Syllepsis in Homotopy Type Theory

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LICS 2022, Haifa, Israel

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Eckmann-Hilton argument

Theorem (Eckmann-Hilton 1962)

If

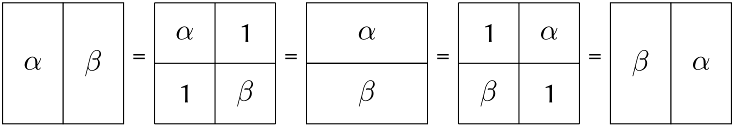
$$(X, \circ, 1) \qquad \textit{and} \qquad (X, \star, 1)$$

are monoid structures on X that satisfy the ***interchange law***

$$(a \circ b) \star (c \circ d) = (a \star c) \circ (b \star d)$$

then

- *both operations are commutative*
- $\circ = \star$



The periodic table of n-categories

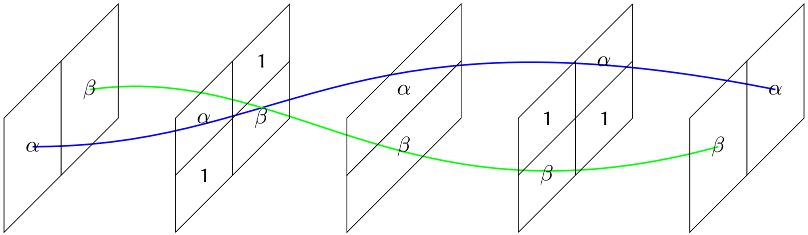
Simple facts:

- a one-object category is a monoid
- a one-object bicategory is a monoidal category
- …

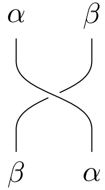
A $(n+k)$-category which has a unique i-cell for $i < k$ is …

k \ n	0	1	2	…
0	set	category	2-category	…
1	monoid	monoidal category	monoidal 2-category	…
2	commutative monoid	braided monoidal cat.	braided monoidal 2-cat.	…
3	“	symmetric monoidal cat.	sylleptic monoidal 2-cat.	…
4	“	“	symmetric monoidal 2-cat.	…
⋮	⋮	⋮	⋮	⋮

Eckmann-Hilton as a higher cell

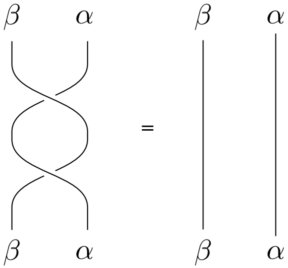


Looking from above the proof can be imagined as a braid:



Syllepsis

Syllepsis is the statement that



An equation between 3-dimensional objects, i.e. a 4-cell.

…but an extra dimension is necessary to untangle the braid!

Type-theoretic Syllepsis

Let A be a type.

For $x, y : A$ write $x = y$ for the **intensional identity type**.

Write $1_x : x = x$ for the reflexivity at $x : A$.
For $p : a = b$ and $q : b = c$ write $p \cdot q : a = c$ for their composition.

By HoTT, we know that these generate an ∞-groupoid!

Given $* : A$ and 2-loops $p, q : 1_* = 1_*$ we can construct

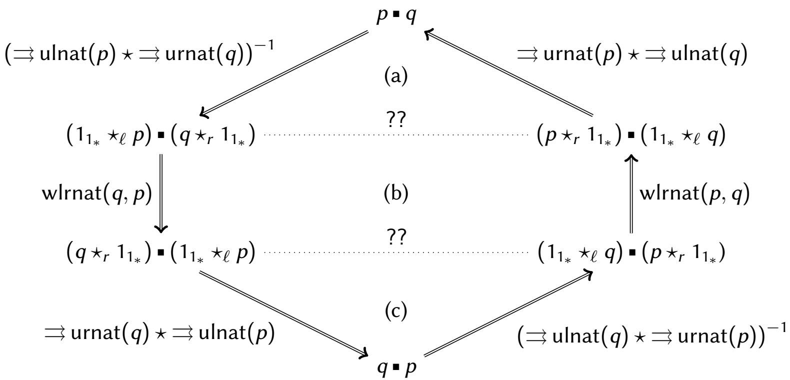
$$\mathsf{EH}(p, q) : p \cdot q = q \cdot p$$

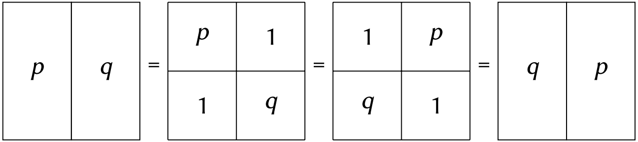
essentially by following one of the two classical proofs.

Go one dimension higher: given 3-loops $p, q : 1_{1_*} = 1_{1_*}$ construct

$$\mathsf{EH}(p, q) \cdot \mathsf{EH}(q, p) = 1_{p \cdot q}$$

Essence de proof





Concluding remarks

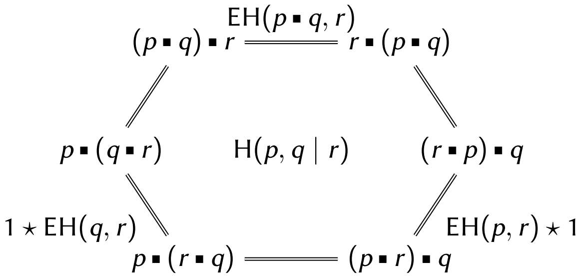
- We have constructed the type-theoretic syllepsis…
- …and some of the coherence required to call it as such.
- This essentially exhausts the known entries of the periodic table.
- Lesson: path induction actually scales fairly well.

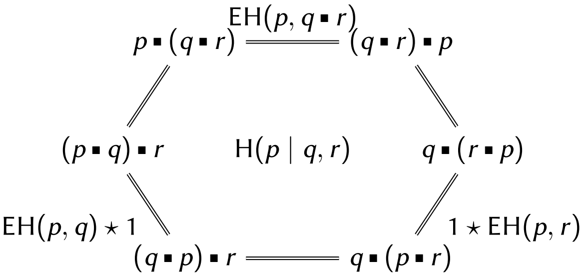
Future work: constructing the **Hopf element** of $\pi_3(S^2) \cong \mathbb{Z}$.

Thank you!

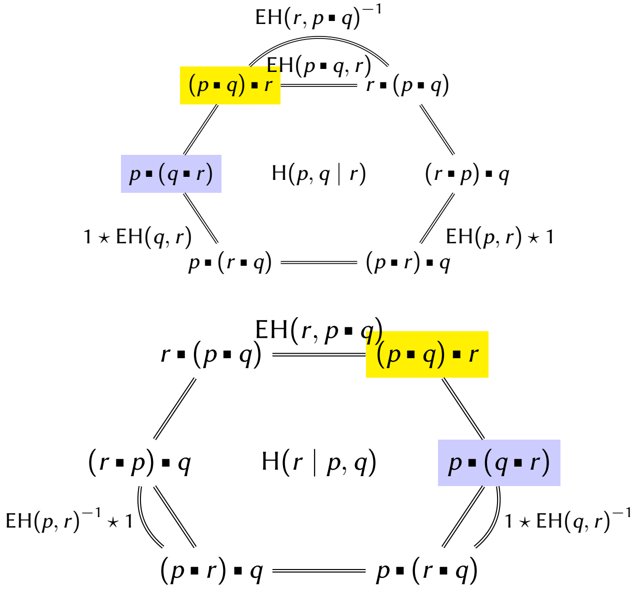
I. Bonus Slides

Hexagon equations





Coherence for the hexagons



Symmetric Monoidal 2-Category