A QUANTUM OF DIRECTION

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ABSTRACT. I argue that the correct primitives for abstract directed homotopy theory have not yet been identified. This assertion is corroborated by examining the directed structure of small categories qua directed homotopy 1-types.

CONTENTS

Despite the enormous success that the research programme of *Homotopy Type Theory (HoTT)* [The13] has met with over the last decade, the solution to an important problem remains elusive: we are not yet in posession of a fully satisfying account of *directed type theory*. There are many reasons to wish for such an account:

(i) There is a well-known connection between directed algebraic topology and concurrency theory: see e.g. [Faj+16]. One would hope that a directed type theory may somehow provide new means of reasoning about concurrent programs.
(ii) Some forms of directed type theory can be used in order to provide a *synthetic* presentation of ∞-categories. This presentation may be non-model-specific, and/or hide a number of frightful technical details under the lustre of type-theoretic principles. For this approach, see the work of Riehl and Shulman [RS17].
(iii) A directed type theory that supports a *directed univalence axiom* would provide powerful new reasoning principles. One of them, which I learned

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from Dan Licata, is the following. Assume an (internal) 'type constructor' $F : \mathcal{U} \to \mathcal{U}$ in book HoTT. Given an equivalence $A \cong B$, univalence yields $A =_{\mathcal{U}} B$. Applying F yields $F(A) =_{\mathcal{U}} F(B)$, which then induces an equivalence $F(A) \cong F(B)$. Thus, univalence implies that *type constructors preserve equivalence*, which has the potential to save work in theorem proving. In analogy with the above, directed univalence would assert that all the functions $A \to B$ can be cast as "directed paths" $A \rightsquigarrow B$ (which may or may not be all functions) induce a function $F(A) \to F(B)$. That is: directed univalence implies that *type constructors preserve (well-behaved) functions*.

(iv) Finally, one may imagine that a form of *directed higher inductive types* (dHITs) may lead to all sorts of interesting new mathematics. For example, one may axiomatize reduction in the λ-calculus by a dHIT which includes the path $(\lambda x.M)N \rightsquigarrow M[N/x]$, and hence conduct a 'directed homotopical' study of reduction. We are free to dream!

There have been a few attempts to formulate a directed type theory. Amongst others, they include the unpublished work of Michael Warren [War11], the 2-dimensional type theory of Licata and Harper [LH11; Lic11], the master's thesis of Nuyts [Nuy15], the type theory of Riehl and Shulman [RS17] and the type theory of Paige North [Nor19c]. With the exception of [RS17], none of these type theories have a semantics in higher-dimensional category theory.

One cannot help but notice that—at least when compared with the explosive development of HoTT—the rate of progress in this topic is rather slow. The purpose of this article is to examine the root causes of this delay. I will implicitly try to argue that the fundamental issue is that *we have failed to identify the correct primitives of directed homotopy theory*, and hence do not have the correct semantic tools that would help us guide the development of good syntax.

Perhaps the greatest insight that propelled the development of HoTT is the relationship between identity types and model categories, as first described by Awodey and Warren [AW09]. In an attempt to point towards a parallel development, I will argue that the primitives of model categories do not adapt well to the directed setting. There are a few constructions of model categories for directed homotopy, see e.g. [Gau03; Gau19]. However, these constructions are more about generating a

directed [homotopy theory]

i.e. a homotopy theory of directed spaces. Instead, I propose that we need a new

[directed homotopy] theory

i.e. a theory that contains fundamentally contains direction.[1]

To corroborate this argument, one must provide it with a firm footing in a specific model of directed homotopy. While there are many models of directed topology that one could employ—see e.g. Grandis' book [Gra09]—I will err on the side of simplicity and concentrate on the category **Cat** of categories. This choice is historically informed: the pre-history of HoTT begins with the groupoid model of Hofmann and Streicher [HS98]. It is also topologically informed: there is a well-known model structure on groupoids—see e.g. [JT08, §2.2]—which shows that they are a presentation of *homotopy 1-types*. It is thus conceivable that (some subcategory of) **Cat** may act as both a toy model of directed type theory, but also a model of *directed homotopy 1-types*.

After introducing some preliminary material in §1, I will argue in §2 that the technology of model categories is inherently undirected. This argument will be

[1] I am grateful to Paige North for this parenthesisation.

corroborated by demonstrating that what we intuitively understand as the basic building blocks of directed homotopy—i.e. directed cylinders, directed path spaces, and so on—are very much at odds with what we understand as intuitive categorical notions of cofibrations, fibrations, etc. Thus, the need for some new primitives will arise.

In order to identify those primitives, in §3 I turn to study of the directed refinements of the weak factorisation systems found in the model structure on groupoids. While these do demonstrate many interesting directed patterns—including a directed notion filling—they by no means suffice to replicate even standard arguments from model categories. The reason is the same as that given in §2, viz. that there is no good notion of 'class of morphisms' to which directed cylinders and paths belong. Instead, they seem to belong to well-behaved *classes of spans and cospans.*

Hence, to regain those standard arguments we turn to the study of well-behaved (co)spans in §4. Two candidate notions immediately arise, one homotopical and one geometric. The first one is well-known in 2-category theory, and is that of *two-sided (co)fibrations.* The second one is also known—mainly through the work of Lawvere—and it is that of *adjoint cylinders and adjoint reflexive graphs.* We demonstrate that both of these are useful: in §4.6 we prove a novel *two-sided lifting result,* which shows that adjoint cylinders ('good cospans') lift against two-sided discrete fibrations ('very good spans') in a specific way. This result is just powerful enough to replicate the standard abstract argument that left homotopy implies right homotopy in model categories.

1. Preliminary material

This section covers some necessary preliminary material on *weak factorisation systems* and *model categories.* The definitions and theorems in this section can be found most standard presentations of model categories: see e.g. [Hov07], [JT08, §A.2.1], or [Cis19, §2.1]. A particularly lucid and comprehensive exposition of weak factorisation systems can be found in [Nor17, §1].

Definition 1.1. A *lifting problem* in a category $\mathcal{E}$ is a commuting diagram

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "j"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "k"'] & \cdot \end{tikzcd}$$

A *solution* to this lifting problem is a *diagonal filler,* i.e. a (non-unique) morphism d such that

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "j"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "k"'] \arrow[ur, dotted, "d"] & \cdot \end{tikzcd}$$

commutes.

Throughout this paper, we will use dashed lines $\cdot \dashrightarrow \cdot$ to denote *unique* arrows that make a diagram commute, and dotted lines $\cdot \cdots\!\!\!\rightarrow \cdot$ to denote the existence of an arrow that makes a diagram commute, but which is not necessary unique.

Definition 1.2. Let $\mathcal{L}$ and $\mathcal{R}$ be classes of morphisms of $\mathcal{E}$. If every lifting problem

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "j"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "k"'] \arrow[ur, dotted, "d"] & \cdot \end{tikzcd}$$

where $j \in \mathcal{L}$ and $f \in \mathcal{R}$ has a filler d, then we write $\mathcal{L} \pitchfork \mathcal{R}$.

We say that $\mathcal{L}$ has the *left lifting property* against $\mathcal{R}$, and that $\mathcal{R}$ has the *right lifting property* against $\mathcal{L}$.

Definition 1.3. Let $\mathcal{L}$, $\mathcal{R}$ be classes of morphisms of a category $\mathcal{E}$.

(1) The class of morphisms that have the left lifting property against $\mathcal{R}$ is denoted by ${}^{\pitchfork}\mathcal{R}$.
(2) The class of morphisms that have the left lifting property against $\mathcal{L}$ is denoted by $\mathcal{L}^{\pitchfork}$.

Definition 1.4. A *weak factorisation system* in a category $\mathcal{E}$ is a pair $(\mathcal{A}, \mathcal{B})$ of classes of morphisms of $\mathcal{E}$ such that

(1) any $f : A \to B$ can be factorised as $f = b \circ a$ where $a \in \mathcal{A}$ and $b \in \mathcal{B}$
(2) $(\mathcal{A}, \mathcal{B})$ are a *lifting pair*; that is, $\mathcal{A} = {}^{\pitchfork}\mathcal{B}$ and $\mathcal{A}^{\pitchfork} = \mathcal{B}$

Given any category X we can construct its arrow category X^2, whose objects are morphisms $\alpha : x \to y$ of X, and whose morphisms are pairs (h, k) that fit in a commutative square between objects, i.e.

$$\begin{tikzcd} x \arrow[r, "h"] \arrow[d, "\alpha"'] & x' \arrow[d, "\alpha'"] \\ y \arrow[r, "k"'] & y' \end{tikzcd}$$

In other words, X^2 is the comma category $\{\mathrm{Id}_X, \mathrm{Id}_X\}$. An object $\alpha \in X^2$ is a *retract* of an object β exactly when it is a retract in X^2. This amounts to the existence of a commuting diagram

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\alpha"'] \arrow[rr, bend left, equal] & \cdot \arrow[r] \arrow[d, "\beta"'] & \cdot \arrow[d, "\alpha"] \\ \cdot \arrow[r] \arrow[rr, bend right, equal] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

in X, where the top and bottom rows compose to identities.

Proposition 1.5. *If $(\mathcal{A}, \mathcal{B})$ are a weak factorisation system, then both $\mathcal{A}$ and $\mathcal{B}$ are closed under composition and retracts; $\mathcal{A}$ is closed under pushouts and coproducts; and $\mathcal{B}$ is closed under pullbacks and products.*

For a proof see e.g. [Nor17, Lemma 1.1.5].

Definition 1.6. A *model category* is a category $\mathcal{C}$ that has finite limits and colimits, and which comes with three distinguished classes of morphisms: the *fibrations* $\mathcal{F}$, the *cofibrations* $\mathcal{C}$, and the *weak equivalences* $\mathcal{W}$. These classes must satisfy the following properties:

(1) $\mathcal{W}$ has the *2-out-of-3 property*: if any two of f, g and $g \circ f$ are in $\mathcal{W}$, then so is the third.
(2) $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ are weak factorisation systems.

Recall that morphisms in $\mathcal{C} \cap \mathcal{W}$ are called *acyclic cofibrations* (or *trivial cofibrations*), and morphisms in $\mathcal{F} \cap \mathcal{W}$ are called *acyclic fibrations* (or *trivial fibrations*).

2. MODEL CATEGORIES ARE NOT DIRECTED

Speaking very broadly, categorical homotopy theory begins as soon as we have the structure of a *homotopical category*: that is, a category $\mathcal{C}$ with a distinguished wide (i.e. including all objects) subcategory $\mathcal{W}$ of *weak equivalences*, which satisfy a 2-out-of-6 property (which implies the 2-out-of-3 property). For details, see [Rie14, §2.1]. As soon as we have these maps, we may *localise* at them to obtain the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of $\mathcal{C}$, in which the hom-sets have been replaced by homotopy classes of morphisms. The additional classes of fibrations and cofibrations provided in a model category are not necessary for this construction, but merely provide a certain amount of technology, which—amongst other things—allows us to state when two homotopy theories are "the same."

The fact that homotopic maps can be collapsed *without* providing an explicit relation of homotopy between morphisms pinpoints a salient feature of homotopy theory, viz. that it is *internal*: the fact that a morphism $f : X \to Y$ is homotopic to $g : X \to Y$ is evidenced by a third morphism in the very same category. In classical homotopy theory [Ark11, §1.3] this evidence is a continuous function $H : X \times [0,1] \to Y$ such that $H(-,0) = f$ and $H(-,1) = g$. The space $X \times [0,1]$ is called the *cylinder* of X. The key thing to notice is that there is a homotopy equivalence between X and its cylinder $X \times [0,1]$. This is a weak equivalence in the Quillen model structure on topological spaces, so when we localise $\mathcal{W}$ it so happens that f and g fall in the same homotopy class, and are formally identified; see e.g. [Cis19, Cor 2.2.18].

The axiomatic approach of model category aims to replicate this situation in other settings. We define a *cylinder object of* A to be a factorisation

$$A + A \xrightarrow{[\mathrm{id}_A, \mathrm{id}_A]} A \;=\; A + A \xrightarrow{[i_0, i_1]} I(A) \xrightarrow{w} A$$

of the codiagonal into a cofibration $[i_0, i_1] : A + A \to I(A)$ and a weak equivalence $w : I(A) \to A$.[3] The idea is that the cospan $[i_0, i_1]$ is a 'good inclusion' at the two ends of the cylinder, and w is an 'abstract homotopy equivalence' between the cylinder object $I(A)$ and A. When we localise at $\mathcal{W}$, the latter turns into an isomorphism. Of course, there is a dual story to be told about the *path objects of* X, which are defined to be factorisations

$$X \xrightarrow{\langle \mathrm{id}_X, \mathrm{id}_X \rangle} X \times X \;=\; X \xrightarrow{w} P(X) \xrightarrow{\langle p_0, p_1 \rangle} X \times X$$

of the diagonal into a weak equivalence $w : X \to P(X)$ and a fibration $\langle p_0, p_1 \rangle : P(X) \to X \times X$. Again, the idea is $P(X)$ is the path space of X, p_0 and p_1 project the endpoints of a path, and w witnesses a homotopy equivalence between $P(X)$ and X.

Whether homotopies are expressed through a cylinder object or a path object is irrelevant—see e.g. [Cis19, Lemma 2.2.12]: if A is cofibrant and X is fibrant, these two coincide; if they are not, we can *replace* them with a cofibrant and fibrant object respectively. This argument is carried entirely through factorisations of morphisms. It does not require that $I(-)$ and $P(-)$ are given functorially, nor that I be left adjoint to P.

The aim is to capture the simplicity and succinctness of this approach in a directed setting. Unfortunately, most of the aforementioned patterns immediately break down. We will illustrate this with two intuitive examples in the category of

[3][DS95] only require w to be a weak equivalence. When $[i_0, i_1]$ is a cofibration they use the name *good cylinder*. If w is also a fibration, they call it a *very good cylinder*.

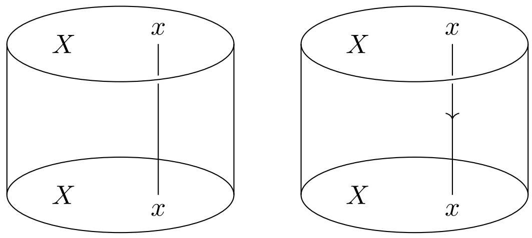


FIGURE 1. A cylinder and a directed cylinder for the space X.

categories. The first one shows that weak equivalences are no longer the right relationship between a space and its cylinder/path object. The second one shows that fibrations and cofibrations are more complex objects than before.

Let

$$\mathbb{1} = \bullet \qquad \mathbb{2} = 0 \longrightarrow 1 \qquad \mathbb{I} = 0 \xrightarrow{\cong} 1$$

be the terminal category, the walking arrow, and the walking isomorphism respectively. The walking isomorphism $\mathbb{I}$ is a *groupoid*. It is easy to see that, if X is a groupoid, $X \times \mathbb{I}$ has the flavour of a cylinder: any functor $F : X \times \mathbb{I} \to Y$ encodes a natural isomorphism $F(-,0) \overset{\cong}{\Rightarrow} F(-,1)$, and any natural isomorphism (cf. homotopy) can be encoded that way. In fact, it is easy to check that $X \times \mathbb{I}$ is a cylinder object in the model structure on groupoids,[5] as $X \times \mathbb{I}$ is equivalent to X. One can picture $X \times \mathbb{I}$ as the left object in Figure 1.

If anything deserves to be called a 'directed cylinder' in $\mathbf{Cat}$, it is definitely the category $X \times \mathbb{2}$: as before, functors $F : X \times \mathbb{2} \to Y$ uniquely encode natural transformations $F(-,0) \Rightarrow F(-,1)$. However, it is difficult to picture a notion of 'weak equivalence' between X and $X \times \mathbb{2}$. Taking $X = \mathbf{1}$, a categorical equivalence would assert that $\mathbf{1} \simeq \mathbf{1} \times \mathbf{2} \cong \mathbf{2}$ which would amount to a categorical equivalence between the terminal category and the walking arrow. Intuitively, taking the product of a category with $\mathbb{2}$ introduces a *quantum of direction*, which is directed-homotopically non-trivial. Grandis [Gra05; Gra09] has developed notions of equivalence for categories seen as directed spaces, which he callls *past and future equivalences* (pf-equivalences). For example, in [Gra05, §3.8] he notes that the inclusion $X \to X^{\mathbb{2}}$ is a special kind of such an equivalence, namely a structural pf-injection.

Dually, if anything deserves to be called a 'directed path space' in $\mathbf{Cat}$, that must be the arrow category $X^{\mathbb{2}}$, which we incidentally defined in §1. We would then expect the arrow $\langle \mathrm{dom}, \mathrm{cod} \rangle : X^{\mathbb{2}} \to X$ to be a 'fibration.' What this should mean is also not evident, but it is clear that there should be some form of *lifts*: if $(p : x \to z, q : w \to y) \in X$, so that an object $\alpha : x \to y$ of $X^{\mathbb{2}}$ is *above* it with respect to $\langle \mathrm{dom}, \mathrm{cod} \rangle$, then there should be a way to lift (p,q) to an arrow $(h,k) : \alpha' \to \alpha$ of $X^{\mathbb{2}}$ so that $\langle \mathrm{dom}, \mathrm{cod} \rangle (h,k) = (p,q)$. This lift may also have other properties: for example it could be cartesian, which would make $\langle \mathrm{dom}, \mathrm{cod} \rangle$ a Grothendieck fibration. However, the mere existence of such lifts is enough to guarantee that X is a groupoid, i.e. an undirected 1-type.

Proposition 2.1. *If $\langle \mathrm{dom}, \mathrm{cod} \rangle : X^{\mathbb{2}} \longrightarrow X \times X$ has any form of lifts at all, then X is a groupoid.*

[5] The model structure on groupoids has categorical equivalences as weak equivalences, Grothendieck fibrations (= isofibrations) as fibrations, and functors that are injective on objects as cofibrations. See e.g. [JT08, §2.2] has

Proof. We want to show that every arrow $\alpha : x \to y$ of X is invertible. We have that $(\mathrm{id}_y : y \to y, \alpha : x \to y)$ is a morphism $(y,x) \to (y,y)$ of $X \times X$, and $\mathrm{id}_y : y \to y$ is above its codomain. Hence, there is an object $\beta : y \to x$ of X^2 that makes the following diagram commute:

$$\begin{tikzcd} y \arrow[r, equal] \arrow[d, dotted, "\beta"'] & y \arrow[d, equal] \\ x \arrow[r, "\alpha"'] & y \end{tikzcd}$$

A similar argument gives us that there is a $\gamma : x \to y$ that makes the following diagram commute:

$$\begin{tikzcd} x \arrow[r, equal] \arrow[d, dotted, "\gamma"'] & x \arrow[d, equal] \\ y \arrow[r, "\beta"'] & x \end{tikzcd}$$

Thus β has a left inverse γ and a right inverse α, so $\alpha = \gamma$. □

Hence, $\langle \mathrm{dom}, \mathrm{cod} \rangle$ cannot be a 'fibration,' in the sense of having any form of lifts for all arrows, so is not a Grothendieck fibration. In the folk model structure on **Cat** [Rez00] this is resolved by taking fibrations to be the *isofibrations*, i.e. the functors for which isomorphisms in the base lift to isomorphisms in the total category. Considering the case of an isomorphism $(p,q) : (x,y) \xrightarrow{\cong} (u,v) \in X \times X$, and an object $\alpha : u \to v$ above its codomain w.r.t. $\langle \mathrm{dom}, \mathrm{cod} \rangle$, it is easy to see that it lifts to a morphism $\alpha' \to \alpha$ in X. In fact, this amounts to *computing the lid of the 'open box'*

$$\begin{tikzcd} x \arrow[r, "p", "\cong"'] \arrow[d, dotted] & u \arrow[d, "\alpha"] \\ y \arrow[r, "\cong", "q"'] & v \end{tikzcd}$$

Indeed, we can put $q^{-1} \circ \alpha \circ p$ for the dotted arrow, which relies on the invertibility of q. The invertibility of p is required for the resultant lift (p,q) to be an isomorphism itself.

Nevertheless, it is easy to prove that isofibrations and Grothendieck fibrations coincide on groupoids: a lift that is an isomorphism is automatically cartesian. This makes it clear that the folk model structure on **Cat** once again encodes a homotopy theory of directed spaces: it merely repurposes a well-behaved tool for groupoids (= undirected spaces) in the world of categories (= directed spaces).

It is thus evident that we have to search beyond the usual axioms of model categories in order to formulate a truly directed theory.

3. One-sided structure

We can perhaps learn more about basic directional strucure by examining how the factorisations used in the model structure on groupoids adapt when we relax them to categories.

3.1. **The first factorisation.** In order to construct the $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ WFS on groupoids, which is a kind of *mapping path space factorisation*, we factor any functor f as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & \{B, f\} \arrow[ur, "p"'] & \end{tikzcd}$$

where i is an injective-on-objects ($\mathcal{C}$) categorical equivalence ($\mathcal{W}$), and p is a Grothendieck fibration ($\mathcal{F}$). $\{B, f\}$ is the usual comma category: its objects are 'paths' $p : y \to f(x)$ of B,[6] and its morphisms are commutative squares[7]

$$\begin{tikzcd} y \arrow[r, "h"] \arrow[d, "\alpha"'] & y' \arrow[d, "\alpha'"] \\ f(x) \arrow[r, "f(k)"'] & f(x') \end{tikzcd}$$

$\{B, f\}$ can also be described as the pullback of the codomain functor along f:

(1)
$$\begin{tikzcd} \{B, f\} \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & B^2 \arrow[d, "\mathsf{cod}"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

$p : \{B, f\} \longrightarrow B$ is defined to be the composite $\{B, f\} \to B^2 \xrightarrow{\mathrm{dom}} B$. More explicitly, p maps

$$\begin{tikzcd} y \arrow[r, "h"] \arrow[d, "\alpha"'] & y' \arrow[d, "\alpha'"] \\ f(x) \arrow[r, "f(k)"'] & f(x') \end{tikzcd} \quad \overset{p}{\longmapsto} \quad y \xrightarrow{h} y'$$

p is sometimes called the *free fibration on* f. It is not hard to show that it is a Grothendieck fibration.

The functor $i : A \longrightarrow \{B, f\}$ is constructed through the universal property of the pullback, where $\mathsf{refl} : B \to B^2$ is the functor that maps y to $\mathrm{id}_y : y \to y$:

(2)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, dotted, "i"] \arrow[ddr, bend right, equal] & B \arrow[dr, bend left, "\mathsf{refl}"] & & \\ & \{B, f\} \arrow[r, "v"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "p"] & B^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\mathsf{cod}"] & B \\ & A \arrow[r, "f"'] & B & \end{tikzcd}$$

Thus, i is a section of q. Explicitly, q sends a path $\alpha : y \to f(x)$ to $x \in A$, and i sends $\gamma : x \to x' \in A$ to the 'degeneracy square'

$$\begin{tikzcd} f(x) \arrow[r, "f(\gamma)"] \arrow[d, equal] & f(x') \arrow[d, equal] \\ f(x) \arrow[r, "f(\gamma)"'] & f(x') \end{tikzcd}$$

Clearly, $p \circ i = \mathrm{dom} \circ v \circ i = \mathrm{dom} \circ \mathsf{refl} \circ f = f$. As i is a section, it is faithful, and injective-on-objects. Finally, it is full.

Thus, i is an *embedding*: it witnesses A as a subcategory of $\{B, f\}$. However, the most important property of i is that $q \dashv i$. In the terminology of John Gray [Gra66] it is a *rari*, i.e. a *right-adjoint-right-inverse*. Thus, i essentially demonstrates that A is a *reflective subcategory* of $\{B, f\}$. The following depiction of a

[6]More rigorously: tuples $(x \in A, y \in B, \alpha : x \to f(y))$.
[7]More rigorously: tuples $(h : y \to y' \in B, k : x \to x' \in A)$ such that $\alpha' \circ h = f(k) \circ \alpha$.

morphism $\alpha \to i(x')$ in $\{B, f\}$ suffices to convince us that morphisms $\alpha \to i(x')$ and $x \to x'$ naturally correspond:

$$\begin{tikzcd} y \arrow[r, "\beta"] \arrow[d, "\alpha"'] & f(x') \arrow[d, equal] \\ f(x) \arrow[r, "f(\gamma)"'] & f(x') \end{tikzcd}$$

Indeed, β is determined as $f(\gamma) \circ \alpha$, so the only non-trivial datum is $\gamma : x \to x'$.

We have so far only assumed that A and B are categories. In order to obtain the $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ factorisation system for groupoids, let us assume that A and B are groupoids. It is then easy to see that $i : A \longrightarrow \{B, f\}$ is a categorical equivalence, as it is additionally essentially surjective: we can recover every object $\alpha : y \xrightarrow{\cong} f(x)$ as $i(x) = \mathrm{id}_x$ up to the isomorphism α itself, now seen as a morphism:

$$\begin{tikzcd} y \arrow[r, "\alpha", "\cong"'] \arrow[d, "\alpha"'] & f(x) \arrow[d, equal] \\ f(x) \arrow[r, equal, "\cong"] & f(x) \end{tikzcd}$$

Hence, taking $\mathcal{C}$ to be functors injective on objects, $\mathcal{W}$ to be categorical equivalences, and $\mathcal{F}$ to be Grothendieck fibrations (which coincide with isofibrations in groupoids), this would conclude the argument.

In order to adapt this to a directed setting, we need to identify the left and right classes of morphisms that turn this factorization into a WFS. The most natural choice would be $\mathcal{L} \overset{\text{def}}{=}$ right adjoint right inverses, and $\mathcal{R} \overset{\text{def}}{=}$ Grothendieck fibrations. However, *neither of those classes are closed under retraction.*[9] We must hence weaken both of these classes in order to define a WFS.

There are a couple of simpler alternative solutions. We could notice that the above factorisation naturally forms an *algebraic weak factorisation system*: see [GT06, §4.4], [Gar09, §§2.10–2.13, 2.16] and [BG16, Ex. 29] for this particular example. An only slightly weaker one would be to notice that it forms a *cloven WFS*, as in [BG12, Def. 3.2.1]. Each of these structures generates an 'underlying WFS' by essentially taking the classes to be morphisms for which there exists some additional algebraic structure, which amounts to a 'witnessed' form of *retract closure*. However, we prefer to spell out the details of the morphism classes, as they lead to certain geometric intuitions.

3.1.1. *The missing classes.* We begin with the left class, as it is simpler.

Suppose $r \dashv i$. It then follows that we have a coherent unit $\eta : \mathrm{Id}_A \Rightarrow i \circ r$ and counit $\epsilon : r \circ i \Rightarrow \mathrm{Id}_X$ such that the *triangle identities*

$$\begin{tikzcd} i \arrow[r, Rightarrow, "\eta * i"] \arrow[dr, equal, "1_i"'] & i \circ r \circ i \arrow[d, Rightarrow, "i * \epsilon"] \\ & i \end{tikzcd} \qquad \begin{tikzcd} r \arrow[r, Rightarrow, "r * \eta"] \arrow[dr, equal, "1_r"'] & r \circ i \circ r \arrow[d, Rightarrow, "\epsilon * r"] \\ & r \end{tikzcd}$$

hold. Our case is special, in that we know two extra facts: not only is $r \circ i = \mathrm{Id}_A$, but we also have that the components of ϵ consist of identities. The triangle identities therefore reduce to

$$\eta * i = 1_i \qquad\qquad r * \eta = 1_r$$

[9] This is well-known for fibrations; see e.g. [LR19, Rem. 2.2.6].

Suppose now that $j : B \longrightarrow Y$ is a retract of $i : A \longrightarrow X$. We can leverage $r : X \longrightarrow A$ to construct a retraction $q : Y \longrightarrow B$ to j, and 'transport' η to a unit $\eta' : \mathrm{Id}_B \Rightarrow j \circ q$. However, only one of these coherence conditions holds of η', i.e. the first one. We hence define the left class to be the following class.

Definition 3.1. $i : A \longrightarrow X$ is a *future section* if there exists a retraction $r : X \longrightarrow A$ and a $\eta : \mathrm{Id}_X \Rightarrow i \circ r$ such that $\eta * i : i \Rightarrow i$ is equal to $1_i : i \Rightarrow i$.

The fibrations are a little more difficult to capture. North has described them as arising from a strictly symmetric internal relation [Nor17, Example 3.2.7], and also as those having the *enriched right lifting property*[11] [Nor19b] with respect to

$$\begin{tikzcd} \mathbb{1} \arrow[d, "i_1"] & & * \arrow[dr] & \\ \mathbb{2} & 0 \arrow[rr] & & 1 \end{tikzcd}$$

We unfold this property. First, recall that Grothendieck fibrations satisfy a lifting property.

Definition 3.2 (Liftings)**.** Let $p : E \longrightarrow B$, and let $\sigma : i \Rightarrow j$ be a natural transformation for functors $i, j : A \longrightarrow B$. Let $v : A \longrightarrow E$ be *over* j, in the sense that $p \circ v = j$. A *lifting* of σ along p consists of a functor $u : A \longrightarrow E$ over i, and a $\breve{\sigma} : u \Rightarrow v$ over σ, in the sense that $p * \breve{\sigma} = \sigma$.

Pictorially:

$$\begin{tikzcd} & & E \arrow[dd, "p"] \\ & & \\ A \arrow[uurr, dotted, bend left=40, "u"] \arrow[uurr, "v"'] \arrow[uurr, phantom, bend left=20, "\Downarrow\breve{\sigma}"] \arrow[rr, bend left=15, "i"] \arrow[rr, bend right=40, "j"'] \arrow[rr, phantom, "\Downarrow\sigma"] & & B \end{tikzcd}$$

The following lemma is given in [Pav90, §II.1.7], and is a simplified version of a much stronger result which can be found in the seminal paper of Gray [Gra66, Theorem 2.10].

Lemma 3.3 (Gray)**.** *A functor $p : E \longrightarrow B$ is a (cloven) Grothendieck fibration iff every $\sigma : i \Rightarrow j$ and $v : A \longrightarrow E$ with v over j has a lifting $\breve{\sigma}$ along p which is cartesian, in the sense that every component of $\breve{\sigma}$ is a cartesian morphism in E with respect to p.*

Thus, Grothendieck fibrations are the fibrations that have cartesian liftings. This is the basis for Street's abstract definition of a fibration internal to a 2-category [Str74; LR19]. By relaxing the requirement that the lift is cartesian we arrive at the following definition.

Definition 3.4 (Basic fibration)**.** A *basic fibration* is a functor $p : E \longrightarrow B$ along which chosen liftings of any $\sigma : i \Rightarrow j$ and $v : A \to B$ over j exist. Moreover, these liftings are required to be

(1) *natural*, in that for any $w : C \longrightarrow A$ we have $\widetilde{\sigma * w} = \breve{\sigma} * w$, and
(2) *normalised*, in that $\breve{1_j} = 1_v$.

[11] The enriched lifting property holds whenever diagonal fillers are given naturally with respect to an enrichment—in this particular case, the cartesian product. See [Rie14, Exercise 11.1.9] or [Awo18, Lemma 2.15] for more details.

In pictures:

$$
\begin{tikzcd}
 & & E \arrow[d, "p"] \\
C \arrow[r, "w"'] \arrow[urr, bend left, "v\circ w"'] \arrow[urr, dotted, bend left=60, "u\circ w"] \arrow[urr, phantom, bend left=40, "\Downarrow \breve{\sigma * w}"] & A \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow\sigma"] \arrow[ur, "v"'] \arrow[ur, dotted, bend left=40, "u"] \arrow[ur, phantom, "\breve{\sigma}"] & B
\end{tikzcd}
$$

It follows immediately by Gray's result that

Corollary 3.5. *Every Grothendieck fibration is a basic fibration.*

This definition of the right class is sufficiently weak so that

Lemma 3.6. *The class of basic fibrations is closed under retraction.*

Proof. Suppose

$$
\begin{tikzcd}
A \arrow[r, bend right, "v"'] \arrow[r, dotted, bend left, "g\circ h"] \arrow[r, phantom, "\Downarrow g*\breve{s*\sigma}"] \arrow[d, equal] & D \arrow[r, "f"] \arrow[d, "q"] \arrow[rr, equal, bend left] & E \arrow[r, "g"] \arrow[d, "p"] & D \arrow[d, "q"] \\
A \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow\sigma"] & B \arrow[r, "s"] \arrow[rr, equal, bend right] & A \arrow[r, "r"] & B
\end{tikzcd}
$$

Given v over j and $\sigma : i \Rightarrow j$, we whisker with s to obtain

$$s * \sigma : s \circ i \Rightarrow s \circ j$$

Then $p \circ (f \circ v) = s \circ q \circ v = s \circ j$, so we can lift $s * \sigma$ to $\breve{s * \sigma} : h \Rightarrow f \circ v$ for some $h : A \longrightarrow E$ over $s \circ i$, with $p * \breve{s * \sigma} = s * \sigma$. Then

$$g * \breve{s * \sigma} : g \circ h \Rightarrow g \circ f \circ v$$

But $g \circ f \circ v = v$, which is over j (w.r.t q), and

$$q \circ (g \circ h) = r \circ p \circ h = r \circ s \circ i = i$$

so $g \circ h$ is over i (w.r.t q), and

$$q * (g * \breve{s * \sigma}) = (q \circ g) * \breve{s * \sigma} = (r \circ p) * \breve{s * \sigma} = r * (p * \breve{s * \sigma}) = r * (s * \sigma) = \sigma$$

It is clear that this construction is natural. □

Letting $\mathcal{F}$ be the class of basic fibrations, we can now show that $\mathcal{FS} \pitchfork \mathcal{F}$.

Theorem 3.7. *If i is a future section, p is a basic fibration, and*

$$
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
C \arrow[r, "k"'] \arrow[ur, dotted, "d"] & B
\end{tikzcd}
$$

commutes, there is a diagonal filler d that makes the diagram commute.

Proof. Let $r \circ i = \mathrm{Id}$ and $\eta : \mathrm{Id}_C \Rightarrow i \circ r$, so $k * \eta : k \Rightarrow k \circ i \circ r$. But, as $p \circ (h \circ r) = k \circ i \circ r$, we can lift $k * \eta$ along the fibration:

$$
\begin{tikzcd}
 & E \arrow[d, "p"] \\
C \arrow[ur, dotted, bend left=60, "d"] \arrow[ur, "h \circ r"'] \arrow[ur, phantom, bend left=30, "\Downarrow \breve{k*\eta}"] \arrow[r, bend left, "k"] \arrow[r, bend right, "k \circ i \circ r"'] \arrow[r, phantom, "\Downarrow k*\eta"] & B
\end{tikzcd}
$$

This gives us the requisite filler d, with $p \circ d = k$. It remains to show that $d \circ i = h$; but—noting that $h \circ r \circ i = h$—we can also pre-whisker $k * \eta$ by i to get

$$
\begin{tikzcd}
 & & E \arrow[d, "p"] \\
A \arrow[r, "i"'] \arrow[urr, bend left, "h"'] \arrow[urr, dotted, bend left=60, "d'"] \arrow[urr, phantom, bend left=45, "\Downarrow \breve{k*\eta*i}"] & C \arrow[ur, dotted, bend left=60, "d"] \arrow[ur, "h \circ r"'] \arrow[ur, phantom, bend left=30, "\Downarrow \breve{k*\eta}"] \arrow[r, bend left, "k"] \arrow[r, bend right, "k \circ i \circ r"'] \arrow[r, phantom, "\Downarrow k*\eta"] & B
\end{tikzcd}
$$

As p is a basic fibration, we have the following naturality equation:

$$\breve{k * \eta} * i = \breve{k * \eta * i} : d' \Rightarrow h$$

Thus $d \circ i = d'$. But recall that, by the definition of future section, $\eta * i = 1_i$, so

$$\breve{k * \eta} * i = \breve{k * \eta * i} = \breve{k * 1_i} = \breve{1_{k \circ i}} = 1_h : d' \Rightarrow h$$

which is an identity, hence $d \circ i = d' = h$. □

Thus,

Theorem 3.8. *$(\mathcal{FS}, \mathcal{F})$ is a weak factorisation system on* $\mathbf{Cat}$.

Precedents in homotopy theory. To those familiar with classical homotopy theory, this factorisation is no surprise.

To begin, the comma category $\{B, f\}$ is a direct categorification of the *mapping path space*: see e.g. [Ark11, §3.5] and [May99, §7.2], where it is also defined as a pullback similar to the one in (1). One then directly proceeds to give an analogous factorisation of continuous maps through it.

Furthermore, the definition of basic fibration is essentially a categorification of notion of *Hurewicz fibration*, i.e. a surjective continuous function satisfying the *covering homotopy property* [Ark11, Def. 3.3.4] [May99, §7.1]. Interestingly, these are *not* the fibrations in the Quillen model structure on topological spaces. However, there is an alternative model structure on topological spaces, the *Strøm model structure*, for which they are. Its construction is technically challenging, but has been abstracted and generalised: see [BR13].

This sort of factorization also appears in more modern work on abstract homotopy theory. van den Berg and Garner [BG12] have shown that the same steps can be 'replayed' in any *path object category*. Such a category is equipped with a functorially given choice of 'path space object,' i.e. an internal category with an involution on paths, along with a mechanism for contracting every path to its endpoint. It is worth noticing that at no point during the construction of their cloven WFS do they use the involution structure, and hence that the construction can be carried through in directed settings too. Finally, they use enrichment to derive the same definition of basic fibration as we have: see [BG12, Prop 6.1.5].

In a slightly different setting, Williamson builds a model category out of structured abstractions, such as cylinders, path objects and intervals. The above factorisation is given in [Wil13, §IX], and attributed to [BG12].

Both sources call future sections by the name of *strong deformation retraction*. Williamson calls basic fibrations by the name of *normally cloven fibrations* [Wil13, Def. VIII.33].

3.1.2. *Geometric intuitions.* It is worth considering for a moment the geometric intuitions that underpin the class of future sections.

To begin, suppose that A is a full subcategory of X. We speak of a *(co)reflection* whenever there exists a left (right) adjoint to the inclusion $A \hookrightarrow X$. In that case, A is a *reflective subcategory* of X. Since (co)reflections involve an adjunction, an attendant universal property is induced. If we consider a category as a directed space, this universal property carries very strong spatial intuitions.

First, the components $\eta_x : x \to r(x)$ of the unit $\eta : \mathrm{Id}_X \Rightarrow i \circ r$ provide a *chosen path* from x to its retracted image $i(r(x)) = r(x)$ in the reflective subcategory A. Moreover, this choice of paths is *continuous*: if we think of a morphism $p : x \to y \in X$ as a directed path in X, naturality amounts to commutation of the following square:

$$\begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[d, "p"'] & r(x) \arrow[d, "r(p)"] \\ y \arrow[r, "\eta_y"'] & r(y) \end{tikzcd}$$

That is: r also reflects the path p to the path $r(p) : r(x) \to r(y)$ in a coherent way.

Second, the components of the unit are universal; this is to say that for each directed path $p : x \to a$ with $a \in A$, we have a *unique* factorisation

$$\begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[dr, "p"'] & r(x) \arrow[d, dashed] \\ & a \end{tikzcd}$$

So each path $p : x \to a$ *into* the reflective subcategory factorises uniquely through the chosen path $\eta_x : x \to r(x)$ from the domain into its reflected image.

Finally, it is easy to see that the components of the counit are identities. As in §3.1.1, this means that the triangle equations reduce to

$$\eta * i = 1_i \quad \text{and} \quad r * \eta = 1_r$$

Let us write out the first one: if a is an object of A, we have that

$$\eta_{i(a)} = \mathrm{id}_{i(a)} : i(a) \to i(a)$$

Thus, for a given point a of the subspace, the chosen path that retracts its image $i(a)$ under the inclusion to itself is the constant path that stays put on $i(a)$ itself.

The other triangle identity says that for every $x \in X$ we have

$$r\left(\eta_x : x \to i\left(r(x)\right)\right) = \mathrm{id}_{r(x)} : r(x) \to r(x)$$

That is: if we retract the chosen path $\eta_x : x \to i(r(x))$ into the subspace A, we obtain the constant path on $r(x)$. Geometrically, this essentially means that we are retracting η_x along itself.

Reflective subcategories can thus be seen as a *very orderly subspace* of a category: not only can the entire category be *retracted towards* to the subcategory along the collection of paths $\eta_x : x \to r(x)$, but also every path into it can be factored through such paths.

Future sections $i : A \longrightarrow X$ are weaker than reflective subcategories. First, they are not necessarily full. However, they are sections, so they are faithful and injective on objects. They only come equipped with a 'unit' $\eta : \mathrm{Id} \Rightarrow i \circ r$. This still satisfies the equation $\eta * i = 1_i$, which is to say that the chosen paths on the subcategory induced by i are constant. If we think of paths as being something we can walk along through time in a non-reversible manner, then a future section shows that we can retract the whole of X to A through a natural choice of paths along each point (object of the category). Moreover, the path given on points of the subcategory induced by i stay put during this retraction.

Regarding the universal property, we have that for each path $p : x \to i(a)$ with an endpoint in the image of i (i.e. in the induced 'subcategory') can be factored as the chosen path followed by a path in the subcategory, but *not uniquely*. In the diagram

$$\begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[dr, "p"'] & ir(x) \arrow[d, "i(\hat{p})"] & & r(x) \arrow[d, dotted, "\hat{p}"] \\ & i(a) & & a \end{tikzcd}$$

we can put $\hat{p} \stackrel{\text{def}}{=} r(f) : r(x) \to r(i(a)) = a$, which makes the triangle commute by the naturality of η and $\eta * i = 1_i$.

Notice how we have started employing directional language in the description of future sections: as η is not invertible, X can be retracted to $i(A)$, but is not 'homotopy equivalent' to it. It would thus be an error to employ the terminology of homotopy theory by following [BG12] in calling a future section a 'strong deformation retract': it is a directed analogue of it. Grandis [Gra09, §1.3.1] calls them *strong future deformation retracts*.

Furthermore, notice that this directionality was entirely arbitrary.

3.1.3. *Inverting the direction, I.* Unlike strong deformation retracts, whose attendant 'homotopy' $\mathrm{Id} \Rightarrow i \circ r$ is invertible, the quantum of direction involved in the definition of a future section is entirely arbitrary. In fact, we are free to invert it.

Definition 3.9. $i : A \longrightarrow X$ is a *past section* if there exists a retraction $r : X \longrightarrow A$ and a $\epsilon : i \circ r \Rightarrow \mathrm{Id}_X$ such that $\epsilon * i : i \Rightarrow i$ is equal to $1_i : i \Rightarrow i$.

Past sections are the directional dual ('co') to future sections: they demonstrate how going against the grain and 'running time backwards' allows a category X to recede back into a quasi-coreflective subcategory A. We may analogously ask for *opfibrations*, which come with chosen *oplifitings*.

Definition 3.10 (Oplifting)**.** Let $p : E \longrightarrow B$, and let $\sigma : i \Rightarrow j$ be a natural transformation for functors $i, j : A \longrightarrow B$. Let $u : A \longrightarrow E$ be *over* i, in the sense that $p \circ u = i$. An *oplifting* of σ along p consists of a functor $v : A \longrightarrow E$ over j, and a $\widehat{\sigma} : u \Rightarrow v$ over σ, in the sense that $p * \widehat{\sigma} = \sigma$.

Definition 3.11 (Basic opfibration)**.** A *basic opfibration* is a functor $p : E \longrightarrow B$ along which opliftings of any $\sigma : i \Rightarrow j$ and $u : A \longrightarrow E$ over i exist. Moreover, these liftings are required to be

(1) *natural*, in that for $w : C \longrightarrow A$ we have $\widehat{\sigma * w} = \widehat{\sigma} * w$, and
(2) *normalised*, in that $\widehat{1_i} = 1_u$.

In pictures:

$$\begin{tikzcd}
 & & E \arrow[d, "p"] \\
C \arrow[r, "w"'] \arrow[urr, bend left=60, "u\circ w"] \arrow[urr, dotted, "v\circ w"'] \arrow[urr, phantom, bend left=30, "\Downarrow \widehat{\sigma\circ w} = \hat{\sigma} * w"] & A \arrow[ur, bend left, "u"] \arrow[ur, dotted, "v"'] \arrow[ur, phantom, "\hat{\sigma}"] \arrow[r, bend left, "i"] \arrow[r, bend right, "j"'] \arrow[r, phantom, "\Downarrow\sigma"] & B
\end{tikzcd}$$

Wrting $\mathcal{PS}$ for the class of past sections and $\mathcal{F}^{\mathrm{op}}$ for the class of opfibrations[14] The results of §3.1 immediately dualise to past sections and basic opfibrations: both classes are closed under retraction, and form a lifting pair. It follows that

Theorem 3.12. *$(\mathcal{PS}, \mathcal{F}^{op})$ is a weak factorisation system on* $\mathbf{Cat}$.

We thus discover that in the presence of directionality the previously used factorisation now splits into two distinct ones!

3.1.4. *Strengthening the classes, I.* It is interesting to examine when the class of future (past) sections coincides with the more common categorical concept of inclusion of a (co)reflective subcategory. In fact, there is a very simple criterion that one can demand of a category, which makes the two classes coincide:

Definition 3.13. A *spacelike* category is one where all idempotents are identities.

This criterion may strike one as a bit arbitrary. It seems less so if we recall that something stronger holds in all groupoids: if $e \circ e = e$, then $e = \mathrm{id}$. Also, it encodes a somewhat intuitive geometric intuition: if walking twice along the 'directed homotopy class' e is the same as walking along it once, then we have enough freedom to unfold the loop e, so nothing stops us from doing this unfolding again.

If we have a future section $i : A \longrightarrow X$, then there exists a $r : X \longrightarrow A$ and a $\eta : \mathrm{Id}_X \Rightarrow i \circ r$ such that $\eta * i = 1_i$. By considering the naturality square at the component η_x, we see that

$$\begin{tikzcd}
x \arrow[r, "\eta_x"] \arrow[d, "\eta_x"'] & i(r(x)) \arrow[d, equal] \\
i(r(x)) \arrow[r, "i(r(\eta_x))"'] & i(r(x))
\end{tikzcd}$$

commutes. The right hand side of this diagram is the component $\eta_{i(r(x))}$, which is merely the identity, as $\eta * i = 1_i$. If we apply r to this square, we obtain

$$r(\eta_x) \circ r(\eta_x) = r(\eta_x)$$

which is to say that $r(\eta_x)$ is an *idempotent* in A. Recall that the missing coherence condition is $r(\eta_x) = \mathrm{id}_{r(x)}$: it follows that if our category is spacelike we obtain this coherence for free, and $r \dashv i$. Nevertheless, we can still generate the missing coherence by replacing r, as long as idempotents split.

Theorem 3.14. *If $i : A \longrightarrow X$ is a future section and all idempotents in A split, then i is right-adjoint-right-inverse $q \dashv i$ to some $q : X \longrightarrow A$.*

[14] The 'op' superscript is simply a symbol, and has no formal meaning here.

Proof. Let $i : A \longrightarrow X$ be a future section with respect to $r : X \longrightarrow A$ and $\eta : \mathrm{Id} \Rightarrow i \circ r$, satisfying $\eta * i = 1_i$. We already know that $r(\eta_x)$ is an idempotent, we use the axiom of choice to split it, thus obtaining a functor $q : X \longrightarrow A$ with the requisite properties:

$$\begin{tikzcd} r(x) \arrow[rr, "r(\eta_x)"] \arrow[dr, "r_x"'] & & r(x) \\ & q(x) \arrow[ur, "s_x"'] & \end{tikzcd} \qquad \begin{tikzcd} q(x) \arrow[rr, equal] \arrow[dr, "s_x"'] & & q(x) \\ & r(x) \arrow[ur, "r_x"'] & \end{tikzcd}$$

We have to exercise some care in splitting identities here: if $x = i(a)$, then $r(\eta_x) = r(\eta_{i(a)}) = r(\mathrm{id}_{i(a)}) = \mathrm{id}_a$, and we pick $r_{i(a)} = s_{i(a)} = \mathrm{id}_a$.

This defines the object part of a functor $q : X \longrightarrow A$. The morphism part is

$$x \xrightarrow{f} x' \quad \longmapsto \quad q(x) \xrightarrow{s_x} r(x) \xrightarrow{r(f)} r(x') \xrightarrow{r_x} q(x')$$

This assignment is functorial: letting $f : x \to x'$ and $g : x' \to x''$, we calculate

$$\begin{aligned} q(g) \circ q(f) &= r_{x''} \circ r(g) \circ s_{x'} \circ r_{x'} \circ r(f) \circ s_x \\ &= r_{x''} \circ r(g) \circ r(\eta_{x'}) \circ r(f) \circ s_x \\ &= r_{x''} \circ r(g) \circ r(\eta_{x'} \circ f) \circ s_x \\ &= r_{x''} \circ r(g) \circ r(ir(f) \circ \eta_x) \circ s_x \\ &= r_{x''} \circ r(g) \circ r(f) \circ r(\eta_x) \circ s_x \\ &= r_{x''} \circ r(g \circ f) \circ s_x \circ r_x \circ s_x \\ &= r_{x''} \circ r(g \circ f) \circ s_x \\ &= q(g \circ f) \end{aligned}$$

Obviously $q(i(f : a \to a')) = r_{i(a')} \circ r(i(f)) \circ s_{i(a)} = f$, so $q \circ i = \mathrm{Id}$. It remains to define $\eta' : \mathrm{Id} \Rightarrow i \circ q$. We let

$$\eta'_x \stackrel{\text{def}}{=} x \xrightarrow{\eta_x} i(r(x)) \xrightarrow{i(r_x)} i(q(x))$$

Evidently, $\eta'_{i(a)} = i(r_{ia}) \circ \eta_{ia} = \mathrm{id}_{i(a)}$. To prove naturality, consider the diagram

$$\begin{tikzcd} & x \arrow[r, "f"] \arrow[dl, "\eta'_x"'] \arrow[d, "\eta_x"] & x' \arrow[d, "\eta_{x'}"'] \arrow[dr, "\eta'_{x'}"] & \\ i(q(x)) \arrow[r, "i(s_x)"'] & i(r(x)) \arrow[r, "i(r(f))"'] & i(r(x')) \arrow[r, "i(r_{x'})"'] & i(q(x')) \end{tikzcd}$$

The bottom row is $i(q(f))$, and it commutes: the central square is a naturality square for η, the right triangle commutes by definition, and

$$i(s_x) \circ \eta'_x = i(s_x) \circ i(r_x) \circ \eta_x = i(r(\eta_x)) \circ \eta_x = \eta_x$$

where the last equality is one we showed before, by the naturality of η at η_x itself. It remains to show the 'problematic' triangle identity: we calculate

$$\begin{aligned} q(\eta'_x) &= q(i(r_x) \circ \eta_x) \\ &= r_x \circ q(\eta_x) \\ &= r_x \circ r_{i(r(x))} \circ r(\eta_x) \circ s_x \\ &= r_x \circ r(\eta_x) \circ s_x \\ &= r_x \circ s_x \circ r_x \circ s_x \\ &= \mathrm{id}_{r(x)} \end{aligned}$$ □

The above result can be adapted to show that a past section $i : A \longrightarrow X$ can be strengthened to a *left-adjoint-right-inverse* whenever idempotents split. Moreover, Paige North has shown[16] that Grothendieck fibrations are closed under retraction under the same condition. I believe this would imply that basic fibrations collapse to Grothendieck fibrations, though I shall leave that discussion to future work.

3.2. **The second factorisation.** To construct the $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ WFS on groupoids, which mimics to *mapping cylinder* factorisation, we seek to factor any functor $f : A \longrightarrow B$ as

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & E^f \arrow[ur, "r"'] & \end{tikzcd}$$

where i is injective-on-objects, and r is both a fibration and a categorical equivalence. The objects of the category E^f are the disjoint union of the objects of A and B. The hom-sets are defined by

$$E^f(x,y) \stackrel{\text{def}}{=} \begin{cases} A(x,y) & \text{if } x, y \in A \\ B(f(x), y) & \text{if } x \in A, y \in B \\ B(x,y) & \text{if } x, y \in B \end{cases}$$

This category may also be obtained as the pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_1"'] & B \arrow[d, "j"] \\ A \times \mathbb{2} \arrow[r, "k"'] & E^f \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

E^f is shaped like a barrel: the top of the barrel is A and the bottom is B. A and B are 'connected' by some 'diagonal' morphisms that cross from top to bottom. Such a diagonal morphism $a \to b$ where $a \in A$ and $b \in B$ is a path $f(a) \to b$ in B. Composition follows A at the top of the barrel, and B at the bottom. Post-composition at the diagonals is the same as in B, but pre-composition acts with f: if $u : a' \to a$, $v : b \to b'$, and $p : a \to b$, then writing $\circ_B$ for composition in B we have

$$v \circ p \circ u \stackrel{\text{def}}{=} v \circ_B p \circ_B f(u) : f(a') \to b' \in E(a', b')$$

The factorisation is obtained by the universal property of the pushout:

$$\begin{tikzcd} & A \arrow[r, "f"] \arrow[d, "i_1"'] & B \arrow[d, "j"] \arrow[ddr, equal, bend left] & \\ A \arrow[r, "i_0"] \arrow[urr, phantom, ""] \arrow[rr, bend right, "i"'] & A \times \mathbb{2} \arrow[r, "k"'] \arrow[dr, bend right, "\pi_1"'] & E^f \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "r"] & \\ & & A \arrow[r, "f"'] & B \end{tikzcd}$$

We define the functor $i \stackrel{\text{def}}{=} k \circ i_0 : A \times \mathbb{2} \longrightarrow E^f$. Intuitively, the 'reflector' $r : E^f \longrightarrow B$ is defined to be f at the top of the barrel (the 'A part'), and the identity elsewhere. It follows that

$$r \circ i = r \circ k \circ i_0 = f \circ \pi_1 \circ i_0 = f$$

[16]Private communication.

The functor $j : B \hookrightarrow E^f$ includes B into the 'B part' fully and faithfully, and has a retraction r. We can construct a natural transformation

$$\eta : \mathrm{Id}_B \Rightarrow j \circ r$$

For $x \in A$, $\eta_x : x \to f(x)$ 'crosses' from $x \in A$ to $f(x) \in B$. Hence it must be a morphism in $B(f(x), f(x))$, and we pick $\mathrm{id}_{f(x)}$. For $y \in B$, $\eta_y \stackrel{\text{def}}{=} \mathrm{id}_y : y \to y$. It is not hard to verify that $r \dashv j$.

Again, we have so far only assumed that A and B are categories. The construction of E^f when A and B are groupoids is even simpler: see [JT08, §2.2]. This more general case for arbitrary categories has been described in a blog post of Shulman [Shu12]. However, even in this more general setting it is not hard to see that $r : E^f \longrightarrow B$ is a categorical equivalence if B is a groupoid, and also an isofibration. Moreover, it is evident that $i : A \longrightarrow E^f$ is injective-on-objects, so that it is a cofibration in the model structure on groupoids.

However, if we seek to adapt this to a directed setting we run into the same problem as before. While the natural choice for a left class (injective-on-objects functors) is fine, the candidate right class of *left adjoint left inverses*, is not closed under retraction. Fortunately, dualising our approach in §3.1.1 solves this.

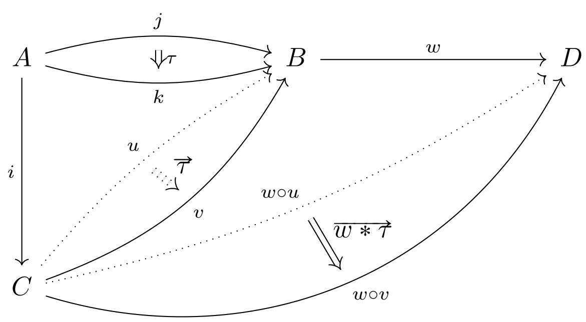
In order to obtain closure under retraction, we will take the dual ('op') of the notions in §3.1.1. First, the left class will be that of *cofibrations*, i.e. the functors that have opextensions.

Definition 3.15 (Extension)**.** Let $i : A \longrightarrow C$, and let $\tau : j \Rightarrow k$ be a natural transformation for functors $j, k : A \longrightarrow B$. Let $v : C \longrightarrow B$ be *under* k, in the sense that $v \circ i = k$. An *extension* of τ along i consists of a functor $u : C \longrightarrow B$ under j, and a $\overrightarrow{\tau} : u \Rightarrow v$ under τ, in the sense that $\overrightarrow{\tau} * i = \tau$.

Definition 3.16 (Basic cofibration)**.** A *basic cofibration* is a functor $i : A \longrightarrow C$ along which extensions of any $\tau : j \Rightarrow k$ and $v : C \longrightarrow B$ under k exist. Moreover, these extensions are required to be

(1) *natural*, in that for $w : B \longrightarrow D$ we have $\overrightarrow{w * \tau} = w * \overrightarrow{\tau}$, and
(2) *normalised*, in that $\overrightarrow{1_k} = 1_v$.

In pictures:



It is not immediately obvious that basic cofibrations are a bigger class than the one we initially envisioned, namely that of embeddings. However, we may adapt a well-known exercise in classical homotopy theory to show that this is indeed the case: see e.g. [Ark11, Prop. 3.2.6].

Lemma 3.17. *Any basic cofibration $i : A \longrightarrow C$ is a (non-full) embedding.*

Proof. Let $i : A \longrightarrow C$ be a basic cofibration. Define the category $\mathbb{C}\mathrm{ocone}(A)$ to consist of A with an 'extra' terminal object $*$ added, along with a unique path

$!_a : a \to *$ from every object $a \in A$. Let $k_0 : A \longrightarrow \mathbb{Cocone}(A)$ be the inclusion, and let $K_* : A \longrightarrow \mathbb{Cocone}(A)$ be the functor that collapses all of A to $*$.

The unique arrows to $*$ evidently form a natural transformation $! : k_0 \Rightarrow K_*$. Moreover, K_* can be trivially extended to all of C. Thus, we can extend $!$ along i to obtain a natural transformation $\overrightarrow{!} : r \Rightarrow K_*$:

$$\begin{tikzcd}
A \arrow[r, bend left, "k_0"] \arrow[r, bend right, "K_*"'] \arrow[r, phantom, "\Downarrow !"] \arrow[d, "i"'] & \mathbb{Cocone}(A) \\
C \arrow[ur, dotted, bend left, "r"] \arrow[ur, bend right, "K_*"'] \arrow[ur, phantom, "\overrightarrow{!}"]
\end{tikzcd}$$

Thus we obtain $r : C \longrightarrow \mathbb{Cocone}(A)$ such that $r \circ i = k_0$, so i is faithful and injective on objects, as $k_0(f : a \to a') = f : a \to a'$. $\square$

It may seem odd that a cofibration is not only injective on objects—as in the model structure on groupoids—but also faithful. Here is a way to think about that: cofibrations in groupoids are injective on objects, for they are not allowed to collapse different 'connected components.' However, if we think of cofibrations as inclusions into a space where *more cells have been attached*, it is clear that for groupoids they need not be faithful. For example, going from A to C may involve attaching a cell that allows us to homotopically contract a non-trivial loop into the identity. However, in the directed case we need to be careful to *not go against the grain*: there might be some directionality about which the 1-type has forgotten. In particular, we are *not* free to attach any cell we please.

In §4.4 we will prove that the functor $i : A \longrightarrow E^f$ defined before is a basic cofibration. Intuitively, this is easy to see: given any natural transformation $\tau : j \Rightarrow k$ with $v : E^f \longrightarrow C$ under k, we may define $u : E^f \longrightarrow C$ to be precisely j on the 'A part' of E^f, and v everywhere else. Then there is a $\overrightarrow{\tau} : u \Rightarrow v$, which follows τ on the 'A part' and is the identity on v everywhere else.

For the right class we will again weaken the definition of a left-adjoint-right-inverse, this time by dropping the other coherence condition.

Definition 3.18. $r : X \to A$ is a *future retraction* if there exists a section $i : A \longrightarrow X$ and a $\eta : \mathrm{Id}_X \Rightarrow i \circ r$ such that $r * \eta : r \Rightarrow r$ is equal to 1_r.

We write $\mathcal{C}$ for the class of basic cofibrations, and $\mathcal{FR}$ for the class of future retractions. We immediately obtain duals of all the results in §3.1, so that

Theorem 3.19. *$(\mathcal{C}, \mathcal{FR})$ is a weak factorisation system on* $\mathbf{Cat}$.

Precedents in homotopy theory. This factorisation is also not surprising from the point of view of classical homotopy theory.

The barrel E^f is a direct categorification of the *mapping cylinder* [Ark11, Def. 3.5.1] [May99, §6.2], where it is also defined as a pushout. Analogously, basic cofibrations are a categorification of *Hurewicz cofibrations*, i.e. a continuous maps satisfying the *homotopy extension property* [Ark11, Def. 3.2.1] [May99, §6.1].

In the process of building a model category from structured abstractions (cylinders, path objects, etc.), Williamson builds an abstract version of this factorisation in [Wil13, §IX]. Therein future retractions are called *strong deformation retractions*, and basic cofibrations are called *normally cloven cofibrations*.

In the directed setting, Grandis calls a very similar notion by the name of *upper fibration* [Gra09, §4.6.3].

3.2.1. *Inverting the direction, II.* As before, we may play the dualisation game in the 'op' direction here as well. On the one hand, we define *opcofibrations*.

Definition 3.20 (Opextension)**.** Let $i : A \longrightarrow C$, and let $\tau : j \Rightarrow k$ be a natural transformation for functors $j, k : A \longrightarrow B$. Let $u : C \longrightarrow B$ be *under* j, in the sense that $u \circ i = j$. An *extension* of τ along i consists of a functor $v : C \longrightarrow B$ under k, and a $\overleftarrow{\tau} : u \Rightarrow v$ under τ, in the sense that $\overleftarrow{\tau} * i = \tau$.

Definition 3.21 (Basic opcofibration)**.** A *basic opcofibration* is a functor $i : A \longrightarrow C$ along which opextensions of any $\tau : j \Rightarrow k$ and $u : C \longrightarrow B$ under j exist. Moreover, these extensions are required to be

(1) *natural*, in that for $w : B \longrightarrow D$ we have $\overleftarrow{w * \tau} = w * \overleftarrow{\tau}$, and
(2) *normalised*, in that $\overleftarrow{1_j} = 1_u$.

In pictures:

$$\begin{tikzcd}
A \arrow[r, bend left, "j"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "i"'] & B \arrow[r, "w"] & D \\
C \arrow[ur, bend left, "u"] \arrow[ur, dotted, bend right, "v"'] \arrow[ur, phantom, "\overleftarrow{\tau}"] \arrow[urr, "w \circ u"] \arrow[urr, dotted, bend right=60, "w \circ v"'] \arrow[urr, phantom, shift right=6ex, "\Downarrow \overleftarrow{w * \tau}"] & &
\end{tikzcd}$$

On the other hand, we define *past retractions*.

Definition 3.22. $r : X \longrightarrow A$ is a *past retraction* if there exists a section $i : A \longrightarrow X$ and a $\eta : i \circ r \Rightarrow \mathrm{Id}_X$ such that $r * \eta : r \Rightarrow r$ is equal to 1_r.

Writing $\mathcal{C}^{\mathrm{op}}$ for the class of basic opcofibrations, and $\mathcal{PR}$ for the class of past retractions, we may then prove that

Theorem 3.23. *$(\mathcal{C}^{op}, \mathcal{PR})$ is a weak factorisation system on* $\mathbf{Cat}$.

3.2.2. *Strengthening the classes, II.* As in §3.1.4, we can also use the triviality or splitting of idempotents here to strengthen the classes of future and past retractions. Suppose $r : A \longrightarrow X$ is a future retraction, with section j and $\eta : \mathrm{Id}_X \Rightarrow j \circ r$. Consider the naturality square

$$\begin{tikzcd}
jr(x) \arrow[r, "\eta_{jr(x)}"] \arrow[d, "\eta_{jr(x)}"'] & jrjr(x) \arrow[d, "\eta_{jrjr(x)}"] \\
jrjr(x) \arrow[r, "jr(\eta_{jr(x)})"'] & jrjrjr(x)
\end{tikzcd}$$

By the fact j is a section of r the right-hand morphism in this diagram is $\eta_{jr(x)}$ as well. By $r * \eta = 1_j$, the bottom morphism is the identity. Thus $\eta_{jr(x)}$ is an idempotent: if the category is spacelike it is an identity, and $r \dashv j$.

We also have the following analogue to Theorem 3.14.

Theorem 3.24. *If $r : X \longrightarrow A$ is a future section, and all idempotents in X split, then r is a left adjoint left inverse $r \dashv j$ to some $q : A \longrightarrow X$.*

3.3. **Strange inclusions.** Finally, we close the section on the one-sided structure by observing that the WFSs we have defined so far are not simply dual, but even more strongly related: in fact, some classes of morphisms are *included* in others. The pattern seems to be that left classes are included in left classes and symmetrically.

Theorem 3.25.

(1) Every past retraction is a basic fibration.
(2) Every future retraction is a basic opfibration.
(3) Every past section is a basic cofibration.
(4) Every future section is a basic opcofibration.

Proof. We prove (1), with the rest being entirely symmetric. Suppose $r : E \longrightarrow B$ is a past retraction, with section $s : B \longrightarrow B$ and 'counit' $\epsilon : s \circ r \Rightarrow \mathrm{Id}_E$. Suppose furthermore that we have the situation

$$\begin{tikzcd}
 & E \arrow[d, "r"] \\
C \arrow[ur, "v"] \arrow[ur, dotted, bend left=50] \arrow[ur, phantom, bend left=25, "\Downarrow\breve{\sigma}"] \arrow[r, bend left=10, "i"] \arrow[r, bend right=40, "j"'] \arrow[r, phantom, yshift=-2ex, "\Downarrow\sigma"] & B \arrow[u, bend right=40, "s"']
\end{tikzcd}$$

Whiskering the counit with v and s with σ, we obtain

$$\begin{aligned} \epsilon * v : s \circ r \circ v &\Rightarrow v &&: C \to E \\ s * \sigma : s \circ i &\Rightarrow s \circ j &&: C \to E \end{aligned}$$

But $s \circ r \circ v = s \circ j$ by assumption, so we can compose these vertically to define

$$\breve{\sigma} \stackrel{\text{def}}{=} (\epsilon * v) \circ (s * \sigma) : s \circ i \Rightarrow v : C \to E$$

Then

$$r * \breve{\sigma} = (r * (\epsilon * v)) \circ (r * (s * \sigma)) = ((r * \epsilon) * v) \circ ((r \circ s) * \sigma) = \sigma$$

as $r \circ s = \mathrm{Id}$ and $r * \epsilon = 1$. This is natural in pre-whiskering at C. □

3.4. **Concluding remarks.** When we add an additional quantum of direction to the usual factorisations used in the model structure on groupoids, we obtain a total of *four* WFSs on **Cat**!

These four factorisations have been identified before by Grandis [Gra09, Theorem 4.6.7]. In that strain of directed homotopy theory they are not discussed as WFSs, but rather as definable factorisations in various increasingly strong settings in which one can do abstract directed homotopy theory, e.g. with various cylinders and connection structures. However, the relationship between the second factorisation system $(\mathcal{PS}, \mathcal{F}^{\mathrm{op}})$ on small categories and directed type theory was first identified by Paige North [Nor19c].

Thus, there is increasing evidence that these four factorisation systems

$$(\mathcal{FS}, \mathcal{F}) \qquad (\mathcal{PS}, \mathcal{F}^{\mathrm{op}}) \qquad (\mathcal{C}, \mathcal{FR}) \qquad (\mathcal{C}^{\mathrm{op}}, \mathcal{PR})$$

with the inclusions

$$\mathcal{FS} \subseteq \mathcal{C}^{\mathrm{op}} \qquad \mathcal{PS} \subseteq \mathcal{C} \qquad \mathcal{FR} \subseteq \mathcal{F}^{\mathrm{op}} \qquad \mathcal{PR} \subseteq \mathcal{F}$$

seem to play an important rôle in directed homotopy. In particular, the first two show us how to 'fill' along future and past sections, which seem to take the place of trivial cofibrations. It seems that these two WFSs might have something interesting to say for the formulation of 'left' and 'right' path induction rules.

It would also be interesting to examine the notion of (co)fibrant replacement. These notions would formerly *replace* an object with a (co)fibrant one up to weak equivalence. The new factoriations give rise to *four* analogous notions. However, this time they have the flavour of *fibrant and cofibrant completion* rather than replacement. For example, using the $(\mathcal{FS}, \mathcal{F})$ system gives

$$A \xrightarrow{!} \mathbf{1} = A \xrightarrow{\text{future section}} A' \xrightarrow{\text{fibration}} \mathbf{1}$$

The first factor has the feel of an inclusion, so we may think of A as a fibrant completion. Analogously, using the $(\mathcal{C}, \mathcal{FR})$ system gives

$$\mathbf{0} \xrightarrow{!} A = A \xrightarrow{\text{cofibration}} A' \xrightarrow{\text{future retraction}} A$$

Again, the second factor is a retraction, so we may think of A' as a bigger object than A, i.e. as its cofibrant completion. Of course, this discussion is moot in $\mathbf{Cat}$ as every object is fibrant and cofibrant, but should acquire an interesting meaning in other settings.

Nevertheless, there is still an awful lot of directed homotopy theory we *cannot* do in this setting. More concretely, we have not yet solved the basic problem we outlined in §2, namely that of finding classes that provide basic notions of directed cylinders and directed path spaces. Consequently, we will proceed to the analysis of what I call the *two-sided structure* on $\mathbf{Cat}$, and which will provide a solution to that problem.

4. Two-sided structure

Even though we have elaborted quite a bit of directed structure corresponding to known notions from classical homotopy theory in $\mathbf{Cat}$, we have not yet found an abstract way to capture *directed cylinders* and *directed path spaces*. In §2 we remarked that the usual arrow $\langle \text{dom}, \text{cod} \rangle : X^2 \longrightarrow X$ cannot be a fibration, as that would imply that X is a groupoid. We are therefore led to the study of more complicated categorical gadgets.

Fortunately, deciding whether we have found the right notions is easy, as there is an elementary *litmus test* to which can put them: at the very least, we would like to be able to reproduce a very basic argument from the theory of model categories, viz. that *left homotopy implies right homotopy*. Recall how this argument works: suppose that $f \sim_\ell g : A \to X$, and that A is cofibrant. Then there exists a left homotopy $H : I(A) \to X$ with respect to some cylinder $A + A \xrightarrow{[i_0, i_1]} I(A) \xrightarrow{w} A$. Because A is cofibrant, i_0 is a cofibration, and in fact an acyclic one (by 2-out-of-3). Given any path object $X \xrightarrow{v} P(X) \xrightarrow{\langle p_0, p_1 \rangle} X \times X$ we form the following commuting diagram, and find a filler $K : I(A) \to P(X)$ for it—as $\langle p_0, p_1 \rangle$ is a fibration:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i_0"'] & X \arrow[r, "v"] & P(X) \arrow[d, "\langle p_0, p_1 \rangle"] \\ I(A) \arrow[rru, dotted, "K"] \arrow[rr, "\langle f \circ w, H \rangle"'] & & X \times X \end{tikzcd}$$

Picturing the horizontal dimension as that of cylinders and the vertical one as that of paths, we can visualise $K : I(A) \to P(X)$ as a *double homotopy* whose four sides are given by $p_i \circ K \circ i_j : A \to X$ for $i, j \in \{0, 1\}$. Given three of those sides, we have constructed the fourth:

$$\begin{tikzcd} f \arrow[r, "\text{constantly} f"] \arrow[d, "\text{constantly } f"'] & f \arrow[d, "H"] \\ f \arrow[r, dotted, "K \circ i_1"'] & g \end{tikzcd}$$

Hence $K \circ i_1 : A \to P(X)$ witnesses $f \sim_r g$.

The purpose of this section is to discover just enough structure on $\mathbf{Cat}$ so that this argument can be reproduced through WFS-like structures alone, without ever admitting that $(-) \times \mathbb{2}$ is left adjoint to $(-)^{\mathbb{2}}$.

This will be achieved in two steps. First, we will see how directed cylinders and path spaces can be viewed as *relations* on a category. This will naturally lead us to the discussion of more complicated categorical structure than what we have used so far, namely those of *two-sided discrete fibrations* and *two-sided codiscrete cofibrations*. Second, we will take a more prosaic view of directed cylinders, which comes from geometric intuitions: we will view them as *adjoint cylinders* in the sense of Lawvere [Law94; Law96]. Directed path spaces will be relaxations of the dual concept, namely *adjoint reflexive graphs*, which is due to Grandis [Gra09, §3.4.6]. These two developments will be joined by presenting a two-sided lifting property in §4.6, which will provide just enough power to prove that left homotopy implies right homotopy in a directed setting.

4.1. **Relations I.** Whereas in homotopy theory path spaces can be captured up to homotopy equivalence, the additional quantum of direction introduced with directed homotopy complicates the situation. In particular, directed path spaces behave more like *relations*: directed paths $a \rightsquigarrow b$ can be seen as relating their source a to their target b.

In particular, we seek some way to examine $(-)^{\mathbb{2}}$ as a *relation internal to* ***Cat***. There are many formalisms for doing so, such as allegories [FS90] and cartesian bicategories [CW87; Car+08]. However, the right formalism for the purposes of higher category theory is that of *(proarrow) equipments*, which can equivalently be expressed as *framed bicategories* [Shu08]; see the blog post [Shu09] for a general discussion.

Equipments can be used to do 'formal category theory' in various settings: they equip a category with *proarrows*, which stand for 'relations' between objects of the category, and which can be reindexed under both base and cobase change. Some standard examples of framed bicategories include

- sets, with functions for morphisms, and relations for proarrows;
- rings, with homomorphisms for morphisms, and modules for proarrows;
- $\mathcal{V}$-categories, with $\mathcal{V}$-functors for morphisms, and $\mathcal{V}$-profunctors (aka $\mathcal{V}$-distributors) for proarrows.

The last example is particularly important to us. Recall that a ($\mathbf{Set}$-)*profunctor* $\phi : A \nrightarrow B$ is just a functor

$$\phi : A^{\text{op}} \times B \to \mathbf{Set}$$

In particular, the *identity profunctor on a category* A, $\mathsf{id}_A : A \nrightarrow A$, is very close to the desired gadget: it is the hom-set functor

$$A(-,-) : A^{\text{op}} \times A \to \mathbf{Set}$$

which acts by precomposition and composition respectively.

One might ask why a profunctor is a relation. Suppose that we replace **Set** by $\mathbb{B} \stackrel{\text{def}}{=} \{0,1\}$, and take A to be a set so that $A^{\text{op}} = A$. Then a functor $A \times A \to \mathbb{B}$ is just a function, so it represents a relation. The basic insight is then due to Lawvere: *the category of sets is a generalised object of truth values*. Indeed, this is a central idea in *Weber 2-toposes* [Web07].

Much research has been interested in generalising the profunctor-like construction of equipments. It is straightforward to do so in the enriched setting: the framed bicategory described above has functors $A^{\text{op}} \otimes B \to \mathcal{V}$ as proarrows. However, it is much harder to construct proarrows in more general settings, such as arbitrary 2-categories: we often have neither a $(-)^{\text{op}}$ nor a tensor! There are two ways to do so: one is *fibrational*, and the other *cofibrational*.

The fibrational way is to consider some structures called *two-sided discrete fibrations* internal to (either a 2-category [Str74] or) a bicategory [Str80]. These are *spans*

$$\begin{tikzcd} & E \arrow[dl, "q"'] \arrow[dr, "p"] & \\ A & & B \end{tikzcd}$$

with appropriate liftings. In particular, p is a fibration, q is an opfibration, and they interact in a coherent way. Recall that spans represent relations: E is the 'relation' object, or *graph* of the relation, and the two morphisms project the source and target of each pair. It is shown in [Str80] that if we have a *fibrational bicategory* we can construct a category of profunctors. As these are spans, they compose by pullback. Unfortunately, this composite is not always a *discrete* two-sided fibration, and the requirements for making sure it can be reflected into one associatively are very strong, and do not work correctly in the enriched setting: see the comment section on the blog post [Shu09].

Alternatively, one can consider certain structures called *two-sided codiscrete cofibrations*, again internal to a bicategory. These are *cospans*

$$\begin{tikzcd} C \arrow[dr, "i"'] & & E \arrow[dl, "j"] \\ & D & \end{tikzcd}$$

with appropriate extensions. Recall that cospans may represent relations: E is the 'barrel,' or *cograph* of the relation. The two morphisms include A and B into it, but the barrel also contains 'cross-category objects' that relate elements of A to elements of B. [Str80] shows that the two-sided codiscrete cofibrations in $\mathcal{V}$-**Cat** exactly capture profunctors $A^{\text{op}} \otimes B \to \mathcal{V}$. Carboni et al. [Car+94] generalize this construction subject to certain axioms on an arbitrary bicategory.

The reader may wonder what all this has to do directed homotopy. Notice that, at least in the particular case of **Cat**, we have established a *triple* presentation of the same gadget:

$$\text{two-sided discrete fibrations} \simeq \text{profunctors} \simeq \text{two-sided codiscrete cofibrations}$$

Consider now the identity profunctor. Up to equivalence, we obtain

$$\begin{tikzcd} & A^{\mathbb{2}} \arrow[dl, "\text{cod}"'] \arrow[dr, "\text{dom}"] & \\ A & & A \end{tikzcd} \simeq A(-,-) : A^{\text{op}} \times A \to \mathbf{Set} \simeq \begin{tikzcd} A \arrow[dr, "i_0"'] & & A \arrow[dl, "i_1"] \\ & A \times \mathbb{2} & \end{tikzcd}$$

We have thus obtained both the directed path space and the directed cylinder as mirror images of the same object under this triple correspondence!

We will now introduce both two-sided fibrations and cofibrations in some generality, and then specialise our discussion to the discrete case. A very good reference for these matters in various 2-categorical settings is the expository article of Loregian and Riehl [LR19].

4.2. **Two-sided fibrations.** Two-sided fibrations are spans whose legs are respectively an opfibration and a fibration that act in a coherent way. They were introduced by Street[21] [Str74]. The original definition is given in terms of pseudo-algebras for a certain 2-monad. In the case of $\mathbf{Cat}$, this simplifies to the following definition [LR19].

Definition 4.1. A span

$$\begin{tikzcd} & E \arrow[dl, "q"'] \arrow[dr, "p"] & \\ A & & B \end{tikzcd}$$

is a *two-sided fibration* whenever

(1) each $f : b \to p(e) \in B$ has a cartesian lift $\check{f}(e) : f^*(e) \to e \in E$ which is q-vertical, i.e. $q(\check{f}) = \mathrm{id}_{q(e)}$,
(2) each $g : q(e) \to a \in A$ has an opcartesian lift $\hat{g}(e) : e \to g_!(e) \in E$ which is p-vertical, i.e. $p(\hat{g}) = \mathrm{id}_{p(e)}$, and
(3) the composite $f^*(e) \to e \to g_!(e)$ induces a canonical comparison morphism $g_!(f^*(e)) \to f^*(g_!(e))$ which is an isomorphism.

It is worth noting how one obtains this 'canonical comparison arrow.' Given $f : b \to p(e) \in B$, one obtains a functor $f^* : E_{-,p(e)} \longrightarrow E_{-,b}$ from the fiber of p over $p(e) \in B$ to the fiber of p over $b \in B$. We apply it to p-vertical oplift of g:

$$\begin{tikzcd} f^*(e) \arrow[r, "\check{f}(e)"] \arrow[dr, dotted, "f^*(\hat{g}(e))"'] & e \arrow[dr, "\hat{g}(e)"] & \\ & f^*(g_!(e)) \arrow[r, "\check{f}(g_!(e))"'] & g_!(e) \end{tikzcd}$$

$$\begin{tikzcd} b \arrow[r, "f"] \arrow[dr, equal] & p(e) \arrow[dr, equal] & \\ & b \arrow[r, "f"'] & p(e) \end{tikzcd}$$

[21] Under the name *bifibration*, which has since evolved to mean a single morphism that is both a fibration *and* an opfibration at once.

Applying q to the top square gives $q\left(f^*(\widehat{g}(e))\right) = g$. Thus $f^*(\widehat{g}(e))$ is q-vertical, so we may factor it oplift of g:

$$
\begin{tikzcd}
f^*(e) \arrow[r, "\check{g}(g_!(f^*(e)))"] \arrow[rrd, bend right, "f^*(\widehat{g}(e))"] & g_!(f^*(e)) \arrow[rd, dotted, "h"] & \\
 & & f^*(g_!(e))
\end{tikzcd}
$$

$$
\begin{tikzcd}
q(e) \arrow[r, "g"] \arrow[rrd, bend right, "g"'] & e \arrow[rd, equal] & \\
 & & e
\end{tikzcd}
$$

It is then this unique h that must be an isomorphism. Note that h is both p-vertical and q-vertical: the second fact may be obtained by applying p to its defining diagram. Alternatively, we may also obtain h by applying $g_!$ first and then factorising through the lift of f.

It is not difficult to prove that condition (3) above is equivalent to the combination of the following two conditions:

(1) Given $f : b \to b' \in B$, the functor $f^* : E_{-,b'} \longrightarrow E_{-,b}$ from the fiber of p over b' to the fiber of p over b maps opcartesian oplifts $\widehat{g}(e) : e \to g_!(e)$ to opcartesian morphisms.

(2) Given $g : a \to a' \in A$, the functor $g_! : E_{a,-} \longrightarrow E_{a',-}$ from the fiber of q over a to the fiber of q over a' maps cartesian lifts $\check{f}(e) : f^*(e) \to e$ to cartesian morphisms.

It is an easy exercise to show that any $h : x \to y \in E$ may be factorised as

$$
x \xrightarrow{h} y = x \xrightarrow{\widehat{q(h)}(x)} (qh)_!(x) \xrightarrow{k} (ph)^*(y) \xrightarrow{\widecheck{p(h)}(y)} y \tag{3}
$$

where k is both p-vertical and q-vertical.

We are interested in the case where the middle component k of this factorisation is actually the identity. This is true exactly for *two-sided discrete fibrations*. We again use the definition given in [LR19].

Definition 4.2. A span
$$
\begin{tikzcd}
 & E \arrow[ld, "q"'] \arrow[rd, "p"] & \\
A & & B
\end{tikzcd}
$$
is a *two-sided discrete fibration* whenever

(1) any $u : q(e) \to a' \in A$ has a *unique* p-vertical q-lift, i.e. there is a unique $\widehat{u} : e \to e' \in E$ with domain e such that both $q(\widehat{u}) = u$ and $p(\widehat{u}) = \mathrm{id}_{p(e)}$;

(2) any $v : b' \to p(e) \in B$ has a *unique* q-vertical p-lift, i.e. there is a unique $\check{v} : e' \to e \in E$ with codomain e such that both $p(\check{v}) = u$ and $q(\check{v}) = \mathrm{id}_{q(e)}$; and

(3) every morphism of E can be written as the composite of the two lifts: for each $f : e \to e' \in E$, cod $\widehat{q(f)}$ = dom $\widecheck{p(f)}$, and

$$
e \xrightarrow{f} e' = e \xrightarrow{\widehat{qf}} \cdot \xrightarrow{\widecheck{pf}} e'
$$

Given $a \in A$ and $b \in B$, define the *doubly-indexed fibre category* $E_{a,b}$ to be the subcategory of morphisms $f \in E$ such that $q(f) = \mathrm{id}_a$ and $p(f) = \mathrm{id}_b$. Evidently, conditions (1) and (2) imply that $E_{a,b}$ is *discrete*: as $p(\mathrm{id}) = \mathrm{id}$ and $q(\mathrm{id}) = \mathrm{id}$, every morphism of $E_{a,b}$ must be an identity morphism.

Condition (3) is a little unusual: it implies that for any $u : a \to a' \in A$ and $v :' b \to b \in B$ there is at most one arrow $f : e \to e' \in E$ that is both over u and v. This is used in a crucial manner in the following alternative characterisation.

Theorem 4.3. $\begin{tikzcd} & E \arrow[dl, "q"'] \arrow[dr, "p"] & \\ A & & B \end{tikzcd}$ *is a two-sided discrete fibration iff*

(1) there exist p-vertical q-opcartesian lifts,
(2) there exist q-vertical p-cartesian lifts, and
(3) each doubly-indexed fibre $E_{a,b}$ is a discrete category.

As a result, the legs of a two-sided discrete fibration are respectively a Grothendieck opfibration and a Grothendieck fibration.

Remark 4.4. Bénabou [Bén00, §6.4] implicitly claims to characterise two-sided fibrations as those spans for which (1') q is a Grothendieck opfibration, (2') p is a Grothendieck fibration, and (3') $E_{a,b}$ is discrete. While (1-3) imply (1'-3'), we cannot see how (1'-3') imply (1) and (2): we certainly have (op)cartesian lifts, but it is not clear why they are 'cross-vertical.'

Example 4.5. The central example of a two-sided discrete fibration is the span

$$\begin{tikzcd} & X^2 \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & \\ X & & X \end{tikzcd}$$

where X is a category, and X^2 is its arrow category, i.e. the functor category $[2, X]$. It is not hard to show that this is a two-sided discrete fibration. Let there be $p : x \to y \in X$, and $f : x' \to x \in X$ with the codomain of f above p wrt the fibration dom, i.e. with $\mathrm{dom}(p) = x$. There is a unique morphism into p that is both above f wrt to dom and also vertical wrt cod, namely

$$\begin{tikzcd} x' \arrow[r, "f"] \arrow[d, dashed, "p \circ f"'] & x \arrow[d, "p"] \\ y \arrow[r, equal] & y \end{tikzcd}$$

A similar situation is the case with dom and cod swapped: the unique morphism above $g : y \to y'$ wrt to cod and vertical with respect to dom is

$$\begin{tikzcd} x \arrow[r, equal] \arrow[d, "p"'] & x \arrow[d, dashed, "g \circ p"] \\ y \arrow[r, "g"'] & y' \end{tikzcd}$$

It is now easy to see that we may factorise a morphism as

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "p"'] & x' \arrow[d, "p'"] \\ y \arrow[r, "g"'] & y' \end{tikzcd} = \begin{tikzcd} x \arrow[r, equal] \arrow[d, "p"'] & x \arrow[r, "f"] \arrow[d, dashed] & x' \arrow[d, "p'"] \\ y \arrow[r, "g"'] & y' \arrow[r, equal] & y' \end{tikzcd}$$

by letting the dashed line be $p' \circ f = g \circ p$.

It is worth remarking that neither dom nor cod are discrete as (op)fibrations themselves: in fact,

$$\begin{tikzcd} x' \arrow[r, "f"] \arrow[d, dashed, "i^{-1}\circ p\circ f"'] & x \arrow[d, "p"] \\ y' \arrow[r, "\cong", "i"'] & y \end{tikzcd}$$

is cartesian over f wrt cod for any isomorphism $i : y' \xrightarrow{\cong} y$.

Example 4.6. Perhaps the simplest example of a two-sided (non-discrete) fibration is

$$\begin{tikzcd} & X^{\mathbb{3}} \arrow[dl, "\mathrm{cod}"'] \arrow[dr, "\mathrm{dom}"] & \\ X & & X \end{tikzcd}$$

where X is a category, and $X^{\mathbb{3}}$ is the category of *composable morphisms* in X, i.e. the functor category $[\mathbb{3}, X]$. The doubly-indexed fiber $X^{\mathbb{3}}_{x,y}$ now has morphisms of the form

$$\begin{tikzcd} x \arrow[r, equal] \arrow[d, "p"'] & x \arrow[d, "u"] \\ z \arrow[r, "h"] \arrow[d, "q"'] & z' \arrow[d, "v"] \\ y \arrow[r, equal] & y \end{tikzcd}$$

While $v \circ u = q \circ p$ implies that the two objects are strongly related they are nonetheless not equal, and there may be many choices of h that make the two squares commute. Working out the factorisation of a morphism of $X^{\mathbb{3}}$ into an opcartesian arrow followed by a doubly-vertical and a cartesian arrow is a jolly exercise that we leave to the reader.

A theorem of Street [Str74] generates two-sided discrete fibrations in $\mathbf{Cat}$ that feel awfully familiar. To begin, let $A \xrightarrow{f} X \xleftarrow{g} B$ be a cospan of functors. Let $\{f, g\}$ be the comma category of that cospan: its objects are morphisms $\alpha : f(x) \to g(y)$ for $x \in A$ and $y \in B$, and its morphisms are commuting squares

$$\begin{tikzcd} f(x) \arrow[r, "f(\beta)"] \arrow[d, "\alpha"'] & f(x') \arrow[d, "\alpha'"] \\ g(y) \arrow[r, "g(\gamma)"'] & g(y') \end{tikzcd} \tag{4}$$

We define two functors $A \xleftarrow{p} \{f, g\} \xrightarrow{q} B$ by

$$\begin{tikzcd} y \xrightarrow{\gamma} y' & \arrow[l, mapsto, "p"'] & f(x) \arrow[r, "f(\beta)"] \arrow[d, "\alpha"'] & f(x') \arrow[d, "\alpha'"] & \arrow[r, mapsto, "q"] & x \xrightarrow{\beta} x' \\ & & g(y) \arrow[r, "g(\gamma)"'] & g(y') & & \end{tikzcd}$$

We then have the

Theorem 4.7 (Street)**.** *The span*

$$\begin{tikzcd} & \{f,g\} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ B & & A \end{tikzcd}$$

is a two-sided discrete fibration.

As a corollary, p is a fibration and q is an opfibration. This puts both the factorisations $(\mathcal{FS}, \mathcal{F})$ and $(\mathcal{PS}, \mathcal{F}^{\mathrm{op}})$ into perspective: the constructions are the special case of this theorem when the cospan is either $B \xrightarrow{\mathrm{Id}} B \xleftarrow{f} A$ or $A \xrightarrow{f} B \xleftarrow{\mathrm{Id}} B$. In that special case, q (resp. p) also happens to be a left adjoint (resp. right adjoint) to an inclusion.

4.2.1. *Factorising spans.* Revisiting the factorization for $(\mathcal{FS}, \mathcal{F})$ as defined by the pullback diagram (2), one immediately sees that it actually provides a factorisation of the *graph* of $f : A \longrightarrow B$, i.e. the span

$$\langle \mathrm{Id}, f \rangle : A \longrightarrow A \times B$$

into a right adjoint right inverse $i : A \longrightarrow \{B, f\}$, and a two-sided discrete fibration $\langle q, p \rangle$ which is the span corresponding to the comma category $\{B, f\}$. Similarly, the pullback diagram used for $(\mathcal{FS}, \mathcal{F}^{\mathrm{op}})$ provides a factorisation of the *opgraph* of $f : A \longrightarrow B$, i.e. $\langle f, \mathrm{Id} \rangle : A \longrightarrow B \times A$ into a left adjoint left inverse and a two-sided discrete fibration.

Finally, these two constructions can be combined to factorise *any* span. Given a span $\langle g, f \rangle : E \to A \times B$, consider the diagram

(5)
$$\begin{tikzcd}
E \arrow[rr, bend left, "f"] \arrow[rr, dashed, "i"] \arrow[rrd, dashed, "i"'] \arrow[rd, dashed, "h"] \arrow[rdd, dashed, "j"] \arrow[dd, bend right, "g"'] & & B \arrow[rd, bend left, "\mathsf{refl}"] & & \\
& (g \mid f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \{B, f\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B^2 \arrow[r, "\mathrm{dom}"] \arrow[d, "\mathrm{cod}"] & B \\
A \arrow[rd, bend right, "\mathsf{refl}"'] & \{g, A\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[r, "f"'] \arrow[d, "g"] & B & \\
& A^2 \arrow[r, "\mathrm{dom}"'] \arrow[d, "\mathrm{cod}"'] & A & & \\
& A & & &
\end{tikzcd}$$

where i is defined essentially as above, and j similarly. These two arrows make the square formed by the top-left opspan commute, and we hence obtain h by the universal property of the pullback.

The span $\langle u, v \rangle : (g \mid f) \to A \times B$ formed by the left-and-downwards and the top-and-across composites is *not* a discrete fibration, but a proper two-sided fibration. The category $(g \mid f)$ can be described more succinctly[24] as $A^2 \times_A E \times_B B^2$. That is: its objects consist of an object $e \in E$, a path $q : g(e) \to a \in A$, and a path $p : b \to f(e) \in B$. Evidently

$$E \xrightarrow{\langle g,f \rangle} A \times B = E \xrightarrow{h} (g \mid f) \xrightarrow{\langle u,v \rangle} A \times B$$

[24]In fact, two-sided fibrations are algebras for this 2-monad.

4.2.2. *A note on composing two-sided fibrations: span repair.* If two-sided fibrations are meant to have a (higher-dimensional) relational flavour, then we should also expect them to admit a notion of *relational composition*. This is indeed the case, and it works by pullback. More specifically, in the diagram

$$\begin{tikzcd}
 & & E \times_B F \arrow[dl, "j"'] \arrow[dr, "i"] \arrow[dd, phantom, very near start, "\lrcorner"] & & \\
 & E \arrow[dl, "p"'] \arrow[dr, "q"] & & F \arrow[dl, "h"'] \arrow[dr, "k"] & \\
B & & A & & C
\end{tikzcd}$$

if the spans $\langle p, q\rangle$ and $\langle h, k\rangle$ are two-sided fibrations, then so is $\langle q \circ j, k \circ i\rangle$. This is shown in detail in the context of ∞-cosmoi in the upcoming book by Riehl and Verity [RV19, p. 11.2.6].

However, this composition does *not* preserve discreteness: we might have $\langle p, q\rangle$ and $\langle h, k\rangle$ are two-sided discrete fibration, yet the composite fibration need not be discrete. We can see this if we compose $\langle \text{dom}, \text{cod}\rangle : A^2 \longrightarrow X$ along itself: the resulting fibration, which is equivalent to $A^3 \longrightarrow A \times A$ as given in Example 4.6, is certainly not discrete.

What is needed in this scenario, and which was already discussed by Street [Str74], is what I like to call *mechanism of repair*. The total category A^3 is indeed a relation over points of A, but it is a composite one: it retains information about the *midpoint*. What is needed is a 2-categorical step analogous to that used to rectify the composition of relations in *regular categories*.

This kind of repair on two-sided fibrations is a known thorn on the side of higher category theorists. Street [Str74], who mentions it in passing, calls it a "tensor product of bimodules coequalizer." Hermida [Her01] simply calls it a coequalizer. Riehl and Verity identify the requisite general notion as a "homotopy coinverter," which is a type of colimit that is not available in the ∞-cosmos setting: see [RV19, §12]. To solve this, they move to the setting of *virtual double categories*, which allows them 'multiarrows' from many two-sided discrete fibrations to one, thus never having to confront composition directly.

I am not sure which of the two concepts—discrete or non-discrete—is the right one for a 'directed homotopy' theory. On the one hand, the discrete one is well-understood, and—as shown by Riehl and Verity—neatly expresses bimodules for $(\infty, 1)$-categories. Nevertheless, it does not behave well under composition, and the discreteness seems limited: recall, for example, that discrete objects in the ∞-cosmos of quasi-categories are actually the *Kan complexes* [RV19, p. 1.2.24], viz. the undirected spaces. Thus, it seems that discreteness eliminates all undirected structure.

4.3. **Relations II.** In §4.1 we remarked that, at least in the case of $\mathbf{Cat}$, there is a triple correspondence between profunctors, two-sided discrete fibrations, and two-sided codiscrete cofibrations. In order to motivate our discussion we will now sketch the details of this correspondence.

Let $\phi : A \longrightarrow\!\!\!\!\!\!\mapsto B$ be a profunctor, i.e. a functor $A^{\text{op}} \times B \rightarrow \mathbf{Set}$. ϕ assigns to each pair of objects $a \in A$, $b \in B$ a set $\phi(a, b)$. This set can be construed as the set of *evidence* that a is related to b by ϕ. ϕ also provides an *action* of morphisms on relation witnesses. For example, if $e \in \phi(a, b)$ and $f : a' \rightarrow a \in A$, then

$$e \cdot f \stackrel{\text{def}}{=} \phi(f, \mathbf{id}_b)(e) \in \phi(a', b)$$

is a witness that a' and b are related. Similarly, if $g : b \to b'$ we have that

$$g \cdot e \stackrel{\text{def}}{=} \phi(\mathrm{id}_a, g)(e) \in \phi(a, b')$$

Functoriality, which in the above notation takes the simple form

$$e \cdot (f_2 \circ f_1) = (e \cdot f_2) \cdot f_1 \qquad (g_2 \circ g_1) \cdot e = g_2 \cdot (g_1 \cdot e) \qquad (g \cdot e) \cdot f = g \cdot (e \cdot f)$$

serves to ensure that this action is coherent with respect to A and B. Profunctors form both a category, but also the paradigmatic example of a bicategory, whose identity 1-cells are the hom-functors$X(-,-) : X^{\mathrm{op}} \times X \longrightarrow \mathbf{Set}$. General feferences include [Bor94, §7.7], [Bén00], and Joyal's CatLab.[26]

4.3.1. *The two-sided Grothendieck construction.* Given a two-sided discrete fibration $\langle q, p\rangle : E \longrightarrow A \times B$, the construction of a profunctor $\phi_E : A^{\mathrm{op}} \times B \longrightarrow \mathbf{Set}$ is reasonably evident. The pair $(a \in A, b \in B)$ is sent to the doubly-indexed fibre $E_{a,b}$ (which is a *set*, as the fibration is discrete). Furthermore, we have to define the functorial action of ϕ on morphisms; this is easy, as there is at most one lift over two morphisms in A and B.

Conversely, one can construct a two-sided discrete fibration from a profunctor. In fact, this is a *two-sided Grothendieck construction.* Given $\phi : A \rightarrow\!\!\!\!\!\!\mapsto B$, we let $\int \phi$ be the category with

objects: $(a \in A, b \in B, e \in \phi(a, b))$

morphisms: $(f, g) : (a, b, e) \to (a', b', e')$ are pairs of morphisms $f : a \to a'$ and $g : b \to b'$ such that $g \cdot e = e' \cdot f$.

That is: (f, g) is a morphism from $e \in \phi(a, b)$ to $e' \in \phi(a', b')$ just if pulling e' back along $f : a \to a'$ to get $e' \cdot f \in \phi(a, b')$ yields the same result as pushing e forwards along $g : b \to b'$ to obtain $g \cdot e \in \phi(a, b')$. Then, the span defined by

$$\begin{tikzcd} & \int\phi \arrow[dl, "q"'] \arrow[dr, "p"] & \\ B & & A \end{tikzcd} \quad \stackrel{\text{def}}{=} \quad \begin{tikzcd} & e \in \phi(a,b) \arrow[dl, mapsto, "q"'] \arrow[dr, mapsto, "p"] & \\ b & & a \end{tikzcd}$$

is a two-sided discrete fibration. For example, pulling back along $f : a \to a'$ provides a lift

$$\begin{tikzcd} e \cdot f \in \phi(a,b) \arrow[r, dashed, "{(f, \mathrm{id}_b)}"] & e \in \phi(a', b) & & \int\phi \arrow[d, "p"] \\ a \arrow[r, "f"] & a' & & A \end{tikzcd}$$

which is unique above f with respect to p, and vertical with respect to q.

These constructions extend to an equivalence between the subcategory of spans that are two-sided discrete fibrations and the functor category $[A^{\mathrm{op}} \times B, \mathbf{Set}]$: see [LR19, Theorem 2.3.2].

4.3.2. *The collage.* Far more well-known than the aforementioned perspective is the construction of the *collage* of a profunctor $\phi : A \rightarrow\!\!\!\!\!\!\mapsto B$. It is the category $A \star_\phi B$ whose objects are the disjoint union of objects of A and B, with morphisms are

$$(A \star_\phi B)(x, y) \stackrel{\text{def}}{=} \begin{cases} A(x, y) & \text{if } x, y \in A \\ \phi(x, y) & \text{if } x \in A,\ y \in B \\ B(x, y) & \text{if } x, y \in B \end{cases}$$

[26] At the time of writing (November 9, 2019) many equations on the CatLab are not rendered correctly.

The collage has a useful pictorial representation:

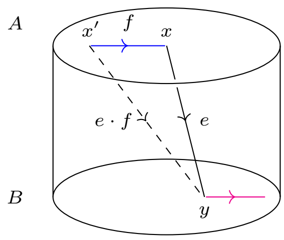


FIGURE 2. A collage

The 'top' of this structure is the category A, and the 'bottom' is the category B. The black diagonal arrow depicts an element $e \in \phi(a, b)$, which in this setting is sometimes called a *heteromorphism*. Near the top is a blue line, which stands for a morphism $f : x \to x' \in A$. The diagonal dashed arrow is the result of the contravariant action of f on e, namely $e \cdot f$. The magenta line at the bottom represents an arbitrary morphism of B.

Joyal captures such shapes by the following definition.

Definition 4.8 (Barrel)**.** A *barrel* is functor $f : X \longrightarrow \mathbb{2}$.

The idea is that the fibre $f^{-1}(0)$ over 0—the *top* of the barrel[28]—stands for A in the above picture, whereas the fibre $f^{-1}(1)$ over q—the *bottom* of the barrel—stands for B. Moreover, the top of the barrel is a *sieve*, in that precomposing any arrow to an arrow at the top yields again an arrow at the top. The bottom of the barrel has the dual property, i.e. is a *cosieve*. The heteromorphisms are all then sent to the walking arrow $0 \to 1 \in \mathbb{2}$. Joyal demonstrates an equivalence between the category[29] of profunctors and the slice category $\mathbf{Cat}/\mathbb{2}$: up to iso, every barrel is a collage of some profunctor.

In the next section we will show that every such collage is a *two-sided codiscrete cofibration*, i.e. a cospan with a structure dual to the two-sided discrete fibrations of §4.2.

4.3.3. *Representable profunctors.* Given a morphism $f : a \to b \in A$, one can immediately obtain a presheaf $A(-, f) : A^{\mathrm{op}} \longrightarrow \mathbf{Set}$. In a similar manner, a functor $f : A \longrightarrow B$ gives rise to *two* profunctors. The first one is the profunctor

$$B(-, f(-)) : B^{\mathrm{op}} \times A \longrightarrow \mathbf{Set}$$

which we denote by $\phi_f : B \twoheadrightarrow A$. The second one is the profunctor

$$B(f(-), -) : A^{\mathrm{op}} \times B \longrightarrow \mathbf{Set}$$

which we denote by $\phi^f : A \twoheadrightarrow B$.

Using the results of §4.2 we see that, when seen as two-sided discrete fibrations, the profunctors ϕ_f and ϕ^f correspond the comma categories $\{B, f\}$ and $\{f, B\}$. Indeed, the objects of $\int \phi^f$ are exactly the pairs $(a \in A, b \in B, f \in B(f(a), b))$.

In an entirely dual manner, the collage of the profunctor ϕ^f is

$$(A \star_{\phi^f} B)(x, y) \stackrel{\text{def}}{=} \begin{cases} A(x, y) & \text{if } x, y \in A \\ B(f(x), y) & \text{if } x \in A, y \in B \\ B(x, y) & \text{if } x, y \in B \end{cases}$$

[28]What we call top Joyal calls bottom and vice versa.

[29]Not to be confused with the bicategory of profunctors.

which is exactly the construction of E^f that we used for $(\mathcal{C}, \mathcal{FR})$ in §3.2.

Finally, in the case of the identity profunctor $\mathsf{id}_X : X \longrightarrow\!\!\!\!\!\!\!\!\;\mid\;\; X$, $\int \mathsf{id}_X$ is exactly $X^{\mathbb{2}} = \{\mathrm{Id}_X, \mathrm{Id}_X\}$, and $X \star_{\mathsf{id}_X} X$ is isomorphic to $X \times \mathbb{2}$. Thus, we obtain the path space and the cylinder of a space from the identity profunctor.

To conclude, not only do we get the one-sided structure as a special case of the two-sided structure for the representables, but we are also given cylinders and path spaces 'for free.'

4.4. **Two-sided cofibrations.** Dualising the two-sided fibrations of §4.2 leads to *two-sided cofibrations*. We may define them as being two-sided fibrations on $\mathbf{Cat}^{\mathrm{op}}$, or pseudo-algebras for the 2-monad

$$\begin{tikzcd} B \arrow[dr, "j"'] & & A \arrow[dl, "i"] \\ & C & \end{tikzcd} \quad \mapsto \quad \begin{tikzcd} B \arrow[dr] & & A \arrow[dl] \\ & (A \times \mathbb{2}) +_A C +_B (B \times \mathbb{2}) & \end{tikzcd}$$

In the interest of brevity and simplicity we proceed directly to the codiscrete variant, which has the following simpler description.

Definition 4.9. The cospan $\begin{tikzcd} B \arrow[dr, "j"'] & & A \arrow[dl, "i"] \\ & C & \end{tikzcd}$ is a *two-sided codiscrete cofibration* just if

(1) Given $\tau : f \Rightarrow g$ where $f, g : A \longrightarrow D$ and any $v : C \longrightarrow D$ below g wrt j there exists a *unique* j-covertical i-extension $\overrightarrow{\tau} : u \Rightarrow v$ of τ, i.e. a unique extension such that both $\overrightarrow{\tau} * i = \tau$ and $\overrightarrow{\tau} * j = \mathrm{Id}_{v \circ j}$.
(2) Given $\tau : f \Rightarrow g$ where $f, g : A \longrightarrow D$ and any $u : C \longrightarrow D$ below f wrt i there exists a *unique* j-covertical i-opextension $\overleftarrow{\tau} : u \Rightarrow v$ of τ, i.e. a unique opextension such that both $\overleftarrow{\tau} * j = \tau$ and $\overleftarrow{\tau} * i = \mathsf{Id}_{v \circ i}$.
(3) Every $\tau : u \Rightarrow v : C \longrightarrow D$ can be factorised as

$$u \xRightarrow{\overleftarrow{\tau * j}} w \xRightarrow{\overrightarrow{\tau * i}} v$$

The diagrams are the usual pictures for basic cofibrations and opcofibrations:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "i"'] & D \\ C \arrow[ur, dotted, bend left, "u"] \arrow[ur, bend right, "v"'] \arrow[ur, phantom, "\overrightarrow{\tau}"] & \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, "j"'] & D \\ C \arrow[ur, bend left, "u"] \arrow[ur, dotted, bend right, "v"'] \arrow[ur, phantom, "\overleftarrow{\tau}"] & \end{tikzcd}$$

where those extensions are *(op)cocartesian* and *cross-covertical.*

Example 4.10. The main interesting example of a two-sided codiscrete cofibration is the cospan

$$\begin{tikzcd} X \arrow[dr, "i_0"'] & & X \arrow[dl, "i_1"] \\ & X \times \mathbb{2} & \end{tikzcd}$$

which is one way of viewing the *directed cylinder* on a category X.

Example 4.11. As with two-sided fibrations, it is interesting to consider the simplest case which is *not codiscrete*. Analogously, that would be the cospan

$$\begin{tikzcd}
X \arrow[dr, "i_0"'] & & X \arrow[dl, "i_2"] \\
& X \times \mathbb{3} &
\end{tikzcd}$$

This can be pictured as a 'three-level cylinder' that consists of three copies of the category X, with the arms of the cospan including X at the top and at the bottom. Dually to 4.6, it is the cylinder-like object that we obtain as the pushout of two cylinders, where we identify the bottom of the first with the top of the second.

Street [Str80] proved that if we view $\mathcal{V}$-**Cat** as a bicategory and relax the above definition in an appropriately bicategorical way then we obtain exactly the $\mathcal{V}-\mathbf{Cat}$ profunctors: see [LR19, Theorem 4.3.2]. Instead, we prove the following more elementary result.

Proposition 4.12. *Any barrel $q : C \longrightarrow \mathbb{2}$ induces a two-sided codiscrete cofibration, viz.*

$$\begin{tikzcd}
q^{-1}(1) \arrow[dr, hook, "i_1"'] & & q^{-1}(0) \arrow[dl, hook', "i_0"] \\
& C &
\end{tikzcd}$$

Proof. Let $A \stackrel{\text{def}}{=} q^{-1}(0)$ and $B \stackrel{\text{def}}{=} q^{-1}(1)$. We show that i_0 is a cofibration. Suppose

$$\begin{tikzcd}
A \arrow[r, bend left, "j"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\tau"] \arrow[d, hook, "i_0"'] & D \\
C \arrow[ur, dotted, "u"] \arrow[ur, bend right, "v"'] \arrow[ur, phantom, "\overleftarrow{\tau}"] &
\end{tikzcd}$$

The barrel structure $q : C \longrightarrow \mathbb{2}$ on C partitions the morphisms into those in $A = q^{-1}(0)$, those in $B = q^{-1}(1)$, and the heteromorphisms that cross from the former to the latter. With that in mind, we define the necessary functor u by case analysis; we make it coincide with j on A, v on B, and use τ for heteromorphisms:

$$\begin{aligned}
u : C \quad & \longrightarrow \quad D \\
f : a \to a' \in A \quad & \longmapsto \quad j(f) : j(a) \to j(a') \\
g : b \to b' \in B \quad & \longmapsto \quad v(f) : v(b) \to v(b') \\
e : a \to b \quad & \longmapsto \quad v(e) \circ \tau_a : j(a) \to v(b)
\end{aligned}$$

Functoriality for the case of heteromorphisms follows from $v \circ i_0 = k$ and the naturality of τ in the third. One can extend τ to $\overleftarrow{\tau}$ by defining it to be τ on A, and identity elsewhere, i.e.

$$\overleftarrow{\tau}_a \stackrel{\text{def}}{=} \tau_a : j(a) \to k(a)$$

$$\overleftarrow{\tau}_b \stackrel{\text{def}}{=} \mathrm{id}_b : v(b) \to v(b)$$

Evidently, this is covertical w.r.t. i_1. For a heteromorphism $e : a \to b$ we have that

$$\begin{tikzcd}
j(a) \arrow[r, "u(e)"] \arrow[d, "\tau_a"'] & v(b) \arrow[d, equal] \\
k(a) \arrow[r, "v(e)"'] & v(b)
\end{tikzcd}$$

commutes by definition of $u(e) \overset{\text{def}}{=} v(e) \circ \tau_a$. Similarly, i_0 is an opcofibration.

It remains to show the factorisation property. Given $\alpha : u \Rightarrow v$ for $u, v : C \longrightarrow D$, we 'restrict' α to A and B by whiskering with i_0, i_1 to obtain

$$\alpha * i_0 : u \circ i_0 \Rightarrow v \circ i_0 : A \longrightarrow D$$
$$\alpha * i_1 : u \circ i_1 \Rightarrow v \circ i_1 : B \longrightarrow D$$

We can then extend those transformations to ones on C:

$$\begin{tikzcd}
A \arrow[r, bend left, "u \circ i_0"] \arrow[r, bend right, "v \circ i_0"'] \arrow[r, phantom, "\Downarrow \alpha * i_0"] \arrow[d, hook, "i_0"'] & D \\
C \arrow[ur, bend right, "v"'] \arrow[ur, dotted, bend left, "w"] \arrow[ur, phantom, "\overrightarrow{\alpha * i_0}"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, bend left, "u \circ i_1"] \arrow[r, bend right, "v \circ i_1"'] \arrow[r, phantom, "\Downarrow \alpha * i_1"] \arrow[d, hook, "i_1"'] & D \\
C \arrow[ur, bend left, "u"] \arrow[ur, dotted, bend right, "z"'] \arrow[ur, phantom, "\overleftarrow{\alpha * i_1}"] &
\end{tikzcd}$$

It is the case that $v = w$: chasing through the details of the extensions defined before, we have that both z and w coincide with u on A and with v on B. On heteromorphisms, w maps $e : a \to b$ to

$$v(e) \circ (\alpha * i_0)_a = v(e) \circ \alpha_a$$

and z maps it to

$$(a * i_1)_b \circ u(e) = \alpha_b \circ u(e)$$

which are equal by the naturality of α. We can then obtain $\overrightarrow{\alpha * i_0} \circ \overleftarrow{\alpha * i_1}$, and see that it is componentwise are equal to α. $\square$

I still owe the reader a result from §3.2, which now follows as a corollary of this proposition. As remarked in §4.3.3, the category E^f used in the 'mapping cylinder' factorisation

$$A \xrightarrow{f} B = A \xrightarrow{i} E^f \xrightarrow{r} B$$

can be obtained as the collage of a representable profunctor, and hence induces a two-sided codiscrete cofibration $[i, j] : A + B \longrightarrow E^f$. It thus follows that i is indeed a basic cofibration.

In a manner analogous to that of §4.2, the pushout that defines the factorisation through E^f, i.e.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "i_1"'] & B \arrow[d, "j"] \arrow[ddr, equal, bend left] & \\
A \arrow[r, "i_0"] \arrow[urr, phantom, ""] \arrow[rr, bend right, "i"'] & A \times 2 \arrow[r, "k"] \arrow[dr, bend right, "\pi_1"'] \arrow[ur, phantom, "\ulcorner"] & E^f \arrow[dr, dashed, "r"] & \\
 & & A \arrow[r, "f"'] & B
\end{tikzcd}$$

shows that can factor the cospan $[\mathrm{id}_B, f] : B + A \longrightarrow B$, i.e. the *cograph* of f, as

$$B + A \xrightarrow{[j,i]} E^f \xrightarrow{r} B$$

The dual diagram factors the *opcograph* $[f, \mathrm{id}_B]$.

A factorisation of general cospans $[g, f] : B + A \longrightarrow C$ as

$$B + A \xrightarrow{[x,y]} B \star A \longrightarrow C$$

where $[x, y]$ is the *free two-sided cofibration* may be obtained in a way dual to that of diagram (5). The category $B \star A$ is complicated to define, but an explicit description has been given by Fosco Loregian [Lor]: it is a barrel whose top is A, whose bottom is B, and whose heteromorphisms from $a \in A$ to $b \in B$ are given by the coend

$$\int^c A(f(a), c) \times B(c, g(b))$$

4.4.1. *Another note on composition: cospan repair.* Like with two-sided discrete fibrations, the composition of two-sided codiscrete cofibrations is also not codiscrete. However, their cospan-like nature seems more amenable to mechanisms of repair. In particular, all that is required is a particular kind of (2-categorical) orthogonal factorisation system: see abbreviated discussions of this point in [Shu09] and [LR19], and the original paper [Car+94]. A similar mechanism of repair has appeared in the context of Fong's corelations [Fon18].

Nevertheless, it is more clear in this case that the codiscrete variant is the ill-behaved one for the purposes of directed homotopy theory: we would definitely like to prove that the pushout of two cylinders is again a cylinder, exactly because this would imply that we may compose two (left) directed homotopies.

4.5. **Adjoint cylinders and reflexive graphs.** We have examined two-sided (discrete) fibrations and cofibrations *qua* cylinders and path spaces in a directed setting. However, a considerably less sophisticated structure is also a contender, as it seems to capture the correct geometric intuitions.

Definition 4.13. An *adjoint cylinder* for a category X is a diagram of adjunctions

$$\begin{tikzcd} X \arrow[d, bend right=60, "i_0"'] \arrow[d, bend left=60, "i_1"] \arrow[d, phantom, shift right=4ex, "\dashv"] \arrow[d, phantom, shift left=4ex, "\dashv"] \\ \mathbb{I}(X) \arrow[u, "r"'] \end{tikzcd}$$

where r is a common retraction to i_0 and i_1, i.e. $r \circ i_0 = r \circ i_1 = \mathrm{Id}_X$.

Adjoint cylinders were introduced by Lawvere [Law94; Law96], who also calls them *unity and identity for adjoint opposites (UIAO)*. Their defining diagram is usually drawn upside down (i.e. with $\mathbb{I}(X)$ on top) but we have reversed that in order to emphasize geometric intuition.

Indeed, an adjoint cylinder presents X as both a *coreflective* and *reflective* subcategory of $\mathbb{I}(X)$. In that way it almost perfectly captures the image of the directed cylinder in Figure 1, viz. two copies of X with a quantum of direction adjoined in the middle.

Example 4.14. $X \times \mathbb{2}$ is an adjoint cylinder for any category X. We define the two inclusions $i_k : X \longrightarrow X \times \mathbb{2}$ for $k \in \{0, 1\}$ by

$$x \xrightarrow{p} y \qquad \mapsto \qquad (x, k) \xrightarrow{(p, \mathrm{id}_k)} (x, k)$$

In the opposite direction we have the projection functor $\pi_1 : X \times \mathbb{2} \longrightarrow X$. which 'collapses' the cylinder by forgetting the cylindrical dimension. It is evident that $\pi_1 \circ i_k = \mathrm{id}_X$, and that $i_0 \dashv \pi_1 \dashv i_1$.

As a special case, this includes the walking morphism

$$\mathbb{1} \underset{i_1}{\overset{i_0}{\rightrightarrows}} \mathbb{2}$$

with the unique arrow $\mathbb{2} \longrightarrow \mathbb{1}$ as a common retraction.

Example 4.15. Unlike the previous example, here is one which is *not* the collage of a profunctor. It so happens that $X \times \mathbb{3}$ is an also an adjoint cylinder for X. We define the two inclusions $i_k : X \longrightarrow X \times \mathbb{2}$ for $k \in \{0, 2\}$ in the same way as before, and take $\pi_1 : X \times \mathbb{2} \longrightarrow X$ to be the common retraction. We have $i_0 \dashv \pi_1 \dashv i_2$. This barrel contains a 'middle tier' of additional evidence, viz. remnants of relational composition. The mechanisms of repair discussed in §4.4.1 are there to eliminate this middle tier, and turn it back into $X \times \mathbb{2}$.

Corresponding to the weakened classes of future and past sections of §3, we could *weaken* the adjunctions in the definition of an adjoint cylinder, so that it would be a cospan whose arms are a future and past section respectively, but with a common retraction (viz. cospans with some additional structure. My preferred name for this structure is that of *cyclical section*. As in §§3.1.4, 3.2.2, appropriate conditions on the underlying categories (i.e. being spacelike, or Cauchy complete) would allow us to strengthen a cyclical section to an adjoint cylinder.

Unlike two-sided codiscrete cofibrations, adjoint cylinders are closed under composition by pushout.[32]

Proposition 4.16. *Adjoint cylinders are closed under cospan composition.*

Proof. Let there be adjoint cylinders

$$\begin{tikzcd} D \arrow[d, bend right=60, "i_1"'] \arrow[d, bend left=60, "i_0"] \arrow[d, phantom, shift right=3, "\dashv"] \arrow[d, phantom, shift left=3, "\dashv"] \\ A \arrow[u, "r"] \end{tikzcd} \qquad \begin{tikzcd} D \arrow[d, bend right=60, "j_1"'] \arrow[d, bend left=60, "j_0"] \arrow[d, phantom, shift right=3, "\dashv"] \arrow[d, phantom, shift left=3, "\dashv"] \\ B \arrow[u, "q"] \end{tikzcd}$$

for D, which come with natural transformations

$$\begin{aligned} \eta &: \mathrm{Id}_A \Longrightarrow i_1 \circ r \\ \epsilon &: i_0 \circ r \Longrightarrow \mathrm{Id}_A \\ \theta &: \mathrm{Id}_B \Longrightarrow j_1 \circ q \\ \zeta &: j_0 \circ q \Longrightarrow \mathrm{Id}_B \end{aligned}$$

that satisfy the triangle equations.

[32]Moreover, this theorem also restricts to the weaker notion of cyclical section.

Cat is strictly 2-cocomplete, so we construct the composite adjoint cylinder by taking the strict 2-pushout $C \stackrel{\text{def}}{=} A +_D B$, which effectively sticks the second cylinder at the end of the first:

$$\begin{tikzcd}
D \arrow[dr, "i_0"'] \arrow[dddrr, equal, bend right] & & D \arrow[dl, "i_1"'] \arrow[dr, "j_0"] & & D \arrow[dl, "j_1"] \arrow[dddll, equal, bend left] \\
& A \arrow[dr, "k"] \arrow[ddr, bend right, "r"'] & & B \arrow[dl, "l"'] \arrow[ddl, bend left, "q"] & \\
& & C \arrow[d, dotted, "h"] & & \\
& & D & &
\end{tikzcd}$$

By the universal property of the pushout, we obtain a unique $h : C \longrightarrow D$ that is a common retract to both $k \circ i_0$ and $l \circ j_1$.

It remains to provide natural transformations that witness the reflection and coreflection. We only show the first, the other one being entirely dual. For succinctness we suppress composition and whiskering, and write 1 for the identity natural transformation. By the universal property of the strict 2-pushout, to *uniquely* define a natural transformation

$$\lambda : (ki_0)h \Rightarrow \mathrm{Id}_C$$

it suffices to define its 'components,' which are prewhiskered by k and l, viz.

$$\lambda_A : ki_0hk \Rightarrow k \quad \text{and} \quad \lambda_B : ki_0hl \Rightarrow l$$

subject to the restriction $\lambda_A * i_1 = \lambda_B * j_0$. The resulting λ then satisfies

$$\lambda * k = \lambda_A \qquad\qquad \lambda * l = \lambda_B \tag{6}$$

Noting that $ki_0hk = ki_0r$, we let

$$\lambda_A \stackrel{\text{def}}{=} k\epsilon : ki_0r \Rightarrow k$$

Noting that $ki_0hl = k_iq$, we let

$$\lambda_B \stackrel{\text{def}}{=} ki_0q \xRightarrow{k(\epsilon i_1)q} ki_1q = lj_0q \stackrel{l\zeta}{\Rightarrow} l$$

It is easy to see that

$$\lambda_A i_1 = k\epsilon i_1 = \lambda_B j_0$$

We can now check the first coherence condition, viz. that

$$\lambda k i_0 = \lambda_A i_0 = k\epsilon i_0 = 1$$

as $\epsilon i_0 = 1$ by the triangle equation of the adjunction. Furthermore, by naturality of the 2-pushout, the natural transformation $h\lambda : h \Rightarrow h$ is the unique natural transformation induced by $h\lambda_A$ and $h\lambda_B$. We calculate that

$$h\lambda_A = hk\epsilon = r\epsilon = 1$$

by the triangle equation, and similarly that

$$h\lambda_B = h(l\zeta \circ hk(\epsilon i_1)q) = hl\zeta \circ hk\epsilon i_1 q = q\zeta \circ r\epsilon i_1 q = 1 \circ 1 = 1$$

Thus it must be that $h\lambda = 1$. □

The dual gadget has the flavour of a directed path space.

Definition 4.17. An *adjoint reflexive graph* for a category X is a diagram of adjunctions

$$\begin{tikzcd} \mathbb{P}(X) \arrow[d, bend right=60, "r_0"'] \arrow[d, bend left=60, "r_1"] \\ X \arrow[u, "i"', "\dashv \quad\quad \dashv" description] \end{tikzcd}$$

where i is a common section to r_0 and r_1, i.e. $r_0 \circ i = r_1 \circ i = \mathrm{Id}_X$.

Here r_0 and r_1 can be read as mapping a 'path' in $\mathbb{P}(X)$ to its domain and codomain respectively. The data of the adjunction is evidence that we can 'retract' these paths in two directions: either backwards along the counit $\epsilon : i \circ r_0 \Rightarrow \mathrm{Id}_X$, or forwards along the unit $\eta : \mathrm{Id}_X \Rightarrow i \circ r_1$. Either one of these processes ends with the 'reflexivity' path on the (co)domain, as given by i.

Example 4.18. The arrow category $X^{\mathbb{2}}$ is an adjoint reflexive graph for every category X. The usual functors $\mathrm{dom}, \mathrm{cod} : X^{\mathbb{2}} \longrightarrow X$ satisfy $\mathrm{dom} \circ \mathsf{refl} = \mathrm{cod} \circ \mathsf{refl} = \mathrm{Id}_X$, where $\mathsf{refl} : X \longrightarrow X^{\mathbb{2}}$ is the usual functor $x \mapsto \mathrm{id}_x : x \to x$. Finally, it is the case that $\mathrm{dom} \dashv \mathsf{refl} \dashv \mathrm{cod}$.

Example 4.19. Recall the category $X^{\mathbb{3}}$ of *composable morphisms* of X defined in Example 4.6. It is also an adjoint reflexive graph: the common section $\mathsf{refl}_{\mathbb{3}} : X \longrightarrow X^{\mathbb{3}}$ to the domain and codomain functors is of course given by the 'degeneracy' $x \mapsto (x \xrightarrow{\mathrm{id}_x} x \xrightarrow{id_x} x)$. In Example 4.6 we showed that this is a two-sided fibration, but that it is *not* discrete. The mechanisms of repair outlined in §4.2.2 are there to 'compose' this object back into $X^{\mathbb{2}}$.

A result dual to Proposition 4.16 shows that adjoint reflexive graphs also compose. Moreover, this proof restricts to the weaker variant of a *cyclical retraction*, which would lack two of the coherence conditions (i.e. r_0 would be a past retraction and r_1 would be a future retraction, with a common section i).

One might reasonably ask whether span consisting of the two retractions of an adjoint reflexive graph is, in fact, a two-sided fibration. I am not so certain it is. As with the strange inclusions of coherent sections and retractions into basic (op/co)fibrations given in §3.3, I believe there must be a weaker notion of two-sided fibration (i.e. non-Grothendieck) into which adjoint reflexive graphs could be included.

4.6. **A two-sided lifting property.** We now have enough technology in place to formulate a lifting property which suffices to reproduce the argument from the start of this section, i.e. that left homotopy implies right homotopy.

In order to see from where this lifting propery comes, let us consider its simplest case. Recall from Example 4.14 the inclusions

$$\begin{tikzcd} \mathbb{1} & & * \arrow[dl, "i_0"'] \arrow[dr, "i_1"] & \\ \mathbb{2} & 0 \arrow[rr] & & 1 \end{tikzcd} \tag{7}$$

which satisfied $i_0 \dashv\, ! \dashv i_1$. Additionally, let the span $\langle q, p \rangle : E \longrightarrow B \times A$ be a two-sided discrete fibration. Given an arrow $g : d \to p(i) \in B$ for some $i \in E$, there is a unique lift $\breve{g}$ above g that is vertical wrt to q:

$$
\begin{array}{lccr}
 & d' \overset{\breve{g}}{\dashrightarrow} i & \\[2em]
\text{over} & d \xrightarrow{g} p(i) & \text{in } B \\[2em]
\text{and} & q(i) = q(i) & \text{in } A
\end{array}
$$

Every aspect of this save for the cross-verticality wrt A can be captured as a diagonal filler for the diagram

$$
\begin{tikzcd}
\mathbb{1} \arrow[r, "i"] \arrow[d, "i_1"'] & E \arrow[d, "p"] \\
\mathbb{2} \arrow[r, "g"'] \arrow[ur, dotted, "\breve{g}"] & B
\end{tikzcd}
$$

Similarly, we may find a unique oplift of $f : q(i) \to e \in A$ that is vertical wrt to p:

$$
\begin{array}{lccr}
 & i \overset{\hat{f}}{\dashrightarrow} e' & \\[2em]
\text{over} & p(i) = p(i) & \text{in } B \\[2em]
\text{and} & q(i) \xrightarrow{f} e & \text{in } A
\end{array}
$$

and we can capture this as a diagonal filler in the diagram

$$
\begin{tikzcd}
\mathbb{1} \arrow[r, "i"] \arrow[d, "i_0"'] & E \arrow[d, "q"] \\
\mathbb{2} \arrow[r, "f"'] \arrow[ur, dotted, "\hat{f}"] & B
\end{tikzcd}
$$

We can now place these diagrams side by side:

$$
\begin{tikzcd}
d' \arrow[r, dotted, "\breve{g}"] \arrow[rr, dotted, bend left, "h"] & i \arrow[r, dotted, "\hat{f}"] & e'
\end{tikzcd}
$$

$$
\begin{array}{lccr}
\text{over} & d \xrightarrow{g} p(i) = p(i) & \text{in } B \\[2em]
\text{and} & q(i) = q(i) \xrightarrow{f} e & \text{in } A
\end{array}
$$

Thus, the composite $h \overset{\text{def}}{=} \hat{f} \circ \breve{g}$ is over both f and g. However, *the object i has been 'lost' after composition.* Put simply, if we lift and oplift 'at the same time,' we lose the 'object control' that was afforded previously by the commutation of the upper triangles (i.e. that the codomain or domain of the lift was indeed the object i). Instead, we now have the diagrams

$$
\begin{tikzcd}
\mathbb{1} \arrow[r, "i"] \arrow[d, "i_0"'] & E \arrow[d, "q"] \\
\mathbb{2} \arrow[r, "f"'] \arrow[ur, dotted, "h"'] \arrow[u, phantom, "\Nwarrow \breve{g}"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{1} \arrow[r, "i"] \arrow[d, "i_1"'] \arrow[dr, phantom, "\Searrow \breve{f}"] & E \arrow[d, "p"] \\
\mathbb{2} \arrow[r, "g"'] \arrow[ur, dotted, "h"'] & B
\end{tikzcd}
$$

where the lower triangles commute, but the upper triangles are filled with natural transformations—which in this particular case coincide with morphisms.

By Example 4.14, the cospan $\mathbb{1} \to \mathbb{2} \leftarrow \mathbb{1}$ is an adjoint cylinder. We can in fact generalise this lifting property to any other adjoint cylinder.

In order to prove that, we first need a result that rephrases the definition of two-sided discrete fibrations (Def. 4.2) in terms of liftings and factorisations of natural transformations. A similar characterisation has been obtained in a bicategorical setting by Carboni et al. [Car+94]: see [LR19, Lemma 4.2.2].

Theorem 4.20. $\begin{tikzcd} & E \arrow[dl, "q"'] \arrow[dr, "p"] & \\ A & & B \end{tikzcd}$ *is a two-sided discrete fibration iff*

(1) *There is a unique q-vertical p-lift $\breve{\sigma} : u \Rightarrow v$ of $\sigma : i \Rightarrow j$, where $i, j : C \longrightarrow B$ and $v : C \longrightarrow E$ is over j w.r.t p.*
(2) *There is a unique p-vertical q-oplift $\hat{\sigma} : u \Rightarrow v$ of $\sigma : i \Rightarrow j$, where $i, j : C \longrightarrow A$ and $u : C \longrightarrow E$ is over i w.r.t q.*
(3) *Every $\sigma : u \Rightarrow v$ for $u, v : C \longrightarrow E$ can be factorised as*

$$u \overset{\widehat{q*\sigma}}{\Longrightarrow} d \overset{\widecheck{p*\sigma}}{\Longrightarrow} v$$

Proof. First we prove the backwards direction: taking $C = \mathbb{1}$, it is not hard to see that the above specialises to the usual definition of two-sided discrete fibration.

Now for the forwards direction. (1) and (2) follow in the same way as Gray's lemma (Lemma 3.3). For (3), we factorise σ_a and $\sigma_{a'}$ (whilst ignoring the dotted arrow for a moment) as

$$
\begin{tikzcd}
u(a) \arrow[r, "\widehat{q\sigma_a}"] \arrow[dr, "u(f)"'] & d(a) \arrow[rr, "\widecheck{p\sigma_a}"] \arrow[drr, dotted] & & v(a) \arrow[dr, "v(f)"] & \\
& u(a') \arrow[rr, "\widehat{q\sigma_{a'}}"'] & & d(a') \arrow[r, "\widecheck{p\sigma_{a'}}"'] & v(a')
\end{tikzcd}
\tag{8}
$$

Focus on the right hand side open box, which w.r.t. p is over the open box

$$
\begin{tikzcd}
pu(a) \arrow[rr, "p(\sigma_a)"] \arrow[dr, dotted, "p(u(f))"'] & & pv(a) \arrow[dr, "p(v(f))"] & \\
& pu(a') \arrow[rr, "p(\sigma_{a'})"'] & & pv(a')
\end{tikzcd}
$$

Taking $p(u(f))$ as a lid for this box makes the diagram commute, as it is the p-image of a naturality square of σ. As $\widecheck{p\sigma_{a'}}$ is cartesian over $p\sigma_{a'}$, there is a unique arrow $h : d(a) \to d(a')$ that can go in the position of the dotted arrow in (8) that makes the right hand square commute. We record that $p(h) = pu(f)$, and $q(h) = qv(f)$; the latter we get by applying q to the right hand square of (8).

In an entirely symmetric way, we use the opcartesian arrow $\widehat{q\sigma_a}$ to obtain a $k : d(a) \to d(a')$ that makes the left hand square commute. But then $q(k) = qv(f)$ and $q(k) = pu(f)$, so in fact $k = h$. We pick this to be $d(f)$, and the uniqueness of this choice makes it functorial. Finally, placing it in the dotted position in (8) makes the entire diagram commute.

We thus obtain a factorisation of σ as $\widetilde{p\sigma} \circ \widehat{q\sigma}$. Before we conclude the proof, let us note for later use that $p \circ d = p \circ u$ and $q \circ d = q \circ v$. ☐

We can now prove our main

Theorem 4.21. *Given an adjoint cylinder and a two-sided discrete fibration*

$$\begin{tikzcd} D \arrow[d, bend right=60, "i_0"'] \arrow[d, bend left=60, "i_1"] \arrow[d, phantom, shift right=3ex, "\dashv"] \arrow[d, phantom, shift left=3ex, "\dashv"] \\ C \arrow[u, "r"'] \end{tikzcd} \qquad \begin{tikzcd} & E \arrow[dl, "p"'] \arrow[dr, "q"] & \\ B & & A \end{tikzcd}$$

such that the diagrams

$$\begin{tikzcd} D \arrow[r, "i"] \arrow[d, "i_0"'] & E \arrow[d, "q"] \\ C \arrow[r, "f"'] & A \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "i"] \arrow[d, "i_1"'] & E \arrow[d, "p"] \\ C \arrow[r, "g"'] & B \end{tikzcd}$$

commute, then there is a $d : C \to E$ and two natural transformations such that the lower triangles of the diagrams

$$\begin{tikzcd} D \arrow[r, "i"] \arrow[d, "i_0"'] & E \arrow[d, "q"] \\ C \arrow[r, "f"'] \arrow[ur, dotted, "d"'] \arrow[u, phantom, shift left=3ex, "\Nwarrow"] & A \end{tikzcd} \qquad \begin{tikzcd} D \arrow[r, "i"] \arrow[d, "i_1"'] \arrow[d, phantom, shift left=3ex, "\Searrow"] & E \arrow[d, "p"] \\ C \arrow[r, "g"'] \arrow[ur, dotted, "d"'] & B \end{tikzcd}$$

commute.

Proof. We have two natural transformations

$$\begin{aligned} \epsilon : i_0 \circ r &\Rightarrow \mathrm{Id}_C \\ \eta : \mathrm{Id}_C &\Rightarrow i_1 \circ r \end{aligned}$$

We whisker them to get

$$\begin{aligned} f * \epsilon : f \circ i_0 \circ r &\Rightarrow f \\ g * \eta : g &\Rightarrow g \circ i_1 \circ r \end{aligned}$$

Note that $f \circ i_0 \circ r = q \circ i \circ r$, and $g \circ i_1 \circ r = p \circ i \circ r$. Then, as q and p are respectively a Grothendieck opfibration and a Grothendieck fibration, we can use Gray's lemma (Lemma 3.3 to construct the lifts:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ C \arrow[ur, "i \circ r"'] \arrow[ur, dotted, bend left=50, "k"] \arrow[ur, phantom, bend left=25, "\Searrow \widetilde{g * \eta}"] \arrow[r, bend left=15, "g"] \arrow[r, bend right=30, "p \circ i \circ r"'] \arrow[r, phantom, "\Downarrow g * \eta"] & B \end{tikzcd} \qquad \begin{tikzcd} & E \arrow[d, "q"] \\ C \arrow[ur, bend left=40, "i \circ r"] \arrow[ur, dotted, "h"'] \arrow[ur, phantom, bend left=20, "\Searrow \widehat{f * \epsilon}"] \arrow[r, bend left=15, "q \circ i \circ r"] \arrow[r, bend right=30, "f"'] \arrow[r, phantom, "\Downarrow f * \epsilon"] & A \end{tikzcd}$$

We can compose these lifts along their common boundary to obtain

$$\begin{array}{llll} \alpha \stackrel{\text{def}}{=} & k & \stackrel{\check{g\eta}}{\Longrightarrow} i \circ r & \stackrel{\widehat{f\epsilon}}{\Longrightarrow} h \\ \text{over} & g & \stackrel{g\eta}{\Longrightarrow} p \circ i \circ r & = \; p \circ i \circ r \\ \text{and} & q \circ i \circ r & = \; q \circ i \circ r & \stackrel{f\epsilon}{\Longrightarrow} f \end{array}$$

Thus $p(\alpha) = g\eta$ and $q(\alpha) = f\epsilon$. By Theorem 4.20, we can factorise that as

$$\begin{array}{lll} \alpha = & k & \stackrel{\widehat{q\alpha}}{\Longrightarrow} d \stackrel{\check{p\alpha}}{\Longrightarrow} h \\ \text{over} & g & = \; g \stackrel{g\eta}{\Longrightarrow} p \circ i \circ r \\ \text{and} & q \circ i \circ r & \stackrel{f\epsilon}{\Longrightarrow} f = \; f \end{array}$$

so we obtain the desired $d : C \to E$ over f and g, which makes the two lower triangles commute.

Consider now $\check{p\alpha} : d \Rightarrow h$. We may whisker it to obtain $\check{p\alpha} * i_0 : d \circ i_0 \Rightarrow h \circ i_0$. But recall how h was obtained: q is a basic opfibration, so—as in Theorem 3.7—

$$\widehat{f\epsilon} * i_0 = \widehat{f\epsilon i_0} = \hat{1} = 1$$

and hence that $h \circ i_0 = i \circ r \circ i_0 = i$. Hence $\check{p\alpha} * i_0 : d \circ i_0 \Rightarrow i$, and similarly $\widehat{q\alpha} * i_1 : i \Rightarrow d \circ i_1$. Therefore, these two transformations fill the upper triangles. ☐

This theorem fits our needs exactly. Suppose we have a 'left directed homotopy' $\alpha : X \times 2 \longrightarrow Y$ which witnesses $f \rightsquigarrow g$, i.e. $\alpha \circ i_0 = f$ and $\alpha \circ i_1 = g$ for the inclusions $i_0, i_1 : X \longrightarrow X \times 2$. We can then use the Theorem 4.21 to find a common filler d to the diagrams

$$\begin{tikzcd} X \arrow[r, "\mathsf{refl} \circ f"] \arrow[d, "i_0"'] & Y^2 \arrow[d, "\mathsf{cod}"] \\ X \times 2 \arrow[r, "\alpha"'] \arrow[ur, dotted, "d"] \arrow[u, phantom, shift left=6, "\Nwarrow"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\mathsf{refl} \circ f"] \arrow[d, "i_1"'] & Y^2 \arrow[d, "\mathsf{dom}"] \\ X \times 2 \arrow[r, "f \circ \pi_1"'] \arrow[ur, dotted, "d"] \arrow[u, phantom, shift left=6, "\Searrow"] & Y \end{tikzcd}$$

The 'right directed homotopy' of interest is then $d \circ i_1 : X \to Y^2$. We calculate

$$\mathsf{dom} \circ (d \circ i_1) = f \circ \pi_1 \circ i_1 = f$$
$$\mathsf{cod} \circ (d \circ i_1) = \alpha \circ i_1 = g$$

Given that any morphism in the total category of a (non-discrete) two-sided fibration can be factorised into three morphisms, as claimed in equation (3), I believe that this theorem can be generalised to also cover any two-sided fibration. The common lift would be replaced with two different lifts d_1, d_2 with a natural transformation $d_1 \Rightarrow d_2$ deforming one to the other. Moreover, this 'middle' natural transformation would be vertical with respect to both legs of the fibration.

5. Conclusions

By this point we have examined some of the structure of **Cat** that seems to either arise as a directed generalisation of the usual homotopical structure on groupoids, or simply appears to be useful and relevant for directed homotopy. We consider this a first attempt some 1-categorical machinery that should underlie an abstract model-category-like presentation of (∞, ∞) categories.

In §3 we examined how the usual mapping path space and mapping cylinder factorisations used in the model structure on groupoids generalise to all small categories. We found that both of them split into two WFSs—a *forwards/future* and a *backwards/past* version—to make a total of 4 WFSs. The WFSs where the right class has a (op)fibrational flavour provide a notion of filling along *future and past sections*, which are slighly weaker than full *reflections and coreflections*. Thus, we can extend any functor from a (co)reflective subcategory to a full category, just by 'following it backwards (forwards)' along the (co) reflection. It is evident that this filling is a directed analogue to filling along acyclic cofibrations. Finally, we noticed that these morphism classes include into each other in an unusual pattern.

Following that, we realised that despite having all these WFSs we still lacked quite a bit of expressive power. Consequently, in §4 we examined what could possibly take the place of cylinders and path spaces as previously used in model categories. It seems that the right notion at this point is to consider not just classes of morphisms, but *classes of spans and cospans*. We discussed two contenders for what the right class of (co)spans is: two-sided (co)fibrations, which are a generalisation of (op)fibrations that permit both a fibrant and opfibrant direction, and adjoint cylinders/reflexive graphs, which are a generalisation of both reflections and coreflections. We concluded by showing that both are useful, for they lift against each other.

It is a blessing and a curse that this work raises more questions than it answers.

What does 'directed localisation' mean? As discussed in §2, one of the standard constructions in model categories is to construct the *homotopy category* of a model category. Since we no longer have weak equivalences, the notion of *localisation* seems to lose its meaning. I believe that there are two things we can do, but possess no evidence on which one is right.

One idea would be *localise along future/past sections*, or even reflections and coreflections. That way, we could identify a category A with its image either at the 'start' or at the 'end' of the cylinder $A \times \mathbb{2}$, obtaining the *forward localisation* and the *backward localisation* respectively. It is unclear whether this approach leads to any interesting results.

However, another route would be to accept that this theory is fundamentally two-dimensional. The right way would thus be to construct a *directed homotopy 2-category* instead, with the left (or right!) directed homotopies as 2-cells. This route is also better understood: it is consistent with the use of the homotopy 2-category as a central device in the upcoming book by Riehl and Verity [RV19].

What is a 'two-sided factorisation system'? It became evident in §4 that we have found a way to consider the *factorisation of spans and cospans*. In the case of those corresponding to representable profunctors—e.g. the graph $\langle \mathrm{id}, f\rangle$—this leads to a factorisation into a right-adjoint-right-inverse followed by a two-sided discrete fibration (or a two-sided codiscrete cofibration followed by a left-adjoint-left-inverse in the cograph case). In the more general case of any span, this led to a factorisation into a single morphism and a general two-sided fibration.

Coupled with the two-sided lifting property of Theorem 4.21, this raises an obvious question: is there a notion of a *two-sided factorisation system* that we have missed, and that could be an invaluable tool for directed homotopy? We believe this to be quite possible, but we have no intuitive descriptions of the single-morphism left class. Paige North has been working a variation of this idea, but her two-sided lifting property looks very different: see the talk [Nor19a].

Should there be a type theory of spans? The Awodey-Warren slogan *'types are fibrations'* was instrumental to developing a higher-dimensional semantics for HoTT. Given our results in §4, we should carefully consider what the slogan

'types are two-sided fibrations'

has to offer to directed type theory.

There are indeed elements of this—but formulated internally—in the directed type theory of Shulman and Riehl [RS17]. However, it might be worthwhile to consider the formulation of a type theory where a type is not a display map, but a *display span*, which would implicitly contain the idea that it is fibred over a part of its context, and opfibred with respect to the rest.

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