Two-dimensional Kripke Semantics

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The study of modal logic has witnessed tremendous development following the introduction of Kripke semantics. However, recent developments in programming languages and type theory have led to a second way of studying modalities, namely through their categorical semantics. We show how the two correspond.

1. Introduction

The development of modal logic has undergone many phases [20, 13, 37, 79]. It is widely accepted that one of the most important developments was the relational semantics of Kripke [54, 55, 56] [13, §1] [37, §4.8]. Kripke semantics has proven time and again that it is intuitive and technically malleable, thereby exerting sustained influence over Computer Science.

However, over the last 30 years another way of studying modalities has evolved: looking at modal logic through the prism of the *Curry-Howard-Lambek correspondence* [58, 74, 80] yields new computational intuitions, often with surprising applications in both programming languages and formal proof. The tools of the trade here are type theory and category theory.

Up to now, these two ways of looking at modalities have appeared mostly unrelated. The purpose of this paper is to establish a connection: I will show that the Kripke and categorical semantics of modal logic are part of a *duality*. It is well-known that dualities between Kripke and algebraic semantics exist: the *Jónsson-Tarski duality* is one of the cornerstones of modern modal logic [13, §5]. The contribution of this paper is to show that such dualities can be elevated to the categorical level of *proofs*.

There are two obstacles to overcome. The first is that we must work over an *intuitionistic* substrate: most research on types and categories is forced to do so, for well-known and unavoidable reasons. Hence, we will develop a duality for *intuitionistic modal logic*. However, there is no consensus on what a minimal intuitionistic modal logic is! The problem is particularly acute in the presence of $\Diamond$ [23]. I will avoid this problem by making canonical choices at each step. First, I will formulate a Kripke semantics based on *bimodules*, i.e. relations that are canonically compatible with a poset. Then, I will show how *Kan extension* uniquely determines two adjoint modalities, $\blacklozenge$ and $\Box$, from any bimodule. The fact these arise automatically is evidence that they are the canonical choice of intuitionistic modalities.

The second obstacle has to do with the appearance of proofs. The Jónsson-Tarski duality establishes a precise correspondence between Kripke and algebraic semantics—in the classical setting. The jump from algebraic to categorical semantics involves adding an extra 'dimension' of proofs. Consequently, in order to re-establish a duality, an additional dimension must be added to Kripke semantics as well. We call the result a *two-dimensional Kripke semantics.* Category theorists will find it anticlimactic: it amounts to the folklore observation that Kripke semantics is a semantics in a presheaf category.

Indeed, a proportion of this paper consists of 'folklore' results that are probably well-known to experts. However, many of them are drawn from related but distinct areas: logic, order theory, category theory, and topos theory. As a result, it does not appear that all of them are known by a *single* expert. Thus, the synthesis presented here appears to be new.

In §2 we recall the Kripke and algebraic semantics of intuitionistic logic, and consider the duality Kripke semantics and certain complete Heyting algebras. We extend this to duality to intuitionistic modal logic in §3, where we show how a relation that is compatible with the intuitionistic order (a bimodule) gives rise to two modalities through Kan extension. In §4 we add proofs to intuitionistic logic, and elevate the duality to one between two-dimensional frames and presheaf categories. We then repeat the process for intuitionistic modal logic in §5. This is achieved by promoting the bimodule to a *profunctor* on the relational side, and adding an *adjunction* on the algebraic side.

For general background in orders we refer to the book by Davey and Priestley [24]. Given a poset $(D, \sqsubseteq_D)$ we define the *opposite* poset D^{op} by reversing the partial order; that is, $x \sqsubseteq_{D^{\mathsf{op}}} y$ iff $y \sqsubseteq_D x$. A *lattice* has all finite meets and joins. A *complete lattice* has arbitrary ones. A complete lattice is *infinitely distributive* just if the law $a \wedge \bigvee_i b_i = \bigvee_i a \wedge b_i$ holds. Such lattices are variously called *frames*, *locales*, or *complete Heyting algebras* [47, 62, 66].

2. Intuitionistic Logic I

There are many types of semantics for intuitionistic logic, including Kripke, Beth, topological, and algebraic semantics. Bezhanishvili and Holliday [10] argue that these form a strict hierarchy, with Kripke being the least general, and algebraic the most. I will briefly review the elements of both these extreme points.

The Kripke semantics of intuitionistic logic are given by Kripke frames [20, §2.2]. *Kripke frame* is a partially-ordered set $(W, \sqsubseteq)$. We refer to W as the set of *worlds* and to $\sqsubseteq$ as the *information order*. A world $w \in W$ is a 'state of knowledge,' and $w \sqsubseteq v$ means that moving from world w to world v possibly entails an increase in the amount of information.

Let $\mathsf{Up}(W) \subseteq \mathcal{P}(W)$ be the set of *upper sets* of W, i.e. the sets $S \subseteq W$ such that $w \in S$ and $w \sqsubseteq v$ implies $v \in S$. A *Kripke model* $\mathfrak{M} = (W, \sqsubseteq, V)$ consists of a Kripke frame $(W, \sqsubseteq)$ as well as a function $V : \mathsf{Var} \to \mathsf{Up}(W)$. The *valuation* V assigns to each propositional variable $p \in \mathsf{Var}$ an upper set $V(p) \subseteq W$, which is the set of worlds in which p is true. The idea is that, once a proposition becomes true, it must remain true as information increases.

We are now able to inductively define a relation $\mathfrak{M}, w \vDash \varphi$ with the meaning that φ is true in world w of model $\mathfrak{M}$. The only interesting clause is that for implication:

$$\mathfrak{M}, w \vDash \varphi \to \psi \stackrel{\text{def}}{\equiv} \forall w \sqsubseteq v.\ \mathfrak{M}, v \vDash \varphi \text{ implies } \mathfrak{M}, v \vDash \varphi$$

This definition is famously monotonic: if $\mathfrak{M}, w \vDash \varphi$ and $w \sqsubseteq v$ then $\mathfrak{M}, v \vDash \varphi$.

The algebraic semantics of intuitionistic logic consist of *Heyting algebras*. These are lattices such that every map $- \wedge x : L \to L$ has a right adjoint, i.e. for $x, y \in L$ there is an element $x \Rightarrow y \in L$ such that $c \wedge x \sqsubseteq y$ iff $c \sqsubseteq x \Rightarrow y$. Then, assuming that we have an interpretation $[\![p]\!] \in L$ of each proposition p, each formula φ of intuitionistic logic is inductively mapped to an element $[\![\varphi]\!] \in L$ using the corresponding algebraic structure. We will not expound on Heyting algebras further; see [20, §7.3] [15, §1.1] [62, §I.8].

2.1. Prime algebraic lattices

Let $(W, \sqsubseteq)$ be any Kripke frame, and let $\mathbb{2} \stackrel{\text{def}}{=} \{0 \sqsubseteq 1\}$. Consider the poset $[W, \mathbb{2}]$ of monotonic functions from W to $\mathbb{2}$, ordered pointwise. This poset has a number of curious properties.

First, the monotonicity of $p : W \to \mathbb{2}$ implies that if $p(w) = 1$ and $w \sqsubseteq v$, then $p(v) = 1$. Hence, the subset $U \stackrel{\text{def}}{=} p^{-1}(1)$ of W is an upper set. Conversely, every upper set $U \subseteq W$ gives rise to a monotonic $p_U : W \to \mathbb{2}$ by setting $p_U(w) = 1$ if $w \in U$, and 0 otherwise. Consequently, there is an order bijection $\mathsf{Up}(W) \cong [W, \mathbb{2}]$, with the order on $\mathsf{Up}(W)$ being inclusion. We will liberally treat upper sets and elements of $[W, \mathbb{2}]$ as the same.

Second, the poset $[W, \mathbb{2}]$ is a *complete lattice*: arbitrary joins and meets are given pointwise. If we view the elements of $[W, \mathbb{2}]$ as upper sets, these joins and meets correspond to arbitrary unions and intersections of upper sets, which are also upper. Moreover, this lattice satisfies the infinite distribution law, so it is a *complete Heyting algebra*, synonymously a *frame*. Given two upper sets $X, Y \subseteq W$ their exponential is given by

$$X \Rightarrow Y \stackrel{\text{def}}{=} \{w \in W \mid \forall w \sqsubseteq v.\ v \in X \text{ implies } v \in Y\}$$

Third, given any $w \in W$, consider its *principal upper set* $\uparrow w \stackrel{\text{def}}{=} \{v \in W \mid w \sqsubseteq v\} \in [W, \mathbb{2}]$. A simple argument shows that $w \sqsubseteq v$ iff $\uparrow v \subseteq \uparrow w$.[1] Thus, we have an *order-embedding* $\uparrow(-) : W^{\mathsf{op}} \to [W, \mathbb{2}]$. This can be shown to preserve meets and exponentials.

Fourth, the principal upper sets $\uparrow w$ are special, in that they are *prime*.[2] An element d of a complete lattice L is prime just if $d \sqsubseteq \bigsqcup X$ implies that $d \sqsubseteq x$ for some $x \in X$. This says that d contains a tiny, indivisible fragment of information: as soon as it approximates a supremum, it must approximate something in the set that is being upper-bounded. The prime elements of $[W, \mathbb{2}]$ are exactly the principal upper sets $\uparrow w$ for some $w \in W$.

[1] This is an order-theoretic consequence of the Yoneda lemma.

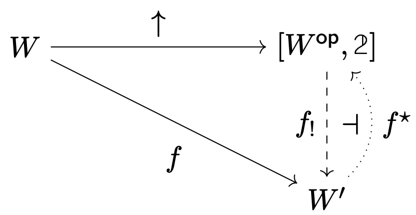
[2] Such elements are variously called *completely join-irreducible* [68], *supercompact* [7] [66, §VII.8], *completely (join-)prime* [81], or simply *join-prime* [32, §1.3].

Fifth, the complete lattice $[W, 2]$ is *prime algebraic*. This means that all its elements can be reconstructed by 'multiplying' or 'sticking together' prime elements. In symbols, a complete lattice L is prime algebraic whenever for every element $d \in L$ we have

$$d = \bigsqcup \{ p \in L \mid p \sqsubseteq d, p \text{ prime } \}$$

Such lattices are variously called *completely distributive, algebraic lattices* [24, §10.29] or *superalgebraic lattices* [66, §VII.8]. In fact, it can be shown that any such lattice is essentially of the form $[W, 2]$, i.e. a lattice of upper sets [68, 64]. See the textbooks by Picado and Pultr [66, §VII.8] and Davey and Priestley [24, §10.29], and the paper by Winskel [81].

Finally, the fact every element can be reconstructed as a supremum of primes means that it is possible to canonically extend any monotonic $f : W \to W'$ to a monotonic $[W^{\mathsf{op}}, 2] \to W'$, as long as W' is a complete lattice. Diagrammatically, in the situation



(1)

there exists a unique $f_!$ which preserves joins and satisfies $f_!(\uparrow w) = f(w)$. It is given by

$$f_!(S) \stackrel{\text{def}}{=} \bigsqcup_i \{ f(w) \mid w \in S \}$$

$f_!$ is called the *(left) Kan extension* of f along $\uparrow(-)$. As it preserves joins, and $[W, 2]$ is complete, it has a right adjoint $f^\star$ by the adjoint functor theorem [24, §7.34] [47, §I.4.2]. For any complete lattice W' this situation amounts to a bijection

$$\mathrm{Hom}_{\mathbf{Pos}}(W, W') \cong \mathrm{Hom}_{\mathbf{CSLatt}}([W^{\mathsf{op}}, 2], W')$$

where **CSLatt** is the category of complete lattices and join-preserving maps.

Suppose then that we have a Kripke model $(W, \sqsubseteq, V)$. Then, the construction given above induces a Heyting algebra $[W, 2]$. Defining $[\![p]\!] = V(p)$, we obtain an algebraic model of intuitionistic logic, which interprets every formula φ as an upper set $[\![\varphi]\!] \in [W, 2]$. This is the upper set of worlds in which a formula is true [20, Theorem 7.20]:

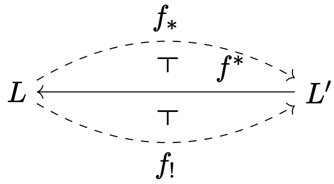
Theorem 2.1. $w \vDash \varphi$ if and only if $w \in [\![\varphi]\!]$

Thus, every Kripke semantics corresponds to a prime algebraic lattice.

2.2. Morphisms

The simplest kind of morphism between frames is a *monotonic* map $f : W \to W'$. Frames and monotonic maps form the category **Pos** of posets. Given a monotonic $f : W \to V$ we may define a monotonic map $f^* : [W', 2] \to [W, 2]$ which maps $p : W' \to 2$ to $p \circ f : W \to 2$. Viewing the elements of $[W', 2]$ as upper sets, f^* maps the upper set $S \subseteq W'$ to the set $f^{-1}(S) \stackrel{\text{def}}{=} \{v \in V \mid f(v) \in S\} \subseteq W$, which is upper by monotonicity of f. f^* preserves arbitrary joins and meets, and hence induces a functor $[-, 2] : \mathbf{Pos}^{\mathsf{op}} \longrightarrow \mathbf{PrAlgLatt}$ to the category **PrAlgLatt** of prime algebraic lattices and complete lattice homomorphisms.

Moreover, $[-, 2]$ is an equivalence! By the adjoint functor theorem any complete lattice homomorphism $f^* : L' \to L$ has a left and right adjoint:



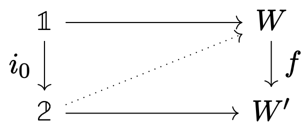
(2)

Given a prime algebraic lattice L, let $\mathsf{Prm}(L) \subseteq L$ be the sub-poset of prime elements. It can be shown that $f_!$ maps primes to primes [32, Lemma 1.23]. We can thus restrict it to a function $\mathsf{Prm}(L) \to \mathsf{Prm}(L')$. This defines a functor $\mathsf{Prm}(-) : \mathbf{PrAlgLatt} \longrightarrow \mathbf{Pos}^{\mathsf{op}}$ with the property that $\mathsf{Prm}([W, 2]) \cong W$. All in all, this amounts to a *duality*

$$\mathbf{Pos}^{\mathsf{op}} \simeq \mathbf{PrAlgLatt} \tag{3}$$

However, monotonic maps are not particularly well-behaved from the perspective of logic, as they do not preserve nor reflect 'local' truth. This is the privilege of *open maps*.

Definition 2.2. Let $i_0 : 1 \to 2$ map the unique point of $1 \stackrel{\text{def}}{=} \{*\}$ to $0 \in 2$. A monotonic map $f : W \to W'$ of Kripke frames is *open* just when it has the right lifting property with respect to $i_0 : 1 \to 2$, i.e. when every commuting diagram of the form



in **Pos** has a diagonal filler (dashed) that makes it commute.

In other words, f is open if whenever $f(w) \sqsubseteq v'$ there exists a $w' \in W$ with $w \sqsubseteq w'$ and $f(w') = v'$.[3] Open maps send upper sets to upper sets [20, Prop. 2.13]. Thus

Lemma 2.3. Let $\mathfrak{M} = (W, \sqsubseteq, V)$ and $\mathfrak{N} = (W', \sqsubseteq, V')$ be Kripke models, and $f : W \to W'$ be open. Suppose $V = f^{-1} \circ V'$, i.e. $w \in V(p)$ iff $f(w) \in V'(p)$. Then $\mathfrak{M}, w \vDash \varphi$ iff $\mathfrak{N}, f(w) \vDash \varphi$.

[3] Such morphisms are often called *p-morphisms* [20, §2.3] or *bounded morphisms* [13, §2.1]. According to Goldblatt [37], open maps were introduced by De Jongh and Troelstra [25] in intuitionistic logic, and by Segerberg [72] in modal logic. More rarely they are called *functional simulations*, and led us to bisimulations [71, §3.2]. The name is chosen because such maps are open with respect to the *Alexandrov topology* on a poset, whose open sets are the upper sets [47, §1.8].

Write $W \vDash \varphi$ to mean that $(W, \sqsubseteq, V), w \vDash \varphi$ for any valuation V and $w \in W$. Then

Lemma 2.4. If $f : W \to W'$ is open and surjective, then $W \vDash \varphi$ implies $W' \vDash \varphi$.

Recall now the induced map $f^* : [W', 2] \to [W, 2]$ for a monotonic $f : W \to W'$. Then

Lemma 2.5.

1. $f : W \to W'$ is open iff $f^* : [W', 2] \to [W, 2]$ preserves exponentials.
2. $f : W \to W'$ is surjective iff $f^* : [W', 2] \to [W, 2]$ is injective.

Consequently, the duality of (3) may be restricted to two wide subcategories:

$$\mathbf{Pos}^{\mathsf{op}}_{\text{open}} \simeq \mathbf{PrAlgLatt}_{\Rightarrow} \qquad\qquad \mathbf{Pos}^{\mathsf{op}}_{\text{open,surj}} \simeq \mathbf{PrAlgLatt}_{\Rightarrow,\text{inj}} \tag{4}$$

The morphisms on the left are open (resp. open surjective) maps, and the morphisms on the right are *complete Heyting homomorphisms*, i.e. complete lattice homomorphisms that preserve exponentials (resp. and are injective).

Finally, let us consider the classical case—as a sanity-check. This amounts to restricting **Pos** to its subcategory of discrete orders, i.e. **Set**. In this case every map is open. The corresponding restriction on the other side is to the category **CABA** of *complete atomic Boolean algebras*, yielding the usual *Tarski duality* $\mathbf{Set}^{\mathsf{op}} \simeq \mathbf{CABA}$ [53].

2.3. Related work

The origins of the construction of a Heyting algebra from a Kripke frame seems to lost in the mists of time. The earliest occurence I have located is in the book by Fitting [30, §1.6], where it is attributed to an exercise in the book by Beth [9].

The duality (3) appears to be somewhat folklore—sufficiently to now be included as an exercise in new textbooks [32, Ex. 1.3.10]; see also Erné [27]. However, I have not been able to find any mention of the dualities of (4) in the literature.

Both the dualities (3) and (4) involve just prime algebraic lattices, which is a far cry from encompassing all Heyting algebras. It is possible to do so, by enlarging the category **Pos** to a class of ordered topological spaces called *descriptive frames* [20, §8.4]. The resulting duality is called *Esakia duality* [28] [32, §4.6] [11, §2.3.4].

3. Modal Logic I

We now wish to extend the results of §2 to *intuitionistic modal logic.*

There is disagreement on what a minimal intuitionistic modal logic is. This arises no matter the methodology we choose—be it relational, algebraic, or proof-theoretic. The situation becomes even more complex if we include a diamond modality ($\Diamond$): see Das and Marin [23] and Wolter and Zakharyaschev [84] for a discussion.

We adopt the *intuitionistic propositional logic with Galois connections* of Dzik et al. [26]. This extends intuitionistic logic with modalities $\blacklozenge$ and $\Box$, and the two inference rules

$$\frac{\blacklozenge\varphi \to \psi}{\varphi \to \Box\psi} \qquad \text{and} \qquad \frac{\varphi \to \Box\psi}{\blacklozenge\varphi \to \psi}$$

These rules correspond to a *Galois connection* [24, §7.23], i.e. an adjunction $\blacklozenge \dashv \Box$ between posets. They imply the derivability of the following rules, amongst others [26, Prop. 2.1].

$$\frac{\varphi \to \psi}{\Box\varphi \to \Box\psi} \qquad \frac{\varphi}{\Box\varphi} \qquad \frac{\varphi \to \psi}{\blacklozenge\varphi \to \blacklozenge\psi} \qquad \frac{}{\blacklozenge(\varphi \vee \psi) \leftrightarrow \blacklozenge\varphi \vee \blacklozenge\psi} \qquad \frac{}{\Box(\varphi \wedge \psi) \leftrightarrow \Box\varphi \wedge \Box\psi}$$

The notation of the 'black diamond' modality $\blacklozenge$ may appear unusual. However, we will argue this logic is, in a way, the canonical intuitionistic modal logic.

The Kripke semantics of classical modal logic is given by a *modal frame* (W, R), which consists of a set W and an *accessibility relation* $R \subseteq W \times W$ [13, §1]. However, if the same set of worlds W is already part of an intuitionistic Kripke frame $(W, \sqsubseteq)$, then we must take care to ensure that $\sqsubseteq$ and R are *compatible*. There are many compatibility conditions that one can consider [67] [73, §3.3]. However, we will take a hint from the category theory literature, and seek a canonical definition of what it means for a relation to be compatible with a poset.

Recall that relations can be presented as functions $R : W \times W \to 2$ which map a pair of worlds (w, v) to 1 whenever $w \mathrel{R} v$. We will ask that R is such function, but with a twist:

Definition 3.1. A *bimodule* $R : W_1 \nrightarrow W_2$ is a monotonic map $R : {W_1}^{\mathsf{op}} \times W_2 \to 2$.

Stated in terms of ordinary relations, $R^{-1}(1) \subseteq W_1 \times W_2$ corresponds to a bimodule just if $w' \sqsubseteq w \mathrel{R} v \sqsubseteq v'$ implies $w' \mathrel{R} v'$. Thus, R is 'compatible' with $\sqsubseteq$, contravariantly (resp. covariantly) on the first (resp. second) component. This is a standard, minimal way to define what it means to be 'a relation in **Pos**.'

We can then define a *modal Kripke frame* $(W, \sqsubseteq, R)$ to be a Kripke frame $(W, \sqsubseteq)$ equipped with a bimodule $R : W \nrightarrow W$. A *modal Kripke model* $\mathfrak{M} = (W, \sqsubseteq, R, V)$ adds to this a function $V : \mathsf{Var} \to \mathsf{Up}(W)$. We extend $\mathfrak{M}, w \vDash \varphi$ to modal formulae:

$$\mathfrak{M}, w \vDash \blacklozenge\varphi \overset{\text{def}}{\equiv} \exists v.\ v \mathrel{R} w \text{ and } \mathfrak{M}, v \vDash \varphi \qquad \mathfrak{M}, w \vDash \Box\varphi \overset{\text{def}}{\equiv} \forall v.\ w \mathrel{R} v \text{ implies } \mathfrak{M}, v \vDash \varphi$$

There are a number of things to note about this definition. First, there is a clear duality between the clauses: we exchange $\forall$ for $\exists$, but we also flip the variance of the relation. As a result, $\blacklozenge$ uses the relation in the *opposite variance* to the more traditional $\Diamond$—hence the notation. Third, the clause for the $\Box$ modality is the traditional one, which is unlike some streams of work on intuitionistic modal logic [67, 73]. Finally, this definition is monotonic: using the bimodule structure of R we can show that if $\mathfrak{M}, w \vDash \varphi$ and $w \sqsubseteq v$ then $\mathfrak{M}, v \vDash \varphi$. Dzik et al. [26, §5] prove that this semantics is sound and complete.

The algebraic semantics of this logic is given by a Heyting algebra H equipped with two monotonic maps $\blacklozenge, \Box : H \to H$ which form an adjunction $\blacklozenge \dashv \Box$, i.e. a Galois connection. Dzik et al. [26, §4] prove that this semantics is also sound and complete.

We are now in a position to discuss how the Kripke and algebraic semantics of this intuitionistic modal logic correspond. Let $(W, \sqsubseteq, R)$ be a modal Kripke frame, and consider the map $\lambda R : W^{\mathsf{op}} \to [W, 2]$ obtained by the cartesian closure of **Pos**. This map takes $w \in W$ to the upper set $\{v \in W \mid w \, R \, v\}$. Putting λR in diagram 1, we obtain through Kan extension the diagram

$$\begin{array}{ccc} W^{\mathsf{op}} & \xrightarrow{\uparrow(-)} & [W, 2] \\ & \searrow^{\lambda R} & \blacklozenge_R \downarrow \dashv \uparrow \Box_R \\ & & [W, 2] \end{array} \tag{5}$$

where we write $\blacklozenge_R$ for $\lambda R_!$ and $\Box_R$ for $\lambda R^\star$. It can be shown that these maps are given by

$$\blacklozenge_R(S) \stackrel{\text{def}}{=} \{w \in W \mid \exists v.\ v \, R \, w \text{ and } v \in S\} \quad \Box_R(S) \stackrel{\text{def}}{=} \{w \in W \mid \forall v.\ w \, R \, v \text{ implies } v \in S\}$$

Thus, any relation R defines an adjunction $\blacklozenge_R \dashv \Box_R$ on $[W, 2]$. Correspondingly, any adjunction $\blacklozenge \dashv \Box$ yields a monotonic map $\blacklozenge \circ \uparrow(-) : W^{\mathsf{op}} \to [W, 2]$, which uniquely corresponds to a bimodule $W^{\mathsf{op}} \times W \to 2$ by the cartesian closure of **Pos**.

Thus, starting from a bimodule, i.e. a relation that is compatible with the information order, we have canonically and uniquely induced two modal operators through Kan extension. These do what we expect them to do: defining $[\![\blacklozenge\varphi]\!] = \blacklozenge_R[\![\varphi]\!]$ and $[\![\Box\varphi]\!] = \Box_R[\![\varphi]\!]$, we get

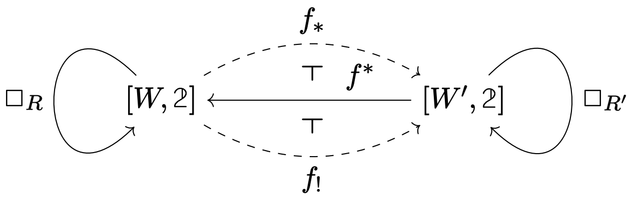
Theorem 3.2. For any modal formula φ, $w \vDash \varphi$ if and only if $w \in [\![\varphi]\!]$.

3.1. Morphisms

Define the category **Bimod** to have bimodules $R : W_1 \nrightarrow W_2$ as objects. A *bimodule morphism* from $R : W_1 \nrightarrow W_2$ to $R' : W_1' \nrightarrow W_2'$ is a pair (f, g) of monotonic maps $f : W_1 \to W_1'$ and $g : W_2 \to W_2'$ such that $R(w, v) \sqsubseteq R'(f(w), g(v))$. Stated in terms of relations, it must be that $w \, R \, v$ implies $f(w) \, R' \, g(v)$.

We define the subcategory **EBimod** to consist of endobimodules $R : W \nrightarrow W$ and pairs of maps (f, f). Thus, objects are bimodules on a single poset W, and morphisms are monotonic maps $f : W \to W'$ that preserve the relation, i.e. $w \, R \, v$ implies $f(w) \, R \, f(v)$. In other words, the objects of **EBimod** are modal Kripke frames.

Recall the adjunctions induced by a monotonic $f : W \to W'$:



(6)

Lemma 3.3. $f : W \to W'$ is a morphism of bimodules $f : R \to R'$ iff $f^* \square_{R'} \subseteq \square_R f^*$.

This constitutes a duality

$$\mathbf{EBimod}^{\mathsf{op}} \simeq \mathbf{PrAlgLattO} \tag{7}$$

where **PrAlgLattO** is the category with objects $(L, \square_L)$, where L is a prime algebraic lattice and $\square_L : L \to L$ is a meet-preserving operator. By the adjoint functor theorem, such operators always have a left adjoint $\blacklozenge_L : L \to L$. Thus, this category contains algebraic models of intuitionistic modal logic—but not all of them. The morphisms are complete lattice homomorphisms $h : L \to L'$ such that $h \square_L \sqsubseteq \square_{L'} h$.

However, as with monotone maps, morphisms of bimodules do not preserve local truth; for that we need a notion of *modally open* maps.

Definition 3.4. Let $(W, \sqsubseteq, R)$ and $(W', \sqsubseteq, R')$ be modal Kripke frames. A bimodule morphism $f : R \to R'$ is *modally open* just if whenever $f(w)\ R'\ v$ then there exists a $w' \in W$ with $w\ R\ w'$ and $f(w') \sqsubseteq v$.

This is similar to Definition 2.2, but ever so slightly weaker: instead of requiring $f(w') = v'$, it requires that the information in $f(w')$ can be increased to v'. Like Definition 2.2, it can also be written homotopy-theoretically, but that requires some ideas from double categories that are beyond the scope of this paper. We have the analogous result about preservation of truth:

Lemma 3.5. Let $\mathfrak{M} = (W, \sqsubseteq, R, V)$ and $\mathfrak{N} = (W', \sqsubseteq, R', V')$ be modal Kripke models, $f : W \to W'$ be open and modally open, and $V = f^{-1} \circ V'$. Then $\mathfrak{M}, w \vDash \varphi$ iff $\mathfrak{N}, f(w) \vDash \varphi$.

Lemma 3.6. Let $f : W \to W'$ be open, modally open, and surjective. If $W \vDash \varphi$ then $W' \vDash \varphi$.

The following result relates the modal openness of f to f^*.

Lemma 3.7. $f : R \to R'$ is modally open iff $\square_R f^* = f^* \square_{R'}$ iff $f_! \blacklozenge_R = \blacklozenge_{R'} f_!$.

Thus, the duality (7) may be restricted to dualities between wide subcategories:

$$\mathbf{EBimod}^{\mathrm{op}}_{\mathrm{moo}} \simeq \mathbf{PrAlgLattO}_{\Rightarrow o} \qquad\qquad \mathbf{EBimod}^{\mathrm{op}}_{\mathrm{moo,\ surj}} \simeq \mathbf{PrAlgLattO}_{\Rightarrow o,\mathrm{inj}} \tag{8}$$

The morphisms on the left are open and modally open (resp. also surjective); and the on the right they preserve exponentials and commute with operators (resp. also injective).

As a sanity check, let us consider the restriction of this duality to the classical setting. A bimodule on a discrete poset is just a relation on a set. The corresponding restriction on the right is to CABAs with operators, and complete homomorphisms which commute with operators. We thus obtain the *Thomason duality* $\mathbf{MFrm}^{\mathrm{op}}_{\mathrm{open}} \simeq \mathbf{CABAO}$ between Kripke frames and modally open maps on the left, and CABAs with operators on the right [77, 53].

3.2. Related work

Many works have presented a Kripke semantics for intuitionistic modal logic. All such semantics assume two accessibility relations: a preorder for the intuitionistic dimension, and a second relation for the modal dimension. What varies is their *compatibility conditions*.

The first person to present such a semantics appears to Fischer Servi [29]. One of the required compatibility conditions is $(\sqsubseteq) \circ R \subseteq R \circ (\sqsubseteq)$. This is weaker than having a bimodule.

The first person to recognise the importance of bimodules was Sotirov in his 1979 thesis. His results are summarised in a conference abstract [75, §4]: they include the completeness of a minimal intuitionistic modal logic with a $\Box$, the K axiom, and the necessitation rule. Božić and Došen [17] repeat the study for the same logic, but for a semantics based on the Fischer Servi compatibility conditions. Wolter and Zakharyaschev [82, §2] argue that bimodule semantics and Fischer Servi semantics are equally expressive.

Plotkin and Stirling [67] attempt to systematise the Kripke semantics of intuitionistic modal logic. This paper and all its descendants—notably the thesis of Simpson [73, §3.3]—adopt a non-standard clause for $\Box$ which uses both $\sqsubseteq$ and R.

The bimodule condition and the *complex algebra* construction (or fragments thereof) have made scattered appearances in the literature: in the early work of Sotirov [75] and Božić and Došen [17]; in Wolter and Zakharyaschev [83, 82, 84], Hasimoto [43, §4], and Orłowska and Rewitzky [65]; and of course in Dzik et al. [26, §7].

With the exception of Dzik et al. [26], none of the above references discuss the $\blacklozenge$ modality. Moreover, in none of these references are the categorical aspects of these structures discussed.

As mentioned before, dualities between frames and algebras have played a significant role in modal logic. Thomason [77] and Goldblatt [36] also considered morphisms of frames, respectively obtaining *Thomason duality* and (categorical) *Jónsson-Tarski duality* between descriptive frames and Boolean Algebras with Operators (BAOs) [37, §6.5]. Kishida [53] surveys a number of (classical) dualities for modal logic.

The duality $\mathbf{EBimod}^{\mathrm{op}} \simeq \mathbf{PrAlgLattO}$ (7) is stated by Gehrke [31, Thm. 2.5] who attributes it to Jonnson and Tarski [50], even though no such theorem appears in that paper.

The duality $\mathbf{EBimod}^{\mathrm{op}}_{\mathrm{moo}} \simeq \mathbf{PrAlgLattO}_{\Rightarrow_o}$ (8) is the direct intuitionistic analogue to that of Thomason. I have not been able to find it anywhere in the literature.

According to the extensive survey of Menni and Smith [63], the idea that the commonly-used modalities $\Box$ and $\Diamond$ are often part of adjunctions $\blacklozenge \dashv \Box$ and $\Diamond \dashv \blacksquare$ is implicitly present throughout the development of modal logic. However, these were not made explicit in a logic until the 2010s, when they appeared in the work of Dzik et al. [26] and Sadrzadeh and Dyckhoff [70]. The same perspective plays a central rôle in the exposition of Kishida [53].

4. Intuitionistic Logic II

In the rest of this paper we will *categorify* [6] the notion of Kripke semantics. The main idea is to replace posets by categories, so that the order $w \sqsubseteq v$ is replaced by a morphism $w \to v$. As there might be multiple morphisms $w \to v$, this allows the recording of not just the fact v may signify more information than v, but also the *manner* in which that is so. The reflexivity and transitivity of the poset are then replaced by the identity and composition laws of the category. This adds a dimension of *proof-relevance* to Kripke semantics.

A corresponding change in our algebraic viewpoint will be that of replacing the set $\mathbb{2}$ of truth values with the category **Set**. This is a classic Lawverean move [59]. While the falsity 0 is only represented by one value, viz. the empty set, the truth 1 can be represented by any non-empty set X. The elements of X can be thought of as a *proofs* of a true statement.

Our goal is to make Kripke semantics proof-relevant. To that end, we trade the frame $(W, \sqsubseteq)$ for an arbitrary category $\mathcal{C}$. Next, we wish to define what it means to have a *proof* that the formula φ holds at a world $w \in \mathcal{C}$. We denote the set of all such proofs by $[\![\varphi]\!]_w$. Assuming we are given a set $[\![p]\!]_w$ for each proposition p and world w, here is a first attempt:

$$[\![\bot]\!]_w \stackrel{\text{def}}{=} \emptyset \qquad [\![\top]\!]_w \stackrel{\text{def}}{=} \{*\} \qquad [\![\varphi \land \psi]\!]_w \stackrel{\text{def}}{=} [\![\varphi]\!]_w \times [\![\psi]\!]_w \qquad [\![\varphi \lor \psi]\!]_w \stackrel{\text{def}}{=} [\![\varphi]\!]_w + [\![\psi]\!]_w$$
$$[\![\varphi \to \psi]\!]_w \stackrel{\text{def}}{=} (v : \mathcal{C}) \to (f : \mathrm{Hom}_{\mathcal{C}}(w, v)) \to [\![\varphi]\!]_v \to [\![\psi]\!]_v$$

where for a family $(B_a)_{a \in A}$ we let $(a : A) \to B_a \stackrel{\text{def}}{=} \{f : A \to \bigcup_{a \in A} B_a \mid \forall a \in A. f(a) \in B_a\}$. This closely follows the usual definition, but adds proofs. For example, a proof in $[\![\varphi_1 \land \varphi_2]\!]_w$ is a pair (x, y) of a proof $x \in [\![\varphi_1]\!]_w$ and a proof $y \in [\![\varphi_2]\!]_w$. Similarly, a proof $F \in [\![\varphi \to \psi]\!]_w$ is a function which maps a proof of 'increase in information' $f : w \to v$ to a function $F(v)(f) : [\![\varphi]\!]_v \to [\![\psi]\!]_v$. In turn, this function maps proofs in $[\![\varphi]\!]_v$ to proofs in $[\![\psi]\!]_v$.

To show that this definition is monotonic we have to do so on proofs: given a proof $x \in [\![\varphi]\!]_w$ and a morphism $f : w \to v$ we have to define a proof $f \cdot x \in [\![\varphi]\!]_v$. Assuming that we are given this operation for propositions, we can extend it by induction; e.g.

$$\begin{aligned} f \cdot (x, y) &\stackrel{\text{def}}{=} (f \cdot x, f \cdot y) && \in [\![\varphi \land \psi]\!]_v \\ f \cdot \alpha &\stackrel{\text{def}}{=} (z : \mathcal{C}) \mapsto (g : \mathrm{Hom}_{\mathcal{C}}(v, z)) \mapsto (x : [\![\varphi]\!]_z) \mapsto \alpha(z)(g \circ f)(x) && \in [\![\varphi \to \psi]\!]_v \end{aligned}$$

Moreover, this definition is compatible with $\mathcal{C}$, in the sense that $g \cdot (f \cdot x) = (g \circ f) \cdot x$ and $\mathsf{id}_w \cdot x = x$. We thus obtain a (covariant) *presheaf* $[\![\varphi]\!] : \mathcal{C} \longrightarrow \mathbf{Set}$ for each formula φ.

It is well-known that the proofs of intuitionistic logic form a *bicartesian closed category* (biCCC), i.e. a category with finite (co)products and exponentials [57]. A biCCC can be seen as a categorification of a Heyting algebra: formulae are objects of the category, and proofs are morphisms. We will not expound on this further; see [58, 22, 4].

It should therefore be the case that the semantics described above form a biCCC. Indeed, it is a well-known fact of topos theory that the *category of presheaves* $[\mathcal{C}, \mathbf{Set}]$ is a biCCC. In fact, the construction of exponentials [62, §I.6] reveals that our definition above is deficient: we should restrict $[\![\varphi \to \psi]\!]_w$ to contain only those functions F that for any $f : w \to v_1$, $g : v_1 \to v_2$, and $x \in [\![\varphi]\!]_{v_1}$ satisfy the *coherence condition* $g \cdot F(v_1)(f)(x) = F(v_2)(g \circ f)(g \cdot x)$.

From now on we will identify two-dimensional Kripke semantics with categorical semantics in a category of presheaves $[\mathcal{C}, \mathbf{Set}]$. Following topos theory, we will call $\mathcal{C}$ a *site*.

4.1. Presheaf categories

The category $[\mathcal{C}, \mathbf{Set}]$ of covariant presheaves is eerily similar to prime algebraic lattices. In a sense they are just the same; but, having traded $\mathbb{2}$ for $\mathbf{Set}$, they have become proof-relevant.

First, letting $P \in [\mathcal{C}, \mathbf{Set}]$, an element $x \in P(w)$ is a proof that P holds at a 'world' $w \in \mathcal{C}$. A morphism $f : w \to v$ of $\mathcal{C}$ then leads to a proof $f \cdot x \stackrel{\text{def}}{=} P(f)(x) \in P(v)$ that P holds at v. Thus, the presheaf P is very much like an upper set.

Second, $[\mathcal{C}, \mathbf{Set}]$ is both complete and cocomplete, with limits and colimits computed pointwise [62, §I]. It is also 'distributive' in an appropriate sense [3, §3.3], which makes it into a *Grothendieck topos*. Amongst other things, this means that it is a cartesian closed category, with the exponential being $(P \Rightarrow Q)(w) \stackrel{\text{def}}{=} \mathrm{Hom}(P \times \mathbf{y}(w), Q)$.

Third, the *representable presheaves* $\mathbf{y}(w) \stackrel{\text{def}}{=} \mathrm{Hom}_{\mathcal{C}}(w, -) : \mathcal{C} \to \mathbf{Set}$ are the proof-relevant analogues of the principal upper set. The Yoneda lemma guarantees that we obtain an *embedding* $\mathbf{y}(-) : \mathcal{C}^{\mathsf{op}} \longrightarrow [\mathcal{C}, \mathbf{Set}]$ which moreover preserves limits and exponentials [4].

Fourth, the representables $\mathbf{y}(w)$ are special, in that they are *tiny* [85].

Definition 4.1. An object $w \in \mathcal{C}$ is *tiny* just if $\mathrm{Hom}(w, -) : \mathcal{C} \to \mathbf{Set}$ preserves colimits.[4]

Tininess is a proof-relevant version of primality: it implies that for any $f : w \to \varinjlim_i v_i$ there exists an i such that f is equal to the composition of a morphism $w \to v_i$ with the injection $v_i \to \varinjlim_i v_i$. By the Yoneda lemma, it follows that all representables $\mathbf{y}(w)$ are tiny.

Fifth, the so-called *co-Yoneda lemma* [61, §III.7] shows that every $P \in [\mathcal{C}, \mathbf{Set}]$ is a colimit of representables. This means that it can be reconstructed by sticking together tiny elements:

$$P \cong \varinjlim_i \mathbf{y}(w_i)$$

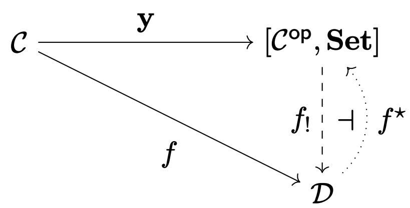
[4]In the literature this property is often referred to as *external tininess* (cf. internal tininess).

Like with prime algebraic lattices, there is a converse to this result: every category which is generated by sticking together tiny elements is in fact a presheaf category:

Theorem 4.2 (Bunge [18])**.** A category which is cocomplete and strongly generated by a small set of tiny objects is equivalent to $[\mathcal{C}, \mathbf{Set}]$ for some small category $\mathcal{C}$.

A textbook presentation of this result can be found in the book by Kelly [52, §5.5].

Finally, the fact every element can be reconstructed as a colimit of tiny objects means that it is possible to uniquely extend any functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ to a cocontinuous functor $[\mathcal{C}^{\mathsf{op}}, \mathbb{2}] \longrightarrow \mathcal{D}$, as long as $\mathcal{D}$ is cocomplete. Diagrammatically, in the situation



(9)

there exists a unique cocontinuous $f_!$ with $f_!(\mathbf{y}(w)) = f(w)$. It is given by

$$f_!\left(\varinjlim_i \mathbf{y}(w_i)\right) = \varinjlim_i f(w_i)$$

$f_!$ is called the *left Kan extension* of f along $\mathbf{y}$. It has a right adjoint $f^\star$ which can be explicitly given by $f^\star(d) = \mathrm{Hom}(f(-), d)$. This amounts to an equivalence

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{D}) \cong \mathrm{Hom}_{\mathbf{Cocont}}([\mathcal{C}^{\mathsf{op}}, \mathbf{Set}], \mathcal{D})$$

where **Cat** is the 2-category of categories, and **Cocont** is the 2-category of cocomplete categories and cocontinuous functors [4, Prop. 9.16] [69, Cor. 6.2.6, Rem. 6.5.9] and [61, § X.3, Cor. 2] [52, Th. 4.51].

All in all, presheaf categories are the categorification of prime algebraic lattices.

4.2. Spacelike sites

Replacing posets with categories does not come for free: the extra dimension of morphisms lead to situations that could not arise in a poset. Some of thse are problematic when thinking of $\mathcal{C}$ as a Kripke frame. Perhaps the most bizarre is the presence of *idempotents*, i.e. morphisms $e : w \to w$ with the property that $e \circ e = e$. Such morphisms represent a non-trivial increase in information which confusingly leaves us in the same world.

The presence of idempotents causes issues. For example, recall that, in prime algebraic lattices, primes and principal upper sets coincide. The astute reader will have noticed we did *not* claim the analogous result in presheaf categories: tiny objects are not necessarily representable in $[\mathcal{C}, \mathbf{Set}]$. For that, we need $\mathcal{C}$ to be *Cauchy-complete* [16, 14].

Definition 4.3. A category is *Cauchy-complete* just if every idempotent splits, i.e. if every idempotent is equal to $s \circ r$ for a section-retraction pair s and r.

Note that every complete category is Cauchy-complete, including **Set** and $[\mathcal{C}, \mathbf{Set}]$.

This leads us to another troublesome situation, namely that of having section-retraction pairs, i.e. $s : w \to v$ and $r : v \to w$ with $r \circ s = \mathsf{id}_w$. In this case w and v contain no more information than each other, but are not isomorphic. We may ask that this does not arise.

Definition 4.4. A category satisfies the *Hemelaer condition* [44, Prop. 5.8] just if every section-retraction pair is an isomorphism.

Combining these two conditions is equivalent to the following definition.

Definition 4.5. A category is *spacelike* if every idempotent is an identity.

In the rest of this paper, we will assume that our sites are at least Cauchy-complete, so that tiny objects coincide with representables.

4.3. Morphisms

The simplest kind of morphism between sites is a functor. Given a $f : \mathcal{C} \longrightarrow \mathcal{D}$ we can define a functor $f^* : [\mathcal{D}, \mathbf{Set}] \longrightarrow [\mathcal{C}, \mathbf{Set}]$ that takes $P : \mathcal{D} \longrightarrow \mathbf{Set}$ to $P \circ f : \mathcal{C} \longrightarrow \mathbf{Set}$. This functor has left and right adjoints, which are given by Kan extension [48, A4.1.4]:

$$\begin{array}{c} f_* \\ [\mathcal{C}, \mathbf{Set}] \;\overset{\top}{\underset{\top}{\xleftarrow{\quad f^* \quad}}}\; [\mathcal{D}, \mathbf{Set}] \\ f_! \end{array} \tag{10}$$

Therefore f^* preserves all limits and colimits, i.e. it is (co)continuous. In short, the presheaf construction gives a functor $[-, \mathbf{Set}] : {\mathbf{Cat}_{\mathrm{cc}}}^{\mathrm{op}} \longrightarrow \mathbf{PshCat}$, where $\mathbf{Cat}_{\mathrm{cc}}$ is the category of small Cauchy-complete categories and functors, and **PshCat** is the category of presheaf categories and (co)continuous functors.

Moreover, this functor is an equivalence. Given a presheaf category we can obtain the site as the subcategory of tiny objects [48, A1.1.10]. But how can we extract $f : \mathcal{C} \longrightarrow \mathcal{D}$ from any (co)continuous functor $f^* : [\mathcal{D}, \mathbf{Set}] \longrightarrow [\mathcal{C}, \mathbf{Set}]$? First, as presheaf categories are locally presentable, the adjoint functor theorem implies that f^* has left and right adjoints, as in (10) [1, §1.66]. This gives what topos theorists call an *essential geometric morphism*. Johnstone [48, §A4.1.5] shows that every such morphism is induced by a $f : \mathcal{C} \longrightarrow \mathcal{D}$, as $f_!$ preserves representables (when $\mathcal{D}$ is Cauchy-complete). We thus obtain a duality[5]

$$\mathbf{Cat}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathbf{PshCat} \tag{11}$$

As with posets, functors here fail to preserve truth; for that we need a notion of openness.

[5] There are some size issues here that are being swept under the rug.

Definition 4.6. $f : \mathcal{C} \longrightarrow \mathcal{D}$ is *open* just if $f^* : [\mathcal{D}, \mathbf{Set}] \longrightarrow [\mathcal{C}, \mathbf{Set}]$ preserves exponentials.

Lemma 4.7. If $f : \mathcal{C} \longrightarrow \mathcal{D}$ is open then there is a natural isomorphism $\theta_w : [\![\varphi]\!]_w \cong [\![\varphi]\!]_{f(w)}$.

Definition 4.6 is somewhat underwhelming, as it does not give explicit conditions that one can check—unlike Definition 2.2. However, obtaining such a description appears difficult.

Some information may be gleaned by considering $(f^*, f_*) : [\mathcal{C}, \mathbf{Set}] \longrightarrow [\mathcal{D}, \mathbf{Set}]$ as a *geometric morphism*. Such a morphism is *open* [46] [49, C3.1] just if both the canonical maps $f^*(c \Rightarrow d) \to f^*(c) \Rightarrow f^*(d)$ and $f^*(\Omega) \to \Omega$ are monic. Johnstone [49, C3.1] proves that (f^*, f_*) is open iff for any $\beta : f(w) \to v'$ in $\mathcal{D}$ there exists an $\alpha : w \to w'$ in $\mathcal{C}$ and a section-retraction pair $s : v' \to f(w')$ and $r : f(w') \to v'$ with $s \circ \beta = f(\alpha)$. This superficially seems like a categorification of Definition 2.2. However, it only guarantees that the canonical map is *sub-cartesian-closed*, whereas we need an isomorphism for Lemma 4.7 to hold.

A stronger condition is to ask that (f^*, f_*) be *locally connected*, i.e. that f^* commute with dependent products [49, C3.3]. All such morphisms are open geometric morphisms. This is stronger than what we need, but sufficient conditions on f can be given [49, C3.3.8].

Finally, an even stronger condition is to ask that (f^*, f_*) be *atomic*, i.e. that f^* is a *logical* functor. This means it preserves exponentials and the subobject classifier [49, A2.1, C3.5]. All atomic geometric morphisms are locally connected. This is again stronger than what we need, and a characterisation in terms of f is elusive: see MathOverflow [76].

It is easier to characterise when (f^*, f_*) is a *surjection*, i.e. when f^* is faithful [48, A2.4.6]. This happens when every $d \in \mathcal{D}$ is the retract of $f(c)$ for some $c \in \mathcal{C}$ [48, A2.4.7]. If $\mathcal{D}$ satisfies the Hemelaer condition, this reduces to f being essentially surjective.

Writing $\mathcal{C} \vDash \varphi$ to mean that $[\![\varphi]\!]_w$ is non-empty for any $w \in \mathcal{C}$ and interpretation of $[\![p]\!]$,

Lemma 4.8. For $\mathcal{D}$ spacelike, $f : \mathcal{C} \to \mathcal{D}$ open and essentially surjective, if $\mathcal{C} \vDash \varphi$ then $\mathcal{D} \vDash \varphi$.

We may thus restrict the duality (11) to

$$\mathbf{Cat}^{\mathrm{op}}_{\mathrm{cc,\ open}} \simeq \mathbf{PshCat}_{\Rightarrow} \qquad\qquad \mathbf{Cat}^{\mathrm{op}}_{\mathrm{sp,\ open,\ es}} \simeq \mathbf{PshCat}_{\Rightarrow,\mathrm{f}} \tag{12}$$

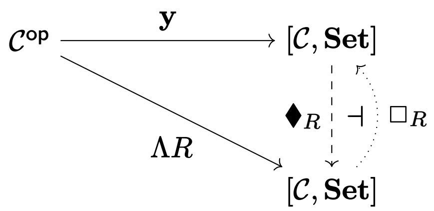
In the first instance, the category on the left is that of Cauchy-complete categories and open functors; and on the right it is presheaf categories and (co)complete, cartesian closed functors. In the second instance, the category on the left is that of spacelike categories and open, essentially surjective functors; and on the right it is presheaf categories and (co)complete, faithful, cartesian closed functors.

5. Modal Logic II

To make a two-dimensional Kripke semantics for modal logic we have to categorify relations. We took the first step by considering with bimodules, i.e. information-order-respecting relations. The second step can be taken by replacing 2 with **Set**; this leads us to the notion of a relation between categories, also known as a *profunctor* or *distributor* [8] [14, §7].

Definition 5.1. A *profunctor* $R : \mathcal{C} \nrightarrow \mathcal{D}$ is a functor $R : \mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathbf{Set}$.

To formulate a two-dimensional Kripke semantics for modal logic, we swap modal Kripke frames $(W, \sqsubseteq, R)$ with a category $\mathcal{C}$ with an (endo)profunctor $R : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathbf{Set}$. To obtain the right definition we can now play the trick we played before: putting $\Lambda R : \mathcal{C}^{\mathsf{op}} \to [\mathcal{C}, \mathbf{Set}]$ into diagram (9) we canonically obtain by Kan extension the following situation:



(13)

We may then define $[\![\blacklozenge\varphi]\!] \stackrel{\text{def}}{=} \blacklozenge_R[\![\varphi]\!] : \mathcal{C} \longrightarrow \mathbf{Set}$ and $[\![\Box\varphi]\!] \stackrel{\text{def}}{=} \Box_R[\![\varphi]\!] : \mathcal{C} \longrightarrow \mathbf{Set}$. It is worth unfolding what a proof of $\Box\varphi$ is at a world w to obtain an explicit description:

$$[\![\Box\varphi]\!]_w = (\Box_R[\![\varphi]\!])(w) = \mathrm{Hom}_{[\mathcal{C},\mathbf{Set}]}(\Lambda R(w), [\![\varphi]\!]) = \mathrm{Hom}_{[\mathcal{C},\mathbf{Set}]}(R(w,-), [\![\varphi]\!]) \tag{14}$$

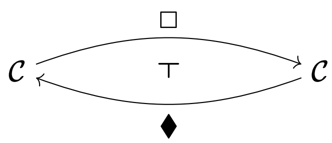
Thus, a proof that φ holds at w is a natural transformation $\alpha : R(w,-) \Rightarrow \varphi$. This has the expected shape of Kripke semantics for $\Box$: for each $v \in \mathcal{C}$ and proof $x \in R(w,v)$ that v is accessible from w, it gives us a proof $\alpha_v(x) \in [\![\varphi]\!]_v$ that φ holds at v.

It is a little harder to see what a proof of $\blacklozenge\varphi$ at a world w is. It becomes more perspicuous if we use the coend formula for the left Kan extension [60, §2.3]:

$$[\![\blacklozenge\varphi]\!] = \Lambda R_![\![\varphi]\!] \cong \int^{v\in\mathcal{C}} \mathrm{Hom}_{[\mathcal{C},\mathbf{Set}]}(\mathbf{y}(v), [\![\varphi]\!]) \times \Lambda R(v) \cong \int^{v\in\mathcal{C}} [\![\varphi]\!]_v \times R(v,-) \tag{15}$$

Hence, a proof that $\blacklozenge\varphi$ holds at w consists of a world $v \in \mathcal{C}$, a proof that $R(v,w)$, and a proof that φ holds at v—which is exactly what we would have expected. The difference is that the coend quotients some of these pairs, according to the action of $\mathcal{C}$ on v.[6]

How well does this fit the categorical semantics of modal logic? As with intuitionistic modal logic, there is also a number of proposals of what that might be. A fairly recent idea is to define it as the semantics of a *Fitch-style calculus*, as studied by Clouston [21]. This is exactly a (bi)cartesian closed category $\mathcal{C}$ equipped with an adjunction:



(16)

[6] See Mac Lane and Moerdijk [62, §VII.2] for a classic textbook exposition on why this construction is a tensor product of $[\![\varphi]\!]$ and ΛR.

The left adjoint $\blacklozenge$ is often written as lock. It does not commonly appear as a modality, but as an operator on contexts that corresponds to 'opening a box' in Fitch-style natural deduction [45, §5.4]. The modality $\Box$ is a right adjoint, so that it automatically preserves all limits, including products. This idea has proven remarkably robust: variations on it have worked well for modal dependent type theories [12, 40, 41, 42, 39]. The fact that an adjunction on a presheaf category corresponds precisely to a two-dimensional Kripke semantics is further evidence that this is the correct notion of categorical model of modal logic.

Finally, note that (14) and (15) look suspiciously like the modal structure of normalization-by-evaluation models for modal type theories. This is explicitly visible in the paper by Valliappan et al. [78, §2], and also implicitly present in the paper by Gratzer [38].

5.1. Morphisms

Define the category **Prof** to have as objects profunctors. A morphism $(f, g, \alpha) : R \to S$ from $R : \mathcal{C} \nrightarrow \mathcal{D}$ to $S : \mathcal{C}' \nrightarrow \mathcal{D}'$ consists of functors $f : \mathcal{C} \longrightarrow \mathcal{C}'$ and $g : \mathcal{D} \longrightarrow \mathcal{D}'$, and a natural transformation $\alpha : R(-,-) \Rightarrow S(f(-), g(-))$. The subcategory **EProf** consists of endoprofunctors $R : \mathcal{C} \nrightarrow \mathcal{C}$, and triples of the form (f, f, α). We synecdochically refer to $\alpha : R(-,-) \Rightarrow S(f(-), f(-))$ as a morphism of **EProf**. Thus, objects are two-dimensional Kripke frames, and morphisms are functors that proof-relevantly preserve the relation.

Lemma 5.2. Morphisms of endoprofunctors $\alpha : R(-,-) \Rightarrow S(f(-), f(-))$ are in bijection with natural transformations $\gamma : f^*\Box_S \Rightarrow \Box_R f^*$.

Proof. Unfolding the definitions, $\gamma : \mathrm{Hom}(S(f(-),-),-) \Rightarrow \mathrm{Hom}(R(-,-), f^*(-))$. As $f_! \dashv f^*$ this is exactly a transformation $\mathrm{Hom}(S(f(-),-),-) \Rightarrow \mathrm{Hom}\big(f_! R(-,-),-\big)$. By the Yoneda lemma, any such transformation arises by precomposition with a unique transformation $f_! R(-,-) \Rightarrow S(f(-),-)$. By $f_! \dashv f^*$ again, this uniquely corresponds to a transformation $\alpha : R(-,-) \Rightarrow f^* S(f(-),-) = S(f(-), f(-))$. $\Box$

We thus obtain a duality

$$\mathbf{EProf}_{\mathrm{cc}}^{\mathrm{op}} \simeq \mathbf{PshCatO} \tag{17}$$

where **PshCatO** is the category of presheaf categories $[\mathcal{C}, \mathbf{Set}]$ equipped with a continuous $\Box : [\mathcal{C}, \mathbf{Set}] \longrightarrow [\mathcal{C}, \mathbf{Set}]$. Note that, as presheaf categories are locally presentable, $\Box$ always has a left adjoint $\blacklozenge$. Thus, the objects are categorical models of modal logic. Morphisms are pairs (f, γ) of a (co)continuous $f : \mathcal{C} \longrightarrow \mathcal{D}$ and a natural transformation $\gamma : f^*\Box \Rightarrow \Box f^*$.

As before, open functors do not preserve truth; for that we need a notion of modal openness. Let $\alpha : R(-,-) \Rightarrow S(f(-), f(-))$. As pointed out in the proof of Lemma 5.2, this uniquely corresponds to a transformation $t_\alpha : f_! R(-,-) \Rightarrow S(f(-),-)$. Its components

$$t_{\alpha,c,v} : \int^{w \in V} R(c,w) \times \mathrm{Hom}_{\mathcal{D}}(f(w), v) \to S(f(c), v)$$

map $x \in R(c,w)$ and $k : f(w) \to v$ to $S(\mathsf{id}_{f(c)}, k)(\alpha_{c,v}(x))$. We can then say that

Definition 5.3. $\alpha : R(-,-) \Rightarrow S(f(-), f(-))$ is *modally open* just if t_α is an isomorphism.

This asks that for every proof $y \in S(f(c), v)$ we should be able to find an object $w \in \mathcal{C}$, a proof $x \in R(c, w)$, and a morphism $k : f(w) \to v$, so that $y = S(\mathsf{id}_{f(c)}, k)(\alpha_{c,v}(x))$. This is clearly a categorification of Definition 3.4, and leads to the following lemma:

Lemma 5.4. α is modally open iff the corresponding $f^*\Box_S \Rightarrow \Box_R f^*$ is an isomorphism.

Proof. The proof of Lemma 5.2 precomposes with t_α to get γ. Thus γ is iso iff t_α is. □

Thus, the duality (17) may be restricted dualities between the wide subcategories

$$\mathbf{EProf}^{\mathrm{op}}_{\mathrm{cc,\ moo}} \simeq \mathbf{PshCatO}_{\Rightarrow o} \qquad\qquad \mathbf{EProf}^{\mathrm{op}}_{\mathrm{sp,\ moo,\ es}} \simeq \mathbf{PshCatO}_{\Rightarrow of} \tag{18}$$

The morphisms on the left are modally open, open maps (resp. also essentially surjective); and the morphisms on the right are (f, γ) where f is cartesian closed (resp. also faithful) and $\gamma : f^*\Box \cong \Box f^*$ is a natural isomorphism.

6. Other related work

Alechina et al. [2] present Kripke and algebraic semantics for constructive S4 and propositional lax logic, and related dualities. Their interpretation of □ is non-standard, cf. [67, 73].

Ghilardi and Meloni [33] explore a presheaf-like interpretation of (predicate) modal logic, which is similar to ours, albeit non-proof-relevant. They work over the identity profunctor $\mathrm{Hom}(-,-)$. They are hence forced to weaken the definition of presheaf. See also [34, 35].

Awodey et al. [5] give a topos-theoretic semantics for a higher-order version of intuitionistic S4 modal logic. They also briefly survey much previous work on presheaf-based and topos-theoretic semantics for first-order modal logic. Their work is not proof-relevant.

Finally, there is clear methodological similarity between the results here and the results of Winskel and collaborators on open maps and bisimulation [51, 19]. One central difference is that Winskel et al. are mainly concerned with open maps between presheaves themselves, whereas I only consider open maps between (two-dimensional) frames.

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