Two-dimensional Kripke Semantics

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Kripke semantics vs. type theory

Modal logic is important in Computer Science:

- temporal logic
- epistemic logic
- dynamic logic
- Hennessy-Milner logic

In most cases, it is given a **Kripke semantics**.

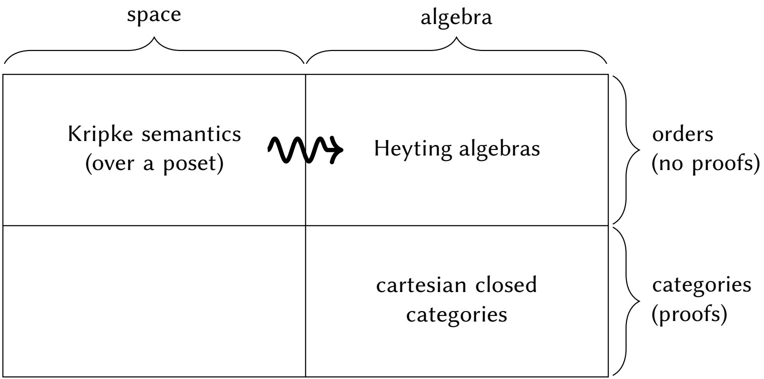
But in type theory **proofs are important** (Curry-Howard-Lambek).

Type-theoretic **modalities** arise *everywhere*:

- ‘logical’ time
- proof-irrelevance
- globality
- information flow

How can we connect these two worlds?

Models of intuitionistic logic



Models of intuitionistic logic

	space	algebra
orders (no proofs)	Kripke semantics (over a poset) ⇝	Heyting algebras
categories (proofs)	two-dimensional Kripke semantics (over a **category**)	cartesian closed categories

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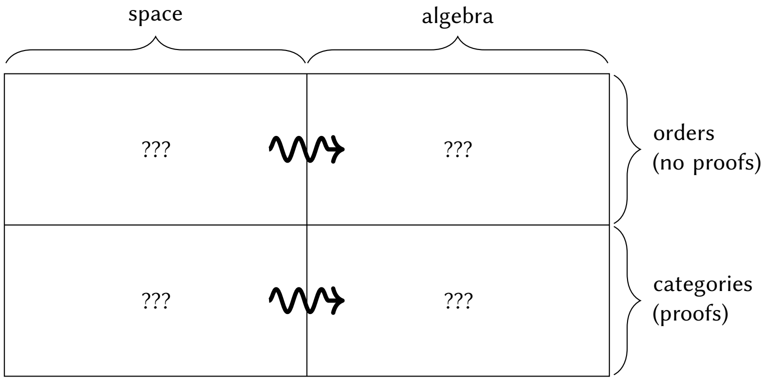
Models of **intuitionistic** modal logic



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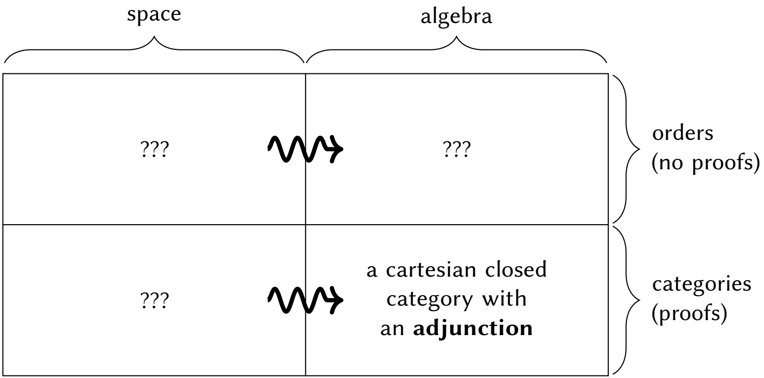

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Models of **intuitionistic** modal logic

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orders (no proofs)	???	a Heyting algebra with a **Galois connection**
categories (proofs)	???	a cartesian closed category with an **adjunction**

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Models of **intuitionistic** modal logic

	space	algebra	
	Kripke semantics over a poset equipped with a **bimodule** ⇝	a Heyting algebra with a **Galois connection**	orders (no proofs)
	two-dimensional Kripke semantics over a category equipped with a **profunctor** ⇝	a cartesian closed category with an **adjunction**	categories (proofs)

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Roadmap

Intuitionistic logic: Space vs. Algebra

Modal logic: bimodules

Coherent semantics

I. Intuitionistic logic: Space vs. Algebra

Kripke semantics of intuitionistic logic

Let $(W, \sqsubseteq)$ be a **Kripke frame**, i.e. a partial order.

$\mathrm{Up}(W)$ = **upper sets** $S \subseteq W$ (where $w \in S$ and $w \sqsubseteq v$ imply $v \in S$)

Let $V : \mathrm{Var} \to \mathrm{Up}(W)$ map each proposition to an upper set. Define

$$
\begin{aligned}
w \vDash p &\overset{\text{def}}{\equiv} w \in V(p) \\
w \vDash \bot &\overset{\text{def}}{\equiv} \text{never} \\
w \vDash \varphi \wedge \psi &\overset{\text{def}}{\equiv} w \vDash \varphi \text{ and } v \vDash \psi \\
w \vDash \varphi \vee \psi &\overset{\text{def}}{\equiv} w \vDash \varphi \text{ or } v \vDash \psi \\
w \vDash \varphi \to \psi &\overset{\text{def}}{\equiv} \forall v.\ w \sqsubseteq v \text{ and } v \vDash \varphi \text{ imply } v \vDash \psi
\end{aligned}
$$

Monotonicity: $w \vDash \varphi$ and $w \sqsubseteq v$ imply $v \vDash \varphi$

Theorem (Kripke)

A formula is valid (in all frames and all words) iff it is a theorem.

Algebraic semantics of intuitionistic logic

A **Heyting algebra** $(H, \leq)$ is a lattice (has finite meets and joins) such that for every $x, y \in H$ there exists $x \Rightarrow y \in H$ with

$$c \wedge x \leq y \quad \iff \quad c \leq x \Rightarrow y \qquad \text{for all } c \in H$$

Suppose that for each proposition p we have an element $[\![p]\!] \in H$. Intuitionistic logic can then be interpreted into H compositionally:

$$\begin{aligned} [\![\bot]\!] &\stackrel{\text{def}}{=} 0 \\ [\![\varphi \wedge \psi]\!] &\stackrel{\text{def}}{=} [\![\varphi]\!] \wedge [\![\psi]\!] \\ [\![\varphi \vee \psi]\!] &\stackrel{\text{def}}{=} [\![\varphi]\!] \vee [\![\psi]\!] \\ [\![\varphi \to \psi]\!] &\stackrel{\text{def}}{=} [\![\varphi]\!] \Rightarrow [\![\psi]\!] \end{aligned}$$

Theorem

A formula is valid ($= 1$ in all algebras) iff it is a theorem.

From space to algebra I

Let $(W, \sqsubseteq)$ be a Kripke frame, and $2 \stackrel{\text{def}}{=} \{0 \sqsubseteq 1\}$.

$[W, 2]$ (= monotone maps $W \to 2$) has many curious properties:

- $[W, 2] \cong \mathsf{Up}(W)$ where the order is inclusion
- It is a **complete Heyting algebra** (arbitrary joins and meets)
- The **principal upper set** $\uparrow w \stackrel{\text{def}}{=} \{v \mid w \sqsubseteq v\}$ gives a map

$$\uparrow : W^{\mathrm{op}} \to [W, 2] \qquad \text{monotone, order-embedding}$$

- An element is a **prime** ($p \sqsubseteq \bigsqcup_i d_i \Rightarrow \exists i.\ p \sqsubseteq d_i$) iff it is $\uparrow w$.
- Every upper set S is a join of primes:

$$S = \bigsqcup \{P \mid P \text{ prime}, P \subseteq S\} = \bigsqcup \{\uparrow w \mid w \in S\}$$

In short: $[W, 2]$ is a **prime algebraic lattice** [Win09].

From space to algebra II

Theorem (Raney; Nielsen, Plotkin, Winskel)

Every prime algebraic lattice is isomorphic to $[W, 2]$ *for some* W.

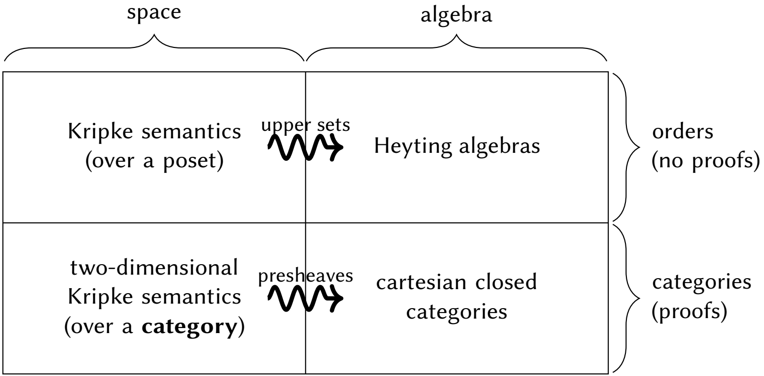
This amounts to a **duality**

$$\mathbf{Pos}^{\mathrm{op}} \simeq \mathbf{PrAlgLatt}$$

The Kripke semantics over $(W, \sqsubseteq)$ can be related to the HA $[W, 2]$:

$$w \vDash \varphi \iff w \in [\![\varphi]\!]$$

Models of intuitionistic logic



Categories as spaces

A category $\mathcal{C}$ has

- **objects** $c, d, \ldots \in \mathcal{C}$
- **morphisms** $f, g, \ldots : c \to d$ between two objects
- a way to **compose** morphisms, and **identity** morphisms

Categories are often used as 'mathematical universes'
(sets, graphs, vector spaces, topological spaces, ...)

But also, a category can be seen as a **partial order with evidence**.

$$\frac{}{x \sqsubseteq x} \qquad \frac{x \sqsubseteq y \quad y \sqsubseteq z}{x \sqsubseteq z}$$

$$\frac{}{\mathrm{id}_x : x \to x} \qquad \frac{f : x \to y \quad g : y \to z}{g \circ f : x \to z}$$

A category can also be seen as a *space* with *direction*.

Two-dimensional Kripke semantics of intuitionistic logic

Take any (small) category $\mathcal{C}$. Given arbitrary sets $[\![p]\!]_w$, define a set

$$[\![\varphi]\!]_w$$

of **proofs** of φ at a world $w \in \mathcal{C}$, by induction on φ.

$$[\![\bot]\!]_w \stackrel{\text{def}}{=} \emptyset$$

$$[\![\varphi \wedge \psi]\!]_w \stackrel{\text{def}}{=} [\![\varphi]\!]_w \times [\![\psi]\!]_w = \{(x, y) \mid x \in [\![\varphi]\!]_w, y \in [\![\psi]\!]_w\}$$

$$[\![\varphi \vee \psi]\!]_w \stackrel{\text{def}}{=} [\![\varphi]\!]_w + [\![\psi]\!]_w = \{(1, a) \mid a \in [\![\varphi]\!]_w\} \cup \{(2, b) \mid b \in [\![\psi]\!]_w\}$$

$$[\![\varphi \to \psi]\!]_w \stackrel{\text{def}}{=} (v : \mathcal{C}) \to \mathrm{Hom}_{\mathcal{C}}(w, v) \to [\![\varphi]\!]_v \to [\![\psi]\!]_v \quad \text{(not exactly)}$$

Monotonicity: for each $f : w \to v$ and $x \in [\![\varphi]\!]_w$ define $f \cdot x \in [\![\varphi]\!]_v$
This defines a **presheaf**

$$[\![\varphi]\!] : \mathcal{C} \longrightarrow \mathbf{Set}$$

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Let $V : \mathrm{Var} \to \mathrm{Up}(W)$ map each proposition to an upper set. Define

$$w \models p \overset{\text{def}}{\equiv} w \in V(p)$$

$$w \models \bot \overset{\text{def}}{\equiv} \text{never}$$

$$w \models \varphi \wedge \psi \overset{\text{def}}{\equiv} w \models \varphi \text{ and } v \models \psi$$

$$w \models \varphi \vee \psi \overset{\text{def}}{\equiv} w \models \varphi \text{ or } v \models \psi$$

$$w \models \varphi \to \psi \overset{\text{def}}{\equiv} \forall v.\ w \sqsubseteq v \text{ and } v \models \varphi \text{ imply } v \models \psi$$

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From space to category

Play the same trick as before, but replace $\mathbb{2}$ by **Set** [Law73].

The category $[\mathcal{C}, \mathbf{Set}]$ of presheaves $\mathcal{C} \longrightarrow \mathbf{Set}$:

- is a **(co)complete cartesian closed category**
- **Representables** $\mathbf{y}(w) \stackrel{\text{def}}{=} \mathrm{Hom}(w, -)$ give an embedding

$$\mathbf{y} : \mathcal{C}^{\mathrm{op}} \longrightarrow [\mathcal{C}, \mathbf{Set}]$$

- A presheaf P is **tiny** just if $\mathrm{Hom}(P, -)$ preserves colimits. All representables are tiny (and vice versa if $\mathcal{C}$ is Cauchy-complete).
- Every presheaf $P : \mathcal{C} \longrightarrow \mathbf{Set}$ is a colimit of tiny objects:

$$P = \varinjlim_{(w,x) \in \mathrm{el}\, P} \mathbf{y}(w)$$

From space to algebra I

Let $(W, \sqsubseteq)$ be a Kripke frame, and $\mathbb{2} \stackrel{\text{def}}{=} \{0 \sqsubseteq 1\}$.

$[W, \mathbb{2}]$ (= monotone maps $W \to \mathbb{2}$) has many curious properties:

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- It is a **complete Heyting algebra** (arbitrary joins and meets)
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In short: $[W, \mathbb{2}]$ is a **prime algebraic lattice** [Win09].

Duality

This construction gives us a duality

$$\mathbf{Cat}_{cc}^{op} \simeq \mathbf{PshCat}$$

between

- (Cauchy-complete, small) categories ($\approx$ '2D' Kripke frames)
- presheaf categories ($\approx$ proof-relevant prime alg. lattices)

In short:

> A two-dimensional Kripke semantics is a categorical semantics in a presheaf category $[\mathcal{C}, \mathbf{Set}]$.

II. Modal logic: bimodules

What is intuitionistic modal logic?

Many possible answers, in particular around $\Diamond$! [DM23]

The Simpson [Sim94] criteria:

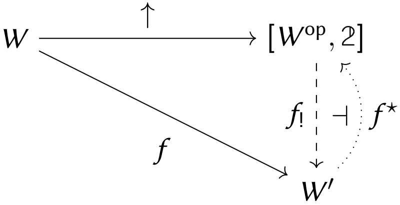
1. It should be conservative over intuitionistic logic.
2. It should prove all intuitionistic theorems (even with modalities).
3. Adding $\varphi \vee \neg\varphi$ should yield a classical modal logic.
4. It should satisfy the disjunction property.
5. $\Box$ and $\Diamond$ should be independent.
6. Its semantics should be 'intuitionistically comprehensible.'

#6 is formalised by translation to intuitionistic first-order logic.

An alternative proposal: **let category theory show you the way**.

Extensions

Let W' be a **complete lattice**, and let $f : W \to W'$ be monotone.



$f_!$: the **unique join-preserving** map satisfying $f_!(\uparrow w) = f(w)$.

$$f_!(S) \stackrel{\text{def}}{=} \bigsqcup \{f(w) \mid w \in S\}$$

As both lattices are complete, this has a right adjoint $f^\star$. Explicitly:

$$f^\star(w') \stackrel{\text{def}}{=} \{w \mid f(w) \sqsubseteq w'\}$$

Then

$$f_!(S) \sqsubseteq w' \quad \Longleftrightarrow \quad S \subseteq f^\star(w')$$

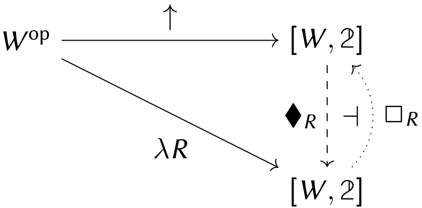
Bimodules and Extensions

Let $(W, \sqsubseteq)$ be a Kripke frame. How to add an accessibility relation?

$R \subseteq W \times W$ is a **bimodule** just if

$$w' \sqsubseteq w \, R \, v \sqsubseteq v' \implies w' \, R \, v'$$

Equivalently: $R : W^{\mathrm{op}} \times W \to 2$. Now extend $\Lambda R : W^{\mathrm{op}} \to [W, 2]$:



Concretely:

$$\blacklozenge_R(S) \stackrel{\text{def}}{=} \{w \in W \mid \exists v. \; v \, R \, w \text{ and } v \in S\}$$

$$\Box_R(S) \stackrel{\text{def}}{=} \{w \in W \mid \forall v. \; w \, R \, v \text{ implies } v \in S\}$$

The logic of Dzik, Jarvinen, and Kondo [DJK10]

A very simple **tense logic** with two modalities, $\blacklozenge$ and $\Box$.

Kripke semantics:

$$w \vDash \blacklozenge\varphi \stackrel{\text{def}}{\equiv} \exists v.\ v\, R\, w \text{ and } v \vDash \varphi$$
$$w \vDash \Box\varphi \stackrel{\text{def}}{\equiv} \forall v.\ w\, R\, v \text{ implies } v \vDash \varphi$$

Algebraic semantics: a Heyting algebra with a Galois connection.

$$\frac{\blacklozenge\varphi \to \psi}{\varphi \to \Box\psi} \quad \text{and} \quad \frac{\varphi \to \Box\psi}{\blacklozenge\varphi \to \psi}$$

Some derivable rules:

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \qquad \frac{\varphi}{\Box\varphi} \qquad \frac{}{\Box\top} \qquad \frac{\blacklozenge\bot}{\bot} \qquad \frac{\varphi \to \psi}{\blacklozenge\varphi \to \blacklozenge\psi}$$

The usual $\Diamond$ is **not monotonic** in this setting.

Lifting to categories

As before we get a duality: $\mathbf{EBimod}^{\mathrm{op}} \simeq \mathbf{PrAlgLattO}$.
To categorify:

- Replace bimodules by **profunctors**
- Use **left Kan extension** along Yoneda

This leads to a duality $\mathbf{EProf}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathbf{PshCatO}$.

Theorem

A two-dimensional Kripke semantics of this logic corresponds to

$$[\mathcal{C}, \mathbf{Set}] \underset{\blacklozenge}{\overset{\Box}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} [\mathcal{C}, \mathbf{Set}]$$

But adjunctions on a cartesian closed category were exactly the models of the modal λ-calculus of Clouston [Clo18]!

III. Coherent semantics

Completeness?

The developments so far only prove **relative completeness**:

- Suppose a formula is valid in all Heyting algebras.
- Then it is valid in all prime algebraic lattices.
- Then it is valid in all Kripke semantics

∴ the algebraic semantics is as complete as the Kripke semantics.

How to get the opposite direction?

The classic proof (Gehrke and van Gool [Gv24, §4.4]):

- Make a Kripke frame of **prime filters** of the algebra.
- Show relative completeness with respect to that.

For this logic: Dzik, Jarvinen, and Kondo [DJK10, §5].

But this is **non-constructive**, and also not very nice.

Coherent semantics

Replace

- the poset of worlds by a **distributive lattice** $(W, \sqsubseteq)$
- upper sets by (non-prime) **filters**

$F \subseteq W$ is a **filter** just if it is an upper set and

$$1 \in F, \qquad x \in F \text{ and } y \in F \text{ imply } x \wedge y \in F$$

$$\begin{aligned}
w \vDash p &\overset{\text{def}}{\equiv} w \in V(p) \in \mathsf{Filt}(W) \\
w \vDash \bot &\overset{\text{def}}{\equiv} (w = 1) \\
w \vDash \varphi \wedge \psi &\overset{\text{def}}{\equiv} w \vDash \varphi \text{ and } v \vDash \psi \\
w \vDash \varphi \vee \psi &\overset{\text{def}}{\equiv} \exists v_1, v_2.\ v_1 \wedge v_2 \sqsubseteq w \text{ and } v_1 \vDash \varphi, v_2 \vDash \psi \\
w \vDash \varphi \to \psi &\overset{\text{def}}{\equiv} \forall v.\ w \sqsubseteq v \text{ and } v \vDash \varphi \text{ imply } v \vDash \psi
\end{aligned}$$

This semantics is also sound and complete for intuitionistic logic!

Modalities and dualities

All previous work on modalities carries through, nearly verbatim.

The main duality is now

$$\mathbf{Stable}^{\mathrm{op}} \simeq \mathbf{Coh}$$

between

- distributive lattices and stable (= $\wedge$-preserving) maps
- **coherent frames** (= algebraic cHAs whose compact elements form a sub-lattice) and Scott-continuous, $\prod$-preserving maps (**not** the usual category from Stone duality)

Then

> The coherent semantics and the Heyting algebra semantics are **equi-complete**, **constructively**.

Categorifying the coherent semantics

Making proofs appear engenders a surprise.

Let $\mathcal{C}$ be a category with finite products and coproducts, which is also **'co-distributive' category**, i.e. a category in which

$$a + (c \times d) \cong (a + c) \times (a + d)$$

Then

> A two-dimensional coherent semantics is a categorical semantics in a **category of algebras**.

Why? Because 'filters' are product-preserving presheaves over $\mathcal{C}$.

If we regard $\mathcal{C}$ as a Lawvere theory, these are **algebras over** $\mathcal{C}$.

See Adámek, Rosický, and Vitale [ARV10].

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