Extremely primitive groups

Tim Burness

School of Mathematics
University of Southampton

Joint work with Cheryl Praeger and Ákos Seress

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Introduction

Let $G \leq \text{Sym}(\Omega)$ be a finite transitive permutation group with nontrivial point stabilizer

$$G_\alpha = \{ x \in G : \alpha x = \alpha \}$$

- $G$ **primitive**: $G_\alpha$ is a maximal subgroup of $G$
- $G$ **2-transitive**: $G_\alpha$ is transitive on $\Omega \setminus \{\alpha\}$
- $G$ **2-primitive**: $G_\alpha$ is primitive on $\Omega \setminus \{\alpha\}$
- $G$ **extremely primitive**: $G$ is primitive, and $G_\alpha$ is primitive on each of its orbits in $\Omega \setminus \{\alpha\}$

i.e. $G_{\alpha,\beta} < G_\alpha$ is maximal for all $\beta \in \Omega \setminus \{\alpha\}$
## Introduction

### Examples

- **$G = S_n$ on $n$ points**

- **$G$ 2-primitive:** By CFSG, all 2-transitive groups are known, hence all 2-primitive groups are known
  
  - e.g. $G = A_n$ or $S_n$ on $n$ points
  
  - e.g. $G = \text{PSL}_2(q)$ on the projective line

- **$G = J_2, G_\alpha = \text{PSU}_3(3): |\Omega| = 100 = 1 + 36 + 63$**

## The problem

Classify the extremely primitive permutation groups
The O’Nan-Scott Theorem

Let $G \leq \text{Sym}(\Omega)$ be a finite primitive permutation group with point stabilizer $H$ and socle $S = T^d$, $T$ simple.

- If $T = Z_p$ is abelian then

  $$G = Z_p^d \rtimes H \leq Z_p^d \rtimes \text{GL}_d(p) = \text{AGL}_d(p)$$

  is affine, with $H \leq \text{GL}_d(p)$ irreducible

- If $T$ is nonabelian then one of the following holds:

  - $G$ is of **diagonal type** or **product type**
  - $G$ is a **twisted wreath product**
  - $G$ is **almost simple**, i.e. $S = T$ and $T \leq G \leq \text{Aut}(T)$
A reduction theorem

Extremely primitive groups are rather more restricted:

**Theorem (Mann-Praeger-Seress, 2007)**

*If $G$ is extremely primitive then $G$ is affine or almost simple.*

Moreover, if $G = \mathbb{Z}_p^d \rtimes H \leq \text{AGL}_d(p)$ is affine then one of the following holds:

- $G$ is solvable: all examples are known
- $p = 2$, $H$ is almost simple and either
  - $G$ is 2-primitive (all examples known), or
  - $G$ is simply primitive and either $(d, H)$ is known, or $(d, H)$ is one of finitely many additional possibilities.
Affine groups

If \( G \) is affine, non-solvable and simply primitive then the known extremely primitive examples \((d, \text{Soc}(H))\) are as follows (each with \( p = 2 \)):

(a) \((10, M_{12}), (10, M_{22}), (11, M_{23}), (11, M_{24}), (22, \text{Co}_3), (24, \text{Co}_1)\)
\((8, \text{L}_2(17)), (8, \text{Sp}_6(2))\)

(b) \((2k, A_{2k+1}) \ k \geq 2, (2k, A_{2k+2}) \ k \geq 3\)

(c) \((2k, \Omega^\pm_{2k}(2)) \ k \geq 3\)

Conjecture (Mann-Praeger-Seress, 2007)

There are no additional extremely primitive affine groups.
Affine groups: A useful lemma

Suppose $G = Z_p^d \rtimes H \leq AGL_d(p)$ is affine and extremely primitive.

$$H \leq GL_d(p) \text{ irreducible} \implies C_{GL_d(p)}(H) = (\mathbb{F}_{p^a})^* \text{ with } a|d$$

$$\implies H \leq GL_{d/a}(p^a).a$$

Lemma

Assume $a < d$. If $h \in H$ has an eigenvalue $\lambda \in \mathbb{F}_{p^a}$ then $\lambda = 1$, so $H$ contains no nontrivial element of order dividing $p^a - 1$.

Suppose $0 \neq u \in Z_p^d$ and $u^h = \lambda u$ with $\lambda \in \mathbb{F}_{p^a}$. Set $U = \langle u \rangle \mathbb{F}_{p^a}$. Then

$$C_H(u) = H_u \leq \langle h, H_u \rangle \leq N_H(U) < H$$

and $H_u < H$ is maximal, so $h \in H_u$ and $\lambda = 1$.

Corollary

If $G$ is non-solvable then $p = 2$. 

Affine groups: Another useful lemma

Suppose $G = \mathbb{Z}_2^d \rtimes H \leq AGL_d(2)$ is affine and extremely primitive.

Let $\mathcal{M}$ be the set of maximal subgroups of $H = G_0$. For $M \in \mathcal{M}$, let $\text{fix}(M)$ be the points in $\Omega = \mathbb{Z}_2^d$ fixed by $M$.

**Lemma**

$$\sum_{M \in \mathcal{M}} (|\text{fix}(M)| - 1) = 2^d - 1,$$
and $|\text{fix}(M)| \leq 2^{d/2}$ for all $M \in \mathcal{M}$.

In particular, $|\mathcal{M}| > 2^{d/2}$.

The lemma quickly follows from two easy observations:

- Suppose $M_1, M_2 \in \mathcal{M}$, $M_1 \neq M_2$ and $v \in \text{fix}(M_1) \cap \text{fix}(M_2)$. Then $v$ is fixed by $\langle M_1, M_2 \rangle = H$, so $v = 0$ since $H$ is irreducible.

- $v \neq 0 \implies H_v \in \mathcal{M}$ (since $G$ is extremely primitive).
Wall’s conjecture

Suppose $G = \mathbb{Z}_2^d \rtimes H \leq AGL_d(2)$ is a primitive affine group.

By the lemma, if $|\mathcal{M}| \leq 2^{d/2}$ then $G$ is not extremely primitive, so bounds on $|\mathcal{M}|$ are important here.

**Conjecture (G.E. Wall, 1961)**

$|\mathcal{M}| \leq |H|$ for any finite group $H$

**Theorem (Liebeck-Martin-Shalev, 2005)**

Wall’s conjecture holds if $H$ is a sufficiently large almost simple group

If $|\mathcal{M}| \leq |H|$ and $G$ is extremely primitive then $2^{d/2} < |H|$ and there are only a small number of explicit $H$-modules over $\mathbb{F}_2$ to consider.
For example, suppose $H = A_n$ or $S_n$ with $n \geq 15$. Let $V$ be a nontrivial irreducible $\mathbb{F}_2 H$-module.

**Theorem (G.D. James, 1983)**

Either $V$ is the fully deleted permutation module for $H$ (of dimension $n - 2$ or $n - 1$), or $\dim V \geq n(n - 5)/2$.

**Theorem (Liebeck-Shalev, 1996)**

If $n$ is sufficiently large then $|\mathcal{M}| \leq n!$

If $n \geq 17$ then $n! < 2^{n(n-5)/4}$, so the following corollary holds:

**Corollary**

There are only finitely many extremely primitive groups of the form $G = \mathbb{Z}_2^d \rtimes H$, with $\text{Soc}(H) = A_n$ and $d \geq n$. 
Almost simple groups

Let $G$ be an almost simple group with socle $T$, so

$$T \leq G \leq \text{Aut}(T)$$

By CFSG, such a group belongs to one of four families:

(i) $G$ is a **symmetric** or **alternating group** (degree $n \geq 5$)

(ii) $G$ is a **classical group**, e.g. $G = L_n(q), \text{PGU}_n(q), \text{PSp}_n(q)$

(iii) $G$ is an **exceptional group**, e.g. $G = G_2(q), ^2E_6(q), E_8(q)$

(iv) $G$ is a **sporadic group**, e.g. $G = M_{22}:2, \text{Co}_1, \text{M}$

**Theorem (B-Praeger-Seress, 2011)**

*The almost simple extremely primitive groups of type (i), (ii) and (iv) have been classified.*
Symmetric and alternating groups

**Theorem**

Let $G$ be an almost simple group with socle $T = A_n$ and point stabilizer $H$. Then $G$ is extremely primitive if and only if $(G, H)$ is one of the following:

<table>
<thead>
<tr>
<th>$H$</th>
<th>Rank</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_G((S_{n/2} \wr S_2) \cap G)$</td>
<td>$(n + 2)/4$</td>
<td>$n \equiv 2 \pmod{4}$</td>
</tr>
<tr>
<td>$N_G(A_{n-1})$</td>
<td>2</td>
<td>$G \leq S_n$</td>
</tr>
<tr>
<td>$N_G(D_{10})$</td>
<td>2</td>
<td>$n = 5$</td>
</tr>
</tbody>
</table>

In the first example, $\Omega$ is the set of partitions of $\{1, \ldots, n\}$ into subsets of size $n/2$. If $n \equiv 0 \pmod{4}$ then

$$G_{\alpha,\beta} = (S_{n/4} \wr V_4) \cap G < (S_{n/4} \wr D_8) \cap G < G_{\alpha} \text{ for}$$

$\alpha = \{1, \ldots, n/2\} \cup \{n/2 + 1, \ldots, n\}$

$\beta = \{1, \ldots, n/4, 3n/4 + 1, \ldots, n\} \cup \{n/4 + 1, \ldots, 3n/4\}$
Sporadic groups

Theorem

If $T$ is a sporadic group then $G$ is extremely primitive, but not 2-primitive, if and only if $(G, H)$ is one of the following ($\alpha = 1$ or 2):

$$(J_{2,\alpha}, U_3(3).\alpha), (HS.\alpha, M_{22}.\alpha), (Suz.\alpha, G_2(4).\alpha)$$

$$(McL.\alpha, U_4(3).\alpha), (Ru, {2F}_4(2)), (Co_2, U_6(2).2), (Co_2, McL)$$

The highest rank in this list is 6, for $(G, H) = (Co_2, McL)$:

$$|\Omega| = 47104 = 1 + 275 + 2025 + 7128 + 15400 + 22275$$

In addition, there are nine 2-primitive almost simple sporadic groups (in fact, every 2-transitive sporadic group is extremely primitive).
Classical groups

**Theorem**

*If* $T$ *is a classical group then* $G$ *is extremely primitive if and only if* $(G, H)$ *is one of the following:*

<table>
<thead>
<tr>
<th>$T$</th>
<th>Type of $H$</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_2(q)$</td>
<td>$P_1$</td>
<td>$q \geq 4$</td>
</tr>
<tr>
<td>$\text{PSp}_n(2)'$</td>
<td>$O_n^{\pm}(2)$</td>
<td>$n \geq 4$</td>
</tr>
<tr>
<td>$L_2(q)$</td>
<td>$D_{2(q+1)}$</td>
<td>$G = T$, $q &gt; 2$, $q + 1$ Fermat</td>
</tr>
<tr>
<td>$L_4(2)$</td>
<td>$A_7$</td>
<td></td>
</tr>
<tr>
<td>$U_4(3)$</td>
<td>$L_3(4)$</td>
<td>$G = T.2^2$ or $G = T.2$</td>
</tr>
<tr>
<td>$L_3(4)$</td>
<td>$A_6$</td>
<td>$G = T.2^2$ or $G = T.2$</td>
</tr>
<tr>
<td>$L_2(11)$</td>
<td>$A_5$</td>
<td>$G = T$</td>
</tr>
</tbody>
</table>
Main ingredients

- Detailed information on the structure and conjugacy classes of maximal subgroups of almost simple groups:
  - Alternating groups: O’Nan-Scott
  - Classical groups: Aschbacher, Kleidman-Liebeck, ...
  - Sporadic groups: Wilson et al.

- By Manning (1927), $H = G_\alpha$ acts faithfully on each orbit in $\Omega \setminus \{\alpha\}$, so we can apply the O’Nan-Scott theorem to $H$. In particular,
  - $\text{Soc}(H)$ is a product of isomorphic simple groups
  - $F(H)$ is either trivial or elementary abelian
  - $Z(H)$ is trivial

- Direct calculation and computation (e.g. using Magma)

- Recent work on bases for primitive permutation groups
Bases

A **base** of a permutation group $G \leq Sym(\Omega)$ is a subset $S$ of $\Omega$ such that the pointwise stabilizer of $S$ in $G$ is trivial.

The **base size**, denoted $b(G)$, is the minimal size of a base for $G$.

**Examples**

- $G = S_n$, $\Omega = \{1, \ldots, n\} \implies b(G) = n - 1$
- $G = GL(V)$, $\Omega = V \implies b(G) = \dim V$

Suppose $G$ is almost simple and extremely primitive. If $\alpha, \beta \in \Omega$, $\alpha \neq \beta$, then $G_{\alpha,\beta} < G_\alpha$ is maximal, so $G_{\alpha,\beta} \neq 1$ and hence $b(G) > 2$.

**The base-two project**

Classify the primitive permutation groups $G$ with $b(G) = 2$
Symmetric and alternating groups

**Theorem (B-Guralnick-Saxl, 2010)**

Let $G$ be an almost simple primitive group with socle $A_n$. Assume $H = G_\alpha$ acts primitively on $\{1, \ldots, n\}$. Then $b(G) = 2$ for all $n > 12$.

Consequently, for extreme primitivity, there are just two cases to deal with:

(i) $H = (S_k \times S_{n-k}) \cap G$ for some $1 \leq k < n/2$;

(ii) $H = (S_k \wr S_{n/k}) \cap G$ with $2 \leq k \leq n/2$.

Consider (i): $G$ is 2-primitive if $k = 1$. If $k > 1$ and $(G, k) \neq (A_n, 2)$ then \(\text{Soc}(H)\) is not a product of isomorphic simple groups.

If $(G, k) = (A_n, 2)$ then $H = G_{\{1,2\}} = S_{n-2}$ and

$$H_{\{2,3\}} = H_{1,2,3} < H_{1,2} < H$$

so $G$ is not extremely primitive.
Sporadic groups

Theorem (B-O’Brien-Wilson, 2010)

The base size of every primitive almost simple sporadic group is known.

In most cases $b(G) = 2$, e.g. if $G = \mathbb{M}$ then the only exception is the case $H = 2.B$ with $b(G) = 3$. We inspect the list of exceptions.

From the structural constraints on $H$, we reduce to a list of cases with $H$ almost simple. Using MAGMA and data in the Web Atlas, we reduce further to 15 specific cases $(G, H)$.

Here $G$ is multiplicity free – every irreducible constituent of $1_H^G$ has multiplicity 1. All such actions of sporadic groups are known (Breuer-Lux, 1996), and all subdegrees have been computed.

In this situation, no extremely primitive examples arise.
Classical groups

Let $G$ be an almost simple classical group with socle $T = \text{Cl}(V)$ and point stabilizer $H$. Roughly speaking, we say $G$ is standard if $H \cap T$ is reducible on $V$, otherwise $G$ is non-standard.

Standard groups have large base sizes, in general.

Example

If $G = \text{PGL}_n(q)$ and $H = P_1$ then $b(G) = n + 1$.

Theorem (B, 2007)

If $G$ is non-standard then $b(G) \leq 5$, with equality if and only if $G = \text{U}_6(2).2$ and $H = \text{U}_4(3).2^2$.

More recently, with Guralnick and Saxl, we have computed $b(G)$ precisely for ‘almost all’ non-standard classical groups $G$. 
Classical groups

Roughly, we get \( b(G) \leq 3 \), with equality only if \( H = C_G(x) \) for an involution \( x \in \text{Aut}(T) \).

We have computed the exact value of \( b(G) \) when \( H \) belongs to Aschbacher’s \( S \) collection of irreducible almost simple subgroups.

In the remaining cases:

- The structure of \( H \) rules out many cases

- We can often work directly with matrices to find an explicit \( x \in G \setminus H \) such that \( H \cap H^x < H \) is non-maximal

- For standard groups, \( \Omega \) is a set of subspaces of \( V \) and we can work with explicit subspaces to find a non-maximal two-point stabilizer \( G_{\alpha,\beta} < G_{\alpha} \)
Suppose $G = \text{PSL}_n(q) = \text{PSL}(V)$ and $H$ is a subfield subgroup of type $\text{GL}_n(q_0)$, where $q = q_0^r$ with $r$ prime. If $r > 2$ then $b(G) = 2$. Suppose $r = 2$.

Here $H = C_G(\sigma)$, with $\sigma \in \text{Aut}(G)$ a field aut of order 2. By fixing a basis $\{v_1, \ldots, v_n\}$ for $V$, we may assume $(a_{ij})^\sigma = (a_{ij}^{q_0})$. Then

$$H \cap H^x = C_H(x^{-1}x^\sigma)$$

for all $x \in G$. Write $\mathbb{F}_q^* = \langle \omega \rangle$ and set

$$x = \begin{pmatrix} 1 & \omega & 0 \\ 0 & 1 & 1 \\ \hline & l_{n-2} \end{pmatrix} \in G, \quad x^{-1}x^\sigma = \begin{pmatrix} 1 & \omega^{q_0} - \omega \\ 0 & 1 \\ \hline & l_{n-2} \end{pmatrix}$$

Then $H \cap H^x = C_H(x^{-1}x^\sigma) < H_U < H$, where $U = \langle v_2 \rangle$, hence $G$ is not extremely primitive.
Some open problems

- Determine the extremely primitive groups of exceptional Lie type.

  Here $b(G) \leq 6$ (by B-Liebeck-Shalev, 2009), but we do not know all the base-two examples.

  Current work with Guralnick and Saxl on bases for exceptional algebraic groups may be useful here.

- Prove the Mann-Praeger-Seress conjecture on affine groups (their list is complete) – a proof of Wall’s conjecture for almost simple groups would be very helpful!

- Classify the base-two almost simple primitive permutation groups

- Calculate the base size of every primitive permutation group...