Two-dimensional Kripke Semantics

I. Presheaves

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Kripke semantics vs. type theory

Modal logic is important in Computer Science:

- temporal logic
- epistemic logic
- dynamic logic
- Hennessy-Milner logic

In most cases, it is given a **Kripke semantics**.

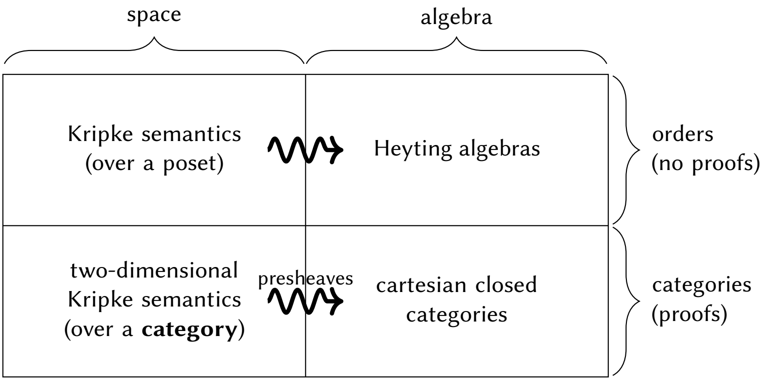
But in type theory **proofs are important** (Curry-Howard-Lambek).

Type-theoretic **modalities** arise *everywhere*:

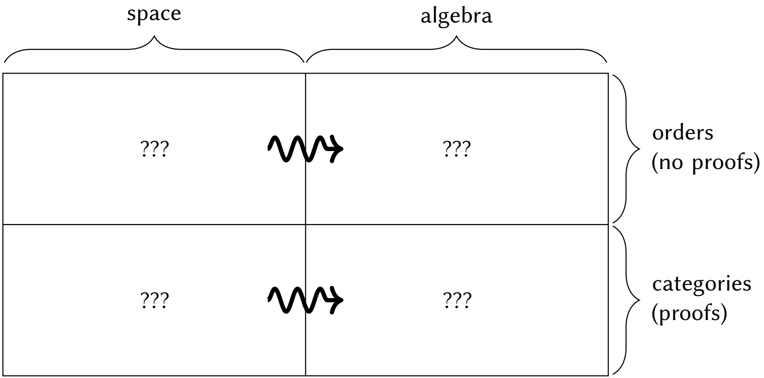
- ‘logical’ time
- proof-irrelevance
- globality
- information flow

How can we connect these two worlds?

Models of intuitionistic logic



Models of **intuitionistic** modal logic



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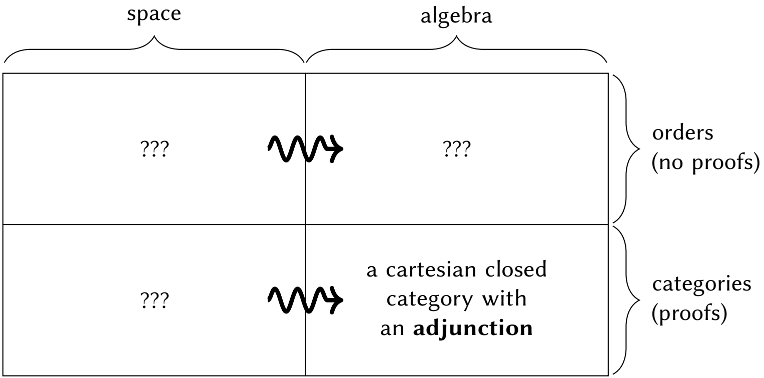

Using an adjunction was proposed by Clouston [Clo18].
It has proven remarkably robust in modal type theory.
This is an **objective** answer in the sense of Lawvere [Law94; LS09].

Models of **intuitionistic** modal logic

	space		algebra
orders (no proofs)	???	⇝	a Heyting algebra with a **Galois connection**
categories (proofs)	???	⇝	a cartesian closed category with an **adjunction**

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Kripke semantics over a poset equipped with a **bimodule**	⇝	a Heyting algebra with a **Galois connection**	orders (no proofs)
???	⇝	a cartesian closed category with an **adjunction**	categories (proofs)

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Models of **intuitionistic** modal logic

	space	algebra
orders (no proofs)	Kripke semantics over a poset equipped with a **bimodule** ⇝	a Heyting algebra with a **Galois connection**
categories (proofs)	two-dimensional Kripke semantics over a category equipped with a **profunctor** ⇝	a cartesian closed category with an **adjunction**

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Kripke semantics of intuitionistic logic

Let $(W, \sqsubseteq)$ be a **Kripke frame**, i.e. a partial order.

$\mathrm{Up}(W)$ = **upper sets** $S \subseteq W$ (where $w \in S$ and $w \sqsubseteq v$ imply $v \in S$)

Let $V : \mathrm{Var} \to \mathrm{Up}(W)$ map each proposition to an upper set. Define

$$
\begin{aligned}
w \vDash p &\overset{\text{def}}{\equiv} w \in V(p) \\
w \vDash \bot &\overset{\text{def}}{\equiv} \text{never} \\
w \vDash \varphi \wedge \psi &\overset{\text{def}}{\equiv} w \vDash \varphi \text{ and } v \vDash \psi \\
w \vDash \varphi \vee \psi &\overset{\text{def}}{\equiv} w \vDash \varphi \text{ or } v \vDash \psi \\
w \vDash \varphi \to \psi &\overset{\text{def}}{\equiv} \forall v.\ w \sqsubseteq v \text{ and } v \vDash \varphi \text{ imply } v \vDash \psi
\end{aligned}
$$

Monotonicity: $w \vDash \varphi$ and $w \sqsubseteq v$ imply $v \vDash \varphi$

Theorem (Kripke)

A formula is valid (in all frames and all words) iff it is a theorem.

Algebraic semantics of intuitionistic logic

A **Heyting algebra** $(H, \leq)$ is a lattice (has finite meets and joins) such that for every $x, y \in H$ there exists $x \Rightarrow y \in H$ with

$$c \wedge x \leq y \quad \iff \quad c \leq x \Rightarrow y \qquad \text{for all } c \in H$$

Suppose that for each proposition p we have an element $[\![p]\!] \in H$. Intuitionistic logic can then be interpreted into H compositionally:

$$\begin{aligned} [\![\bot]\!] &\stackrel{\text{def}}{=} 0 \\ [\![\varphi \wedge \psi]\!] &\stackrel{\text{def}}{=} [\![\varphi]\!] \wedge [\![\psi]\!] \\ [\![\varphi \vee \psi]\!] &\stackrel{\text{def}}{=} [\![\varphi]\!] \vee [\![\psi]\!] \\ [\![\varphi \to \psi]\!] &\stackrel{\text{def}}{=} [\![\varphi]\!] \Rightarrow [\![\psi]\!] \end{aligned}$$

Theorem

A formula is valid ($= 1$ in all algebras) iff it is a theorem.

Prime algebraic lattices: from space to algebra

Let $(W, \sqsubseteq)$ be a **Kripke frame**, and $2 \stackrel{\text{def}}{=} \{0 \sqsubseteq 1\}$.

$[W, 2]$ (= monotone maps $W \rightarrow 2$) has many curious properties:

- $[W, 2] \cong \mathsf{Up}(W)$ where the order is inclusion
- It is a **complete Heyting algebra** (arbitrary joins and meets)
- The **principal upper set** embedding $\uparrow : W^{\text{op}} \rightarrow [W, 2]$ given by $w \mapsto \{v \mid w \sqsubseteq v\}$ preserves meets and exponentials.
- An element is a **prime** ($p \sqsubseteq \bigsqcup_i d_i \Rightarrow \exists i.\ p \sqsubseteq d_i$) iff it is $\uparrow w$.
- Every upper set S is a join of primes:

$$S = \bigsqcup \{P \mid P \text{ prime}, P \subseteq S\} = \bigsqcup \{\uparrow w \mid w \in S\}$$

In short: $[W, 2]$ is a **prime algebraic lattice** [Win09].

There is a **duality**: $\mathbf{Pos}^{\text{op}} \simeq \mathbf{PrAlgLatt}$.

Intuitionistic logic: from space to category

Play the same trick as before, but replace $\mathbb{2}$ by **Set** [Law73].

The category $[\mathcal{C}, \mathbf{Set}]$ of presheaves $\mathcal{C} \longrightarrow \mathbf{Set}$:

- is a **(co)complete cartesian closed category**
- The **Yoneda embedding** $\mathbf{y} : \mathcal{C}^{\mathrm{op}} \longrightarrow [\mathcal{C}, \mathbf{Set}]$ given by $\mathbf{y}(w) \stackrel{\mathrm{def}}{=} \mathrm{Hom}(w, -)$ preserves products and exponentials.
- A presheaf P is **tiny** just if $\mathrm{Hom}(P, -)$ preserves colimits. All representables are tiny [and vice versa if $\mathcal{C}$ is Cauchy-complete].
- Every presheaf $P : \mathcal{C} \longrightarrow \mathbf{Set}$ is a colimit of tiny objects:

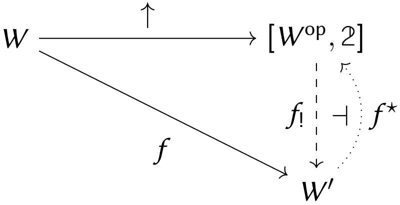
$$P = \varinjlim_{(w,x) \in \mathrm{el}\, P} \mathbf{y}(w)$$

There is a duality: $\mathbf{Cat}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathbf{PshCat}$ (Bunge's theorem).

2D Kripke semantics = semantics in $[\mathcal{C}, \mathbf{Set}]$.

Extensions

Let W' be a **complete lattice**, and let $f : W \to W'$ be monotone.



$f_!$: the **unique join-preserving** map satisfying $f_!(\uparrow w) = f(w)$.

$$f_!(S) \stackrel{\text{def}}{=} \bigsqcup \{f(w) \mid w \in S\}$$

As both lattices are complete, this has a right adjoint $f^\star$. Explicitly:

$$f^\star(w') \stackrel{\text{def}}{=} \{w \mid f(w) \sqsubseteq w'\}$$

Then

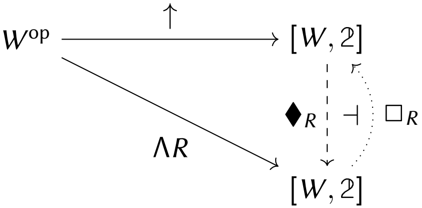
$$f_!(S) \sqsubseteq w' \quad \iff \quad S \subseteq f^\star(w')$$

Bimodules and Extensions

Let $(W, \sqsubseteq)$ be a Kripke frame. $R \subseteq W \times W$ is a **bimodule** just if

$$w' \sqsubseteq w \mathrel{R} v \sqsubseteq v' \Longrightarrow w' \mathrel{R} v'$$

Equivalently: $R : W^{\mathrm{op}} \times W \to 2$. Now extend $\Lambda R : W^{\mathrm{op}} \to [W, 2]$:



Concretely: $\begin{cases} \blacklozenge_R(S) \stackrel{\text{def}}{=} \{w \in W \mid \exists v.\ v \mathrel{R} w \text{ and } v \in S\} \\ \Box_R(S) \stackrel{\text{def}}{=} \{w \in W \mid \forall v.\ w \mathrel{R} v \text{ implies } v \in S\} \end{cases}$

Every such adjunction on $[W, 2]$ corresponds to a bimodule!

Duality: $\mathbf{EBimod}^{\mathrm{op}} \simeq \mathbf{PrAlgLattO}$.

The logic of Dzik, Järvinen, and Kondo [DJK10]

A very simple **tense logic** with two modalities, $\blacklozenge$ and $\Box$.

Kripke semantics:

$$w \models \blacklozenge\varphi \overset{\text{def}}{\equiv} \exists v.\ v\,R\,w \text{ and } v \models \varphi$$

$$w \models \Box\varphi \overset{\text{def}}{\equiv} \forall v.\ w\,R\,v \text{ implies } v \models \varphi$$

Algebraic semantics: a Heyting algebra with a Galois connection.

$$\frac{\blacklozenge\varphi \to \psi}{\varphi \to \Box\psi} \qquad \text{and} \qquad \frac{\varphi \to \Box\psi}{\blacklozenge\varphi \to \psi}$$

Some derivable rules:

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \qquad \frac{\varphi}{\Box\varphi} \qquad \frac{}{\Box\top} \qquad \frac{\blacklozenge\bot}{\bot} \qquad \frac{\varphi \to \psi}{\blacklozenge\varphi \to \blacklozenge\psi}$$

The usual $\lozenge$ is **not monotonic** in this setting.

Lifting to categories

- Replace bimodules by **profunctors**
- Use **left Kan extension** along Yoneda

This leads to a duality $\mathbf{EProf}_{cc}^{op} \simeq \mathbf{PshCatO}$.

Modalities on presheaves $P : \mathcal{C} \longrightarrow \mathbf{Set}$:

$$(\blacklozenge P)(w) = \int^{v \in \mathcal{C}} R(v, w) \times P(v)$$

$$(\square P)(w) = \int_{v \in \mathcal{C}} R(w, v) \to P(v) \cong \mathrm{Hom}_{[\mathcal{C}, \mathbf{Set}]}(R(w, -), [\![\varphi]\!])$$

Theorem

A two-dimensional Kripke semantics over $\mathcal{C}$ uniquely corresponds to

$$[\mathcal{C}, \mathbf{Set}] \underset{\blacklozenge}{\overset{\square}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} [\mathcal{C}, \mathbf{Set}]$$

Key facts

- The Kripke semantics (over a poset) and the algebraic semantics of intuitionistic (modal) logic are related by a **duality**.
- The interpretation of modalities arises canonically by extension (from a bimodule).
- These dualities can be **restricted** to ‘truth-preserving’ and ‘deduction-preserving’ morphisms respectively.
- They can also be **categorified**, relating 2D Kripke semantics (over a category) with cartesian closed categories.
- The interpretation of modalities arises canonically by **Kan extension** (from a profunctor).

References I

[Clo18] Ranald Clouston. “Fitch-Style Modal Lambda Calculi”. In: *Foundations of Software Science and Computation Structures*. Ed. by Christel Baier and Ugo Dal Lago. Vol. 10803. Lecture Notes in Computer Science. Cham: Springer International Publishing, 2018, pp. 258–275. DOI: 10.1007/978-3-319-89366-2_14 (cit. on pp. 4–8).

[DJK10] Wojciech Dzik, Jouni Järvinen, and Michiro Kondo. “Intuitionistic propositional logic with Galois connections”. In: *Logic Journal of IGPL* 18.6 (2010), pp. 837–858. DOI: 10.1093/jigpal/jzp057 (cit. on p. 15).

References II

[Law94] F William Lawvere. "Tools for the Advancement of Objective Logic: Closed Categories and Toposes". In: *The Logical Foundations of Cognition*. Ed. by John Macnamara and Gonzalo E Reyes. Oxford University Press, 1994, pp. 43–56. DOI: 10.1093/oso/9780195092158.003.0004 (cit. on pp. 4–8).

[Law73] F. William Lawvere. "Metric spaces, generalized logic, and closed categories". In: *Rendiconti del Seminario Matematico e Fisico di Milano* 43.1 (1973), pp. 135–166. DOI: 10.1007/BF02924844 (cit. on p. 12).

[LS09] F. William Lawvere and Stephen H. Schanuel. *Conceptual Mathematics: A First Introduction to Categories*. 2nd ed. Cambridge University Press, 2009. DOI: 10.1017/CBO9780511804199 (cit. on pp. 4–8).

References III

[Win09] Glynn Winskel. “Prime algebraicity”. In: *Theoretical Computer Science* 410.41 (2009), pp. 4160–4168. DOI: 10.1016/j.tcs.2009.06.015 (cit. on p. 11).