On the Semantics of Intensionality

Alex Kavvos
Department of Computer Science, University of Oxford

FoSSaCS 2017
Uppsala, Sweden, 26 April 2017

EXTENSION

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What is intensionality?

Extensional equality is something we pick:

see e.g. constructive mathematics…

- for *sets:*

$$A = B \iff \forall x.\ x \in A \longleftrightarrow x \in B$$

- for functions:

$$f = g : A \to B \iff \forall x \in A.\ f(x) = g(x)$$

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To be **intensional** is to be
finer than extensional equality.

What is intensionality?

"The notions of intensionality and extensionality carry symmetric-sounding names, but this apparent symmetry is misleading. Extensionality is enshrined in mathematically precise axioms with a clear conceptual meaning. Intensionality, by contrast, remains elusive. It is a "loose baggy monster" into which all manner of notions may be stuffed, and a compelling and coherent general framework for intensional concepts is still to emerge."

— Samson Abramsky
"Intensionality, Definability and Computation" (2014)

A framework for intensionality

I’ve been looking for a **mathematical setting**, in terms of category theory, where the **same mathematical objects** can be seen both **extensionally** and **intensionally.**

Why?

- a categorical approach to **Gödel numbering**
- **intensional recursion**
- **non-functional computation**

Vignette n^o 1
Categorical Gödel numbering

Lawvere's classic paper from 1969.

Vignette nº 1

Categorical Gödel numbering

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Reprints in Theory and Applications of Categories, No. 15, 2006, pp. 1–13.

DIAGONAL ARGUMENTS
AND
CARTESIAN CLOSED CATEGORIES

F. WILLIAM LAWVERE

Author Commentary

In May 1967 I had suggested in my Chicago lectures certain applications of category theory to smooth geometry and dynamics, reviving a direct approach to function spaces and therefore to functionals. Making that suggestion more explicit led later to elementary topos theory as well as to the line of research now known as synthetic differential geometry. The fuller development of those subjects turned out to involve a truth value object that classifies subobjects, but in the present paper (presented in the 1968 Battelle conference in Seattle) I refer only to weak properties of such an object; it is the other axiom, cartesian

Vignette nº 1
Categorical Gödel numbering

Lawve

with domain 1; for example if (as in the next section) X and Z are set-valued functors, then a natural transformation g is point-surjective if every element of the *inverse limit* of Z comes from an element of the inverse limit of X. In case Z is of the form Y^A, an even weaker notion of surjectivity can be considered, which in fact suffices for our fixed point theorem. Namely

$$X \xrightarrow{g} Y^A$$

will be called *weakly point-surjective* iff for every $f : A \longrightarrow Y$ there is x such that for every $a : 1 \longrightarrow A$

$$\langle a, xg \rangle \epsilon = a.f$$

Finally we say that an object Y has the *fixed point property* iff for every endomorphism $t : Y \longrightarrow Y$ there is $y : 1 \longrightarrow Y$ with $y.t = y$.

1.1. THEOREM. *In any cartesian closed category, if there exists an object A and a weakly point-surjective morphism*

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then Y has the fixed point property.

PROOF. Let $\overline{g}$ be the morphism whose λ-transform is g. Then for any $f : A \longrightarrow Y$ there is $x : 1 \longrightarrow A$ such that for all $a : 1 \longrightarrow A$

$$\langle a, x \rangle \overline{g} = a.f.$$

Now consider any endomorphism t of Y and let f be the composition

$$A \xrightarrow{A\delta} A \times A \xrightarrow{\overline{g}} Y \xrightarrow{t} Y;$$

thus there is x such that for all a

$$A \xrightarrow{A\delta} A \times A \xrightarrow{\overline{g}} Y \xrightarrow{t} Y;$$

thus there is x such that for all a

$$\langle a, x\rangle\overline{g} = \langle a, a\rangle\overline{g}t$$

since $a(A\delta) = \langle a, a\rangle$. But then $y = \langle x, x\rangle\overline{g}$ is clearly a fixed point for t. ■

The famed "diagonal argument" is of course just the contrapositive of our theorem. Cantor's theorem follows with $Y = 2$.

1.2. Corollary. *If there exists $t : Y \longrightarrow Y$ such that $yt \neq y$ for all $y : 1 \longrightarrow Y$ then for no A does there exist a point-surjective morphism*

$$A \longrightarrow Y^A$$

(or even a weakly point-surjective morphism).

Vignette n^o 1

Categorical Gödel numbering

Lawvere proves a version of
Gödel's First Incompleteness Theorem:

Theorem. If *satisfaction* (= weak point-surjection) is definable, then $\neg$ has a fixed point

So Lawvere's fixed points *oughtn't* exist.

Can we tell a story about what *ought* to exist?

Vignette nº 2

Intensional Recursion

In the **untyped λ-calculus**, there are two ways to obtain recursion. Let $\ulcorner u \urcorner$ be the Gödel number of u.

First Recursion Theorem (FRT) $\forall f \in \Lambda. \exists u \in \Lambda.\ u = f\, u$

Second Recursion Theorem (SRT) $\forall f \in \Lambda. \exists u \in \Lambda.\ u = f \ulcorner u \urcorner$

Enumeration Theorem (EN) $\exists \mathbf{E}. \forall u \in \Lambda^0.\ \mathbf{E} \ulcorner u \urcorner = u$

Given EN, the SRT implies the FRT.
But is the SRT really stronger? What does it do?

Vignette nº 3
Non-functional computation

From the perspective of extensional equality, being **intensional** entails **being non-functional**. This is an open issue in **higher-order computability**.

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Non-functional computation

From the perspec
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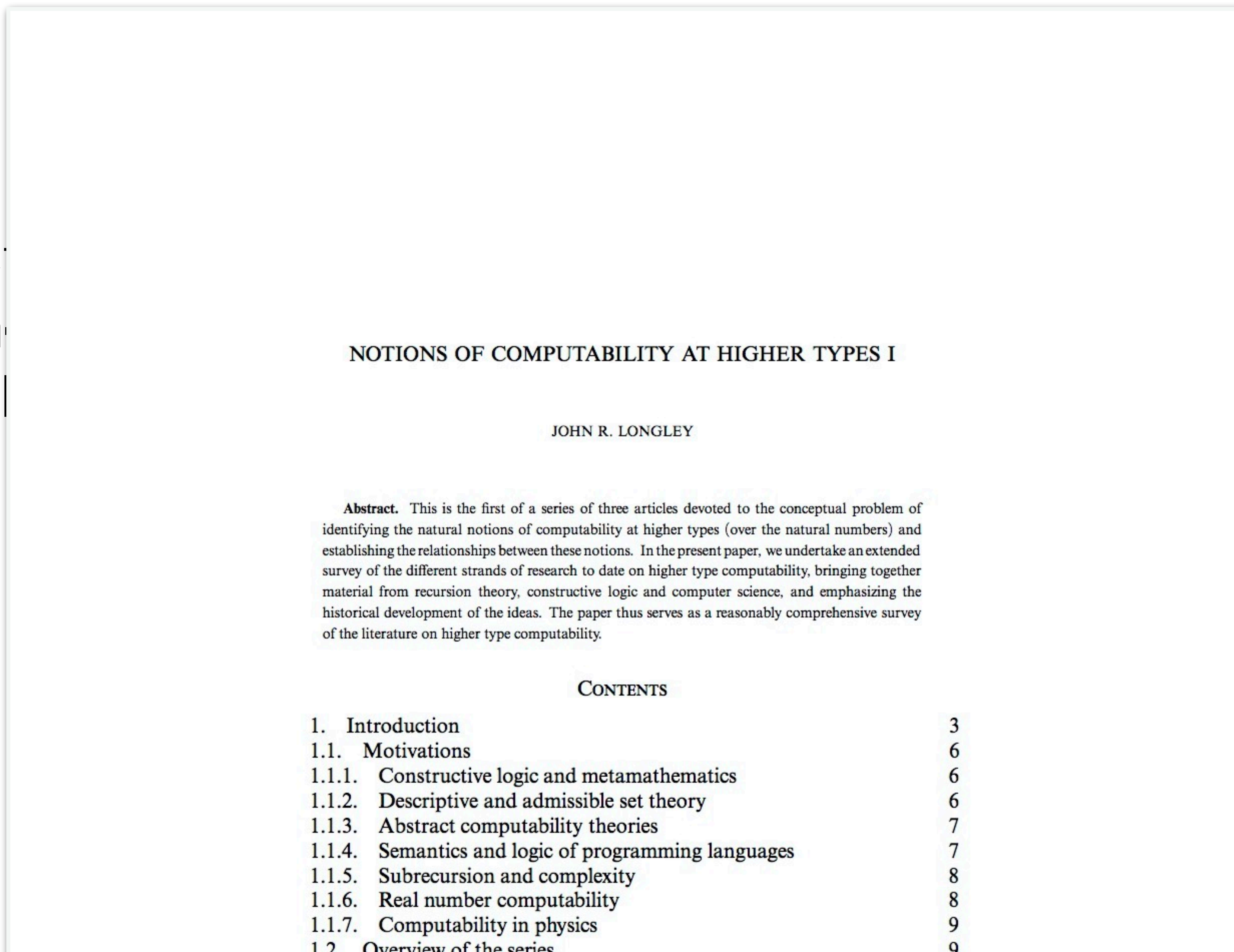

NOTIONS OF COMPUTABILITY AT HIGHER TYPES I

JOHN R. LONGLEY

Abstract. This is the first of a series of three articles devoted to the conceptual problem of identifying the natural notions of computability at higher types (over the natural numbers) and establishing the relationships between these notions. In the present paper, we undertake an extended survey of the different strands of research to date on higher type computability, bringing together material from recursion theory, constructive logic and computer science, and emphasizing the historical development of the ideas. The paper thus serves as a reasonably comprehensive survey of the literature on higher type computability.

CONTENTS

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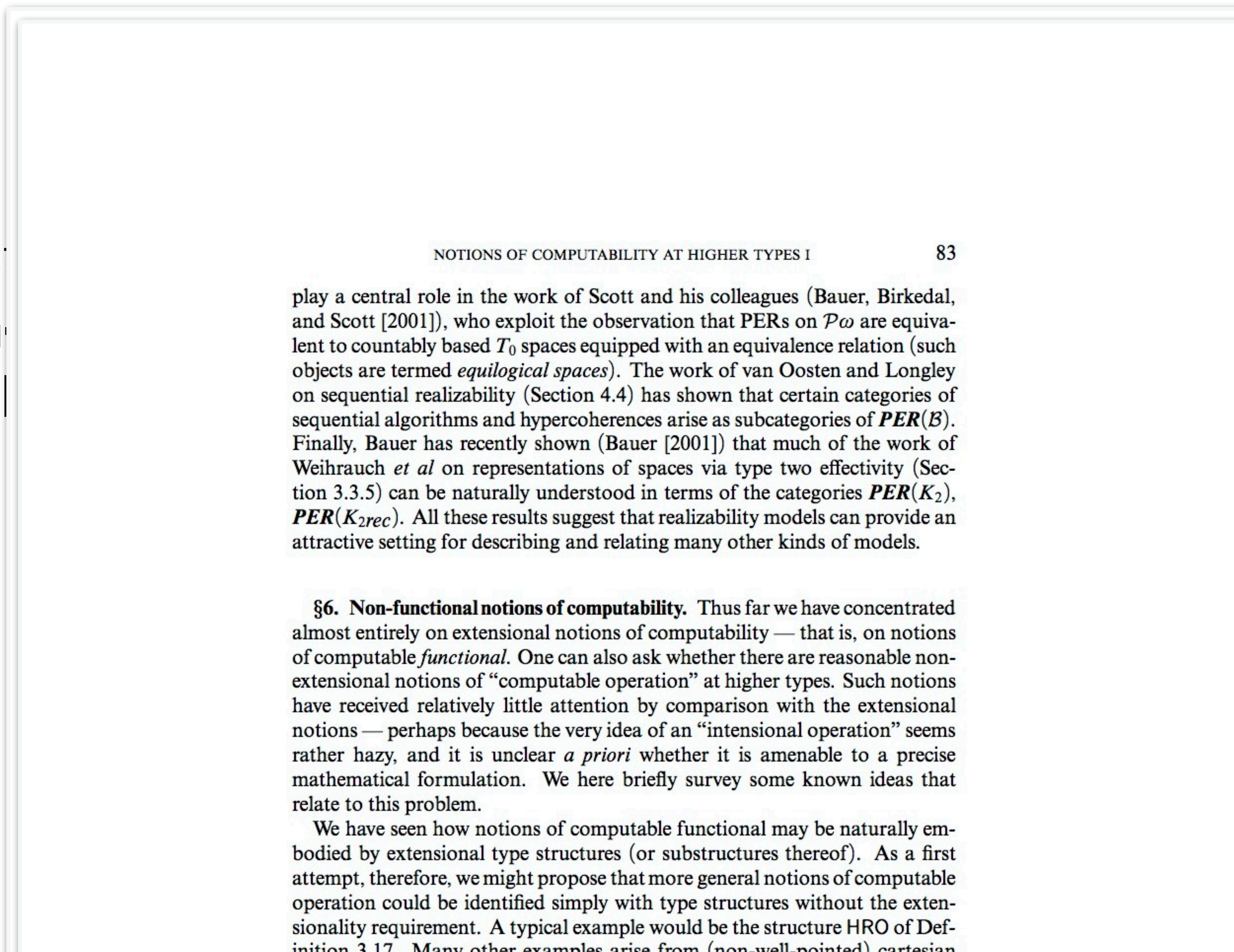

play a central role in the work of Scott and his colleagues (Bauer, Birkedal, and Scott [2001]), who exploit the observation that PERs on $\mathcal{P}\omega$ are equivalent to countably based T_0 spaces equipped with an equivalence relation (such objects are termed *equilogical spaces*). The work of van Oosten and Longley on sequential realizability (Section 4.4) has shown that certain categories of sequential algorithms and hypercoherences arise as subcategories of $\boldsymbol{PER}(\mathcal{B})$. Finally, Bauer has recently shown (Bauer [2001]) that much of the work of Weihrauch *et al* on representations of spaces via type two effectivity (Section 3.3.5) can be naturally understood in terms of the categories $\boldsymbol{PER}(K_2)$, $\boldsymbol{PER}(K_{2rec})$. All these results suggest that realizability models can provide an attractive setting for describing and relating many other kinds of models.

§6. Non-functional notions of computability. Thus far we have concentrated almost entirely on extensional notions of computability — that is, on notions of computable *functional*. One can also ask whether there are reasonable non-extensional notions of "computable operation" at higher types. Such notions have received relatively little attention by comparison with the extensional notions — perhaps because the very idea of an "intensional operation" seems rather hazy, and it is unclear *a priori* whether it is amenable to a precise mathematical formulation. We here briefly survey some known ideas that relate to this problem.

We have seen how notions of computable functional may be naturally embodied by extensional type structures (or substructures thereof). As a first attempt, therefore, we might propose that more general notions of computable operation could be identified simply with type structures without the extensionality requirement. A typical example would be the structure HRO of Definition 3.17. Many other examples arise from (non-well-pointed) cartesian objects are termed *equilogical spaces*). The work of van Oosten and Longley on sequential realizability (Section 4.4) has shown that certain categories of sequential algorithms and hypercoherences arise as subcategories of $\boldsymbol{PER}(\mathcal{B})$. Finally, Bauer has recently shown (Bauer [2001]) that much of the work of Weihrauch *et al* on representations of spaces via type two effectivity (Section 3.3.5) can be naturally understood in terms of the categories $\boldsymbol{PER}(K_2)$, $\boldsymbol{PER}(K_{2rec})$. All these results suggest that realizability models can provide an attractive setting for describing and relating many other kinds of models.

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Code as a type former, or: modality-as-intension

For each type A

let there be a type $\Box A$

whose elements can be understood as:

- programs that—when run—will yield objects of that type
- “codes” of objects of that type
- “intensions” of objects of that type

Modality-as-Intension

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From code for a function, to a map on codes: **intensional substitution,** a.k.a. **the s-m-n theorem**

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- The SRT then says:

“for each $f : \Box A \to A$ we have a $\ulcorner u \urcorner : \Box A$”

The type of the Second Recursion Theorem is the Gödel-Löb axiom

- It's **Löb's rule**, from provability logic!

$$\frac{\Box A \to A}{\Box A}$$

- Equivalent to the **Gödel-Löb axiom**:

$$\Box(\Box A \to A) \to \Box A$$

All is well, except in category theory

- Categories are not intensional
  - Lawvere (1969): some categories are *not well-pointed:*
    $\forall x : \mathbf{1} \to A.\ f \circ x = g \circ x$ yet $f \neq g : A \to B$
  - But, in general, the arrows will be distinguishable.
- Modality is a functor, but intension isn't!
  - Categorical semantics of modal logic (S4):
    a cartesian closed category $\mathcal{C}$ and a monoidal comonad
    $$(\Box : \mathcal{C} \longrightarrow \mathcal{C}, \delta : \Box \Rightarrow \Box^2, \epsilon : \Box \Rightarrow \mathsf{Id})$$
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We cannot use
ordinary category theory.

Enter P-categories

- P-sets: sets, up to a *partial equivalence relation (PER)*

symmetric and transitive

$$A = (|A|, \sim_A)$$

- For $x, y \in |A|$ the relation $x \sim_A y$ intuitively means: x and y are well-defined and extensionally equal.

  P-categories are 'categories' with P-sets

$$\mathcal{C}(A, B) = (|\mathcal{C}(A, B)|, \sim_{\mathcal{C}(A,B)})$$

  instead of hom-sets. The laws of categories hold up to the PERs; e.g.

$$(h \circ g) \circ f \sim_{\mathcal{C}(A,C)} h \circ (g \circ f)$$

What about the modality?

- The modality is **almost a functor,** but not: we might want $\Box f \not\sim \Box g$ even if $f \sim g$.

- We will call this an **exposure**: it's a functor-like map that **reflects** the PER. In symbols,

$$Q : (\mathfrak{B}, \sim) \looparrowright (\mathfrak{C}, \sim)$$

such that $\begin{matrix} Q(g \circ f) \sim Qg \circ Qf \\ Q(id_A) \sim id_{QA} \end{matrix}$ and $Qf \sim Qg \Longrightarrow f \sim g$

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Intensional equality implies extensional equality.

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Comonadic Exposures

There is a notion of **natural transformation of exposures**, so we can **mimic (strong monoidal) comonads**. For that, we also need isomorphisms:

$$m_{\mathbf{0}} : Q\mathbf{1} \to \mathbf{1}$$

$$m_{A,B} : QA \times QB \to Q(A \times B)$$

$a : \mathbf{1} \to A$ —quoting→ $Qa \circ m_0 : \mathbf{1} \to QA$

Intensional Fixpoints

"The notions of intensionality and extensionality carry symmetric-sounding names, but this apparent symmetry is misleading."

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An **intensional fixed point** of $f : QA \to A$

is a point $y : \mathbf{1} \to A$

such that $$\mathbf{1} \xrightarrow{y} A = \mathbf{1} \xrightarrow{m_0} Q\mathbf{1} \xrightarrow{Qy} QA \xrightarrow{f} A$$

Cf. Lawvere's fixed points which **oughtn't** exist: $$\mathbf{1} \xrightarrow{y} A = \mathbf{1} \xrightarrow{y} A \xrightarrow{f} A$$

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PA $\vdash \mathbf{fix}(\psi) \leftrightarrow \psi(\ulcorner \mathbf{fix}(\psi) \urcorner)$ $\lambda\beta \vdash u = f \ulcorner u \urcorner$ $\phi_e(x) \simeq f(e, x)$

Intensional Fixpoints

Theorem. There is an exposure corresponding to a Gödel numbering of PA.

Leads to abstract analogues of Gödel Incompleteness Theorem
and Tarski's Undefinability Theorem.

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Kleene & Lawvere

Definition 45. An arrow $r : X \times A \to Y$ is a *(cartesian) weak point-surjection* if, for every $f : A \to Y$, there exists a $x_f : \mathbf{1} \to X$ such that

$$\forall a : \mathbf{1} \to A.\ r \circ \langle x_f, a \rangle = f \circ a$$

Theorem 44 (Lawvere)**.** *If $r : A \times A \to Y$ is a weak-point surjection, then every arrow $t : Y \to Y$ has a fixed point.*

Theorem 45. *Let $Q : (\mathfrak{B}, \sim) \looparrowright (\mathfrak{B}, \sim)$ be a cartesian exposure, and let $\delta_A : QA \to Q^2 A$ be a reasonable quoting device. If $r : QA \times QA \to Y$ is a weak-point surjection, then every arrow $t : QY \to Y$ has an intensional fixed point.*

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Construct a P-category where…

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Kleene's **Second Recursion Theorem**

P-category of assemblies

- Assembly = a set with realisers (drawn from a PCA)
- The P-category $\mathfrak{Asm}(A)$ for a PCA A has
  - objects: assemblies $X = (|X|, \|\cdot\| : |X| \to \mathcal{P}(A))$

    element $x \in |X|$ is realised by $\|x\| \subseteq A$
  - morphisms $(f : |X| \to |Y|, r \in A) : X \to Y$

    where "r tracks f": $a \in \|x\| \Rightarrow r \cdot a \in \|f(x)\|$
  - $(f, r) \sim (g, s)$ whenever $f = g$

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partial combinatory algebra

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Exposure on assemblies

Define $X = (|X|, \|\cdot\|) \longrightarrow QX = (|QX|, \|\cdot\|)$

$$|QX| = \{(x, a) \mid a \in \|x\|\}$$

$$\|(x, a)\| = \{a\}$$

$$(f, r) : X \to Y \longrightarrow (f_r, r) : QX \to QY$$

$$f_r(x, a) = (f(x), r \cdot a)$$

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In fact, in this P-category for the PCA K1, with this exposure:
First Recursion Theorem = extensional (Lawvere) fixed points
Second Recursion Theorem = intensional fixed points

Coda

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