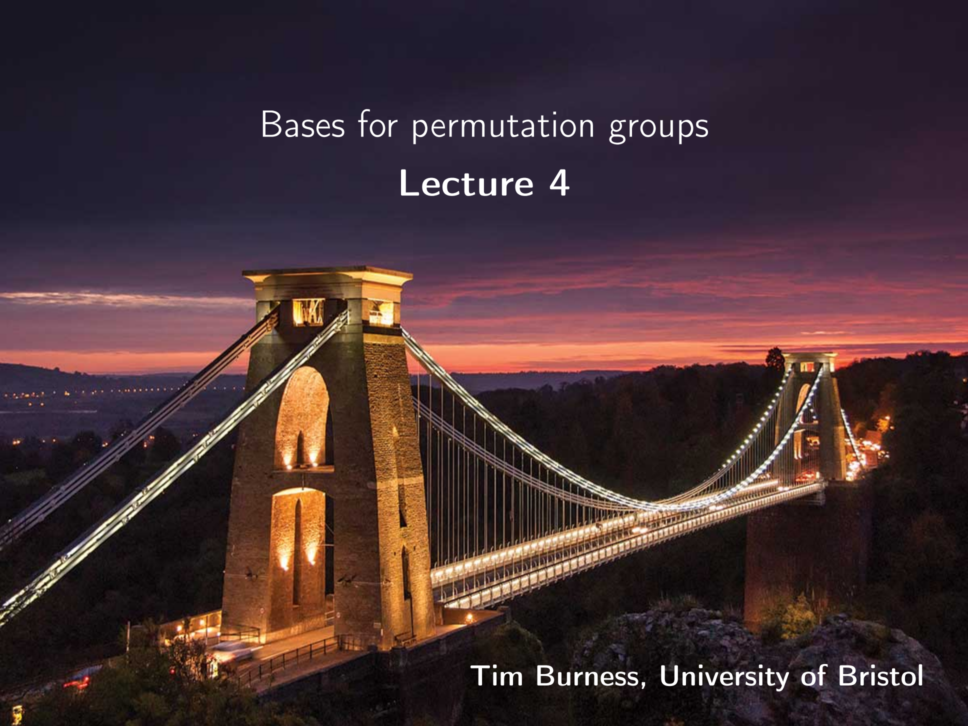
Bases for permutation groups

Lecture 4



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Today

Applications:

- Minimal dimension and the intersection number
- 2-generation of finite groups
- Extreme primitivity

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https://seis.bristol.ac.uk/∼tb13602/padova2021.html

▸ Link

First application

Minimal dimension and the intersection number

Minimal dimension

Let G be a finite group and let $\mathcal{M}$ be a set of maximal subgroups of G.

We say that $\mathcal{M}$ is **irredundant** if

$$\bigcap_{H \in \mathcal{M}} H < \bigcap_{H \in \mathcal{S}} H$$

for every proper subset $\mathcal{S}$ of $\mathcal{M}$.

Definition (Garonzi & Lucchini, 2019)

The **minimal dimension** of G, denoted $\mathrm{Mindim}(G)$, is the minimal size of a maximal irredundant set of maximal subgroups of G.

Note. $\mathrm{Mindim}(G) = 1 \iff G$ is cyclic of prime power order

Example. If $G = S_3$ then $\mathcal{M} = \{\langle(1,2)\rangle, \langle(1,3)\rangle\}$ is maximal irredundant and $\mathrm{Mindim}(G) = 2$.

The base size connection

Let $H < G$ be maximal and set $H_G = \bigcap_{g \in G} H^g \trianglelefteqslant G$.

We may view G/H_G as a permutation group on $\Omega = G/H$. Define

$$b(G,H) = \min\{|S| \,:\, S \subseteq G,\ \bigcap_{g \in S} H^g = H_G\}$$

If $S = \{g_1, \ldots, g_b\} \subseteq G$ is such a set and $b = b(G,H)$, then

$$\mathcal{M} = \{H^{g_1}/H_G, \ldots, H^{g_b}/H_G\}$$

is a maximal irredundant set of maximal subgroups of G/H_G and thus

$$\mathrm{Mindim}(G/H_G) \leqslant b(G,H).$$

In particular, if $H_G = 1$ then $b(G,H)$ is the **base size** of G on Ω.

Alternating groups

Let $G = A_n$ with $n \geqslant 13$ and let $\mathcal{M} = \{\text{maximal subgroups of } G\}$.

If $H \in \mathcal{M}$ acts primitively on $\{1, \ldots, n\}$, then $b(G, H) = 2$ **[BGS, 2011]**.

If $n = 2m$ then $b(G, H) = 3$ for $H = (S_2 \wr S_m) \cap G \in \mathcal{M}$.

Theorem (Garonzi & Lucchini, 2019)

For $n \geqslant 4$,

$$\mathrm{Mindim}(A_n) = \begin{cases} 3 & \text{if } n \in \{6, 7, 8, 11, 12\} \cup \mathcal{A} \\ 2 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \mathcal{A} &= \{2p \,:\, p \neq 11 \text{ prime, } 2p-1 \text{ not a prime power}\} \\ &= \{34, 46, 58, 86, 94, 106, 118, 134, 142, 146, \ldots\} \end{aligned}$$

Note. If $n = q + 1$ for a prime power q, then $\mathrm{L}_2(q) \in \mathcal{M}$ is primitive.

Two related invariants

Let $\mathcal{M}^*$ be the set of maximal subgroups H of G with $H_G = \Phi(G)$, the Frattini subgroup of G. Define

$$\alpha(G) = \min\{|\mathcal{T}| \,:\, \mathcal{T} \subseteq \mathcal{M},\ \bigcap_{H\in\mathcal{T}} H = \Phi(G)\}$$

$$\beta(G) = \begin{cases} \min\{b(G,H) \,:\, H \in \mathcal{M}^*\} & \text{if } \mathcal{M}^* \neq \emptyset \\ \infty & \text{otherwise,} \end{cases}$$

the **intersection number** and **base number** of G, respectively.

Note. $\mathrm{Mindim}(G) \leqslant \alpha(G) \leqslant \beta(G)$.

The proof of **[GL, 2019]** shows that

$$\mathrm{Mindim}(A_n) = \alpha(A_n) = \beta(A_n) \in \{2,3\}$$

for all $n \geqslant 4$.

Simple groups

Theorem (B, Garonzi & Lucchini, 2020)

Let G be a nonabelian finite simple group.

- If G is an alternating, sporadic or exceptional group of Lie type, then

  $$\mathrm{Mindim}(G) = \alpha(G) = \beta(G) \leqslant 3,$$

  with equality if and only if

  $$G \in \{\mathrm{M}_{22}, G_2(2)'\} \cup \{A_n : n \in \{6, 7, 8, 11, 12\} \cup \mathcal{A}\}.$$

- If G is a classical group, then either

  $$\mathrm{Mindim}(G) \leqslant \alpha(G) \leqslant \beta(G) \leqslant 3,$$

  or $G = \mathrm{U}_4(2)$ and $\mathrm{Mindim}(G) = \alpha(G) = 3$, $\beta(G) = 4$.

Corollary

Let G be a nonabelian finite simple group. Then the following hold:

- $\alpha(G) \leqslant 3$, with equality for infinitely many simple groups G.
- $\beta(G) \leqslant 4$, with equality iff $G = \mathrm{U}_4(2)$.
- $\beta(G) - \alpha(G) \leqslant 1$, with equality if $G = \mathrm{U}_4(2)$ or $\mathrm{Sp}_6(4)$.
- $\alpha(G) - \mathrm{Mindim}(G) \leqslant 1$.

Question. Is there a simple group G with $\alpha(G) - \mathrm{Mindim}(G) = 1$?

Note. There exist finite **solvable** groups G such that $\alpha(G) - \mathrm{Mindim}(G)$ is arbitrarily large.

Note. If there exists a maximal subgroup H of G with $b(G, H) = 2$, then

$$\mathrm{Mindim}(G) = \alpha(G) = \beta(G) = 2.$$

Comments on the proof

- **G sporadic:** [B, O'Brien & Wilson, 2010] gives $\beta(G) \leqslant 3$, with equality iff $G = \mathrm{M}_{22}$. Using **Magma**, we get $\mathrm{Mindim}(\mathrm{M}_{22}) = 3$.

- **G alternating:** Let $G = A_n$. For $n \geqslant 13$, [B, Guralnick & Saxl, 2011] gives $\beta(G) = 2$ if G has a maximal primitive subgroup.

  Combining this with work of [James, 2006], we reduce to the case where $n = 2p$ with p a prime.

  If $n = q + 1$ with q a prime power, then $H = \mathrm{L}_2(q) \in \mathcal{M}$, so we may assume $n \in \mathcal{A}$.

  We have $|H|^2 > |G|$ for all $H \in \mathcal{M}$, so $\alpha(G) \geqslant 3$. In addition, $b(G) = 3$ when $H = (S_2 \wr S_p) \cap G$, so $\alpha(G) = \beta(G) = 3$.

  Finally, [Garonzi & Lucchini, 2019] show that for all $A, B \in \mathcal{M}$, there exists $C \in \mathcal{M}$ such that $\{A, B, C\}$ is irredundant.

If G is a finite simple **exceptional group** of Lie type, then

$$\mathrm{Mindim}(G) = \alpha(G) = \beta(G) = \begin{cases} 3 & \text{if } G = G_2(2)' \cong \mathrm{U}_3(3) \\ 2 & \text{otherwise.} \end{cases}$$

Example

Suppose $G = E_8(q)$. Then $H = N_G(T) = C_m{:}C_{30} \in \mathcal{M}$, where

$$m = q^8 - q^7 + q^5 - q^4 + q^3 - q + 1.$$

Since $|x^G| > q^{58}$ for all nontrivial $x \in G$, we deduce that

$$Q(G,2) \leqslant \widehat{Q}(G,2) = \sum_{i=1}^{k} \frac{|x_i^G \cap H|^2}{|x_i^G|} < |H|^2 q^{-58} < 1$$

for all $q \geqslant 2$, so $b(G) = 2$ and thus $\beta(G) = 2$.

B & Guralnick (2019): If $G = G_2(q)$, q even and $H = \mathrm{L}_2(q)^2$ then we constructed an $x \in G$ with $H \cap H^x = 1$.

For **classical groups**, a key ingredient is the following:

Theorem (B & Guralnick)

Let $G = \mathrm{Cl}(V)$ be a finite simple classical group with $\dim V \geqslant 6$.

Let $H \in \mathcal{C}_3$ is a maximal subgroup corresponding to a field extension of prime degree k (e.g. $G = \mathrm{L}_n(q)$ and H is of type $\mathrm{GL}_{n/k}(q^k)$).

Then $b(G) \leqslant 3$. In fact, for $k \geqslant 3$ we have

$$b(G) = \begin{cases} 3 & \text{if } G = \mathrm{PSp}_6(q) \text{ and } H \text{ is of type } \mathrm{Sp}_2(q^3) \\ 2 & \text{otherwise.} \end{cases}$$

The **orthogonal groups** require special attention.

Example. Suppose $G = \Omega_n(q)$ with nq odd and set $k = 2\lfloor (n+1)/4 \rfloor$.

Then $b(G) = 2$ when $H \in \mathcal{M}$ is the stabilizer of a nondegenerate k-space of plus-type, so $\beta(G) = 2$.

Almost simple groups

We can extend these results to **almost simple** groups. Recall that

$$\alpha(G) = \min\{|\mathcal{T}| : \mathcal{T} \subseteq \mathcal{M}, \bigcap_{H \in \mathcal{T}} H = 1\}$$

$$\beta(G) = \min\{b(G,H) : H \in \mathcal{M}\}$$

Theorem (B, Garonzi & Lucchini, 2020)

Let G be a finite almost simple group with socle G_0.

- $\alpha(G) \leqslant 4$, with equality iff $G = \mathrm{U}_4(2).2$
- $\beta(G) \leqslant 4$, with equality iff $G = S_6$ or $G_0 = \mathrm{U}_4(2)$
- $\beta(G) - \alpha(G) \leqslant 1$
- $\alpha(G) - \mathrm{Mindim}(G) \leqslant 1$, with equality if $G = \mathrm{U}_4(2).2$

An example: symmetric groups

Claim. If $G = S_n$ then $\beta(G) \leqslant 4$, with equality iff $n = 6$.

For $n \leqslant 14$ we can use **Magma**, so assume $n \geqslant 15$.

Theorem (BGL, 2020). If $n = ab \geqslant 8$, $a \geqslant b \geqslant 2$ and $H = S_b \wr S_a$, then

$$b(G) = \begin{cases} 2 & \text{if } b \geqslant 3 \text{ and } a \geqslant \max\{b+3, 8\} \\ 3 & \text{otherwise} \end{cases}$$

- $n = 2m$: $H = S_2 \wr S_m$, $b(G) = 3$
- $n = p$ prime: $H = \mathrm{AGL}_1(p)$, $b(G) = 2$
- n odd, p smallest prime divisor: $H = S_p \wr S_{n/p}$, $b(G) \leqslant 3$

Second application

2-generation and the uniform domination number

Suppose $G = \langle x, y \rangle$ is finite and non-cyclic. Set $G^{\#} = G \setminus \{1\}$.

How are the generating pairs $\{x, y\}$ distributed across the group?

More precisely:

- Can we impose conditions on the orders of x and y, or their conjugacy classes?
- What is the probability that two random elements generate G?
- Does G have the $\frac{3}{2}$-**generation** property?

  That is, does every nontrivial element belong to a generating pair?

Theorem (Steinberg, 1962). Every simple group is 2-generated.

Spread and uniform spread

We say that G has **spread** k if for any $x_1, \ldots, x_k \in G^{\#}$ there exists $y \in G$ such that $G = \langle x_i, y \rangle$ for all i.

Let $s(G) \geqslant 0$ be the **exact spread** of G.

G has **uniform spread** k if there exists $C = z^G$ such that for any $x_1, \ldots, x_k \in G^{\#}$ there exists $y \in C$ with $G = \langle x_i, y \rangle$ for all i.

Let $u(G) \geqslant 0$ be the **exact uniform spread** of G.

Theorem (Breuer, Guralnick & Kantor, 2008)

G simple $\implies$ $u(G) \geqslant 2$

The generating graph

The **generating graph** of G, denoted $\Gamma(G)$, has vertices $G^{\#}$, with x, y adjacent iff $G = \langle x, y \rangle$. In this setting,

$$s(G) \geqslant 1 \iff \Gamma(G) \text{ has no isolated vertices}$$
$$s(G) \geqslant 2 \implies \Gamma(G) \text{ is connected with diameter at most 2}$$

Note. If $1 \neq N \trianglelefteq G$ and G/N is non-cyclic, then no element in N belongs to a generating pair, so $s(G) = 0$.

Theorem (B, Guralnick & Harper, 2021)

If G is a finite group, then

$$\begin{aligned} s(G) \geqslant 1 &\iff s(G) \geqslant 2 \\ &\iff G/N \text{ is cyclic for every non-trivial normal subgroup } N \end{aligned}$$

In particular, there is no finite group G with $s(G) = 1$.

The domination numbers

Let G be a finite group with $u(G) \geqslant 1$ and generating graph $\Gamma(G)$.

A **total dominating set** (TDS) of $\Gamma(G)$ is a set S of vertices such that every vertex of $\Gamma(G)$ is adjacent to a vertex in S.

The **total domination number** of G is the minimal size of a TDS:

$$\gamma_t(G) = \min\left\{ |S| \ : \ \begin{array}{l} S \subseteq G^{\#} \text{ such that for all } x \in G^{\#}, \\ \text{there exists } y \in S \text{ with } G = \langle x, y \rangle \end{array} \right\}$$

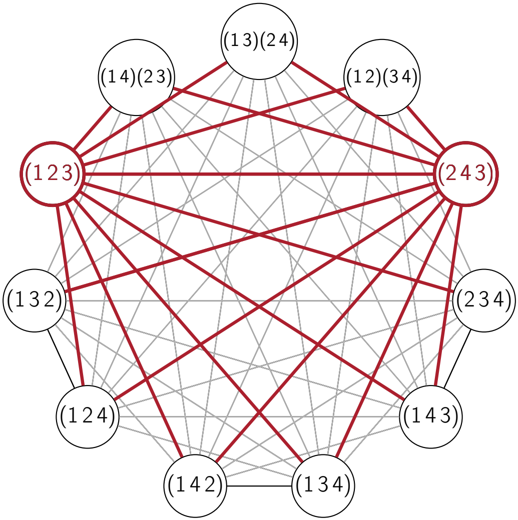
Similarly, the **uniform domination number** $\gamma_u(G)$ is the minimal size of a TDS for $\Gamma(G)$ consisting of **conjugate** elements.

Note that

$$2 \leqslant \gamma_t(G) \leqslant \gamma_u(G) \leqslant |C|$$

for some conjugacy class C of G.

An example: $G = A_4$



Conclusion. $\{(1,2,3),(2,4,3)\}$ is a TDS for G, hence $\gamma_u(G) = 2$

Simple groups

Recall that $u(G) \geqslant 2$ if G is simple, so we can study $\gamma_u(G)$ for simple groups:

- Can we determine "good" bounds on $\gamma_u(G)$?
- Are there any examples with $\gamma_u(G) = 2$? Can we classify them?
- Suppose $\gamma_u(G) = 2$ and $y \in G$.

  What is the probability, denoted $P(G, y)$, that $\{y, y^g\}$ is a TDS for a randomly chosen conjugate y^g?
- What are the asymptotic properties of

$$P(G) = \max\{P(G, y) \,:\, y \in G\}$$

  for sequences of simple groups G with $\gamma_u(G) = 2$?

The base size connection

Let $\mathcal{M}(y)$ be the set of maximal subgroups of G containing $y \in G$.

Lemma. Suppose $\mathcal{M}(y) = \{H\}$, H core-free. Then $\{y^{g_1}, \dots, y^{g_c}\}$ is a TDS if and only if $\bigcap_{i=1}^{c} H^{g_i} = 1$, so $\gamma_u(G) \leqslant b(G, H)$,

Proof. Simply observe that $G = \langle x, y^{g_i} \rangle \iff x \notin H^{g_i}$. ∎

Lemma. Suppose that for all $y \in G^{\#}$ there exists $H \in \mathcal{M}(y)$ with H core-free and $b(G, H) \geqslant c$. Then $\gamma_u(G) \geqslant c$.

Proof. If $y \in G^{\#}$ and $g_1, \dots, g_{c-1} \in G$, then $G \neq \langle x, y^{g_i} \rangle$ for all $x \in \bigcap_i H^{g_i} \neq 1$. ∎

Lemma. Suppose $\mathcal{M}(y) = \{H\}$, H core-free. Then $\{y^{g_1}, \ldots, y^{g_c}\}$ is a TDS if and only if $\bigcap_{i=1}^{c} H^{g_i} = 1$, so $\gamma_u(G) \leqslant b(G, H)$,

Example. Let G be an exceptional simple group of Lie type and assume

$$G \notin \{F_4(2^f), G_2(3^f), {}^2F_4(2)'\}.$$

By **Weigel (1992)**, there exists $y \in G$ with $\mathcal{M}(y) = \{H\}$, so $\gamma_u(G) \leqslant 6$ by applying **B, Liebeck & Shalev (2009)**.

Example. Take $G = E_8(q)$ and $y \in G$ with

$$|y| = q^8 + q^7 - q^5 - q^4 - q^3 + q + 1.$$

Then $\mathcal{M}(y) = \{H\}$, with $H = \langle y \rangle{:}C_{30}$, and $\gamma_u(G) = b(G, H) = 2$.

Lemma. Suppose that for all $y \in G^{\#}$ there exists $H \in \mathcal{M}(y)$ with H core-free and $b(G,H) \geqslant c$. Then $\gamma_u(G) \geqslant c$.

Example. Let $G = A_n$ with $n \geqslant 8$ even, so each $y \in G^{\#}$ is contained in a maximal intransitive subgroup of G.

- By **Halasi (2012)**, $\gamma_u(G) \geqslant b(G,H) \geqslant \lceil \log_2 n \rceil - 1$.
- Set $d = (2, \frac{n}{2} - 1)$, $k = \frac{n}{2} - d$ and $y = (1, \ldots, k)(k+1, \ldots, n) \in G$.

  Then $\mathcal{M}(y) = \{H\}$ with $H = (S_k \times S_{n-k}) \cap G$ and

$$\gamma_u(G) \leqslant b(G,H) \leqslant \left\lceil \log_{\lceil \frac{2n}{n-2d} \rceil} n \right\rceil \cdot \left\lceil \frac{n+2d}{n-2d} \right\rceil \leqslant 2\lceil \log_2 n \rceil.$$

We conclude that

$$\lceil \log_2 n \rceil - 1 \leqslant \gamma_u(G) \leqslant 2\lceil \log_2 n \rceil$$

Probabilistic methods

For $y \in G$, $c \in \mathbb{N}$ we define

$$Q(G, y, c) = \text{Probability } c \text{ random conjugates of } y \text{ do } \textbf{not} \text{ form a TDS}$$

Note. $Q(G, y, c) < 1 \implies \gamma_u(G) \leqslant c$

Lemma. Let $x_1^G, \dots, x_k^G$ be the conjugacy classes of elements of prime order in G. Then

$$Q(G, y, c) \leqslant \sum_{i=1}^{k} |x_i^G| \cdot \left(\sum_{H \in \mathcal{M}(y)} \mathrm{fpr}(x_i, G/H) \right)^c$$

Note. If $\mathcal{M}(y) = \{H\}$, then this upper bound is $\widehat{Q}(G, c)$.

Some results for simple groups

Theorem (B & Harper, 2019)

Let G be a finite simple group.

- G sporadic: $\gamma_u(G) \leqslant 4$ (e.g. $\gamma_u(\mathrm{M}_{11}) = \gamma_u(\mathrm{M}_{12}) = 4$)
- $G = A_n$: $\gamma_u(G) \leqslant c \log_2 n$ (e.g. $c = 77$)
- G exceptional: $\gamma_u(G) \leqslant 5$
- G classical, rank r: $\gamma_u(G) \leqslant 7r + 56$

Example. If $G = \Omega_{2r+1}(q)$, then $r \leqslant \gamma_u(G) \leqslant 7r$

Example. If $G = F_4(q)$, then each $y \in G$ is contained in a maximal parabolic subgroup, or a maximal subgroup of type $B_4(q)$ or ${}^3D_4(q)$.

In particular, $y \in H$ with $b(G, H) \geqslant 3$, so $\gamma_u(G) \geqslant 3$.

Theorem (B & Harper, 2019/20)

If G is simple, then $\gamma_u(G) = 2$ only if G is one of the following:

- A_n, $n \geqslant 13$ prime
- ${}^2B_2(q), {}^2G_2(q), {}^2F_4(q), {}^3D_4(q), {}^2E_6(q), E_6(q), E_7(q), E_8(q)$
- $\mathrm{M}_{23}, \mathrm{J}_1, \mathrm{J}_4, \mathrm{Ru}, \mathrm{Ly}, \mathrm{O'N}, \mathrm{Fi}_{23}, \mathrm{Fi}_{24}', \mathrm{Th}, \mathbb{B}, \mathbb{M}$, or $\mathrm{J}_3, \mathrm{He}, \mathrm{Co}_1, \mathrm{HN}$
- $\mathrm{L}_2(q)$, $q \geqslant 11$ odd
- $\mathrm{L}_n^{\epsilon}(q)$, n odd, $(n, q, \epsilon) \neq (3, 2, +), (3, 4, +), (3, 3, -), (3, 5, -)$
- $G = \mathrm{PSp}_{4m+2}(q)$, $m \geqslant 2$, q odd
- $G = \mathrm{P\Omega}_{4m}^{-}(q)$, $m \geqslant 2$

Recall that $P(G) = \max\{P(G,y) : y \in G\}$, where $P(G,y)$ is the probability that $\{y, y^g\}$ is a TDS for a randomly chosen conjugate y^g.

Note. If $\mathcal{M}(y) = \{H\}$, then

$$P(G,y) = \frac{|\{y^g \in y^G \,:\, H \cap H^g = 1\}|}{|y^G|} = \frac{r|H|^2}{|G|}$$

where r is the number of regular orbits of H on G/H.

Theorem (B & Harper, 2020)

Suppose G is simple and $\gamma_u(G) = 2$. Also assume

$$G \notin \{\mathrm{PSp}_{4m+2}(q) \,:\, m \geqslant 2,\ q \text{ odd}\} \cup \{\mathrm{P\Omega}^-_{4m}(q) \,:\, m \geqslant 2\}.$$

Then

$$P(G) \to \begin{cases} \frac{1}{2} & \text{if } G = \mathrm{L}_2(q) \\ 1 & \text{otherwise} \end{cases} \quad \text{as } |G| \to \infty$$

Third application

Extremely primitive groups

Definition

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a primitive permutation group. Then G is **extremely primitive** if G_α acts primitively on each of its orbits in $\Omega \setminus \{\alpha\}$.

Examples

- $G = S_n$, $\Omega = \{1, \ldots, n\}$ (2-primitive)
- $G = \mathrm{PGL}_2(q)$, $\Omega = \mathbb{F}_q \cup \{\infty\}$ (2-primitive)
- $G = \mathrm{J}_2$, $G_\alpha = \mathrm{U}_3(3)$: $|\Omega| = 100 = 1 + 36 + 63$
- $G = \mathrm{Co}_2$, $G_\alpha = \mathrm{McL}$:

$$|\Omega| = 47104 = 1 + 275 + 2025 + 7128 + 15400 + 22275$$

Problem. Determine all the finite extremely primitive groups.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite primitive group with point stabilizer $H = G_\alpha$.

Theorem (Manning, 1927)

G extremely primitive $\implies$ H acts faithfully on each orbit in $\Omega \setminus \{\alpha\}$

So we can apply the **O'Nan-Scott theorem** to H:

Lemma. If G is extremely primitive, then either

- $F(H) = 1$ and $\mathrm{soc}(H) = T^k$ for some nonabelian simple group T; or
- $H = F(H)K$ is affine, where $\mathrm{soc}(H) = F(H) = (C_p)^d$ acts regularly on each H-orbit in $\Omega \setminus \{\alpha\}$ and $K \leqslant \mathrm{GL}_d(p)$ is irreducible.

Theorem (Mann, Praeger & Seress, 2007)

G extremely primitive $\implies$ G is affine or almost simple

The base size connection

Suppose G is primitive with point stabilizer H. Then G is extremely primitive iff $H \cap H^x < H$ is maximal for all $x \in G \setminus H$.

Lemma. Let G be an almost simple primitive group. If $b(G) = 2$ then G is not extremely primitive.

Theorem

Let G be an almost simple primitive group with socle G_0 and point stabilizer H.

- **B, Praeger & Seress, 2012:** The extremely primitive groups with socle a sporadic, alternating or classical group are known.
- **B, Thomas, 2020:** The extremely primitive groups with socle an exceptional group of Lie type are known.

  (The only examples are $(G, H) = (G_2(4).c, \mathrm{J}_2.c)$ with $c = 1, 2$.)

Some almost simple extremely primitive groups

G_0	$H \cap G_0$	Rank	Conditions
A_n	$(S_{n/2} \wr S_2) \cap G_0$	$\frac{1}{4}(n+2)$	$n \equiv 2 \pmod 4$
A_n	A_{n-1}	2	$G = S_n$ or A_n
A_6	$\mathrm{L}_2(5)$	2	$G = S_6$ or A_6
A_5	D_{10}	2	
$\mathrm{L}_2(q)$	P_1	2	
$\mathrm{L}_2(q)$	$D_{2(q+1)}$	$\frac{1}{2}q$	$G = G_0$, $q+1$ Fermat
$\mathrm{Sp}_n(2)$	$\mathrm{O}_n^{\pm}(2)$	2	$n \geqslant 6$
$\mathrm{U}_4(3)$	$\mathrm{L}_3(4)$	3	–
$\mathrm{L}_3(4)$	A_6	3	–
$\mathrm{L}_2(11)$	A_5	2	$G = G_0$
$G_2(4)$	J_2	3	

Almost simple groups

- The structural conditions on H (via O'Nan-Scott) are restrictive.

  For example, if G is a group of Lie type and $H = QL$ is a parabolic subgroup, then the unipotent radical Q has to be elementary abelian.

- If H is "small" then we aim to show that $b(G,H) = 2$, typically by estimating $\widehat{Q}(G,2)$.

- **B & Thomas (2020):**

  If G_0 is exceptional and H is a maximal subgroup of the form $H = N_G(T)$ for some maximal torus T, then $b(G,H) = 2$.

- In some cases we are forced to construct an explicit element $x \in G$ so that $H \cap H^x < H$ is non-maximal.

Affine groups

Theorem (Mann, Praeger & Seress, 2007)

Let $G = VH \leqslant \mathrm{AGL}(V)$ be extremely primitive with $V = (\mathbb{F}_p)^d$.

- All the examples with H solvable are known.
- If H is non-solvable, then $p = 2$ and H is almost simple.

  Moreover, either G is 2-transitive (all known), or G is simply primitive and the known cases (d, H) are as follows (with $k \geqslant 3$):

  $$(10, \mathrm{M}_{12}),\ (10, \mathrm{M}_{22}),\ (11, \mathrm{M}_{23}),\ (11, \mathrm{M}_{24}),\ (22, \mathrm{Co}_3),\ (24, \mathrm{Co}_1),$$

  $$(8, \mathrm{L}_2(17)),\ (8, \mathrm{Sp}_6(2)), (2k, A_{2k-1}),\ (2k, A_{2k+2}),\ (2k, \Omega^+_{2k}(2))$$

- There are at most finitely many additional affine examples.

  Conjecture (MPS). There are no additional affine EP groups!

Let $G = VH$ be primitive with $V = (\mathbb{F}_2)^d$ and H almost simple.

Let $\mathcal{M}(H)$ be the set of maximal subgroups of H. For $M \in \mathcal{M}(H)$ set

$$\mathrm{fix}(M) = \{v \in V \,:\, v^x = v \text{ for all } x \in M\} = \bigcap_{x \in M} C_V(x).$$

Note. H irreducible $\implies \dim \mathrm{fix}(M) \leqslant \lfloor d/2 \rfloor$ for all $M \in \mathcal{M}(H)$.

Lemma. $\sum_{M \in \mathcal{M}(H)} (|\mathrm{fix}(M)| - 1) \leqslant 2^d - 1$, with equality iff G is EP.

Corollary. $|\mathcal{M}(H)| < 2^{d/2} \implies G$ is not EP

If Wall's conjecture holds, then $|\mathcal{M}(H)| < |H|$ and this reduces the problem to a short list of candidate cases.

Theorem (B & Thomas, 2021). All of these candidates have been eliminated, completing the classification of all EP groups (modulo Wall).

Next week

- Jan Saxl's base-two project
- Summary of the main results
- The Saxl graph of a base-two permutation group
- Saxl graphs: Main results and open problems

Some references

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