University of BRISTOL

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Big Ideas in Science: Symmetry

Perfect Symmetry

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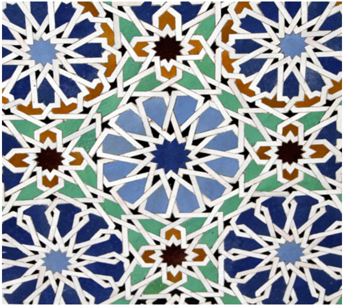



Symmetry in science

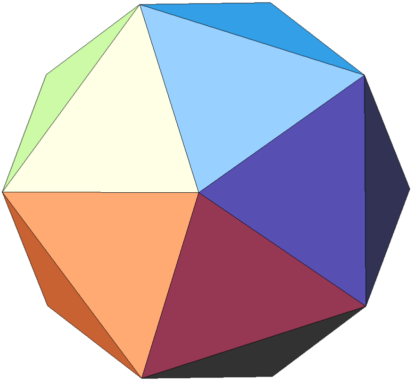
- **Physics (Next week's lecture!):**
  - The physical laws of the universe (e.g. conservation of energy)
  - Relativity and quantum physics
- **Chemistry:** The symmetry of molecules and crystals
- **Biology:** Bilateral symmetry in multicellular organisms
- **Computer science:** The design and implementation of algorithms:

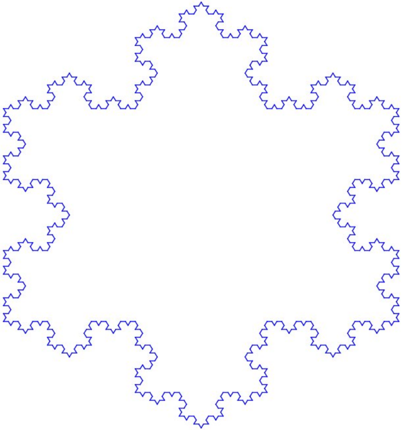
  Symmetry $\rightsquigarrow$ faster, more efficient computation

- **Psychology:** Visual symmetry perception

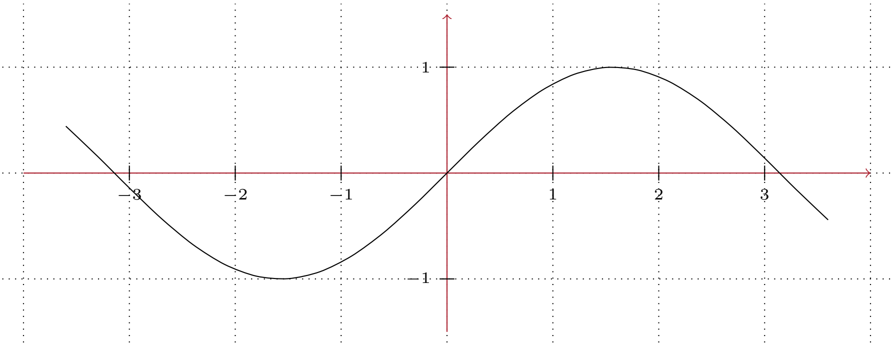
etc. etc. ...

Symmetry in mathematics: Perfect symmetry





Symmetry in mathematics



$$\begin{pmatrix} 2 & 4 & -1 & 9 \\ 4 & 7 & 2 & 3 \\ -1 & 2 & 5 & 0 \\ 9 & 3 & 0 & 3 \end{pmatrix} \qquad x^2 + y^2 + z^2 = 12$$

Example: The symmetry of addition

Problem: *Calculate the sum of the first 50 odd numbers*

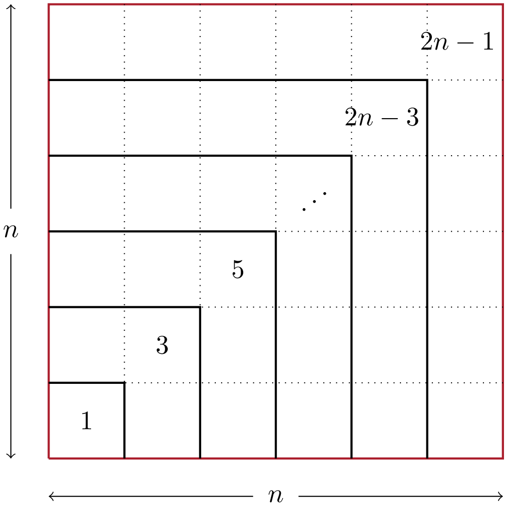
$$
\begin{array}{ccccccc}
1 & 3 & 5 & \cdots & 95 & 97 & 99 \\
+ & + & + & & + & + & + \\
99 & 97 & 95 & \cdots & 5 & 3 & 1 \\
\hline\hline
100 & 100 & 100 & \cdots & 100 & 100 & 100
\end{array}
$$

Answer: $(100 \times 50)/2 = 2500 = 50^2$

Generalisation: *The sum of the first n odd numbers is*

$$1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = (2n \times n)/2 = n^2$$

$$1 + 3 + 5 + \cdots + (2n-3) + (2n-1) = n^2$$



Problem: *Calculate*

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{95} + \frac{1}{97} + \frac{1}{99}$$

Reversing the summation is not helpful, since

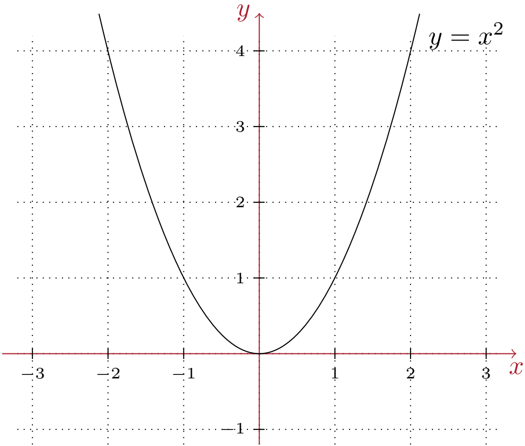
$$1 + \frac{1}{99} \neq \frac{1}{3} + \frac{1}{97}$$

and so on. This **broken symmetry** is reflected in the complexity of the solution:

Answer: $\frac{3200355699626285671281379375916142064964}{1089380862964257455695840764614254743075} \approx 2.94$

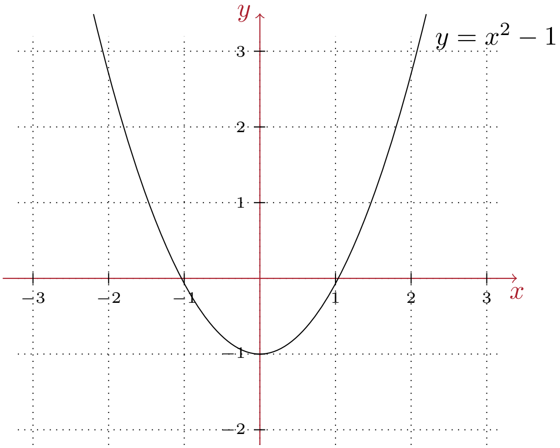
Example: Solving equations

Problem: *Find the solutions to the equation* $x^2 = 0$



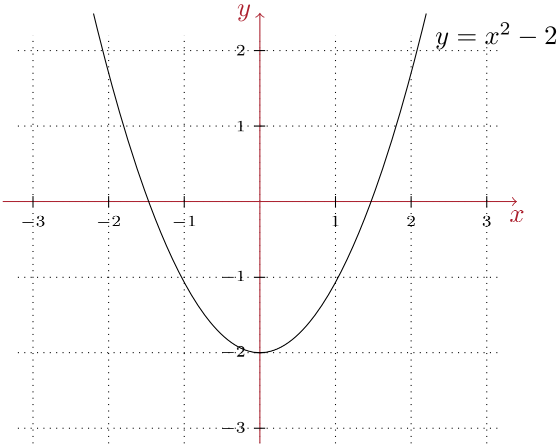
Solutions: $x = 0$

The equation $x^2 - 1 = 0$



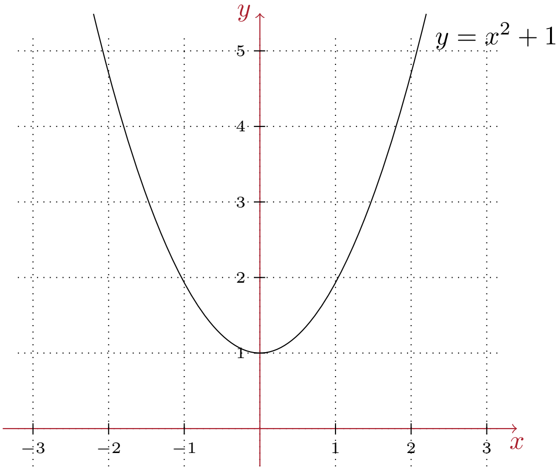
Solutions: $x = 1$ and $x = -1$

The equation $x^2 - 2 = 0$



Solutions: $x = \sqrt{2}$ and $x = -\sqrt{2}$

The equation $x^2 + 1 = 0$



By **symmetry**, we expect to find two solutions, but what are they?

$\sqrt{-1}$

To solve the equation $x^2 + 1 = 0$ we need to “invent” a new number

$$i = \sqrt{-1}$$

such that $i^2 = -1$, so

$$i^2 + 1 = (-i)^2 + 1 = 0$$

Solutions: $x = i$ and $x = -i$

Note: This is similar to how we solve the equation $x + 1 = 0$, by “inventing” the number “-1”. Negative numbers were not widely accepted until the 16th century!

Complex numbers

We now have a new number system, the **complex numbers**

$$\mathbb{C} = \{a + bi \ : \ a \text{ and } b \text{ are real numbers}\}$$

e.g. $2 - 3i$ and $\sqrt{2} + \pi i$ are complex numbers.

We add and multiply in a natural way, remembering $i^2 = -1$, e.g.

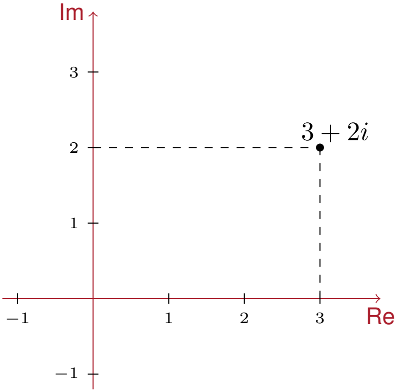
$$(2 - 3i) + (-4 + 7i) = (2 - 4) + (-3i + 7i) = -2 + 4i$$

$$\begin{aligned}(2 - 3i) \times (-4 + 7i) &= (2 \times -4) + (2 \times 7i) + (-3i \times -4) + (-3i \times 7i) \\ &= -8 + 14i + 12i - 21i^2 \\ &= (-8 + 21) + (14i + 12i) \\ &= 13 + 26i\end{aligned}$$

The complex plane

We can associate the complex number $a + bi$ with the point in the plane with coordinates (a, b).

Conversely, any point in the plane corresponds to a complex number.



Applications

Complex numbers have fundamental applications throughout mathematics, science, engineering and technology. For example:

- Quantum physics
- Relativity
- Fluid dynamics
- Electrical engineering
- Digital signal processing

etc. etc.

The quadratic formula

We can use complex numbers to solve **any** quadratic equation. Consider

$$ax^2 + bx + c = 0$$

where a, b and c are numbers (with $a \neq 0$).

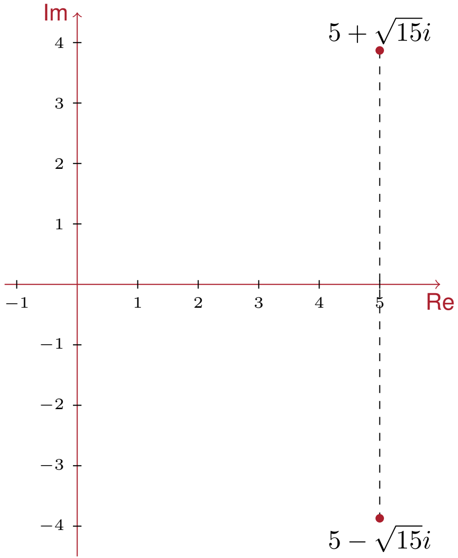
The solutions are given by the familiar **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: $x^2 - 10x + 40 = 0$: $a = 1$, $b = -10$, $c = 40$

$$x = \frac{10 \pm \sqrt{100 - 160}}{2} = 5 \pm \frac{1}{2}\sqrt{-60} = 5 \pm \sqrt{-15} = 5 \pm \sqrt{15}i$$

Symmetry in the solutions



Girolamo Cardano

Higher degree equations

Cardano also studied extensions of the quadratic formula to **cubic** and **quartic** equations. The formulae are **complicated(!)** e.g.

$$x = -\frac{b}{3a} + \sqrt[3]{-\frac{1}{2}\left(\frac{2b^3}{27a^3} - \frac{bc}{a^2} + \frac{d}{a}\right) + \sqrt{\frac{1}{4}\left(\frac{2b^3}{27a^3} - \frac{bc}{a^2} + \frac{d}{a}\right)^2 + \frac{1}{27}\left(\frac{c}{a} - \frac{b^2}{3a^2}\right)^3}}$$
$$+ \sqrt[3]{-\frac{1}{2}\left(\frac{2b^3}{27a^3} - \frac{bc}{a^2} + \frac{d}{a}\right) - \sqrt{\frac{1}{4}\left(\frac{2b^3}{27a^3} - \frac{bc}{a^2} + \frac{d}{a}\right)^2 + \frac{1}{27}\left(\frac{c}{a} - \frac{b^2}{3a^2}\right)^3}}$$

is a solution of the cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

Problem: Is there a similar formula for solutions of the **quintic** equation

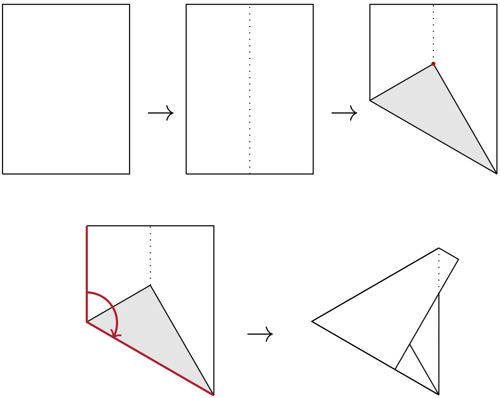
$$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0$$

A mathematical theory of symmetry

By studying the **symmetries** of the solutions, **Évariste Galois** showed that there is no such formula for the quintic equation!

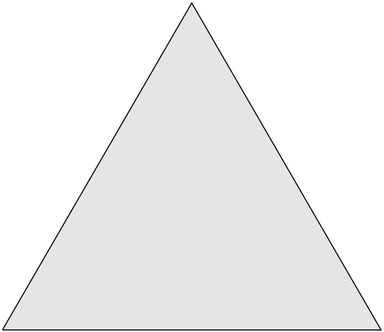
By encoding the symmetries in an algebraic object called a **group**, this incredible breakthrough marked the birth of a **mathematical theory of symmetry.**

Groups and symmetry: An example



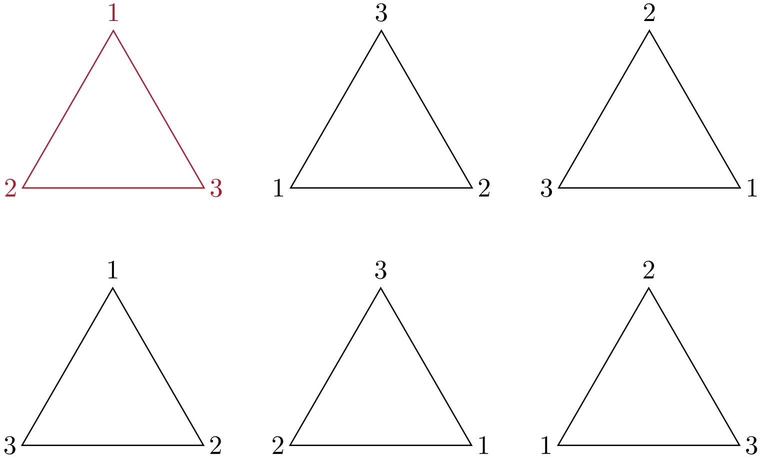
1. Fold paper in half long-ways, then open it out flat
2. Turn bottom left corner up to touch the fold line, making a sharp point with the bottom right corner, and fold
3. Fold the two red edges together, and then tuck in the top corner

4. Label the corners 1,2,3 on both sides, so each corner has the same label front and back
5. Imagine the outline of an equilateral triangle on your desk:



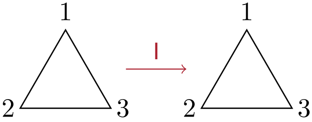
6. Check that there are **six** different ways (keeping track of the corners) to place your paper triangle onto this outline

The six configurations



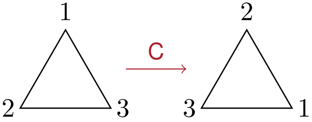
The symmetries of an equilateral triangle

I



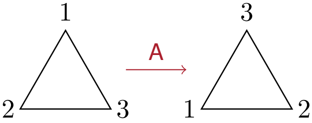
Identity symmetry

C



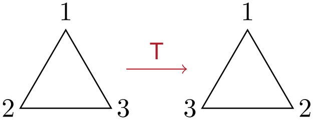
Clockwise rotation

A

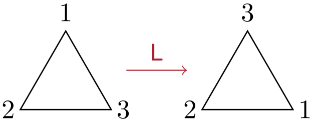


Anticlockwise rotation

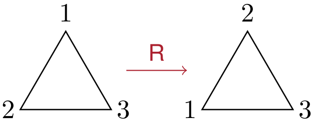
The symmetries of an equilateral triangle

T 

Top corner flip

L 

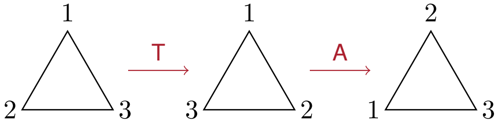
Left corner flip

R 

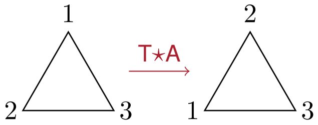
Right corner flip

Combining symmetries

We can "multiply" two symmetries by performing one after the other, e.g.



so



and the "product" T⋆A is itself a symmetry. More precisely,

$$\mathsf{T}\star\mathsf{A} = \mathsf{R}$$

The symmetry group

$\star$	I	C	A	T	L	R
I	I	C	A	T	L	R
C	C	A	I	R	T	L
A	A	I	C	L	R	T
T	T	L	R	I	C	A
L	L	R	T	A	I	C
R	R	T	L	C	A	I

The symmetries of an equilateral triangle are **encoded** by its **symmetry group**

$$(\{\mathrm{I}, \mathrm{C}, \mathrm{A}, \mathrm{T}, \mathrm{L}, \mathrm{R}\}, \star)$$

Big idea: We can study and compare mathematical objects by investigating the (algebraic) properties of their corresponding symmetry groups.

Properties

$\star$	I	C	A	T	L	R
I	I	C	A	T	L	R
C	C	A	I	R	T	L
A	A	I	C	L	R	T
T	T	L	R	I	C	A
L	L	R	T	A	I	C
R	R	T	L	C	A	I

- I$\star$X $=$ X$\star$I $=$ X for any symmetry X
- Each symmetry occurs exactly once in each row and column
- In particular, each symmetry has an "inverse", e.g. C$\star$A $=$ A$\star$C $=$ I, so A is the inverse of C
- Order matters, e.g. C$\star$T $\neq$ T$\star$C

Group Theory

The concept of a symmetry group can be generalised, leading to the notion of an **abstract group**, which are fundamental objects in Pure Mathematics.

Groups arise naturally in many different contexts, e.g.

- $(\mathbb{Z}, +)$ is a group, where $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
- $(\mathbb{C}, +)$ is a group
- $(\{1, -1, i, -i\}, \times)$ is a group. Here is the group table:

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

- Groups of matrices, groups of functions... etc. etc.

“Simple” groups

Let $G = \{$I, C, A, T, L, R$\}$ be the symmetry group of an equilateral triangle.

$\star$	I	C	A	T	L	R
I	I	C	A	T	L	R
C	C	A	I	R	T	L
A	A	I	C	L	R	T
T	T	L	R	I	C	A
L	L	R	T	A	I	C
R	R	T	L	C	A	I

Consider the **subgroups** $H = \{$I, C, A$\}$ and $K = \{$I, T$\}$.

Every element of G is of the form X$\star$Y, where X is in H and Y is in K, so

$$G = H \star K$$

is a “factorisation” of G.

The atoms of symmetry

We have “factorised” $G = H \star K$ as a “product” of H and K.

Here H and K are special because they cannot be factorised any further.

Groups like this are called **simple groups** – they play the role of prime numbers in group theory.

Key fact: Every group can be “factorised” as a “product” of simple groups, so the simple groups are the **basic building blocks** of all groups.

Big idea: Simple groups encode the **atoms of symmetry**.

Big problem: Find all the simple groups!

The Classification Theorem

The **Classification of Finite Simple Groups** is one of the most amazing achievements in the history of mathematics!

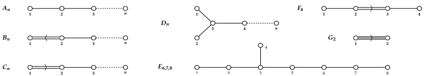
Theorem. *Any finite simple group is one of the following:*

1. *A group with a prime number of elements*
2. *A group of "alternating" or "Lie type"*
3. *One of 26 "sporadic" groups*

- This problem occupied a global team of mathematicians for several decades – the theorem was announced in 1980
- The proof is incredibly complicated – it is over 10000 pages long!
- The theorem provides us with a **periodic table of groups**, which gives a complete description of the atoms of symmetry

The Periodic Table Of Finite Simple Groups

Dynkin Diagrams of Simple Lie Algebras



$0, C_1, \mathbb{Z}_1$ **1** 1																	C_2 2
$A_1(4), A_1(5)$ A_5 60	$A_2(2)$ $A_1(7)$ 168											$^2A_3(4)$ $B_2(3)$ 25 920	$C_3(3)$ 4 585 351 680	$D_4(2)$ 174 182 400	$^2D_4(2^2)$ 197 406 720	$G_2(2)'$ $^2A_2(9)$ 6 048	C_3 3
$A_1(9), B_2(2)'$ A_6 360	$^2G_2(3)'$ $A_1(8)$ 504											$B_2(4)$ 979 200	$C_3(5)$ 228 501 000 000 000	$D_4(3)$ 4 952 179 814 400	$^2D_4(3^2)$ 10 151 968 619 520	$^2A_2(16)$ 62 400	C_5 5
A_7 2 520	$A_1(11)$ 660	$E_6(2)$ 214 841 575 522 005 575 270 400	$E_7(2)$ 7 997 476 042 075 799 759 100 487 262 680 802 918 400	$E_8(2)$ 337 804 753 143 634 806 261 388 190 614 085 595 079 991 692 242 467 651 576 160 959 909 068 800 000	$F_4(2)$ 3 311 126 603 366 400	$G_2(3)$ 4 245 696	$^3D_4(2^3)$ 211 341 312	$^2E_6(2^2)$ 76 532 479 683 774 853 939 200	$^2B_2(2^3)$ 29 120	Tits* $^2F_4(2)'$ 17 971 200	$^2G_2(3^3)$ 10 073 444 472	$B_3(2)$ 1 451 520	$C_4(3)$ 65 784 756 654 489 600	$D_5(2)$ 23 499 295 948 800	$^2D_5(2^2)$ 25 015 379 558 400	$^2A_2(25)$ 126 000	C_7 7
$A_3(2)$ A_8 20 160	$A_1(13)$ 1 092	$E_6(3)$ 7 257 703 347 541 463 210 028 258 395 214 643 200	$E_7(3)$ 1 271 375 236 818 136 742 240 479 751 139 021 644 554 379 203 770 766 254 617 395 200	$E_8(3)$ [illegible]	$F_4(3)$ 5 734 420 792 816 671 844 761 600	$G_2(4)$ 251 596 800	$^3D_4(3^3)$ 20 560 831 566 912	$^2E_6(3^2)$ 14 636 855 916 969 695 633 965 120 680 532 377 600	$^2B_2(2^5)$ 32 537 600	$^2F_4(2^3)$ 264 905 352 699 586 176 614 400	$^2G_2(3^5)$ 49 825 657 439 340 552	$B_2(5)$ 4 680 000	$C_3(7)$ 273 457 218 604 953 600	$D_4(5)$ 8 911 539 000 000 000 000	$^2D_4(4^2)$ 67 536 471 195 648 000	$^2A_3(9)$ 3 265 920	C_{11} 11
A_9 181 440	$A_1(17)$ 2 448	$E_6(4)$ 85 528 710 781 342 640 103 833 619 055 142 765 466 746 880 000	$E_7(4)$ [illegible]	$E_8(4)$ [illegible]	$F_4(4)$ 19 009 825 523 840 945 451 297 669 120 000	$G_2(5)$ 5 859 000 000	$^3D_4(4^3)$ 67 802 350 642 790 400	$^2E_6(4^2)$ 85 696 576 147 617 709 485 896 772 387 584 983 695 360 000 000	$^2B_2(2^7)$ 34 093 383 680	$^2F_4(2^5)$ 1 318 633 155 799 591 447 702 161 609 782 722 560 000	$^2G_2(3^7)$ 239 189 910 264 352 349 332 632	$B_2(7)$ 138 297 600	$C_3(9)$ 54 025 731 402 499 584 000	$D_5(3)$ 1 289 512 799 941 305 139 200	$^2D_4(5^2)$ 17 880 203 250 000 000 000	$^2A_2(64)$ 5 515 776	C_{13} 13
A_n $\frac{n!}{2}$	$PSL_{n+1}(q), L_{n+1}(q)$ $A_n(q)$ $\frac{q^{n(n+1)/2}}{(n+1,q-1)}\prod_{i=1}^{n}(q^{i+1}-1)$	$E_6(q)$ $\frac{q^{36}(q^{12}-1)(q^9-1)(q^8-1)(q^6-1)(q^5-1)(q^2-1)}{(3,q-1)}$	$E_7(q)$ $\frac{q^{63}}{(2,q-1)}\prod_{i=1, i\neq 2,8}^{9}(q^{2i}-1)$	$E_8(q)$ $q^{120}(q^{30}-1)(q^{24}-1)(q^{20}-1)(q^{18}-1)(q^{14}-1)(q^{12}-1)(q^8-1)(q^2-1)$	$F_4(q)$ $q^{24}(q^{12}-1)(q^8-1)(q^6-1)(q^2-1)$	$G_2(q)$ $q^6(q^6-1)(q^2-1)$	$^3D_4(q^3)$ $q^{12}(q^8+q^4+1)(q^6-1)(q^2-1)$	$^2E_6(q^2)$ $\frac{q^{36}(q^{12}-1)(q^9+1)(q^8-1)(q^6-1)(q^5+1)(q^2-1)}{(3,q+1)}$	$^2B_2(2^{2n+1})$ $q^2(q^2+1)(q-1)$	$^2F_4(2^{2n+1})$ $q^{12}(q^6+1)(q^4-1)(q^3+1)(q-1)$	$^2G_2(3^{2n+1})$ $q^3(q^3+1)(q-1)$	$O_{2n+1}(q), \Omega_{2n+1}(q)$ $B_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$PSp_{2n}(q)$ $C_n(q)$ $\frac{q^{n^2}}{(2,q-1)}\prod_{i=1}^{n}(q^{2i}-1)$	$O_{2n}^{+}(q)$ $D_n(q)$ $\frac{q^{n(n-1)}(q^n-1)}{(4,q^n-1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$O_{2n}^{-}(q)$ $^2D_n(q^2)$ $\frac{q^{n(n-1)}(q^n+1)}{(4,q^n+1)}\prod_{i=1}^{n-1}(q^{2i}-1)$	$PSU_{n+1}(q)$ $^2A_n(q^2)$ $\frac{q^{n(n+1)/2}}{(n+1,q+1)}\prod_{i=2}^{n+1}(q^i-(-1)^i)$	$\mathbb{Z}_p$ C_p p

- Alternating Groups
- Classical Chevalley Groups
- Chevalley Groups
- Classical Steinberg Groups
- Steinberg Groups
- Suzuki Groups
- Ree Groups and Tits Group*
- Sporadic Groups
- Cyclic Groups

Alternates† Symbol Order‡

M_{11} 7 920	M_{12} 95 040	M_{22} 443 520	M_{23} 10 200 960	M_{24} 244 823 040	$J(1), J(11)$ J_1 175 560	HJ J_2 604 800	HJM J_3 50 232 960	J_4 86 775 571 046 077 562 880	HS 44 352 000	McL 898 128 000	F_7, HHM, HTH He 4 030 387 200	Ru 145 926 144 000
Sz Suz 448 345 497 600	$O'NS, O\text{–}S$ $O'N$ 460 815 505 920	$\cdot 3$ Co_3 495 766 656 000	$\cdot 2$ Co_2 42 305 421 312 000	$\cdot 1$ Co_1 4 157 776 806 543 360 000	F_5, D HN 273 030 912 000 000	LyS Ly 51 765 179 004 000 000	F_3, E Th 90 745 943 887 872 000	$M(22)$ Fi_{22} 64 561 751 654 400	$M(23)$ Fi_{23} 4 089 470 473 293 004 800	$F_{3+}, M(24)'$ Fi'_{24} 1 255 205 709 190 661 721 292 800	F_2 B 4 154 781 481 226 426 191 177 580 544 000 000	F_1, M_1 M 808 017 424 794 512 875 886 459 904 961 710 757 005 754 368 000 000 000

*The Tits group $^2F_4(2)'$ is not a group of Lie type, but is the (index 2) commutator subgroup of $^2F_4(2)$. It is usually given honorary Lie type status.

The groups starting on the second row are the classical groups. The sporadic suzuki group is unrelated to the families of Suzuki groups.

†For sporadic groups and families, alternate names in the upper left are other names by which they may be known. For specific non-sporadic groups these are used to indicate isomorphims. All such isomorphisms appear on the table except the family $B_n(2^m) \cong C_n(2^m)$.

‡Finite simple groups are determined by their order with the following exceptions:
$B_n(q)$ and $C_n(q)$ for q odd, $n > 2$;
$A_8 \cong A_3(2)$ and $A_2(4)$ of order 20160.

Summary

- Symmetry is a central idea in mathematics, which arises in many different ways
- Symmetries can be exploited to find simple and elegant solutions
- Seeking symmetry has led to many fundamental breakthroughs that have revolutionised science and technology
- Mathematicians have developed the powerful language of group theory to study symmetry in all its forms, with many far reaching applications

Further “reading”

- **Podcast series by Ian Stewart**:
  http://www2.warwick.ac.uk/newsandevents/podcasts/media/more/symmetry
- **Video by Marcus du Sautoy**:
  http://www.ted.com/talks/marcus_du_sautoy_symmetry_reality_s_riddle.html
- **Video by Tim Burness and John Conway**:
  http://www.youtube.com/watch?v=jsSeoGpiWsw
- **Ian Stewart**, *Why Beauty is Truth: The History of Symmetry*, 2008
- **Ian Stewart**, *Symmetry: A Very Short Introduction*, 2013
- **Marcus du Sautoy**, *Finding Moonshine: A Mathematician’s Journey Through Symmetry*, 2009