : A \vdash B$ type is interpreted by a **fibration** $[\![B]\!] : [\![A]\!] \to \mathbf{GPD}$ of groupoids.
- A term of type $x : A \vdash B$ type is a **section** of the fibration $[\![B]\!]$.
- The identity type $\vdash \mathrm{Id}_A(M, N)$ type is interpreted by the set of isomorphisms of the groupoid $[\![A]\!]$, i.e.

$$\mathrm{Hom}_{[\![A]\!]}([\![M]\!], [\![N]\!])$$

In this model there are types with **non-trivial identity types**.

But where do groupoids come from?

Paths

Let X be a (topological) space.

Definition

A **path** in space X is a continuous function $p : [0,1] \to X$.

Write $p : x \rightsquigarrow y$ if $p(0) = x$ and $p(1) = y$.

Given $p : x \rightsquigarrow y$ let $p^{-1} : y \rightsquigarrow x$ by $p^{-1}(t) \stackrel{\text{def}}{=} p(1-t)$.

Given $p : x \rightsquigarrow y$ and $q : y \rightsquigarrow z$ let

$$(p \centerdot q)(t) \stackrel{\text{def}}{=} \begin{cases} p(2t) & \text{if } 0 \leq t \leq 1/2 \\ q(2t-1) & \text{if } 1/2 \leq t \leq 1 \end{cases}$$

Question: given

$$p : x \rightsquigarrow y \qquad q : y \rightsquigarrow z \qquad r : z \rightsquigarrow w$$

is the following true?

$$(p \centerdot q) \centerdot r \stackrel{?}{=} p \centerdot (q \centerdot r)$$

Homotopy

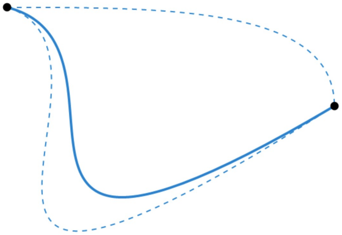
Let $f, g : X \to Y$ be continuous functions.

Definition

A **homotopy** H from f to g is a continuous function

$$H : X \times [0, 1] \to Y$$

such that $H(-, 0) = f$ and $H(-, 1) = g$.



Write $f \sim g$ if there is a homotopy from f to g.
$\sim$ is an equivalence relation.

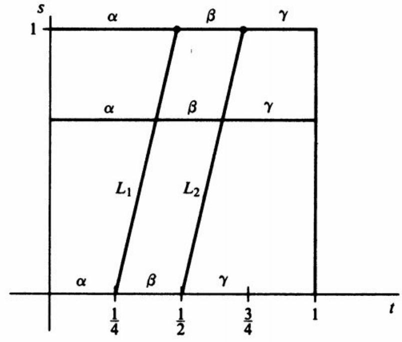
Associativity and Homotopy

Given

$$p : x \rightsquigarrow y \qquad q : y \rightsquigarrow z \qquad r : z \rightsquigarrow w$$

we have that

$$(p \mathbin{\blacksquare} q) \mathbin{\blacksquare} r \sim p \mathbin{\blacksquare} (q \mathbin{\blacksquare} r) : x \rightsquigarrow w$$



If $1_x : x \rightsquigarrow x$ and $1_y : y \rightsquigarrow y$ are constant paths then $p \mathbin{\blacksquare} 1_y \sim p \sim 1_x \mathbin{\blacksquare} p$.

The Fundamental Groupoid

Let X be a space. Its **fundamental groupoid** $\pi(X)$ consists of

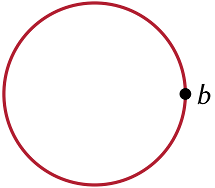
objects — the points of X

isomorphisms — equiv. classes $[p]$ of paths $p : x \rightsquigarrow y$ up to $\sim$

Taking only equivalence classes of **loops** $p : x \rightsquigarrow x$ at $x \in X$ gives the **fundamental group** $\pi(X, x)$ of X at x.

These are essential **algebraic invariants** of the space X.

$\mathbb{S}^1 \quad \stackrel{\text{def}}{=}$



Theorem

$\pi(\mathbb{S}^1, b) \cong \mathbb{Z}$

∞-Groupoids

The fundamental insight:

Why quotient at all?

Definition

A **groupoid** $\mathcal{G}$ consists of

- a set of **objects** $\mathrm{ob}(\mathcal{G})$
- for $x, y \in \mathrm{ob}(\mathcal{G})$ a set of **isomorphisms** $\mathrm{Hom}(x, y)$

⋮

Definition (sort of)

An ∞-**groupoid** $\mathcal{G}$ consists of

- a set of **0-cells** $\mathrm{ob}(\mathcal{G})$
- for $x, y \in \mathrm{ob}(\mathcal{G})$ an ∞-groupoid of **1-cells** $\mathrm{Hom}(x, y)$

⋮

The Fundamental ∞-Groupoid

Let X be a space. Its **fundamental ∞-groupoid** $\pi_\infty(X)$ consists of

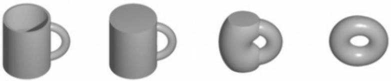
- 0-cells) the points of X
- 1-cells) paths $p : x \rightsquigarrow y$ between points
- 2-cells homotopies $H : p \sim q$ between paths

$\vdots$

Exact definition(s) tiresome to describe **analytically**.

Grothendieck's (1928–2014) dream, aka the **homotopy hypothesis**:

∞-groupoids = topological spaces up to homotopy



Identity Types and Homotopy

The intended pun:

types = spaces = ∞-groupoids elements of the identity type = paths in the space

For example, given $\vdash A$ type we can write down a term

$$\_ \bullet \_ : (x, y, z : A) \to \mathrm{Id}_A(x, y) \to \mathrm{Id}_A(y, z) \to \mathrm{Id}_A(x, z)$$

Informal proof: Suppose $x, y, z : A$, $p : \mathrm{Id}_A(x, y)$, and $q : \mathrm{Id}_A(y, z)$. By the elimination rule we may assume that $x \equiv y$ and $y \equiv z$, so it suffices to define a term of type $\mathrm{Id}_A(x, x)$. Take $\mathrm{refl}(x)$.

Remember that because of the **computation rule** we have

$$\mathrm{refl}(x) \bullet \mathrm{refl}(x) \equiv \mathrm{refl}(x)$$

Associativity of path composition

Given $x, y, z : A$ we can then define a term

$$\mathsf{assoc}_{xyz} : (p : \mathsf{Id}_A(x, y)) \to (q : \mathsf{Id}_A(y, z)) \to (r : \mathsf{Id}_A(z, w)) \to \mathsf{Id}_{\mathsf{Id}_A(x,w)}((p \cdot q) \cdot r, p \cdot (q \cdot r))$$

Informal proof. Given p, q, r as above we may assume that $x \equiv y \equiv z \equiv w$ and $p \equiv q \equiv r \equiv \mathsf{refl}(x)$. Thus, we only need a term of type

$$\mathsf{Id}_{\mathsf{Id}_A(x,x)}(\underbrace{(p \cdot q) \cdot r}_{\equiv\, \mathsf{refl}(x)}, \underbrace{p \cdot (q \cdot r)}_{\equiv\, \mathsf{refl}(x)})$$

and for that we may take $\mathsf{refl}(\mathsf{refl}(x))$.

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and for that we may take $\mathsf{refl}(\mathsf{refl}(x))$.

This can be taken to its logical conclusion—see HoTT book:

> The elimination rule of the identity type generates the structure of an ∞-groupoid.

In other words, MLTT is a **synthetic** theory of ∞-groupoids.

Summary

- **Intensional identity types** allow proofs of non-trivial, non-definitional equalities in MLTT.
- Iterated identity types generate the structure of an ∞-groupoid.
- That is why sometimes the elimination rule for the identity type is known as **path induction**.
- MLTT can be seen as a synthetic theory of ∞-groupoids.

Tomorrow: homotopy levels; equivalence; higher inductive types.

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