Extremely primitive groups

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Introduction

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite transitive permutation group with nontrivial point stabilizer

$$G_\alpha = \{x \in G \,:\, \alpha x = \alpha\}$$

- G **primitive:** G_α is a maximal subgroup of G
- G 2-**transitive:** G_α is transitive on $\Omega \setminus \{\alpha\}$
- G 2-**primitive:** G_α is primitive on $\Omega \setminus \{\alpha\}$
- G **extremely primitive:** G is primitive, and G_α is primitive on each of its orbits in $\Omega \setminus \{\alpha\}$

  i.e. $G_{\alpha,\beta} < G_\alpha$ is maximal for all $\beta \in \Omega \setminus \{\alpha\}$

Introduction

Examples

- $G = S_n$ on n points
- G 2-**primitive:** By CFSG, all 2-transitive groups are known, hence all 2-primitive groups are known

  e.g. $G = A_n$ or S_n on n points

  e.g. $G = \mathrm{PSL}_2(q)$ on the projective line
- $G = \mathrm{J}_2$, $G_\alpha = \mathrm{PSU}_3(3)$: $|\Omega| = 100 = 1 + 36 + 63$

The problem

Classify the extremely primitive permutation groups

The O'Nan-Scott Theorem

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite primitive permutation group with point stabilizer H and socle $S = T^d$, T simple.

- If $T = Z_p$ is abelian then

  $$G = Z_p^d \rtimes H \leqslant Z_p^d \rtimes \mathrm{GL}_d(p) = \mathrm{AGL}_d(p)$$

  is **affine**, with $H \leqslant \mathrm{GL}_d(p)$ irreducible

- If T is nonabelian then one of the following holds:
  - G is of **diagonal type** or **product type**
  - G is a **twisted wreath product**
  - G is **almost simple**, i.e. $S = T$ and $T \leqslant G \leqslant \mathrm{Aut}(T)$

A reduction theorem

Extremely primitive groups are rather more restricted:

Theorem (Mann-Praeger-Seress, 2007)

If G is extremely primitive then G is **affine** *or* **almost simple***.*

Moreover, if $G = Z_p^d \rtimes H \leqslant \mathrm{AGL}_d(p)$ is **affine** *then one of the following holds:*

- *G is solvable: all examples are known*
- *$p = 2$, H is almost simple and either*
  - *G is 2-primitive (all examples known), or*
  - *G is simply primitive and either (d, H) is known, or (d, H) is one of finitely many additional possibilities*

Affine groups

If G is affine, non-solvable and simply primitive then the known extremely primitive examples $(d, \mathrm{Soc}(H))$ are as follows (each with $p = 2$):

(a) $(10, \mathrm{M}_{12})$, $(10, \mathrm{M}_{22})$, $(11, \mathrm{M}_{23})$, $(11, \mathrm{M}_{24})$, $(22, \mathrm{Co}_3)$, $(24, \mathrm{Co}_1)$ $(8, \mathrm{L}_2(17))$, $(8, \mathrm{Sp}_6(2))$

(b) $(2k, A_{2k+1})$ $k \geq 2$, $(2k, A_{2k+2})$ $k \geq 3$

(c) $(2k, \Omega^{\pm}_{2k}(2))$ $k \geq 3$

Conjecture (Mann-Praeger-Seress, 2007)

There are no additional extremely primitive affine groups.

Affine groups: A useful lemma

Suppose $G = Z_p^d \rtimes H \leqslant \mathrm{AGL}_d(p)$ is affine and extremely primitive.

$$H \leqslant \mathrm{GL}_d(p) \text{ irreducible } \implies C_{\mathrm{GL}_d(p)}(H) = (\mathbb{F}_{p^a})^* \text{ with } a|d$$
$$\implies H \leqslant \mathrm{GL}_{d/a}(p^a).a$$

Lemma

Assume $a < d$. If $h \in H$ has an eigenvalue $\lambda \in \mathbb{F}_{p^a}$ then $\lambda = 1$, so H contains no nontrivial element of order dividing $p^a - 1$.

Suppose $0 \neq u \in Z_p^d$ and $u^h = \lambda u$ with $\lambda \in \mathbb{F}_{p^a}$. Set $U = \langle u \rangle_{\mathbb{F}_{p^a}}$. Then

$$C_H(u) = H_u \leqslant \langle h, H_u \rangle \leqslant N_H(U) < H$$

and $H_u < H$ is maximal, so $h \in H_u$ and $\lambda = 1$.

Corollary

If G is non-solvable then $p = 2$.

Affine groups: Another useful lemma

Suppose $G = Z_2^d \rtimes H \leqslant \mathrm{AGL}_d(2)$ is affine and extremely primitive.

Let $\mathcal{M}$ be the set of maximal subgroups of $H = G_0$. For $M \in \mathcal{M}$, let $\mathrm{fix}(M)$ be the points in $\Omega = Z_2^d$ fixed by M.

Lemma

$$\sum_{M \in \mathcal{M}} (|\mathrm{fix}(M)| - 1) = 2^d - 1, \text{ and } |\mathrm{fix}(M)| \leq 2^{d/2} \text{ for all } M \in \mathcal{M}.$$

In particular, $|\mathcal{M}| > 2^{d/2}$.

The lemma quickly follows from two easy observations:

- Suppose $M_1, M_2 \in \mathcal{M}$, $M_1 \neq M_2$ and $v \in \mathrm{fix}(M_1) \cap \mathrm{fix}(M_2)$. Then v is fixed by $\langle M_1, M_2 \rangle = H$, so $v = 0$ since H is irreducible.
- $v \neq 0 \implies H_v \in \mathcal{M}$ (since G is extremely primitive).

Wall's conjecture

Suppose $G = Z_2^d \rtimes H \leqslant \mathrm{AGL}_d(2)$ is a primitive affine group.

By the lemma, if $|\mathcal{M}| \leq 2^{d/2}$ then G is **not** extremely primitive, so bounds on $|\mathcal{M}|$ are important here.

Conjecture (G.E. Wall, 1961)

$|\mathcal{M}| \leq |H|$ for any finite group H

Theorem (Liebeck-Martin-Shalev, 2005)

Wall's conjecture holds if H is a sufficiently large almost simple group

If $|\mathcal{M}| \leq |H|$ and G is extremely primitive then $2^{d/2} < |H|$ and there are only a small number of explicit H-modules over $\mathbb{F}_2$ to consider.

For example, suppose $H = A_n$ or S_n with $n \geq 15$. Let V be a nontrivial irreducible $\mathbb{F}_2H$-module.

Theorem (G.D. James, 1983)

Either V is the fully deleted permutation module for H (of dimension $n-2$ or $n-1$), or $\dim V \geq n(n-5)/2$.

Theorem (Liebeck-Shalev, 1996)

If n is sufficiently large then $|\mathcal{M}| \leq n!$

If $n \geq 17$ then $n! < 2^{n(n-5)/4}$, so the following corollary holds:

Corollary

There are only finitely many extremely primitive groups of the form $G = Z_2^d \rtimes H$, with $\mathrm{Soc}(H) = A_n$ and $d \geq n$.

Almost simple groups

Let G be an almost simple group with socle T, so

$$T \leqslant G \leqslant \mathrm{Aut}(T)$$

By CFSG, such a group belongs to one of four families:

(i) G is a **symmetric** or **alternating group** (degree $n \geq 5$)

(ii) G is a **classical group**, e.g. $G = \mathrm{L}_n(q), \mathrm{PGU}_n(q), \mathrm{PSp}_n(q)$

(iii) G is an **exceptional group**, e.g. $G = G_2(q), {}^2E_6(q), E_8(q)$

(iv) G is a **sporadic group**, e.g. $G = \mathrm{M}_{22}{:}2, \mathrm{Co}_1, \mathbb{M}$

Theorem (B-Praeger-Seress, 2011)

The almost simple extremely primitive groups of type (i), (ii) and (iv) have been classified.

Symmetric and alternating groups

Theorem

Let G be an almost simple group with socle $T = A_n$ and point stabilizer H. Then G is extremely primitive if and only if (G, H) is one of the following:

H	Rank	Conditions
$N_G((S_{n/2} \wr S_2) \cap G)$	$(n+2)/4$	$n \equiv 2 \pmod 4$
$N_G(A_{n-1})$	2	$G \leqslant S_n$
$N_G(D_{10})$	2	$n = 5$

In the first example, Ω is the set of partitions of $\{1, \ldots, n\}$ into subsets of size $n/2$. If $n \equiv 0 \pmod 4$ then

$$G_{\alpha,\beta} = (S_{n/4} \wr V_4) \cap G < (S_{n/4} \wr D_8) \cap G < G_\alpha \text{ for}$$

$$\alpha = \{1, \ldots, n/2\} \cup \{n/2+1, \ldots, n\}$$
$$\beta = \{1, \ldots, n/4, 3n/4+1, \ldots, n\} \cup \{n/4+1, \ldots, 3n/4\}$$

Sporadic groups

Theorem

If T is a sporadic group then G is extremely primitive, but not 2-primitive, if and only if (G, H) is one of the following ($\alpha = 1$ or 2):

$$(\mathrm{J}_2.\alpha, \mathrm{U}_3(3).\alpha), (\mathrm{HS}.\alpha, \mathrm{M}_{22}.\alpha), (\mathrm{Suz}.\alpha, G_2(4).\alpha)$$

$$(\mathrm{McL}.\alpha, \mathrm{U}_4(3).\alpha), (\mathrm{Ru}, {}^2F_4(2)), (\mathrm{Co}_2, \mathrm{U}_6(2).2), (\mathrm{Co}_2, \mathrm{McL})$$

The highest rank in this list is 6, for $(G, H) = (\mathrm{Co}_2, \mathrm{McL})$:

$$|\Omega| = 47104 = 1 + 275 + 2025 + 7128 + 15400 + 22275$$

In addition, there are nine 2-primitive almost simple sporadic groups (in fact, every 2-transitive sporadic group is extremely primitive).

Classical groups

Theorem

If T is a classical group then G is extremely primitive if and only if (G, H) is one of the following:

T	Type of H	Conditions
$\mathrm{L}_2(q)$	P_1	$q \geq 4$
$\mathrm{PSp}_n(2)'$	$O_n^{\pm}(2)$	$n \geq 4$
$\mathrm{L}_2(q)$	$D_{2(q+1)}$	$G = T$, $q > 2$, $q+1$ Fermat
$\mathrm{L}_4(2)$	A_7	
$\mathrm{U}_4(3)$	$\mathrm{L}_3(4)$	$G = T.2^2$ or $G = T.2$
$\mathrm{L}_3(4)$	A_6	$G = T.2^2$ or $G = T.2$
$\mathrm{L}_2(11)$	A_5	$G = T$

Main ingredients

- Detailed information on the structure and conjugacy classes of maximal subgroups of almost simple groups:
  - Alternating groups: O'Nan-Scott
  - Classical groups: Aschbacher, Kleidman-Liebeck, ...
  - Sporadic groups: Wilson et al.

- By Manning (1927), $H = G_\alpha$ acts faithfully on each orbit in $\Omega \setminus \{\alpha\}$, so we can apply the O'Nan-Scott theorem to H. In particular,
  - $\mathrm{Soc}(H)$ is a product of isomorphic simple groups
  - $F(H)$ is either trivial or elementary abelian
  - $Z(H)$ is trivial

- Direct calculation and computation (e.g. using MAGMA)

- Recent work on **bases** for primitive permutation groups

Bases

A **base** of a permutation group $G \leqslant \mathrm{Sym}(\Omega)$ is a subset S of Ω such that the pointwise stabilizer of S in G is trivial.

The **base size**, denoted $b(G)$, is the minimal size of a base for G.

Examples

- $G = S_n$, $\Omega = \{1, \dots, n\} \implies b(G) = n - 1$
- $G = \mathrm{GL}(V)$, $\Omega = V \implies b(G) = \dim V$

Suppose G is almost simple and extremely primitive. If $\alpha, \beta \in \Omega$, $\alpha \neq \beta$, then $G_{\alpha,\beta} < G_\alpha$ is maximal, so $G_{\alpha,\beta} \neq 1$ and hence $b(G) > 2$.

The base-two project

Classify the primitive permutation groups G with $b(G) = 2$

Symmetric and alternating groups

Theorem (B-Guralnick-Saxl, 2010)

Let G be an almost simple primitive group with socle A_n. Assume $H = G_\alpha$ acts primitively on $\{1, \ldots, n\}$. Then $b(G) = 2$ for all $n > 12$.

Consequently, for extreme primitivity, there are just two cases to deal with:

(i) $H = (S_k \times S_{n-k}) \cap G$ for some $1 \leq k < n/2$;
(ii) $H = (S_k \wr S_{n/k}) \cap G$ with $2 \leq k \leq n/2$.

Consider (i): G is 2-primitive if $k = 1$. If $k > 1$ and $(G, k) \neq (A_n, 2)$ then $\mathrm{Soc}(H)$ is not a product of isomorphic simple groups.

If $(G, k) = (A_n, 2)$ then $H = G_{\{1,2\}} = S_{n-2}$ and

$$H_{\{2,3\}} = H_{1,2,3} < H_{1,2} < H$$

so G is not extremely primitive.

Sporadic groups

Theorem (B-O'Brien-Wilson, 2010)

The base size of every primitive almost simple sporadic group is known.

In most cases $b(G) = 2$, e.g. if $G = \mathbb{M}$ then the only exception is the case $H = 2.\mathbb{B}$ with $b(G) = 3$. We inspect the list of exceptions.

From the structural constraints on H, we reduce to a list of cases with H almost simple. Using MAGMA and data in the Web Atlas, we reduce further to 15 specific cases (G, H).

Here G is **multiplicity free** – every irreducible constituent of 1_H^G has multiplicity 1. All such actions of sporadic groups are known (Breuer-Lux, 1996), and all subdegrees have been computed.

In this situation, no extremely primitive examples arise.

Classical groups

Let G be an almost simple classical group with socle $T = \mathrm{Cl}(V)$ and point stabilizer H. Roughly speaking, we say G is **standard** if $H \cap T$ is reducible on V, otherwise G is **non-standard**.

Standard groups have large base sizes, in general.

Example

If $G = \mathrm{PGL}_n(q)$ and $H = P_1$ then $b(G) = n+1$.

Theorem (B, 2007)

If G is non-standard then $b(G) \leq 5$, with equality if and only if $G = \mathrm{U}_6(2).2$ and $H = \mathrm{U}_4(3).2^2$.

More recently, with Guralnick and Saxl, we have computed $b(G)$ precisely for 'almost all' non-standard classical groups G.

Classical groups

Roughly, we get $b(G) \leq 3$, with equality only if $H = C_G(x)$ for an involution $x \in \mathrm{Aut}(T)$.

We have computed the exact value of $b(G)$ when H belongs to Aschbacher's $\mathcal{S}$ collection of irreducible almost simple subgroups.

In the remaining cases:

- The structure of H rules out many cases
- We can often work directly with matrices to find an explicit $x \in G \setminus H$ such that $H \cap H^x < H$ is non-maximal
- For standard groups, Ω is a set of subspaces of V and we can work with explicit subspaces to find a non-maximal two-point stabilizer $G_{\alpha,\beta} < G_\alpha$

Classical groups: An example

Suppose $G = \mathrm{PSL}_n(q) = \mathrm{PSL}(V)$ and H is a subfield subgroup of type $\mathrm{GL}_n(q_0)$, where $q = q_0^r$ with r prime. If $r > 2$ then $b(G) = 2$. Suppose $r = 2$.

Here $H = C_G(\sigma)$, with $\sigma \in \mathrm{Aut}(G)$ a field aut of order 2. By fixing a basis $\{v_1, \ldots, v_n\}$ for V, we may assume $(a_{ij})^\sigma = (a_{ij}^{q_0})$. Then

$$H \cap H^x = C_H(x^{-1}x^\sigma)$$

for all $x \in G$. Write $\mathbb{F}_q^* = \langle \omega \rangle$ and set

$$x = \left(\begin{array}{cc|c} 1 & \omega & \\ 0 & 1 & \\ \hline & & I_{n-2} \end{array}\right) \in G, \quad x^{-1}x^\sigma = \left(\begin{array}{cc|c} 1 & \omega^{q_0} - \omega & \\ 0 & 1 & \\ \hline & & I_{n-2} \end{array}\right)$$

Then $H \cap H^x = C_H(x^{-1}x^\sigma) < H_U < H$, where $U = \langle v_2 \rangle$, hence G is not extremely primitive.

Some open problems

- Determine the extremely primitive groups of exceptional Lie type.

  Here $b(G) \leq 6$ (by B-Liebeck-Shalev, 2009), but we do not know all the base-two examples.

  Current work with Guralnick and Saxl on bases for exceptional algebraic groups may be useful here.

- Prove the Mann-Praeger-Seress conjecture on affine groups (their list is complete) – a proof of Wall's conjecture for almost simple groups would be very helpful!

- Classify the base-two almost simple primitive permutation groups

- Calculate the base size of every primitive permutation group...