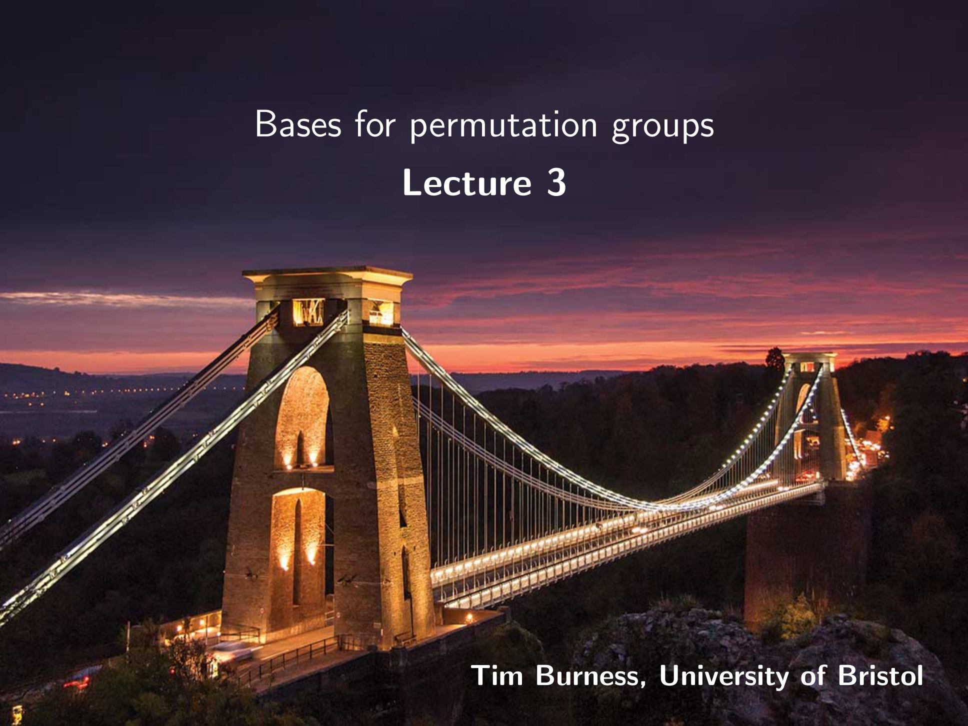

Bases for permutation groups

Lecture 3

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Today

- **Cameron's base size conjecture:**
  - The probabilistic method
  - Main results
  - Key steps and techniques in the proof
  - Fixed point ratio estimates
- **Extensions and related problems**

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https://seis.bristol.ac.uk/∼tb13602/padova2021.html
Link

Cameron's conjecture

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a **non-standard** almost simple primitive permutation group with socle G_0 and point stabilizer H.

Recall. Roughly speaking, "non-standard" means that

- H acts primitively on $\{1, \ldots, m\}$ if $G_0 = A_m$; and
- $H \cap G_0$ acts irreducibly on V if $G_0 = \mathrm{Cl}(V)$ is a classical group.

Conjecture (Cameron, 1999)

We have $b(G) \leqslant 7$, with equality iff $G = \mathrm{M}_{24}$ in its natural action.

Theorem (B et al., 2007-11)

Cameron's conjecture is true.

Moreover, almost every 6-tuple of points in Ω form a base for G.

Probabilistic methods

Recall that $b(G)$ is the minimal number of elements $g_1, \ldots, g_b$ in G s.t.

$$H^{g_1} \cap H^{g_2} \cap \cdots \cap H^{g_b} = 1.$$

For a non-standard group $G \leqslant \mathrm{Sym}(\Omega)$, we may not be able to equip $\Omega = G/H$ with a natural geometric or combinatorial structure, which makes **constructive** methods difficult to apply.

In this setting, our main tool is a **probabilistic** method introduced by **Liebeck & Shalev (1999):**

For a positive integer c, let

$$P(G,c) = \frac{|\{(\alpha_1, \ldots, \alpha_c) \in \Omega^c \,:\, \bigcap_i G_{\alpha_i} = 1\}|}{|\Omega|^c}$$

be the probability that a random c-tuple of points in Ω is a base for G.

Set $Q(G,c) = 1 - P(G,c)$, so $b(G) \leqslant c$ iff $Q(G,c) < 1$.

Observation. A c-tuple in Ω fails to be a base iff it is fixed by some $x \in G$ of prime order.

The probability that a random c-tuple is fixed by $x \in G$ is $\mathrm{fpr}(x)^c$, where

$$\mathrm{fpr}(x) = \frac{|C_\Omega(x)|}{|\Omega|} = \frac{|x^G \cap H|}{|x^G|}$$

is the **fixed point ratio** of x. Therefore,

$$Q(G,c) \leqslant \sum_{x \in \mathcal{P}} \mathrm{fpr}(x)^c = \sum_{i=1}^{k} |x_i^G| \cdot \mathrm{fpr}(x_i)^c =: \widehat{Q}(G,c),$$

where $\mathcal{P}$ is the set of elements of prime order in G and the x_i are representatives of the distinct G-classes in $\mathcal{P}$.

In particular, $\widehat{Q}(G,c) < 1 \implies b(G) \leqslant c$.

Symmetric and alternating groups

Suppose $G_0 = A_m$, so $G = A_m$ or S_m (for $m \neq 6$).

Cameron & Kantor (1993): A sketch proof (with an undetermined constant), based on two properties of primitive $H \leqslant S_m$, $H \neq A_m, S_m$:

- $|H|$ is "small".
- Nontrivial elements in H have "large" support on $\{1, \ldots, m\}$.

Therefore, if $x \in H$ has prime order, then $|x^G \cap H| \leqslant |H|$ is "small" and $|x^G|$ is "large", which implies that $\mathrm{fpr}(x)$ is also "small".

Using explicit estimates, this yields $\widehat{Q}(G, c) < 1$ for some constant c.

The main result

Theorem (B, Guralnick & Saxl, 2011)

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a non-standard group with socle $G_0 = A_m$ and point stabilizer H. Then the following hold:

- $b(G) \leqslant 5$, with equality iff $G = S_6$ and $H = \mathrm{PGL}_2(5)$.
- $b(G)$ is known in all cases. In particular, $b(G) = 2$ if $m \geqslant 13$.
- $P(G,2) \to 1$ as $m \to \infty$.

Main ingredients:

- **Maróti (2002)** on the order of H.
- **Guralnick & Magaard (1998)** on the **minimal degree** of H.
- Computational methods (using **Magma**) for small m.

Assume $m \geqslant 40$ and the action of H on $\{1,\dots,m\}$ is **not** one of:

(a) $H = (S_a \wr S_b) \cap G$ in its product action (with $m = a^b$);

(b) $H = A_\ell$ or S_ℓ acting on k-sets in $\{1,\dots,\ell\}$ (with $m = \binom{\ell}{k}$); or

(c) H is an almost simple orthogonal group over $\mathbb{F}_2$ acting on a set of hyperplanes of the natural module for H.

Maróti: $|H| \leqslant m^{1+\lfloor \log_2 m \rfloor} =: f(m)$

Guralnick & Magaard: $\mu(H) := \min\{\mathrm{supp}(x) \,:\, 1 \neq x \in H\} \geqslant m/2$

The bound on $\mu(H)$ implies that

$$|C_G(x)| \leqslant 2^{m/4} \lceil m/4 \rceil ! \lceil m/2 \rceil ! =: g(m)$$

for all $1 \neq x \in H$. Therefore,

$$\widehat{Q}(G,2) = \sum_{i=1}^{k} \frac{|x_i^G \cap H|^2}{|x_i^G|} \leqslant \frac{f(m)^2 g(m)}{m!} < 1$$

Sporadic groups

Theorem (B, O'Brien & Wilson, 2010)

Let $G \leqslant \mathrm{Sym}(\Omega)$ be an almost simple primitive group with sporadic socle G_0 and point stabilizer H. Then the following hold:

- $b(G) \leqslant 7$, with equality iff $G = \mathrm{M}_{24}$ and $H = \mathrm{M}_{23}$.
- The exact value of $b(G)$ is known in all cases.

Note. For two cases with $G = \mathbb{B}$, we proved $b(G) \leqslant 3$ and the exact base size was determined in later work by **Neunhöffer et al. (2011)**.

Computational methods play an essential role in the proof:

- **WebAtlas:** Permutation reps and generators for maximal subgroups
- **GAP Character Table Library:** character tables; fusion maps
- Random search and double coset computations

Assume $G \neq \mathbb{M}$. Then the maximal subgroups of G have been determined up to conjugacy and in most cases we can access the character tables of G and H in **GAP**.

Moreover, the corresponding **fusion map** from H-classes to G-classes is usually available, which allows us to compute $\widehat{Q}(G, c)$ precisely.

Set $c = \lceil \log_{|\Omega|} |G| \rceil$, so $b(G) \geqslant c$.

- If $\widehat{Q}(G, c) < 1$, then $b(G) = c$.
- If not, we almost always have $\widehat{Q}(G, c+1) < 1$, so $b(G) \in \{c, c+1\}$.
- In these cases, we use the **WebAtlas** and **Magma** to construct $H < G < S_m$ for some m.
    - By random search, we seek $g_1, \ldots, g_{c-1} \in G$ such that $H \cap H^{g_1} \cap \cdots \cap H^{g_{c-1}} = 1$, which will yield $b(G) = c$.
    - If $|G : H| < 10^7$, it may be feasible to construct the action of G on G/H and determine the order of every c-point stabilizer.

Double coset methods

There are some cases where $|\Omega|$ is large and $\log_{|\Omega|}|G| = c - \epsilon$ with $\epsilon > 0$ small; in this situation, the previous methods may be ineffective.

Here we expect $b(G) = c + 1$, but it is not feasible to compute the order of every c-point stabilizer.

Let's assume $c = 2$ (this is typical). To show $b(G) = 3$, we need to rule out the existence of a regular H-orbit on $\Omega = G/H$.

Each H-orbit on Ω is an (H, H) double coset. By random search, we seek a set $T \subseteq G$ representing **distinct** (H, H) double cosets s.t.

(a) $|HxH| < |H|^2$ for all $x \in T$; and

(b) $\sum_{x \in T} |HxH| > |G| - |H|^2$.

If T exists, then H does not have a regular orbit on Ω and thus $b(G) \geqslant 3$.

For $G \neq \mathbb{M}$, these methods are effective unless $G = \mathbb{B}$ and

$$H = 2^{2+10+20}.(\mathrm{M}_{22}{:}2 \times S_3) \ \text{ or } \ [2^{30}].\mathrm{L}_5(2).$$

In both cases, $\lceil \log_{|\Omega|} |G| \rceil = 2$, $\widehat{Q}(G,3) < 1$ and $|\Omega| > 10^{17}$.

Using further computational methods, including the **Orb** package due to **Müller et al. (2007)**, one can show that $b(G) = 3, 2$ respectively.

Finally, suppose $G = \mathbb{M}$ is the Monster.

There are 44 known classes of maximal subgroups of G and any additional candidate has socle $\mathrm{L}_2(13)$ or $\mathrm{L}_2(16)$ (**Wilson, 2017**).

The p-local maximal subgroups for $p = 2, 3$ require special attention, e.g. work is needed to bound $\mathrm{fpr}(x)$ when $x \in G$ is an involution.

Conclusion. If $G = \mathbb{M}$, then $b(G) \leqslant 3$, with equality iff $H = 2.\mathbb{B}$.

Groups of Lie type

Let $G \leqslant \mathrm{Sym}(\Omega)$ be non-standard, with socle G_0 and point stabilizer H.

Assume G_0 is a group of Lie type over $\mathbb{F}_q$ ($q = p^f$), so either

- G_0 is an exceptional group; or
- $G_0 = \mathrm{Cl}(V)$ is a classical group and $H \cap G_0$ acts irreducibly on V.

Recall that $G_0 \leqslant G \leqslant \mathrm{Aut}(G_0)$ and $\mathrm{Out}(G_0) = \mathrm{Aut}(G_0)/G_0$ is solvable. Suppose $x \in G$ has prime order r.

- x is an **inner or diagonal** automorphism: **semisimple** ($r \neq p$) or **unipotent** ($r = p$)
- x is a **graph** automorphism ($r = 2, 3$ only)
- x is a **field** automorphism ($q = q_0^r$)
- x is a **graph-field** automorphism ($q = q_0^r$, $r = 2, 3$ only)

Example

Suppose $G_0 = \mathrm{L}_n(q) = \mathrm{SL}_n(q)/Z$, where $q = p^f$ and $n \geqslant 3$. Then

$$\mathrm{Aut}(G_0) = \langle \mathrm{PGL}_n(q), \varphi, \gamma \rangle,$$

where

$\varphi : Z(a_{ij}) \to Z(a_{ij}^p)$ is a **field** automorphism of order f

$\gamma : Z(a_{ij}) \mapsto Z(a_{ij})^{-\mathsf{T}}$ is the inverse-transpose **graph** automorphism

Elements in $\mathrm{PGL}_n(q) \setminus G_0$ are the **diagonal** automorphisms.

There is an extensive literature on conjugacy classes in almost simple groups of Lie type, e.g. representatives, centralizer structure, etc.

Fixed point ratios

To apply the probabilistic method for bounding $b(G)$, we need upper bounds on fixed point ratios. The most general result is the following:

Theorem (Liebeck & Saxl, 1991)

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive almost simple group of Lie type over $\mathbb{F}_q$ with socle G_0. Assume $G_0 \neq \mathrm{L}_2(q)$. Then either

$$\max_{1 \neq x \in G} \mathrm{fpr}(x) \leqslant \frac{4}{3q}$$

or $G_0 \in \{\mathrm{L}_4(2), \mathrm{PSp}_4(3), \mathrm{P}\Omega_4^-(3)\}$.

Example. Suppose $G = \mathrm{PGL}_m(q)$, q odd and $\Omega = P(V)$ is the set of 1-dim subspaces of V. Let $x = Z\hat{x}$ with $\hat{x} = [-I_1, I_{m-1}]$.

Then $|C_\Omega(x)| \sim q^{m-2}$ and $|\Omega| \sim q^{m-1}$, so $\mathrm{fpr}(x) \sim q^{-1}$.

This is already sufficient to prove a version of Cameron's conjecture for groups of bounded rank (with a large constant).

e.g. if G_0 is exceptional then $|G| < q^{249}$ and thus $b(G) \leqslant 500$ since

$$\widehat{Q}(G, 500) \leqslant \left(\frac{4}{3q}\right)^{500} \sum_{i=1}^{k} |x_i^G| < \left(\frac{4}{3q}\right)^{500} |G| < \left(\frac{4}{3q}\right)^{500} q^{249} < \frac{1}{q}$$

Question. Can we improve the constant? Classical groups of large rank?

- The bound $\mathrm{fpr}(x) < \frac{4}{3q}$ is independent of the rank of G and the element x: we need stronger bounds to attack Cameron's conjecture.
- For G classical we need to exploit the "non-standard" hypothesis, which means that $|H|$ is "small" and this should force

$$\mathrm{fpr}(x) = \frac{|x^G \cap H|}{|x^G|}$$

to be small as well.

Classical groups

Let's assume G_0 is a classical group over $\mathbb{F}_q$, where $q = p^f$ and the natural module V is m-dimensional.

The key tool for proving Cameron's conjecture for classical groups (with an undetermined constant) is the following bound:

Theorem (Liebeck & Shalev, 1999)

There exists an absolute constant $\epsilon > 0$ such that

$$\mathrm{fpr}(x) < |x^G|^{-\epsilon}$$

for every element x of prime order in a non-standard classical group G.

Note. This is **false** for standard classical groups.

e.g. if $G = \mathrm{PGL}_m(q)$, $\Omega = P(V)$ and $x = [-I_1, I_{m-1}]$ (mod scalars), then $\mathrm{fpr}(x) \sim q^{-1}$ and $|x^G| \sim q^{2m-2}$.

Theorem (Liebeck & Shalev, 1999)

We have $b(G) \leqslant \lceil 11/\epsilon \rceil$ for every non-standard classical group G.

Proof. The proof combines the previous $\mathrm{fpr}(x)$ bound with two facts on conjugacy classes (where $m = \dim V$):

- G has at most q^{4m} conjugacy classes of elements of prime order.
- $|x^G| \geqslant q^{m/2}$ for all $x \in G$ of prime order.

Set $c = \lceil 11/\epsilon \rceil$. Then

$$\widehat{Q}(G,c) = \sum_{i=1}^{k} |x_i^G| \cdot \mathrm{fpr}(x_i)^c < \sum_{i=1}^{k} |x_i^G|^{-10} \leqslant k \cdot (q^{m/2})^{-10} \leqslant q^{-m}$$

so $b(G) \leqslant c$. Moreover, $P(G,c) \to 1$ as $|G|$ tends to infinity. ∎

Sharper bounds

The following is an explicit version of the Liebeck-Shalev bound on $\mathrm{fpr}(x)$.

Theorem (B, 2007)

Let G be a non-standard classical group with point stabilizer H and $\dim V = m$. Then

$$\mathrm{fpr}(x) < |x^G|^{-\frac{1}{2}+\eta}$$

for all $x \in G$ of prime order, where $\eta \to 0$ as $m \to \infty$.

- We can take $\eta = \frac{1}{m} + \delta$ in the exponent, where $\delta = 0$, or (G, H, δ) is one of a small number of known exceptions.
- The bound is essentially best possible. For small values of m, say $m \leqslant 5$, more accurate bounds can be obtained.

An example

Let $G = \mathrm{L}_m(q)$, where m is even and q is odd, and let H be a maximal subgroup of type $\mathrm{O}_m^+(q)$.

Let $x = [-I_k, I_{m-k}] \in G$ (mod scalars) with $k < m/2$ even. Then

$$|x^G \cap H| = \frac{|\mathrm{O}_m^+(q)|}{|\mathrm{O}_k^+(q)||\mathrm{O}_{m-k}^+(q)|} + \frac{|\mathrm{O}_m^+(q)|}{|\mathrm{O}_k^-(q)||\mathrm{O}_{m-k}^-(q)|} \sim q^{k(m-k)}$$

and

$$|x^G| = \frac{|\mathrm{GL}_m(q)|}{|\mathrm{GL}_k(q)||\mathrm{GL}_{m-k}(q)|} \sim q^{2k(m-k)}$$

Conclusion. $\mathrm{fpr}(x) \sim q^{-k(m-k)} \sim |x^G|^{-\frac{1}{2}}$.

Note. If $G = \mathrm{L}_m(q)$, $q = q_0^2$ and H is a maximal subgroup of type $\mathrm{GL}_m(q_0)$, then $\mathrm{fpr}(x) \sim |x^G|^{-\frac{1}{2}}$ for all $x \in H$ of prime order.

Subgroup structure

Aschbacher's theorem (1984) provides a framework for the proof.

Let G be an almost simple classical group with socle $G_0 = \mathrm{Cl}(V)$ and let H be a maximal subgroup with $G = HG_0$. Aschbacher proves that either

- H is **geometric**, contained in one of 8 subgroup collections $\mathcal{C}_1, \ldots, \mathcal{C}_8$ that are defined in terms of the geometry of V; or
- $H \in \mathcal{S}$ is **non-geometric**: almost simple with socle H_0 and some covering group of H_0 acts absolutely irreducibly on V.

For example, the subgroups in $\mathcal{C}_1$ stabilize subspaces of V, while those in $\mathcal{C}_2$ stabilize direct sum decompositions of V.

Kleidman & Liebeck (1990), **Bray, Holt & Roney-Dougal (2013)**:

The structure, conjugacy and maximality of the geometric subgroups is known. BHR also handle the non-geometric subgroups for $m \leqslant 12$.

Geometric subgroups

Let $H \in \mathcal{C}_i$ be geometric, $x \in H$ prime order. Since the embedding of H in G is transparent, it is usually straightforward to identify the G-class of x (e.g. by considering the action of x on V).

Moreover, we can study the decomposition of $x^G \cap H$ into H-classes and by combining an upper bound on $|x^G \cap H|$ with a lower bound on $|x^G|$, we obtain an upper bound on $\mathrm{fpr}(x)$.

Algebraic groups. Write $G_0 = (\bar{G}_\sigma)'$, where $\bar{G}$ is a simple algebraic group over $\bar{\mathbb{F}}_q$ and σ is a Steinberg endomorphism. In many cases, H is of the form $N_G(\bar{H}_\sigma)$ for some σ-stable closed subgroup $\bar{H}$ of $\bar{G}$. In this setting,

$$\mathrm{fpr}(x) = \frac{|x^G \cap H|}{|x^G|} \sim q^{\dim(x^{\bar{G}} \cap \bar{H}) - \dim x^{\bar{G}}}$$

and thus bounds at the algebraic group level can be applied.

Non-geometric subgroups

A different approach is required when H is non-geometric.

Let $\rho : \widehat{H}_0 \to \mathrm{GL}(V)$ be absolutely irreducible.

- Typically, the embedding of H in G is **not** transparent and it is difficult to identify the G-class of $x \in H$.
- The possibilities for H are only known in small dimensions.

To bound $\mathrm{fpr}(x)$, we apply two key tools (cf. proof for $G_0 = A_m$):

- **Liebeck (1985):** Either $|H| < q^{6m}$ or (G, H, ρ) is known.
- **Guralnick & Saxl (2003):**

  Either $\nu(x) > \max\{2, \sqrt{m}/2\}$ for all $1 \neq x \in H \cap \mathrm{PGL}(V)$, or (G, H, ρ) is known, where $\nu(x)$ is the codimension of the largest eigenspace of $\hat{x} \in \mathrm{GL}(\bar{V})$ for $\bar{V} = V \otimes \bar{\mathbb{F}}_q$.

An example

Suppose $G = \mathrm{L}_m(q)$, $H \in \mathcal{S}$ and $x \in H$ has prime order.

Assume the **L** and **G-S** bounds apply, so $|x^G \cap H| < |H| < q^{6m}$ and $\nu(x) = s \geqslant \lceil \sqrt{m}/2 \rceil = \beta$.

The latter bound yields $|x^G| > \frac{1}{2}q^{2\beta(m-\beta)}$ and thus $\mathrm{fpr}(x) < |x^G|^{-\frac{1}{2}}$ if

$$2q^{12m} < q^{2\beta(m-\beta)}.$$

This inequality holds if $m > 144$, so we may assume $m \leqslant 144$.

At the expense of some additional known cases, we can take $|H| < q^{2m+4}$, which reduces the problem to $m \leqslant 16$.

We can now apply work of **Lübeck (2001)** (defining characteristic) and **Hiss & Malle (2001)** (non-defining characteristic) to determine the possibilities for (G, H, ρ).

Cameron's conjecture for classical groups

Theorem (B, 2007)

Let G be a non-standard classical group with point stabilizer H. Then $b(G) \leqslant 5$, with equality iff $G = \mathrm{U}_6(2).2$ and $H = \mathrm{U}_4(3).2^2$.

Let $x_1, \ldots, x_k$ represent the G-classes of elements of prime order and set

$$\zeta_G(t) = \sum_{i=1}^{k} |x_i^G|^{-t}, \quad t \in \mathbb{R}$$

Fact. If $m \geqslant 6$, then $\zeta_G(1/3) < 1$.

By combining this with the generic bound $\mathrm{fpr}(x) < |x^G|^{-\frac{1}{2}+\frac{1}{m}}$, we get

$$\widehat{Q}(G,4) < \sum_{i=1}^{k} |x_i^G|^{1+4(-\frac{1}{2}+\frac{1}{m})} \leqslant \zeta_G(1/3) < 1$$

for $m \geqslant 6$, so $b(G) \leqslant 4$. The remaining groups can be handled directly.

Exceptional groups

Theorem (B, Liebeck & Shalev, 2009)

Let G be an almost simple primitive group with socle G_0, an exceptional group of Lie type. Then $b(G) \leqslant 6$. Moreover, $P(G,6) \to 1$ as $|G| \to \infty$.

The goal is to show that $\widehat{Q}(G,6) < 1$ and $\widehat{Q}(G,6) \to 0$ as $q \to \infty$.

Subgroups. The maximal subgroups of G are known for $G_0 = {}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$, ${}^3D_4(q)$, $G_2(q)$, $F_4(q)$, $E_6^{\epsilon}(q)$. In general, we apply an Aschbacher-type subgroup structure theorem due to **Liebeck & Seitz**.

FPRs. **Lawther, Liebeck & Seitz (2002)** obtain detailed fixed point ratio estimates. In some cases, we work with the minimal and adjoint modules to understand the fusion of H-classes in G.

For H parabolic, we can use tools from character theory to study the permutation character, noting that $1_H^G(x) = |C_{\Omega}(x)|$ for $\Omega = G/H$.

Suppose $G = E_8(q)$ and $H = \Omega_{16}^{+}(q)$ with q even. Set $\bar{G} = E_8$, $\bar{H} = D_8$ and let $\bar{V} = \mathrm{Lie}(\bar{G})$ be the adjoint module for $\bar{G}$.

Since $|H| < q^{120}$, the contribution to $\widehat{Q}(G,6)$ from elements $x \in G$ with $|x^G| > q^{145}$ is at most $q^{145}(q^{-25})^6 = q^{-5}$. Now assume $|x^G| \leqslant q^{145}$.

If x has odd order, then $C_{\bar{G}}(x) = E_7T_1$ is the only possibility, which means that the 1-eigenspace of x on $\bar{V}$ is 134-dimensional. Now

$$\bar{V} \downarrow \bar{H} = \mathrm{Lie}(\bar{H}) \oplus \bar{U},$$

where $\mathrm{Lie}(\bar{H})$ is the adjoint module for $\bar{H}$ and $\bar{U}$ is a spin module.

Using this, we calculate that $x = (I_{12}, \lambda I_2, \lambda^{-1} I_2)$ on the natural module for $\bar{H}$, so $|x^G \cap H| \sim q^{50}$ and $|x^G| \sim q^{114}$.

For $|x| = 2$, we get $|x^G \cap H| \sim q^{26}$ and $|x^G| \sim q^{58}$ if x is a long root element, otherwise $|x^G| > q^{92}$ and we note that $i_2(H) \sim q^{64}$.

Further results

By applying similar methods, stronger results can be established.

Theorem (B, 2018)

Let G be a non-standard group with socle G_0 and point stabilizer H. Then $b(G) = 6$ iff (G, H) is one of the following:

$$(\mathrm{M}_{23}, \mathrm{M}_{22}),\ (\mathrm{Co}_3, \mathrm{McL}.2),\ (\mathrm{Co}_2, \mathrm{U}_6(2).2),\ (\mathrm{Fi}_{22}.2, 2.\mathrm{U}_6(2).2)$$

or $G_0 = E_7(q)$ and $H = P_7$, or $G_0 = E_6(q)$ and $H = P_1$ or P_6.

Theorem (B, Guralnick & Saxl, 2014)

Let G be a non-standard classical group with point stabilizer $H \in \mathcal{S}$. Then the exact base size $b(G)$ is known.

Ongoing work with Guralnick seeks to extend this to **all** non-standard classical groups.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite primitive group with $H = G_\alpha$.

Seress (1996): G solvable $\implies b(G) \leqslant 4$

Theorem (B, 2020)

If H is solvable then $b(G) \leqslant 5$ is best possible. In addition, the exact value of $b(G)$ is known when G is almost simple.

- If $G = VH$ is affine, then G is solvable and thus $b(G) \leqslant 4$ by Seress.
- G almost simple: $b(G) \leqslant 5$, with equality iff $(G, H) = (S_8, S_4 \wr S_2)$ or $(G_0, H) = (\mathrm{L}_4(3), P_2)$, $(\mathrm{U}_5(2), P_1)$.

Conjecture (Vdovin, Kourovka Notebook 17.41)

Let G be a finite group and let H be a core-free solvable subgroup. Then $b(G) \leqslant 5$ with respect to the action of G on $\Omega = G/H$.

Comments on the proof

- By O'Nan-Scott, G is either affine, almost simple or product type.

- Suppose G is **almost simple** with socle G_0. The possibilities for (G, H) were determined by **Li & Zhang (2011)**.

  - G_0 sporadic: Apply **[B, O'Brien & Wilson, 2010]**.
  - $G_0 = A_m$: If $m \geqslant 17$ then $m = p$ is prime, $H = \mathrm{AGL}_1(p) \cap G$ and $b(G) = 2$.
  - $G_0 = \mathrm{L}_2(q)$: Needs special attention (to determine exact $b(G)$).
  - G_0 Lie type, H parabolic: We estimate $\widehat{Q}(G, c)$, noting that the rank of G_0 and the underlying field $\mathbb{F}_q$ are small.
  - G_0 Lie type, H non-parabolic:

    We estimate $\widehat{Q}(G, c)$; for G_0 exceptional we apply recent base size results of **[B & Thomas, 2020]**.

Comments on the proof

- Finally, suppose $G \leqslant \mathrm{Sym}(\Omega)$ is **product type**.

  Here $T^k \trianglelefteqslant G \leqslant L \wr P$ where $L \leqslant \mathrm{Sym}(\Gamma)$ is almost simple with socle T, $P \leqslant S_k$ is transitive and $\Omega = \Gamma^k$.

- In addition, P and L_γ are solvable (for $\gamma \in \Gamma$).

- As in the proof of Pyber's conjecture: $b(G) \leqslant \log_{|\Gamma|} d(P) + b(L) + 1$.

- **Seress (1996)** gives $d(P) \leqslant 5$ and the result for almost simple groups gives $b(L) \leqslant 5$. Note that $|\Gamma| \geqslant 5$.

- We reduce to the cases with $b(L) = 5$, so $|\Gamma| \leqslant 130$.

  By **Bailey & Cameron (2011)**, we have $b(L \wr P) \leqslant 5$ if L has at least 5 regular orbits on Γ^5. This is easily checked using **Magma**.

Next week: Applications

- The minimal dimension of a finite group
- 2-generation and the uniform domination number
- Extremely primitive groups

Some references

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