Fixed point ratios in actions of finite classical groups, IV

Timothy C. Burness
St John's College
Oxford OX1 3JP
England

June 7, 2006

Abstract

This is the final paper in a series of four on fixed point ratios in non-subspace actions of finite classical groups. Our main result states that if G is a finite almost simple classical group and Ω is a non-subspace G-set then either $\mathrm{fpr}(x) \lesssim |x^G|^{-\frac{1}{2}}$ for all elements $x \in G$ of prime order, or (G, Ω) is one of a small number of known exceptions. In this paper we complete the proof by assuming G_ω is either an almost simple irreducible subgroup in Aschbacher's $\mathscr{S}$ collection or a subgroup in a small additional set $\mathscr{N}$ which arises when G has socle $\mathrm{Sp}_4(q)'$ (q even) or $\mathrm{P\Omega}_8^+(q)$.

1 Introduction

If G is a permutation group on a finite set Ω then the *fixed point ratio* of an element $x \in G$, which we denote by $\mathrm{fpr}(x)$, is defined to be the proportion of points in Ω fixed by x. Our main result on fixed point ratios, which we refer to as Theorem 1, states that if Ω is a faithful, transitive, non-subspace G-set, where G is a finite almost simple classical group with socle G_0, then

$$\mathrm{fpr}(x) < |x^G|^{-\frac{1}{2}+\frac{1}{n}+\iota}$$

for all elements $x \in G$ of prime order, where either $\iota = 0$ or (G_0, Ω, ι) belongs to a short list of known exceptions (see [4, Table 1] for the list of exceptional cases). Here a transitive G-set is said to be *non-subspace* if a point stabilizer G_ω is a non-subspace subgroup of G, i.e. $G_\omega \cap G_0$ is contained in a maximal subgroup of G_0 which acts irreducibly on the natural G_0-module (see [4, Definition 1]). In almost all cases $n = \dim V$ (see Remark 1.2).

The proof of Theorem 1 is based on Aschbacher's main theorem on the subgroup structure of finite classical groups. Recall that in [1], eight collections of subgroups of G are defined, labelled $\mathscr{C}_i$ for $1 \leqslant i \leqslant 8$, and in general it is shown that if H is a maximal subgroup of G not containing G_0 then either H is contained in one of the $\mathscr{C}_i$ collections, or it belongs to a family $\mathscr{S}$ of almost simple groups which satisfy various irreducibility conditions (see [19] for a detailed description of the $\mathscr{C}_i$ collections, and [19, §1.2] for more details on the subgroups in $\mathscr{S}$). Due to the existence of certain outer automorphisms, a small additional collection $\mathscr{N}$ of subgroups arises when G_0 is $\mathrm{Sp}_4(q)'$ (q even) or $\mathrm{P\Omega}_8^+(q)$ (see Table 3.1 and [5, 3.1]).

This is the final paper in a series of four. In [4] we provided some background and motivation, stated our main results and described applications to the study of minimal bases and monodromy groups. In [5] and [6] we established Theorem 1 in the case where G_ω is a non-subspace subgroup contained in a member of one of the $\mathscr{C}_i$ collections. Therefore, it just remains to consider the collections $\mathscr{S}$ and $\mathscr{N}$. In this paper we complete the proof of Theorem 1 via Theorem 1.1 below.

E-mail address: burness@maths.ox.ac.uk

G_0	type of H	ι
$\mathrm{P}\Omega_8^+(q)$	$\Omega_7(q)$	.219
$\Omega_7(q)$	$G_2(q)$	.108
$\Omega_{10}^-(2)$	A_{12}	.087
$\mathrm{Sp}_8(2)$	A_{10}	.062
$\Omega_8^+(2)$	A_9	.124
$\mathrm{P}\Omega_8^+(3)$	$\Omega_8^+(2)$	.081
$\Omega_7(3)$	$\mathrm{Sp}_6(2)$	.065
$\mathrm{PSU}_6(2)$	$\mathrm{PSU}_4(3)$	.076
$\mathrm{Sp}_6(2)$	$\mathrm{SU}_3(3)$	.054
$\mathrm{PSU}_4(3)$	$\mathrm{PSL}_3(4)$	.011
$\mathrm{SL}_4(2)$	A_7	.164

Table 1.1: The exceptional cases with $\iota > 0$

Theorem 1.1. *Let G be a finite almost simple classical group acting transitively and faithfully on a set Ω with point stabilizer $G_\omega \leqslant H$, where $H \leqslant G$ is a maximal non-subspace subgroup in one of the collections $\mathscr{S}$ or $\mathscr{N}$. Then*

$$\mathrm{fpr}(x) < |x^G|^{-\frac{1}{2}+\frac{1}{n}+\iota}$$

for all elements $x \in G$ of prime order, where $\iota = 0$ or (G_0, H, ι) is listed in Table 1.1, where G_0 denotes the socle of G.

Remark 1.2. The integer n in the statement of Theorem 1.1 is defined as follows: if $G_0 \in \{\mathrm{Sp}_4(2)', \mathrm{SL}_3(2)\}$ then $n = 2$, otherwise n is the minimal degree of a non-trivial irreducible $K\widehat{G}_0$-module, where $\widehat{G}_0$ is a covering group of G_0 and K is the algebraic closure of $\mathbb{F}_q$. We also note that each of the subgroups appearing in Table 1.1 is a member of the collection $\mathscr{S}$; the *type of H* refers to the socle of the almost simple group $H \cap G_0$.

Notation. Our notation for classical groups is standard (see [19] for example) and other notation and terminology is consistent with the previous papers [4], [5] and [6] in this series. In particular, if $H \leqslant G$ is a non-subspace subgroup and $x \in H$ has prime order then

$$f(x, H) := \frac{\log |x^G \cap H|}{\log |x^G|}$$

and thus Theorem 1 states that $f(x, H) < 1/2 + 1/n + \iota$ (see [5, (1)]). In addition, we adopt the standard Aschbacher-Seitz [2] notation for representatives of unipotent classes of involutions in symplectic and orthogonal groups and we define the *associated partition* of a general unipotent element $x \in \mathrm{PGL}(V)$ to be the partition of the integer $\dim V$ which corresponds to the Jordan normal form of x on V (see [5, §3.3]). Also, for each $x \in \mathrm{PGL}(V)$ we define $\nu(x)$ to be the codimension of the largest eigenspace of x on V (see [5, 3.16]). For any $r \in \mathbb{N}$ and subset $S \subseteq X$ of a finite group X we write $i_r(S)$ for the number of elements of order r in S.

2 Proof of Theorem 1.1: $H \in \mathscr{S}$

Let G be a finite almost simple classical group over $\mathbb{F}_q$, with socle G_0 and natural module V of dimension n. We write $q = p^f$, where p is prime. If H is a maximal subgroup of G in Aschbacher's $\mathscr{S}$ collection then $H \cap G_0$ is almost simple, with socle H_0. Moreover, if $\widehat{H}_0$ is the full covering group of H_0 then $\widehat{H}_0$ acts absolutely irreducibly on V and is defined over no proper subfield of $\mathbb{F}_{q^u}$, where $u = 2$ if G_0 is unitary, otherwise $u = 1$. In addition, $\widehat{H}_0$ fixes a non-degenerate form on V only if G_0 fixes a form of the same type (see [19, §1.2] for example).

	d	p	G_0
$(\mathscr{A}1)$	arbitrary	odd	$\begin{cases} \mathrm{P}\Omega^{\epsilon}_{d-1}(p) & \text{if } (d,p)=1 \\ \mathrm{P}\Omega^{\epsilon}_{d-2}(p) & \text{otherwise} \end{cases}$
$(\mathscr{A}2)$	$d \equiv 2\,(4)$	2	$\mathrm{Sp}_{d-2}(2)$
$(\mathscr{A}3)$	$d \equiv 0\,(4)$	2	$\begin{cases} \Omega^{+}_{d-2}(2) & \text{if } d \equiv 0\,(8) \\ \Omega^{-}_{d-2}(2) & \text{if } d \equiv 4\,(8) \end{cases}$
$(\mathscr{A}4)$	odd	2	$\begin{cases} \Omega^{+}_{d-1}(2) & \text{if } d \equiv \pm 1\,(8) \\ \Omega^{-}_{d-1}(2) & \text{if } d \equiv \pm 3\,(8) \end{cases}$

Table 2.1: The collection $\mathscr{A}$, $H_0 = A_d$

	H_0	G_0	representation of H_0
$(\mathscr{B}1)$	$\mathrm{PSL}_d(q)$ $d \geqslant 5$	$\mathrm{PSL}_{\frac{1}{2}d(d-1)}(q)$	$\bigwedge^2 V_d$
$(\mathscr{B}2)$	$\begin{cases} \Omega_7(q) & p>2 \\ \mathrm{Sp}_6(q) & p=2 \end{cases}$	$\mathrm{P}\Omega^{+}_{8}(q)$	spin representation
$(\mathscr{B}3)$	$\begin{cases} \Omega_9(q) & p>2 \\ \mathrm{Sp}_8(q) & p=2 \end{cases}$	$\mathrm{P}\Omega^{+}_{16}(q)$	spin representation
$(\mathscr{B}4)$	$\mathrm{P}\Omega^{+}_{10}(q)$	$\mathrm{PSL}_{16}(q)$	one of the two spin representations
$(\mathscr{B}5)$	$E_6(q)$	$\mathrm{PSL}_{27}(q)$	$M(\lambda_1)$ or $M(\lambda_6)$
$(\mathscr{B}6)$	$E_7(q)$	$\begin{cases} \mathrm{PSp}_{56}(q) & p>2 \\ \Omega^{+}_{56}(q) & p=2 \end{cases}$	$M(\lambda_7)$
$(\mathscr{B}7)$	M_{24}	$\mathrm{SL}_{11}(2)$	
$(\mathscr{B}8)$	Co_1	$\Omega^{+}_{24}(2)$	

Table 2.2: The collection $\mathscr{B}$

Our strategy is as follows. First we define three sets of irreducible inclusions $H < G$, denoted by the letters $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ (see Tables 2.1-2.3). In $\mathscr{A}$, H_0 is an alternating group, $q = p$ is prime and V is the fully deleted permutation module for H_0 over $\mathbb{F}_p$. We establish Theorem 1.1 for these inclusions in Proposition 2.5; the collections $\mathscr{B}$ and $\mathscr{C}$ are considered in Propositions 2.6 and 2.10 respectively. If the irreducible embedding of H in G is not one of the inclusions in $\mathscr{A}$, $\mathscr{B}$ or $\mathscr{C}$ then Theorems 2.2 and 2.4 imply that the following hold:

(i) If $n \geqslant 6$ then $\nu(x) > \max(2, \frac{1}{2}\sqrt{n})$ for all $1 \neq x \in H \cap \mathrm{PGL}(V)$.
(ii) $|H| < |\mathbb{F}|^{2n+4}$, where V is defined over the field $\mathbb{F}$.

(Here $\nu(x)$ denotes the codimension of the largest eigenspace of x on V - see [5, 3.16].) In particular, if $x \in H \cap \mathrm{PGL}(V)$ has prime order then the bound on $\nu(x)$ in (i) yields a lower bound for $|x^G|$ via [5, 3.38]; an upper bound for $f(x,H)$ now follows since (ii) gives $|x^G \cap H| < |\mathbb{F}|^{2n+4}$. This leaves a small number of inclusions which we can deal with on a case-by-case basis (see Tables 2.10 and 2.11) and we consider the remaining cases with $n < 6$ in Proposition 2.22.

Definition 2.1. Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be the set of irreducible inclusions $H < G$ listed in Tables 2.1, 2.2 and 2.3 respectively. In Table 2.1, we have $H_0 = A_d$ with $d \geqslant 5$. We write $M(\lambda)$ for the unique irreducible $\mathbb{F}_q H_0$-module of highest weight λ and we follow [3] in labelling the fundamental dominant weights $\{\lambda_i\}$.

Theorem 2.2 ([22, 4.2]). *If $H \in \mathscr{S}$ then one of the following holds:*

(i) H_0 is alternating, embedded in G as in $\mathscr{A}$;
(ii) H_0 is embedded in G_0 as in $\mathscr{B}$;
(iii) $|H| < |\mathbb{F}|^{2n+4}$, where V is defined over $\mathbb{F}$ and $n = \dim V$.

	H_0	G_0	
$(\mathscr{C}1)$	$\mathrm{PSL}_3^\epsilon(q)$ $\quad p > 2$	$\mathrm{PSL}_6^\epsilon(q)$	S^2V_3
$(\mathscr{C}2)$	$\begin{cases} \Omega_7(q) & p > 2 \\ \mathrm{Sp}_6(q) & p = 2 \end{cases}$	$\mathrm{P}\Omega_8^+(q)$	spin representation
$(\mathscr{C}3)$	${}^3D_4(q_0)$ $\quad q = q_0^3$	$\mathrm{P}\Omega_8^+(q)$	minimal module
$(\mathscr{C}4)$	$G_2(q)'$	$\begin{cases} \Omega_7(q) & p > 2 \\ \mathrm{Sp}_6(q) & p = 2 \end{cases}$	$M(\lambda_1)$
$(\mathscr{C}5)$	$G_2(q)$ $\quad p = 3$	$\Omega_7(q)$	$M(\lambda_2)$
$(\mathscr{C}6)$	A_6	$\mathrm{PSL}_6^\epsilon(p)$	$p \equiv \epsilon\,(3),\ p \geqslant 5$
$(\mathscr{C}7)$	A_7	$\mathrm{PSL}_6^\epsilon(p)$	$p \equiv \epsilon\,(3),\ p \geqslant 5$
$(\mathscr{C}8)$	$\mathrm{PSL}_2(7)$	$\mathrm{P}\Omega_6^\epsilon(p)$	$p \neq 2, 7$
$(\mathscr{C}9)$	$\mathrm{PSL}_3(4)$	$\mathrm{PSL}_6^\epsilon(p)$	$p \equiv \epsilon\,(3),\ p \geqslant 5$
$(\mathscr{C}10)$	$\mathrm{PSL}_3(4)$	$\mathrm{P}\Omega_6^-(3)$	
$(\mathscr{C}11)$	$\mathrm{SU}_3(3)$	$\mathrm{PSp}_6(p)$	$p \neq 3$
$(\mathscr{C}12)$	$\mathrm{SU}_3(3)$	$\mathrm{PSL}_7^\epsilon(p)$	$p \equiv \epsilon\,(3),\ p \geqslant 5$
$(\mathscr{C}13)$	$\mathrm{SU}_3(3)$	$\Omega_7(p)$	$p \geqslant 5$
$(\mathscr{C}14)$	$\mathrm{SU}_4(2)$	$\mathrm{P}\Omega_6^\epsilon(p)$	$p \equiv \epsilon\,(3),\ p \geqslant 5$
$(\mathscr{C}15)$	$\mathrm{PSU}_4(3)$	$\mathrm{PSL}_6^\epsilon(p)$	$p \equiv \epsilon\,(3),\ p \geqslant 5$
$(\mathscr{C}16)$	$\mathrm{PSU}_4(3)$	$\mathrm{PSU}_6(2)$	
$(\mathscr{C}17)$	$\mathrm{SU}_5(2)$	$\mathrm{PSp}_{10}(p)$	$p \geqslant 3$
$(\mathscr{C}18)$	$\mathrm{Sp}_6(2)$	$\Omega_7(p)$	$p \geqslant 3$
$(\mathscr{C}19)$	$\Omega_8^+(2)$	$\mathrm{P}\Omega_8^+(p)$	$p \geqslant 3$
$(\mathscr{C}20)$	M_{12}	$\mathrm{PSL}_6(3)$	
$(\mathscr{C}21)$	M_{22}	$\mathrm{PSU}_6(2)$	
$(\mathscr{C}22)$	J_2	$\mathrm{PSp}_6(q)$	$p \geqslant 3$

Table 2.3: The collection $\mathscr{C}$

Remark 2.3. If $G_0 = \mathrm{PSU}_n(q)$ then part (iii) of Theorem 2.2 reads $|H| < q^{4n+8}$ because the natural G_0-module is a vector space over the field $\mathbb{F}_{q^2}$ and not $\mathbb{F}_q$.

Theorem 2.4 ([12, 7.1]). *If $H \in \mathscr{S}$ and $n = \dim V \geqslant 6$ then one of the following holds:*

(i) H_0 is alternating, embedded in G as in $\mathscr{A}$;

(ii) H_0 is embedded in G_0 as in $\mathscr{C}$;

(iii) $\nu(x) > \max(2, \frac{1}{2}\sqrt{n})$ for all non-trivial elements $x \in H \cap \mathrm{PGL}(V)$.

2.1 The $\mathscr{A}$ collection

Proposition 2.5. *The conclusion to Theorem 1.1 holds for the collection $\mathscr{A}$.*

Proof. Let $H < G$ be an inclusion in the collection $\mathscr{A}$, where H has socle $H_0 = A_d$. Let V denote the fully deleted permutation module for H_0 over $\mathbb{F}_p$, i.e. $V = U/(U \cap W)$, where U and W are the submodules of $\mathbb{F}_p^d$ defined as follows

$$U = \{(a_1, \ldots, a_d) : \sum_{i=1}^{d} a_i = 0\},\ W = \{(a, \ldots, a) : a \in \mathbb{F}_p\}$$

with respect to the natural action of the symmetric group S_d on the coordinates of $\mathbb{F}_p^d$. Observe that $H \leqslant S_d \leqslant \mathrm{PGL}(V)$. Let $x \in H$ be an element of prime order r and let h denote the number of r-cycles in the cycle-shape of x. Let K denote the algebraic closure of $\mathbb{F}_p$ and let $\bar{G}$ be a simple algebraic group of adjoint type over K such that $\bar{G}_\sigma$ has socle G_0, where σ is a suitable Frobenius morphism of $\bar{G}$. According to [5, 3.3] we may assume G is without triality if $G_0 = \mathrm{P}\Omega_8^+(p)$. To establish Theorem 1.1 for the inclusions listed in Table 2.1 we show that $f(x, H) < 1/2 + 1/n$, with the exception of the following cases:

H_0	A_{12}	A_{10}	A_9	A_7
G_0	$\Omega_{10}^-(2)$	$\mathrm{Sp}_8(2)$	$\Omega_8^+(2)$	$\Omega_6^+(2)$
$f(x,H) <$	.687*	.687*	.749*	.914*

(Here $n = n(G)$ is the integer defined in Remark 1.2.) These bounds are obtained through direct calculation and agree with the relevant entries in Table 1.1. (Note that $\Omega_6^+(2) \cong \mathrm{SL}_4(2)$ and we list the case $(\Omega_6^+(2), A_7)$ in Table 1.1 under $G_0 = \mathrm{SL}_4(2)$.) The asterisks indicate that these cases are exceptions to the main statement of Theorem 1.1.

First consider ($\mathscr{A}$2). Here $d \equiv 2\,(4)$ and we may assume $d \geqslant 10$ since $H_0 \cong G_0$ if $d = 6$. Referring to a general d-tuple $(a_1, \ldots, a_d)$, one can check that the elements defined by

$$\begin{aligned} e_i &: \quad a_{2i-1} = a_{2i} = 1,\ a_j = 0 \text{ for all other } j; \\ f_i &: \quad a_j = a_d = 1 \text{ for all } j \leqslant 2i-1, \text{ otherwise } a_k = 0; \\ g &: \quad a_j = 1 \text{ for all } j, \end{aligned}$$

where $1 \leqslant i \leqslant \frac{1}{2}(d-2)$, form a basis for U. Since p divides d we have $\dim V = d - 2$ and it is easy to see that the elements in the set

$$\{e_i + (U \cap W),\, f_i + (U \cap W) : 1 \leqslant i \leqslant (d-2)/2\} = \{\bar{e}_i, \bar{f}_i : 1 \leqslant i \leqslant (d-2)/2\}$$

form a standard symplectic basis for V with respect to the form on V induced from the symmetric bilinear form f on U defined by

$$f((a_1, \ldots, a_d), (b_1, \ldots, b_d)) = \sum_{i=1}^{d} a_i b_i.$$

If r is odd then we calculate that x is $\bar{G}$-conjugate to $[I_{d-2-h(r-1)}, \omega I_h, \ldots, \omega^{r-1} I_h]$, where $\omega \in K$ is a primitive r^{th} root of unity. If $r = 2$ and $h < d/2$ then replacing x by a suitable conjugate we may assume that x interchanges the first two coordinates, while fixing the last two. This implies that $f(\bar{f}_1, \bar{f}_1 x) = f(\bar{f}_1, \bar{e}_1 + \bar{f}_1) = 1$ and we conclude that x is $\bar{G}$-conjugate to either b_h or c_h, the precise type depending on the parity of h. (As remarked in the Introduction, in this paper we adopt the standard Aschbacher-Seitz [2] notation for representatives of unipotent classes of involutions). If $h = d/2$ then the action of x on the above basis for V is given by

$$x:\ \bar{e}_i \mapsto \bar{e}_i,\ \ \bar{f}_i \mapsto \bar{f}_i + \sum_{j \neq i} \bar{e}_j$$

and therefore x is G-conjugate to $a_{d/2-1}$. It follows that $x^G \cap H \subseteq x^{S_d}$ for all elements $x \in H$ of prime order and thus

$$|x^G \cap H| \leqslant |x^{S_d}| = \frac{d!}{h!(d-hr)!r^h}. \tag{1}$$

If $r = 2$ and $h < d/2$ then $|x^G| > 2^{h(d-h-1)-1}$ (see [5, 3.22]) and (1) implies that $f(x, H) < 1/2 + 1/(d-2)$ with the exception of the following cases, where the upper bounds for $f(x, H)$ are obtained through direct calculation.

(h, d)	$(4, 10)$	$(3, 10)$	$(2, 10)$	$(1, 10)$
$f(x, H) <$	$.615$	$.636^*$	$.666^*$	$.687^*$

For instance, if $(h, d) = (1, 10)$ then $f(x, H) < .687^*$ since $|x^G \cap H| = 45$ and $|x^G| = 255$. As before, the asterisks indicate that the case $(G_0, H_0) = (\mathrm{Sp}_8(2), A_{10})$ appears in Table 1.1. If $h = d/2$ then $|x^G| > 2^{d^2/4-d}$ and (1) is sufficient unless $d = 10$, where direct calculation yields $f(x, H) < .619$. If r is odd then

$$|x^G| > \frac{1}{2}\left(\frac{2}{3}\right)^{\frac{1}{2}(r-1)} 2^{\frac{1}{2}h(r-1)(2d-hr-3)}$$

and we are left to deal with the following cases:

(r, h, d)	$(3, 1, 10)$	$(3, 2, 10)$	$(3, 3, 10)$	$(5, 2, 10)$
$f(x, H) <$	$.590$	$.598$	$.614$	$.593$

The cases ($\mathscr{A}3$) and ($\mathscr{A}4$) are similar. For example, in ($\mathscr{A}4$) d is odd and x is given as follows up to $\bar{G}$-conjugacy:

$$x = \begin{cases} [I_{d-1-h(r-1)}, \omega I_h, \ldots, \omega^{r-1} I_h] & \text{if } r > 2 \\ b_h \text{ or } c_h & \text{if } r = 2. \end{cases}$$

Note that we may assume $d \geqslant 7$ since $G_0 = \Omega_4^-(2)$ if $d = 5$ and thus $n = 2$ (see Remark 1.2). If $d = 7$ then $n = 4$ since $G_0 = \Omega_6^+(2)$ and we obtain the following results through direct calculation. Here $\zeta = 1$ if $G = \mathrm{O}_6^+(2)$, otherwise $\zeta = 0$.

(h, r)	$(1, 2)$	$(2, 2)$	$(3, 2)$	$(1, 3)$	$(2, 3)$	$(1, 5)$	$(1, 7)$
$\lvert x^G \cap H\rvert$	21	105	105	70	280	504	$2^{\zeta}.360$
$\lvert x^G\rvert$	28	210	420	112	1120	1344	$2^{\zeta}.2880$
$f(x, H) <$	$.914^*$	$.871^*$	$.771^*$	$.901^*$	$.803^*$	$.864^*$	$.760^*$

(If $(G_0, H_0) = (\Omega_{10}^-(2), A_{12})$ then $f(x, H) \leqslant (\log 10395)/(\log 706860) \approx .687^*$ for all $x \in H$ of prime order, with equality if and only if $x \in A_{12}$ has cycle-shape (2^6). Similarly, if $(G_0, H_0) = (\Omega_8^+(2), A_9)$ then $f(x, H) \leqslant (\log 36)/(\log 120) \approx .749^*$, with equality if and only if $x \in S_9$ has cycle-shape $(2, 1^7)$; if $G = \Omega_8^+(2)$ then $f(x, H) \leqslant (\log 378)/(\log 3780)$.)

Now assume p is odd. If $r = p$ and d and p are coprime then x is $\bar{G}$-conjugate to $[J_p^h, I_{d-1-hp}]$; if p divides d then x is given as follows (up to $\bar{G}$-conjugacy)

$$x = \begin{cases} [J_p^h, I_{d-2-hp}] & \text{if } h < d/p \\ [J_p^{h-2}, J_{p-1}^2] & \text{if } h = d/p > 1 \\ [J_{p-2}] & \text{if } p = d \end{cases}$$

and the desired result quickly follows. (Here J_i denotes a standard Jordan block of size i.) For example, if p divides d and $h = d/p > 1$ then

$$|x^G| > \frac{1}{4}\left(\frac{p}{p+1}\right)p^{\frac{1}{2}(p-1)(2dh-5h-ph^2)}$$

(see [5, 3.21]) and (1) is sufficient unless $(h, p) = (3, 3)$, where direct calculation yields $f(x, H) < .619$. If x is semisimple and r is odd then x is conjugate to $[I_{d-e-h(r-1)}, \omega I_h, \ldots, \omega^{r-1}I_h]$, where $e = 1$ if d and p are coprime, otherwise $e = 2$. Therefore (1) holds and the desired result quickly follows. Finally, let us assume x is a semisimple involution. If d is coprime to p then x is $\mathrm{PO}_{d-1}(K)$-conjugate to $[-I_h, I_{d-1-h}]$; in particular, if d is even and $h \geqslant d/2 - 1$ then

$$|x^G \cap H| \leqslant \frac{d!}{(d/2-1)!2^{d/2}} + \frac{d!}{(d/2)!2^{d/2}} = \frac{(d/2+1)d!}{(d/2)!2^{d/2}}$$

and the bound $|x^G| > \frac{1}{4}(p+1)^{-1}p^{d^2/4-d/2+1}$ is always sufficient. If not, then (1) holds and the result quickly follows. Similar reasoning applies when p divides d. □

2.2 The $\mathscr{B}$ collection

Proposition 2.6. *The conclusion to Theorem 1.1 holds for the collection $\mathscr{B}$.*

Proof. Let $H < G$ be an inclusion in the collection $\mathscr{B}$. For easy reference, we partition the proof into a number of separate lemmas, beginning with the embedding labelled ($\mathscr{B}$2).

Lemma 2.7. *The conclusion to Theorem 1.1 holds for ($\mathscr{B}$2).*

Proof. This embedding is obtained by restricting a spin representation of $G_0 = \mathrm{P}\Omega_8^+(q)$ to the stabilizer of a 1-dimensional non-singular subspace of V. Fix a spin representation ψ of $G_0 = \mathrm{P}\Omega_8^+(q)$ and define $\bar{G} = \mathrm{PSO}_8(K)$, where K is the algebraic closure of $\mathbb{F}_q$. We may assume that G does not contain a triality automorphism (see [5, 3.3]). We claim that

$$f(x, H) \leqslant \frac{\log 672}{\log 2240} \approx .844^* \tag{2}$$

for all elements $x \in H$ of prime order, and hence this case is included in Table 1.1.

Case 1. $p = 2$
Here $H_0 = \mathrm{Sp}_6(q)$ and we define $\bar{H} = \mathrm{Sp}_6(K)$. If $x \in H - \mathrm{PGL}(V)$ is a field automorphism of prime order r then $q = q_0^r$ and (2) follows since [5, 3.43, 3.48] imply that

$$|x^G \cap H| \leqslant |\mathrm{Sp}_6(q) : \mathrm{Sp}_6(q^{1/r})| < 2q^{21\left(1-\frac{1}{r}\right)}, \ \ |x^G| \geqslant |\Omega_8^+(q) : \Omega_8^+(q^{1/r})| > \frac{1}{2}q^{28\left(1-\frac{1}{r}\right)}.$$

If x is an involutory graph-field automorphism then similar bounds hold (with $r = 2$). For the remainder of Case 1 let us assume $x \in H \cap \mathrm{PGL}(V)$ has prime order r. If $r = 2$ then using [7, Table 6] and the proof of [5, 3.22] we obtain the following results:

$\mathrm{Sp}_6(q)$-class	$\mathrm{O}_8^+(q)$-class	$\lvert x^G \cap H\rvert \leqslant$	$\lvert x^G\rvert \geqslant$	$f(x, H) <$
a_2	a_2	$(q^2+1)(q^6-1)$	$(q^2+1)^2(q^6-1)$	$.800^*$
b_1, c_2	a_4	$q^4(q^6-1)$	$q^2(q^4-1)(q^6-1)$	$.840^*$
b_3	c_4	$q^2(q^4-1)(q^6-1)$	$q^2(q^4-1)^2(q^6-1)$	$.753^*$

Now assume r is odd. Let $i \geqslant 1$ be minimal such that r divides $q^i - 1$ and write θ for the natural embedding of $\mathrm{Sp}_6(q)$ in $\Omega_8^+(q)$ as the stabilizer of a 1-dimensional non-singular subspace of the natural module (see [19, 4.1.7]). If $x \in H$ is $\bar{H}$-conjugate to the diagonal matrix $\mathrm{diag}[\mu_1, \mu_2, \mu_3] \in$

$\mathrm{GL}_3 < \bar{H}$ then $\theta(x)$ is $\bar{G}$-conjugate to $\mathrm{diag}[1, \mu_1, \mu_2, \mu_3] \in \mathrm{GL}_4 < \bar{G}$ and using [5, 3.55(iv)] we see that the possibilities for $C_{\bar{H}}(x)$ and $C_{\bar{G}}(x)$ are as follows:

i	$C_{\bar{H}}(x)$	$C_{\bar{G}}(x)$
$6,3$	GL_1^3	$\mathrm{SO}_2 \times \mathrm{GL}_1^3$
4	$\mathrm{Sp}_2 \times \mathrm{GL}_1^2$	GL_2^2
$2,1$	$\mathrm{Sp}_4 \times \mathrm{GL}_1$	GL_4
	$\mathrm{Sp}_2 \times \mathrm{GL}_2$	$\mathrm{SO}_4 \times \mathrm{GL}_2$
	$\mathrm{Sp}_2 \times \mathrm{GL}_1^2$	GL_2^2
	GL_3	$\mathrm{SO}_2 \times \mathrm{GL}_3$ or $\mathrm{GL}_3 \times \mathrm{GL}_1$
	$\mathrm{GL}_2 \times \mathrm{GL}_1$	$\mathrm{SO}_2 \times \mathrm{GL}_2 \times \mathrm{GL}_1$ or $\mathrm{GL}_2 \times \mathrm{GL}_1^2$
	GL_1^3	$\mathrm{SO}_2 \times \mathrm{GL}_1^3$ or GL_1^4

It is now straightforward to check that (2) holds. For instance, when $q = 2$ we obtain the following results. (Here we adopt the notation of [8] for labelling $\mathrm{Sp}_6(2)$-classes.)

$\mathrm{Sp}_6(2)$-class	i	$C_{\bar{H}}(x)$	$C_{\bar{G}}(x)$	$\lvert x^G \cap H\rvert$	$\lvert x^G\rvert$	$f(x,H) <$
$3A$	2	$\mathrm{Sp}_4 \times \mathrm{GL}_1$	GL_4	672	2240	$.844^*$
$3B$	2	GL_3	$\mathrm{SO}_2 \times \mathrm{GL}_3$	2240	89600	$.677^*$
$3C$	2	$\mathrm{Sp}_2 \times \mathrm{GL}_2$	$\mathrm{SO}_4 \times \mathrm{GL}_2$	13440	268800	$.761^*$
$5A$	4	$\mathrm{Sp}_2 \times \mathrm{GL}_1^2$	GL_2^2	48384	580608	$.813^*$
$7A$	3	GL_1^3	$\mathrm{SO}_2 \times \mathrm{GL}_1^3$	207360	24883200	$.719^*$

We conclude that (2) holds, with equality if x belongs to the $\mathrm{Sp}_6(2)$-class $3A$.

Case 2. $p \neq 2$

Here $H_0 = \Omega_7(q)$ and arguing as before we easily deduce that (2) holds if $x \in H - \mathrm{PGL}(V)$. Let us assume $x \in H \cap \mathrm{PGL}(V)$ has prime order r. Recall that if $r = p$ then the *associated partition of* x (with respect to V) is the partition of $n = \dim V$ which encodes the Jordan normal form of x on V (see [5, §3.3]). If λ' (resp. λ) denotes the associated partition of $x \in H$ (resp. $\psi(x) \in G$) then from [7, Table 7] we deduce that the possibilities for λ' and λ are as follows. Here the symbol † (resp. ‡) signifies the condition $p \geqslant 7$ (resp. $p \geqslant 5$).

λ'	$(7)^\dagger$	$(5,1^2)^\ddagger$	$(3^2,1)$	$(3,2^2)$	$(3,1^4)$	$(2^2,1^3)$
λ	$(7,1)$	(4^2)	$(3^2,1^2)$	$(3,2^2,1)$	(2^4)	$(2^2,1^4)$
$f(x,H) <$	$.773^*$	$.818^*$	$.778^*$	$.782^*$	$.835^*$	$.795^*$

We now explain how these bounds are derived. If $p = 7$ and $\lambda = (7,1)$ then [5, 3.18] implies that

$$|x^G \cap H| \leqslant \frac{|\mathrm{O}_7(q)|}{|\mathrm{O}_1(q)|q^3}, \;\; |x^G| \geqslant \frac{1}{2}\frac{|\mathrm{O}_8^+(q)|}{|\mathrm{O}_1(q)|^2q^4}$$

and thus $f(x,H) < .773^*$ as claimed. The case $\lambda = (4^2)$ is similar. For $p = 3$ we require precise values for $|x^G \cap H|$ and $|x^G|$. For example, suppose $\lambda = (2^4)$. First observe that the partition $\lambda' = (3,1^4)$ corresponds to precisely two distinct H_0-classes, represented by x_+ and x_-, where

$$|x_\epsilon^{H_0}| = \frac{|\mathrm{O}_7(q)|}{|\mathrm{O}_4^\epsilon(q)||\mathrm{O}_1(q)|q^5} = \frac{1}{2}q^2(q^2+\epsilon)(q^6-1).$$

In the natural embedding $H_0 \hookrightarrow G_0$, the images of the elements x_+ and x_- represent the two distinct G_0-classes with associated partition $(3,1^5)$. These G_0-classes fuse in $\mathrm{Inndiag}(G_0)$ and

$$|\psi(x_\epsilon)^{G_0}| = |x_\epsilon^{G_0}| = \frac{|\mathrm{O}_8^+(q)|}{|\mathrm{O}_5(q)||\mathrm{O}_1(q)|q^6} = \frac{1}{2}q^2(q^4-1)(q^6-1).$$

(Here $\mathrm{Inndiag}(G_0)$ is the group of *inner-diagonal* automorphisms of G_0, see [11, 2.5.10].) Hence

$$f(x,H) \leqslant \max_{\epsilon=\pm}\left\{\frac{\log|x_\epsilon^{H_0}|}{\log|x_\epsilon^{G_0}|}, \frac{\log(|x_+^{H_0}|+|x_-^{H_0}|)}{\log 2|x_\epsilon^{G_0}|}\right\} < .835^* \qquad (3)$$

for all $q \geqslant 3$. The other bounds are derived in a similar fashion.

Now assume x is a semisimple involution. Here [5, 3.55(iii)] implies that $\psi(x)$ is $\bar{G}$-conjugate to either $[-iI_4, iI_4]$ or $[-I_4, I_4]$, where $i \in K$ satisfies $i^2 = -1$, and the following results hold:

x	$\psi(x)$	$f(x,H) <$
$[-I_2, I_5], [-I_6, I_1]$	$[-iI_4, iI_4]$	$.829^*$
$[-I_4, I_3]$	$[-I_4, I_4]$	$.778^*$

For example, if $\psi(x)$ is conjugate to $[-iI_4, iI_4]$ and $C_G(x)$ is of type $\mathrm{GL}_4^{\epsilon}(q)$ then

$$|x^G \cap H| \leqslant |\mathrm{O}_7(q) : \mathrm{O}_6^{\epsilon}(q)\mathrm{O}_1(q)| + |\mathrm{O}_7(q) : \mathrm{O}_5(q)\mathrm{O}_2^{\epsilon}(q)|, \;\; |x^G| \geqslant \frac{1}{2}|\mathrm{SO}_8^+(q) : \mathrm{GL}_4^{\epsilon}(q)|,$$

and we deduce that $f(x,H) < .829^*$ if $\epsilon = +$ and $f(x,H) < .826^*$ if $\epsilon = -$. For semisimple elements of odd prime order we argue as in Case 1 and the result quickly follows. □

Lemma 2.8. *The conclusion to Theorem 1.1 holds for* $(\mathscr{B}3)$ *and* $(\mathscr{B}4)$*.*

Proof. First consider $(\mathscr{B}4)$. Fix a spin representation which embeds $H_0 = \mathrm{P}\Omega_{10}^+(q)$ in G, where G has socle $G_0 = \mathrm{PSL}_{16}(q)$, and suppose $x \in H$ has prime order. If $x \in H \cap \mathrm{PGL}(V)$ then $|x^G| > \frac{1}{2}q^{95}$ since $\nu(x) \geqslant 4$ (see the proof of [12, 7.5]) and [5, 3.49] implies that the same bound holds if $x \in H - \mathrm{PGL}(V)$. The desired result now follows since $|x^G \cap H| < |H| < 2\log_2 q.q^{45}$.

For the remainder, let us consider the embedding $(\mathscr{B}3)$. This is obtained by restricting a spin representation ψ of $\mathrm{P}\Omega_{10}^+(q)$ to the stabilizer of a 1-dimensional non-singular subspace of the natural $\mathrm{P}\Omega_{10}^+(q)$-module. If $x \in H - \mathrm{PGL}(V)$ has prime order r then x induces a field automorphism on both H_0 and G_0 and therefore [5, 3.43, 3.48] imply that

$$|x^G \cap H| \leqslant |\mathrm{Sp}_8(q) : \mathrm{Sp}_8(q^{1/r})| < 2q^{36\left(1-\frac{1}{r}\right)}, \;\; |x^G| > \frac{1}{4}q^{120\left(1-\frac{1}{r}\right)}.$$

These bounds are always sufficient. Now assume $x \in H \cap \mathrm{PGL}(V)$ has prime order r. If $r = p = 2$ then we easily derive the following bounds:

$\mathrm{Sp}_8(q)$-class	a_2	b_1, c_2	b_3, a_4, c_4
$\mathrm{O}_{16}^+(q)$-class	a_4	a_8	c_8
$f(x,H) <$	.278	.250	.319

If $r = p > 2$ and $\dim x^{\mathrm{SO}_9} > 22$ then the $\mathrm{PSO}_{16}(K)$-class of $\psi(x)$ is given in [7, Table 5] and it is very easy to check that $f(x,H) < 9/16$ for all such elements x. To deal with the remaining classes, we simply extend [7, Table 5] to all unipotent elements of prime order: each remaining class has a representative in a Levi D_4 subgroup of B_4; the spin module restricts to a direct sum of two non-isomorphic spin modules for D_4 and [7, Table 7] applies. In this way we obtain the following results (up to conjugacy):

x	$\psi(x)$	$\dim x^{\mathrm{SO}_9}$	$\dim \psi(x)^{\mathrm{SO}_{16}}$
$[J_3^2, I_3]$	$[J_3^4, I_4]$	22	76
$[J_3, J_2^2, I_2]$	$[J_3^2, J_2^4, I_2]$	20	70
$[J_3, I_6]$	$[J_2^8]$	14	56
$[J_2^4, I_1]$	$[J_3, J_2^4, I_5]$	16	60
$[J_2^2, I_5]$	$[J_2^4, I_8]$	12	44

We leave the reader to check that the subsequent bounds on $|x^G \cap H|$ and $|x^G|$ arising from [5, 3.18] are always sufficient. The argument for semisimple elements is straightforward and left to the reader. □

Lemma 2.9. *The conclusion to Theorem 1.1 holds for the remaining embeddings in* $\mathscr{B}$*.*

Proof. We begin with ($\mathscr{B}1$). Let $x \in H$ be an element of prime order r and observe that $H \cap \mathrm{PGL}(V) \leqslant \mathrm{PGL}_d(q)$. If $x \in H - \mathrm{PGL}(V)$ then [5, 3.49] implies that the trivial bound $|x^G \cap H| < |H| < 2\log_2 q.q^{d^2-1}$ is always sufficient so let us assume $x \in H \cap \mathrm{PGL}(V)$. If $y \in H$ is a long root element (with respect to H_0) then an easy calculation reveals that $\nu(y) = d-2$ (with respect to V) and thus [7, 2.8] implies that $\nu(x) \geqslant d-2$ if $r = p$. Moreover, [5, 3.22] gives $|x^G| > \frac{1}{2}q^{d^3-5d^2+10d-8}$ and the result follows since $|x^G \cap H| < q^{d^2-1}$. Similarly, if x is semisimple then $\nu(x) \geqslant d-1$ (minimal if $\nu(x) = 1$ with respect to the natural H_0-module) and this time the result follows via [5, 3.36].

The other cases are just as easy. For example, consider ($\mathscr{B}7$), so $(G,H) = (\mathrm{SL}_{11}(2), M_{24})$. If $x \in H$ has odd prime order then inspection of the corresponding Brauer character (see [16, p.267]) reveals that $\nu(x) \geqslant 6$, whence $|x^G| > 2^{65}$ and the bound $|x^G \cap H| < |M_{24}|$ is sufficient. Alternatively, if x is an involution then Theorem 2.4 gives $\nu(x) \geqslant 3$, hence $|x^G| > 2^{47}$ and the result follows since $i_2(M_{24}) = 43263$. Similarly, for ($\mathscr{B}8$) we have $(G,H) = (\Omega_{24}^+(2), Co_1)$ and $\nu(x) \geqslant 5$ for all non-trivial elements $x \in H$ (see [12, Table 1]). In particular, if x has odd prime order r then $\nu(x) \geqslant 6$, hence $|x^G| > \frac{1}{3}2^{102}$ and the bound $|x^G \cap H| \leqslant i_r(Co_1)$ is always sufficient, where $i_r(Co_1)$ denotes the number of elements of order r in Co_1. The remaining cases are left to the reader. □

This completes the proof of Proposition 2.6. □

2.3 The $\mathscr{C}$ collection

Proposition 2.10. *The conclusion to Theorem 1.1 holds for the collection $\mathscr{C}$.*

Proof. Let $H < G$ be an inclusion in the collection $\mathscr{C}$. As before, we partition the proof into a number of separate lemmas. We begin by assuming H_0 is a simple group of Lie type in defining characteristic; these are the inclusions labelled ($\mathscr{C}1$)-($\mathscr{C}5$) in Table 2.3. Note that we have already considered the embedding labelled ($\mathscr{C}2$) in Lemma 2.7.

Lemma 2.11. *The conclusion to Theorem 1.1 holds for ($\mathscr{C}1$).*

Proof. Let $\rho : \mathrm{SL}_3^\epsilon(q) \to \mathrm{SL}_6^\epsilon(q)$ be the corresponding irreducible representation and note that q is odd (see Table 2.3). We begin by assuming $x \in H \cap \mathrm{PGL}(V) \leqslant \mathrm{PGL}_3^\epsilon(q)$ has prime order r. Now, if $\nu(x) \geqslant 3$ then [5, 3.38] implies that $|x^G| > \frac{1}{2}(q+1)^{-2}q^{19}$ and in this case the trivial bound $|x^G \cap H| < q^8$ is always sufficient. A straightforward calculation reveals that ρ sends a long root element to the class containing $[J_3, J_2, I_1]$ and thus [7, 2.8] implies that $\nu(x) \geqslant 3$ whenever x is unipotent. In fact, it is easy to see that $\nu(x) \geqslant 3$ for all elements of odd prime order. On the other hand, if $r = 2$ then $\rho(x)$ is conjugate to $[-I_2, I_4]$ and the result follows since $|x^G \cap H| < 2q^4$ and $|x^G| > \frac{3}{8}q^{16}$.

Now assume $x \in H - \mathrm{PGL}(V)$. If x is a field automorphism of prime order r then $q = q_0^r$ and the desired result follows since $|x^G \cap H| < |H| < 2\log_3 q.q^8$ and $|x^G| > \frac{1}{12}q^{35/2}$. The same bounds apply if $\epsilon = +$, $q = q_0^2$ and x is an involutory graph-field automorphism. Finally, if x is an involutory graph automorphism then

$$|x^G \cap H| \leqslant \frac{|\mathrm{PGL}_3^\epsilon(q)|}{|\mathrm{SO}_3(q)|} < 2q^5, \quad |x^G| \geqslant \frac{|\mathrm{PSL}_6^\epsilon(q)|}{|\mathrm{Sp}_6(q)|} > \frac{1}{2}(q+1)^{-1}q^{14}$$

(see [5, Table 3.10]) and the result follows. □

Lemma 2.12. *The conclusion to Theorem 1.1 holds for ($\mathscr{C}3$).*

Proof. Here $q = q_0^3$ and $H_0 = C_{G_0}(\psi)$, where $\psi \in \mathrm{Aut}(G_0)$ is a triality graph-field automorphism (see the proof of [19, 2.3.4]). According to [18, Table 1], the maximality of H in G implies that

$G \cap \mathrm{PGL}(V) = G_0$. We begin by assuming $x \in H - \mathrm{PGL}(V)$ has prime order r. If $r > 3$ then x is a field automorphism of G_0, $q_0 = q_1^r$ and the result follows since

$$|x^G \cap H| \leqslant |{}^3D_4(q_0) : {}^3D_4(q_0^{1/r})| < 2q_0^{28\left(1-\frac{1}{r}\right)}, \;\; |x^G| > \frac{1}{4}q_0^{84\left(1-\frac{1}{r}\right)}.$$

If $r = 2$ or 3 then [21, 1.3] gives

$$|x^G \cap H| \leqslant i_r(\mathrm{Aut}({}^3D_4(q_0))) < 2(q_0+1)q^{15+4\delta_{3,r}} \tag{4}$$

and the desired result follows since [5, 3.49] implies that $|x^G| > \frac{1}{8}q_0^{42}$.

Now suppose $x \in H \cap \mathrm{PGL}(V)$ has prime order r. If $r = p > 2$ and x has associated partition $\lambda = (2^2, 1^4)$ then x lies in the H_0-class labelled A_1 in [26]. Moreover, from [26, p.677] we deduce that

$$|x^G \cap H| = (q_0^2-1)(q_0^8+q_0^4+1), \;\; |x^G| = (q_0^6+1)^2(q_0^{18}-1)$$

and thus $f(x, H) < 1/3$ for all $q_0 \geqslant 3$. On the other hand, if $\lambda \neq (2^2, 1^4)$ then [5, 3.55(i)] implies that $|x^G| > \frac{1}{8}q_0^{48}$ (minimal if $\lambda = (3, 2^2, 1)$) and the desired result follows since $|x^G \cap H| < q_0^{28}$. Next suppose $r = p = 2$. Since x is centralized by ψ, it follows that x is G_0-conjugate to a_2 or c_4 (see [5, 3.55(ii)]), whence $|x^G| > \frac{1}{2}q_0^{30}$ and (4) implies that $f(x, H) < .607$. Similarly, if x is a semisimple involution then $C_G(x)$ is of type $\mathrm{O}_4^+(q)^2$, so $|x^G| > \frac{1}{8}q_0^{48}$ and (4) is always sufficient. Finally, if x is a semisimple element of odd prime order then $\nu(x) \geqslant 4$ since no element with $\nu(x) = 2$ is fixed by ψ (see [5, 3.55(iv)]). For the same reason $C_{\bar{G}}(x)$ is not of type GL_4, whence $\dim x^{\bar{G}} \geqslant 18$ and the subsequent bounds

$$|x^G \cap H| < |{}^3D_4(q_0)| < q_0^{28}, \;\; |x^G| > \frac{1}{2}(q_0^3+1)^{-1}q_0^{57}$$

are always sufficient. □

Lemma 2.13. *The conclusion to Theorem 1.1 holds for ($\mathscr{C}4$) and ($\mathscr{C}5$).*

Proof. First observe that if $p = 3$ then $H_0 = G_2(q)'$ admits an involutory graph automorphism τ which interchanges the two 7-dimensional $G_2(q)$-modules $M(\lambda_1)$ and $M(\lambda_2)$. Therefore we need only consider the embedding ($\mathscr{C}4$) and thus $n = 7 - \delta_{2,p}$. We claim that

$$f(x, H) < .750^* \tag{5}$$

for all elements $x \in H$ of prime order, so this case is included in Table 1.1.

If $x \in H - \mathrm{PGL}(V)$ has prime order r then x is a field automorphism of G_0 and $q = q_0^r$. Applying [5, 3.43] we deduce that

$$|x^G \cap H| \leqslant |G_2(q) : G_2(q_0)| < 2q^{14\left(1-\frac{1}{r}\right)}, \;\; |x^G| \geqslant (2, q-1)^{-1}|\mathrm{Sp}_6(q) : \mathrm{Sp}_6(q_0)| > \frac{1}{4}q^{21\left(1-\frac{1}{r}\right)}$$

and one can check that these bounds are sufficient unless $(r, q) = (2, 4)$, where direct calculation yields $f(x, H) < .670^*$. Let us assume for the remainder that $x \in H \cap \mathrm{PGL}(V) = G_2(q)$ has prime order r.

Case 1. $r = p$

In [20, Table 1], Lawther gives the Jordan normal form on $\bar{V} = V \otimes K$ for representatives of each unipotent class in G_2 (the algebraic group). For $p > 3$, the size of each unipotent class in $G_2(q)$ is given in [15, p.158] and we derive the results in Table 2.4. Here λ denotes the associated partition of x with respect to $\bar{V}$ and we adopt Lawther's notation for labelling the unipotent classes in the algebraic group G_2. The symbol † appearing in the final row denotes the condition $p \geqslant 7$.

Now assume $p \leqslant 3$. Here detailed information on the unipotent classes in $G_2(q)$ is given by Enomoto in [9]. In particular, centralizer orders for unipotent elements are listed in [9,

G_2-class	λ	$\lvert x^G \cap H\rvert \leqslant$	$\lvert x^G\rvert \geqslant$	$f(x,H) <$
A_1	$(2^2,1^3)$	q^6-1	$(q^2+1)(q^6-1)$	$.750^*$
$\widetilde{A}_1$	$(3,2^2)$	$q^2(q^6-1)$	$\frac{1}{2}q^2(q^4-1)(q^6-1)$	$.692^*$
$G_2(a_1)$	$(3^2,1)$	$q^2(q^2-1)(q^6-1)$	$\frac{1}{2}q^3(q-1)(q^4-1)(q^6-1)$	$.743^*$
$G_2{}^{\dagger}$	(7)	$q^4(q^2-1)(q^6-1)$	$\frac{1}{2}q^6(q^2-1)(q^4-1)(q^6-1)$	$.680^*$

Table 2.4: $(\mathscr{C}4)$, $r=p>3$

p	G_2-class	G-class	$\lvert x^G \cap H\rvert \leqslant$	$\lvert x^G\rvert \geqslant$	$f(x,H) <$
3	A_1	$(2^2,1^3)$	q^6-1	$(q^2+1)(q^6-1)$	$.750^*$
	$\widetilde{A}_1$, $\widetilde{A}_1^{(3)}$	$(3,2^2)$	$q^2(q^6-1)$	$\frac{1}{2}q^2(q^4-1)(q^6-1)$	$.705^*$
	$G_2(a_1)$	$(3^2,1)$	$\frac{1}{2}q^2(q^2-1)(q^6-1)$	$\frac{1}{2}q^3(q-1)(q^4-1)(q^6-1)$	$.714^*$
2	A_1	a_2	q^6-1	$(q^2+1)(q^6-1)$	$.750^*$
	$\widetilde{A}_1$	b_3	$q^2(q^6-1)$	$q^2(q^4-1)(q^6-1)$	$.672^*$

Table 2.5: $(\mathscr{C}4)$, $r=p\leqslant 3$

Tables 1-2] and using this data, together with [20, Table 1], we derive the results listed in Table 2.5. The fourth row in Table 2.5 is worth noting. Here $p=3$ and there are precisely two distinct $G_2(q)$-classes corresponding to the G_2-class $G_2(a_1)$, with representatives x_+ and x_-, where $|C_{G_2(q)}(x_\epsilon)| = 2q^4$ and

$$|x_\epsilon^{\Omega_7(q)}| = \frac{|\mathrm{O}_7(q)|}{|\mathrm{O}_2^\epsilon(q)||\mathrm{O}_1(q)|q^6} = \frac{1}{2}q^3(q-\epsilon)(q^4-1)(q^6-1),$$

i.e. the elements x_+ and x_- represent the two distinct $\Omega_7(q)$-classes with associated partition $\lambda=(3^2,1)$. The entries in the fourth row follow immediately. As a final remark, we note that (5) is best possible since

$$\lim_{q\to\infty}\frac{\log[q^6-1]}{\log[(q^2+1)(q^6-1)]} = \frac{3}{4}.$$

Case 2. $r \neq p$

If $q<5$ then the values of the associated Brauer character are listed in [16] and we can compute precise values for $f(x,H)$. Indeed, the reader can check that $f(x,H) < 1/2+1/n$ with the exception of the cases listed in Table 2.6.

For $q \geqslant 5$, we follow the proof of [7, 7.7]. By replacing x by a suitable conjugate we may assume that $x \in A_2.2 < G_2$ (as algebraic groups over $K=\bar{\mathbb{F}}_q$) where A_2 is generated by long root subgroups and

$$V\downarrow A_2 = \begin{cases} V_3 \oplus V_3^* & \text{if } p=2 \\ V_3\oplus V_3^* \oplus 0 & \text{if } p\neq 2\end{cases} \tag{6}$$

q	$G_2(q)$-class of x	$f(x,H) <$
4	$3B$	$.717^*$
	$5A, 5B$	$.719^*$
	$13A, 13B$	$.684^*$
3	$2A$	$.701^*$
	$7A$	$.680^*$
	$13A, 13B$	$.646^*$
2	$3B$	$.685^*$

Table 2.6: $(\mathscr{C}4)$, $r\neq p$

H_0-class of x	G_0-class of x	$\lvert x^G \cap H\rvert$	$\lvert x^G\rvert$	$f(x,H) <$	
$2A$, $2E$	$2D$	58275	14926275	.665*	
$(2B, 2C, 2D)$	$(2A, 2B, 2C)$	$3^{\zeta}.3780$	$3^{\zeta}.189540$	.678*	(.705*)
$2F$	$2E$	120	1080	.686*	
$2G$	$2F$	37800	7960680	.664*	
$(5A, 5B, 5C)$	$(5A, 5B, 5C)$	$3^{\zeta}.580608$	$3^{\zeta}.2751211008$	.611	(.630*)
$7A$	$7A$	24883200	176863564800	.658*	

Table 2.7: ($\mathscr{C}19$), $r \neq p$, $p = 3$

where V_3 and 0 denote the natural and trivial A_2-modules respectively. First assume r is odd. If $x \in G_0$ is a regular semisimple element then (6) implies that x is also regular as an element of $G_2(q)$ and thus $f(x,H) \leqslant \log_b a$, where

$$a = f.\frac{|G_2(q)|}{(q-1)^2}, \quad b = f.\frac{|\mathrm{Sp}_6(q)|}{(q+1)^3}$$

and $f = \log_p q$. This yields $f(x,H) < .704^*$ for all $q \geqslant 5$. If we assume x is not regular then using (6) we calculate that $f(x,H)$ is maximal if $x = [\mu, \mu^{-1}, 1] \in A_2$ for some $1 \neq \mu \in K$. Here $C_{G_2}(x) = A_1T_1$ and we calculate that $f(x,H) < .732^*$ for all $q \geqslant 5$ since $f(x,H) \leqslant \log_\beta \alpha$, where

$$\alpha = f.\frac{|G_2(q)|}{|\mathrm{GL}_2(q)|}, \quad \beta = f.\frac{|\mathrm{Sp}_6(q)|}{|\mathrm{Sp}_2(q)||\mathrm{GU}_2(q)|}.$$

Finally, suppose x is a semisimple involution. Now, there is a unique class of involutions in $G_2(q)$ and (6) implies that x acts on V as $[-I_4, I_3]$. Therefore

$$|x^G \cap H| = q^4(q^4+q^2+1), \quad |x^G| = \frac{|\mathrm{O}_7(q)|}{|\mathrm{O}_4^+(q)||\mathrm{O}_3(q)|} = \frac{1}{2}q^6(q^2+1)(q^4+q^2+1)$$

(since $x \in G_0$) and we conclude that $f(x,H) < .691^*$ for all $q \geqslant 5$. □

To complete the proof of Proposition 2.10, let us assume H_0 is not a simple group of Lie type in defining characteristic. These are the cases labelled ($\mathscr{C}6$)-($\mathscr{C}22$) in Table 2.3.

Lemma 2.14. *The conclusion to Theorem 1.1 holds for* ($\mathscr{C}19$).

Proof. Here $H_0 = \Omega_8^+(2)$, $G_0 = \mathrm{P}\Omega_8^+(p)$ and the embedding $\rho : H \to G$ corresponds to the reduction modulo p of the complex 8-dimensional representation of H_0 which arises from the natural action of the Weyl group $W(E_8) = 2.\Omega_8^+(2).2$ on a maximal torus $T_8 < E_8$, where the algebraic group E_8 is defined over the complex numbers. Now $H \cap \mathrm{PGL}(V) \leqslant \mathrm{O}_8^+(2)$ and we claim that

$$f(x,H) < \begin{cases} .706^* & \text{if } p = 3 \\ .500 & \text{if } p \geqslant 5 \end{cases} \tag{7}$$

for all elements $x \in H$ of prime order. In particular, the case $p = 3$ is an exception to the main statement of Theorem 1.1 and is therefore listed in Table 1.1. To justify (7), let us begin by assuming $x \in H \cap \mathrm{PGL}(V)$ has prime order r. Throughout, K denotes the algebraic closure of $\mathbb{F}_p$ and $\bar{G} = \mathrm{PSO}_8(K)$.

Case 1. $x \in H \cap \mathrm{PGL}(V)$, $r \neq p$

For semisimple elements, we can use the values of the associated Brauer character χ to compute precise values for $f(x,H)$. The values of χ are listed in [16] for $p \leqslant 7$ and in [8] for $p > 7$. When $p = 3$ we obtain the results displayed in Table 2.7. Here we adopt the standard Atlas notation for labelling class representatives (see [8]) and we set $\zeta = 1$ if G contains a triality automorphism, otherwise $\zeta = 0$.

p	λ	$\lvert x^G \cap H\rvert$	$\lvert x^G\rvert \geqslant$	$f(x,H) <$	
3	(2^4), $(3,1^5)$	$3^\zeta.2240$	$3^\zeta.262080$	.619	(.650*)
	$(3,2^2,1)$	89600	10483200	.706*	
	$(3^2,1^2)$	268800	377395200	.634*	
5	(4^2), $(5,1^3)$	$3^\zeta.580608$	$3^\zeta.47528208000000$	.422	(.441)
7	$(7,1)$	24883200	93756760664555520000	.371	

Table 2.8: ($\mathscr{C}19$), $r = p$

The third row of Table 2.7 merits some explanation. The notation here indicates that if $x \in H_0$ is H_0-conjugate to $2B$ (resp. $2C$) then $\rho(x)$ is G_0-conjugate to $2A$ (resp. 2B) and so on. In particular, if $x \in \{2B, 2C, 2D\}$ then $f(x,H) < .678^*$ if $\zeta = 0$, otherwise $f(x,H) < .705^*$. Similar notation applies for $x \in \{5A, 5B, 5C\}$. We conclude that (7) holds when $p = 3$. For $p \geqslant 5$, we can do entirely similar calculations and the reader can check that $f(x,H) < 1/2$. In particular, we note that if $p \geqslant 5$ and h is H_0-conjugate to $3E$ then $\chi(\rho(h)) = 2$ and thus $\rho(h)$ is $\bar{G}$-conjugate to $[I_4, \omega I_2, \omega^2 I_2]$, where $\omega \in K$ is a primitive cube root of unity.

Case 2. $x \in H \cap \mathrm{PGL}(V)$, $r = p$
In this case Lagrange's Theorem implies that $p \in \{3,5,7\}$ and we derive the results presented in Table 2.8. Here λ denotes the associated partition of $x \in G$ and ζ is defined as before. Note that we only list those partitions λ for which $x^G \cap H$ is non-empty. We now explain how we derive the results in Table 2.8. First observe that a triality graph automorphism τ of G_0 induces a triality automorphism on H_0. If x is H_0-conjugate to $3A$ then $x \in A_9 < \Omega_8^+(2)$ has cycle-shape $(3,1^6)$ (see ($\mathscr{A}4$) in Table 2.1). Without loss, we may assume that the restriction of ρ to A_9 factors through $\Omega_7(3)$ as follows

$$\rho : A_9 \xrightarrow{\rho_1} \Omega_7(3) \xrightarrow{\rho_2} \mathrm{P}\Omega_8^+(3), \tag{8}$$

where ρ_1 is the irreducible representation afforded by the fully deleted permutation module for A_9 over $\mathbb{F}_3$ (see ($\mathscr{A}1$) in Table 2.1) and ρ_2 is the restriction of a spin representation of $\mathrm{P}\Omega_8^+(3)$ (see ($\mathscr{B}2$) in Table 2.2). From the proofs of Proposition 2.5 and Lemma 2.7 we deduce that $\rho(x)$ acts on V with associated partition $\lambda = (2^4)$ and thus [5, 3.55(i)] implies that $\rho(x)$ and $\rho(x)^\tau$ belong to distinct G_0-classes. The three H_0-classes $\{3A, 3B, 3C\}$ are permuted by a triality automorphism of H_0 (see [5, 3.55(iv)]) and therefore they are fused in G if and only if G contains a triality automorphism. This explains the entries in the second row and the case $p = 5$ is entirely similar. The entries in rows 3 and 4 are also derived via (8). Finally, if $p = 7$ then $x \in A_9 < \Omega_8^+(2)$ has cycle-shape $(7,1^2)$ and we may assume that the restriction of ρ to A_9 is the irreducible representation afforded by the fully deleted permutation module for A_9 over $\mathbb{F}_7$. Then the proof of Proposition 2.5 implies that $\lambda = (7,1)$ and the entries in the final row follow at once.

Case 3. $x \in H - \mathrm{PGL}(V)$
Here x is a triality graph automorphism since G_0 is defined over the prime field. As previously stated, x induces a triality graph automorphism on H_0 and if we assume $p > 7$ then the obvious bounds (see [5, Table 3.10])

$$|x^G \cap H| \leqslant 2|\Omega_8^+(2) : G_2(2)| + 2|\Omega_8^+(2) : \mathrm{PGU}_3(2)| = 1641600, \quad |x^G| > \frac{1}{8}p^{14}$$

are always sufficient. Now assume $p \in \{5,7\}$. Let τ be a G_2-type triality automorphism of H_0 (see [5, 3.47]) and observe that Lagrange's Theorem implies that $C_{G_0}(\tau) = G_2(p)$. According to [11, p.215], the four distinct H_0-classes of triality automorphisms in $\mathrm{Aut}(H_0)$ are represented by the elements $\tau^\pm$ and $(h\tau)^\pm$, where $h \in H_0$ lies in the H_0-class $3E$ and $[h,\tau] = 1$. As remarked in Case 1, $\rho(h)$ is $\bar{G}$-conjugate to $[I_4, \omega I_2, \omega^2 I_2]$, i.e. $\rho(h)$ lies in the G_0-class $3E$ and thus $h\tau$ is a non-G_2 triality of G_0. We conclude that the centralizers $C_{H_0}(x)$ and $C_{G_0}(x)$ are of the same type. In particular, if x is a G_2-type triality then $f(x,H) < .486$ since

$$|x^G \cap H| = 2^{\xi}|\Omega_8^+(2) : G_2(2)| = 2^{\xi}.14400, \quad |x^G| > 2^{\xi-3}p^{14}$$

(see [5, 3.48]) where $\xi = 1$ if G contains an involutory graph automorphism, otherwise $\xi = 0$. In the same way we deduce that $f(x,H) < 1/2$ if x is a non-G_2 triality. Finally, let us assume $p = 3$. Now, if $y \in x^G \cap H$ and $C_{H_0}(y) = G_2(2)$ then $C_{G_0}(y) = G_2(3)$ since $|G_2(2)| > 3^5|\mathrm{SL}_2(3)|$. Furthermore, if $C_{H_0}(y) = \mathrm{PGU}_3(2)$ and $C_{G_0}(y) = G_2(3)$ then $|x^{G_0} \cap H_0| \geqslant |\Omega_8^+(2) : \mathrm{PGU}_3(2)| = 806400$ and thus $\mathrm{fpr}(x) > .691$ since $|x^{G_0}| = |\mathrm{P}\Omega_8^+(3) : G_2(3)| = 1166400$. This contradicts [23, Theorem 1] (see [4, (2)]) and thus $C_{H_0}(x)$ and $C_{G_0}(x)$ are of the same type. In particular, if x is a G_2-type triality then $f(x,H) < .701^*$ since $|x^G \cap H| = 2^{\xi}.14400$ and $|x^G| = 2^{\xi}.1166400$. Similarly, if x is a non-G_2 triality then $f(x,H) < .673^*$. □

Lemma 2.15. *The conclusion to Theorem 1.1 holds for ($\mathscr{C}$18).*

Proof. Here $H = \mathrm{Sp}_6(2) \leqslant \mathrm{PGL}(V)$ and $\rho : \mathrm{Sp}_6(2) \to \Omega_7(p)$ is the restriction of the map in ($\mathscr{C}$19). More precisely, ρ factors through $\Omega_8^+(2)$ as follows:

$$\rho : \mathrm{Sp}_6(2) \xrightarrow{\rho_1} \Omega_8^+(2) \xrightarrow{\rho_2} \mathrm{P}\Omega_8^+(p), \tag{9}$$

where ρ_1 is the restriction of a spin representation (see ($\mathscr{B}$2) in Table 2.2) and ρ_2 is the embedding ($\mathscr{C}$19). Let $x \in H$ be an element of prime order r. We claim that

$$f(x,H) < \begin{cases} .707^* & \text{if } p = 3 \\ .500 & \text{if } p \geqslant 5 \end{cases}$$

and thus the case $p = 3$ appears in Table 1.1.

Let $x \in H$ be an element of prime order r and let χ be the associated Brauer character. Since χ is given in [8, 16], we can compute precise values for $f(x,H)$ when $r \neq p$. For example, when $p = 3$ we derive the following results:

$\mathrm{Sp}_6(2)$-class of x	$\Omega_7(3)$-class of x	$\lvert x^G \cap H\rvert$	$\lvert x^G\rvert$	$f(x,H) <$
$2A$	$2A$	63	351	.707*
$2B$, $2D$	$2C$	4095	331695	.655*
$2C$	$2B$	945	22113	.685*
$5A$	$5A$	48384	38211264	.618
$7A$	$7A$	207360	327525120	.625

We can do entirely similar calculations when $p \geqslant 5$ and the bound $f(x,H) < 1/2$ quickly follows. Now assume $r = p$. Here $r \in \{3, 5, 7\}$ and in view of (9) and our earlier work we derive the following results, where λ denotes the associated partition of $\rho(x) \in G$.

p	$\mathrm{Sp}_6(2)$-class of x	λ	$\lvert x^G \cap H\rvert$	$\lvert x^G\rvert \geqslant$	$f(x,H) <$
3	$3A$	$(3,1^4)$	672	26208	.640
	$3B$	$(3,2^2)$	2240	262080	.619
	$3C$	$(3^2,1)$	13440	1572480	.667*
5	$5A$	$(5,1^2)$	48384	30466800000	.447
7	$7A$	(7)	207360	797251366195200	.357

For example, suppose x is $\mathrm{Sp}_6(2)$-conjugate to $3A$, i.e. $x = [I_4, \omega, \omega^2]$, where $\omega \in K$ is a primitive cube root of unity. Then the proof of Lemma 2.7 implies that $y = \rho_1(x)$ is $\mathrm{O}_8^+(2)$-conjugate to $[\omega I_4, \omega^2 I_4]$, the proof of Lemma 2.14 gives $\rho_2(y) = [J_3, I_5]$ and thus $\lambda = (3, 1^4)$. The other results are obtained in a similar fashion. □

Lemma 2.16. *The conclusion to Theorem 1.1 holds for ($\mathscr{C}$16) and ($\mathscr{C}$21).*

Proof. Both of these cases can be analysed using GAP [10]. For the embedding ($\mathscr{C}16$) we find that $f(x,H) \leqslant (\log 666)/(\log 6336) \approx .743^*$, with equality possible if x lies in the G_0-class $2D$. Therefore this case is recorded in Table 1.1. For ($\mathscr{C}21$) we calculate that $f(x,H) < .663$ for all elements $x \in H$ of prime order. □

Lemma 2.17. *The conclusion to Theorem 1.1 holds for ($\mathscr{C}11$).*

Proof. Here $H_0 = \mathrm{SU}_3(3)$ and $G_0 = \mathrm{PSp}_6(p)$ with $p \neq 3$. If $p = 2$ then using GAP [10] we deduce that $f(x,H) \leqslant (\log 63)/(\log 315) \approx .721^*$ for all elements x of prime order, with equality if x is G-conjugate to a_2. In particular, the case $p = 2$ is listed in Table 1.1. Now assume $p \geqslant 5$. Let $x \in H$ be an element of prime order r. We claim that $f(x,H) < 1/2$. If $x \in H_0$ is an involution then x is G-conjugate to $[-I_4, I_2]$, whence $|x^G \cap H| = 63$, $|x^G| = p^4(p^4+p^2+1)$ and thus $f(x,H) < .321$. The values of the corresponding Brauer character imply that $\nu(x) \geqslant 3$ for all other elements $x \in H$ of prime order $r \neq p$, so [5, 3.36] gives $|x^G| > \frac{1}{4}(p+1)^{-1}p^{13}$ and the bound $|x^G \cap H| \leqslant i_r(H)$ is sufficient. Finally, if $r = p$ then Lagrange's Theorem implies that $p = 7$ and we deduce that $f(x,H) < .401$ since $|x^G \cap H| \leqslant i_7(H) = 1728$ and

$$|x^G| \geqslant \frac{|\mathrm{Sp}_6(7)|}{|\mathrm{Sp}_2(7)||\mathrm{O}_2^-(7)|7^7} = 123530400$$

since [17, Theorem II] implies that x is not a long root element. □

Lemma 2.18. *The conclusion to Theorem 1.1 holds for the remaining embeddings in $\mathscr{C}$.*

Proof. In each of the remaining cases we claim that

$$f(x,H) < \frac{1}{2} + \frac{1}{n} \tag{10}$$

for all elements $x \in G$ of prime order, where n is defined as in Remark 1.2. Let V denote the natural G_0-module, write χ for the Brauer character of the corresponding representation $\rho : \widehat{H}_0 \to \mathrm{GL}(V)$ and let $x \in H$ be an element of prime order r. If $x \in H - \mathrm{PGL}(V)$ then the reader can check that the bound $|x^G \cap H| \leqslant i_r(\mathrm{Aut}(H_0) - H_0)$, with the lower bound on $|x^G|$ from [5, 3.49], is always sufficient.

Now suppose $x \in H \cap \mathrm{PGL}(V)$. If $r \neq p$ then $\chi(x)$ is listed in [8, 16] and (10) is easily checked. Now assume $r = p$ and let λ denote the associated partition of x with respect to V. Now [17] implies that x is not a long root element, i.e. $\lambda \neq (2^2, 1^{n-4})$ if G_0 is orthogonal, otherwise $\lambda \neq (2, 1^{n-2})$. From this observation we derive a lower bound for $|x^G|$ and we find that the upper bound $|x^G \cap H| \leqslant i_r(H \cap \mathrm{PGL}(V))$ is always sufficient. For instance, in ($\mathscr{C}14$) we have $H_0 = \mathrm{SU}_4(2)$, $n = 4$ and Lagrange's Theorem implies that $p = 5$. Moreover, since x is not a long root element, we have

$$|x^G| \geqslant \frac{|\mathrm{O}_6^-(5)|}{|\mathrm{O}_3(5)||\mathrm{O}_1(5)|5^4} = 196560$$

and we conclude that $f(x,H) < .702$ since $i_5(H) = 5184$. □

This completes the proof of Proposition 2.10. □

2.4 The remaining cases

Now assume that the irreducible embedding of H in G is not in $\mathscr{A}$, $\mathscr{B}$ or $\mathscr{C}$. Then Theorems 2.2 and 2.4 apply and we use work of Lübeck [25] and Hiss-Malle [14] to quickly reduce to a small collection of irreducible embeddings which we can deal with on a case-by-case basis. This is the collection $\mathscr{D}$ (see Tables 2.10 and 2.11). Of course, Theorem 2.4 only applies if $\dim V \geqslant 6$ and the remaining cases are considered in Proposition 2.22.

G_0	$\mathrm{PSL}_n(q)$	$\mathrm{PSU}_n(q)$	$\mathrm{PSp}_n(q)$	$\mathrm{P}\Omega_n^\epsilon(q)$
N	10	64	64	64

Table 2.9: The values $N = N(G_0)$

Proposition 2.19. *If* $\dim V \geqslant 6$ *and the inclusion* $H < G$ *is not a member of one of the collections* $\mathscr{A}$, $\mathscr{B}$ *or* $\mathscr{C}$ *then the conclusion to Theorem 1.1 holds.*

Proof. First observe that Theorem 2.2 implies that

$$|x^G \cap H| < |H| < \begin{cases} q^{4n+8} & \text{if } G_0 = \mathrm{PSU}_n(q) \\ q^{2n+4} & \text{otherwise.} \end{cases} \tag{11}$$

If $x \in H \cap \mathrm{PGL}(V)$ has prime order then Theorem 2.4 and [5, 3.38] imply that $|x^G| > g(n,q)$ for some function g. For example, if $G_0 = \mathrm{PSp}_n(q)$ then

$$g(n,q) = \frac{1}{8}(q+1)^{-1}\max(q^{\alpha(n-\alpha)+1}, q^{3n-8}),$$

where $\alpha = \lceil \frac{1}{2}\sqrt{n}\rceil + \beta$ and $\beta = 1$ if $n = 4m^2$ for some $m \in \mathbb{N}$, otherwise $\beta = 0$. Let $N = N(G_0)$ be the smallest integer such that the bounds (11) and $|x^G| > g(n,q)$ imply that $f(x,H) < 1/2 + 1/n$ for all $n \geqslant N$ and all values of q. Then N is given in Table 2.9. If $x \in H - \mathrm{PGL}(V)$ then lower bounds for $|x^G|$ are given in [5, 3.49] and it is easy to check that (11) is always sufficient if $n \geqslant N(G_0)$. For the remainder we may assume $6 \leqslant n < N$.

Case 1. *H_0 is a simple group of Lie type in defining characteristic*

Let $\rho : \widehat{H}_0 \to \mathrm{GL}(V)$ be the absolutely irreducible representation corresponding to the embedding $H < G$, where $\widehat{H}_0$ is the full covering group of H_0. Assume to begin with that $H_0 = \mathrm{PSL}_2(q')$, where q' is a power of p. Then G_0 is either symplectic or orthogonal and [19, 5.4.6(i)] implies that $q' = q^i$ for some $i \geqslant 1$ and that $n = l^i \geqslant 2^i$, where l is the dimension of some irreducible $K\widehat{H}_0$-module and K is the algebraic closure of $\mathbb{F}_q$. Of course

$$|x^G \cap H| < |\mathrm{Aut}(H_0)| \leqslant i\log_2 q.q^i(q^{2i}-1) \tag{12}$$

and applying Theorem 2.4 and [5, 3.38, 3.49] we deduce that

$$|x^G| > \frac{1}{8}(q+1)^{-1}q^{3n-11} \tag{13}$$

for all elements $x \in H$ of prime order. One can check that these bounds are sufficient unless $(i,n) \in \{(3,8),(1,6)\}$. Suppose $(i,n) = (3,8)$. If $p = 2$ then $G_0 = \Omega_8^+(q)$ and H is not maximal (see [18]) so we may assume p is odd and thus $G_0 = \mathrm{PSp}_8(q)$. If x has odd prime order then the desired result follows by applying (12) since Theorem 2.4 gives $\nu(x) \geqslant 3$ and thus $|x^G| > \frac{1}{4}q^{18}$ (minimal if x is unipotent with associated partition $\lambda = (2^3, 1^2)$); on the other hand, if x is an involution then [5, 3.37] gives $|x^G| > \frac{1}{4}q^{16}$ and one can easily check that the bound

$$|x^G \cap H| \leqslant i_2(\mathrm{Aut}(\mathrm{PSL}_2(q^3))) < 2(1+q^{-3})q^6$$

(see [5, 3.14]) is always sufficient. Now assume $(i,n) = (1,6)$. Here p must be odd since n is divisible by 3. Now, if $G_0 = \mathrm{PSp}_6(q)$ then $|x^G| > \frac{1}{4}(q+1)^{-1}q^{13}$ and (12) is sufficient. If G_0 is orthogonal then we require $f(x,H) < 3/4$ (see Remark 1.2) and we find that the above bounds (12) and (13) are sufficient for all $q \geqslant 7$. If $q < 7$ then $H_0 \cong A_5$ since $\mathrm{PSL}_2(3)$ is not simple. However, we can rule out this case since neither A_5 nor $2.A_5$ admits an irreducible representation of degree 6 in characteristic 2 or 5.

Now assume $H_0 \neq \mathrm{PSL}_2(q')$. Here we apply Lübeck's work [25, Tables A.6-48] on the small degree irreducible representations of simple groups of Lie type. To illustrate the method, suppose

	H_0	G_0	representation of H_0
$(\mathscr{D}1)$	$\mathrm{PSL}_3^{\epsilon}(q)$	$\begin{cases} \Omega_7(q) & p=3 \\ \mathrm{P}\Omega_8^+(q) & p\neq 3 \end{cases}$	V_{adj}
$(\mathscr{D}2)$	$\mathrm{PSL}_4^{\epsilon}(q)$ $p=2$	$\Omega_{14}^{\epsilon'}(q)$	V_{adj}
$(\mathscr{D}3)$	$\mathrm{PSU}_5(q)$	$\mathrm{PSU}_{10}(q)$	$\bigwedge^2 V_5$
$(\mathscr{D}4)$	$\mathrm{PSL}_6^{\epsilon}(q)$	$\begin{cases} \mathrm{PSp}_{20}(q) & p>2 \\ \Omega_{20}^{\epsilon'}(q) & p=2 \end{cases}$	$\bigwedge^3 V_6$
$(\mathscr{D}5)$	$\mathrm{PSp}_4(q)$	$\mathrm{P}\Omega_{10}^{\epsilon}(q)$	$M(2\lambda_2)$
$(\mathscr{D}6)$	$\mathrm{PSp}_4(q^2)$	$\mathrm{P}\Omega_{16}^+(q)$	$M(\lambda_1)\otimes M(\lambda_1)^{(1)}$
$(\mathscr{D}7)$	$\mathrm{PSp}_6(q)$	$\begin{cases} \mathrm{P}\Omega_{14}^{\epsilon}(q) & p\neq 3 \\ \Omega_{13}(q) & p=3 \end{cases}$	$M(\lambda_2)$
$(\mathscr{D}8)$	$\mathrm{PSp}_6(q)$ $p>2$	$\mathrm{PSp}_{14}(q)$	$M(\lambda_3)$
$(\mathscr{D}9)$	$\mathrm{PSp}_8(q)$	$\begin{cases} \Omega_{27}(q) & p>2 \\ \Omega_{26}^{\epsilon}(q) & p=2 \end{cases}$	$M(\lambda_2)$
$(\mathscr{D}10)$	$\mathrm{P}\Omega_8^-(q_0)$ $q=q_0^2$	$\mathrm{P}\Omega_8^+(q)$	one of the two spin representations
$(\mathscr{D}11)$	$\begin{cases} \Omega_{11}(q) & p>2 \\ \mathrm{Sp}_{10}(q) & p=2 \end{cases}$	$\begin{cases} \mathrm{PSp}_{32}(q) & p>2 \\ \Omega_{32}^+(q) & p=2 \end{cases}$	spin representation
$(\mathscr{D}12)$	$\mathrm{P}\Omega_{12}^+(q)$	$\begin{cases} \mathrm{PSp}_{32}(q) & p>2 \\ \Omega_{32}^+(q) & p=2 \end{cases}$	one of the two spin representations
$(\mathscr{D}13)$	$F_4(q)$	$\begin{cases} \mathrm{P}\Omega_{26}^+(q) & p\neq 3 \\ \Omega_{25}(q) & p=3 \end{cases}$	$M(\lambda_1)$
$(\mathscr{D}14)$	$F_4(q)$ $p=2$	$\Omega_{26}^+(q)$	$M(\lambda_4)$
$(\mathscr{D}15)$	${}^2E_6(q)$	$\mathrm{PSU}_{27}(q)$	$M(\lambda_1)$ or $M(\lambda_6)$

Table 2.10: The collection $\mathscr{D}$, I

$H_0 = \mathrm{PSL}_m(q')$, where $m \geqslant 3$. Then [19, 5.4.6(i)] implies that $q' = q^i$ and $n = l^{i/u} \geqslant m^{i/u}$ for some integer $i \geqslant 1$, where l is the dimension of an irreducible $K\widehat{H}_0$-module and $u = 2$ if G_0 is unitary, otherwise $u = 1$. In particular, i is even if G_0 is unitary and we have

$$|x^G \cap H| < |\mathrm{Aut}(H_0)| < 2i\log_2 q.q^{i(m^2-1)}. \tag{14}$$

Now, if ρ is self-dual then G_0 is symplectic or orthogonal, so (13) holds and we may assume $n < N(G_0) = 64$. Using [25] we calculate that (14) is sufficient unless ρ is one of the embeddings labelled $(\mathscr{D}1)$, $(\mathscr{D}2)$ or $(\mathscr{D}4)$ in Table 2.10. If ρ is not self-dual and $G_0 = \mathrm{PSL}_n(q)$ then we may assume $n < 10$ and we find that (14) is always sufficient since Theorem 2.4 and [5, 3.38] imply that $|x^G| > \frac{1}{2}q^{6n-19}$. Finally, if ρ is not self-dual and $G_0 = \mathrm{PSU}_n(q)$ then i is even, $|x^G| > \frac{1}{2}(q+1)^{-1}q^{6n-18}$ and close inspection of [25] reveals that (14) is always sufficient.

Proceeding in this way in each of the other cases, we find that we are left to deal with the set of inclusions listed in Table 2.10. Here $M(\lambda)^{(l)}$ denotes the twisted module $M(\lambda)^{\phi^l}$, where ϕ is a field automorphism of $\widehat{H}_0$ induced by the map $\mu \mapsto \mu^p$ on field elements. We write V_{adj} for the non-trivial composition factor of the adjoint module for H_0.

Lemma 2.20. *The conclusion to Theorem 1.1 holds for the embeddings in Table 2.10.*

Proof. Let K denote the algebraic closure of $\mathbb{F}_q$ and let $\bar{G}$ be a simple classical algebraic group over K of adjoint type with the property that there exists a Frobenius morphism σ of $\bar{G}$ such that $\bar{G}_\sigma$ has socle G_0. Let ρ be the absolutely irreducible representation corresponding to the embedding of H in G. We claim that (10) holds for all elements $x \in G$ of prime order.

Case i. *The irreducible embedding* $(\mathscr{D}1)$
Here ρ is the representation afforded by the non-trivial composition factor of the adjoint module for $\mathrm{SL}_3^{\epsilon}(q)$. If $p \neq 3$ then $q \equiv \epsilon\,(3)$ (see [18, 2.3.3]) and therefore we may assume $q > 2$. Let

$\gamma \in \mathrm{Aut}(H_0)$ be the inverse-transpose graph automorphism and observe that γ acts on V since ρ has highest weight $\lambda = \lambda_1 + \lambda_2$ and this is fixed by γ with respect to the induced action on the set of weights. Therefore $H \cap \mathrm{PGL}(V) \leqslant \mathrm{PGL}_3^{\epsilon}(q).\langle\gamma\rangle = \widetilde{H}$. We also note that if $p \neq 3$ then $\widetilde{H}$ is the centralizer in $\mathrm{PGO}_8^+(q)$ of a non-G_2 triality graph automorphism (see [5, 3.47]).

Let $x \in H \cap \mathrm{PGL}(V)$ be an element of prime order r, write $\bar{H} = \mathrm{PSL}_3(K)$ and choose unipotent $\bar{H}$-class representatives $u_1 = [J_2, I_1]$ and $u_2 = [J_3]$. If $r = p$ then a straightforward calculation with the adjoint representation reveals that representatives for the $\mathrm{PO}_n(K)$-class of $\rho(x)$ can be chosen as follows

	$p \geqslant 5$	$p = 3$	$p = 2$
$x = u_1$	$[J_3, J_2^2, I_1]$	$[J_3, J_2^2]$	c_4
u_2	$[J_5, J_3]$	$[J_3^2, I_1]$	$-$

and the desired result follows via [5, 3.18, 3.22]. Similarly, if $r = 2$ then $\rho(x)$ is given as follows (up to conjugacy)

	$p \geqslant 5$	$p = 3$	$p = 2$
$x = [-I_2, I_1]$	$[-I_4, I_4]$	$[-I_4, I_3]$	$-$
γ	$[-I_5, I_3]$	$[-I_4, I_3]$	b_3

and it is easy to check that (10) holds. For example, if $p = 3$ and x is conjugate to $[-I_4, I_3]$ then

$$|x^G \cap H| \leqslant \frac{|\mathrm{PGL}_3^{\epsilon}(q)|}{|\mathrm{SO}_3(q)|} + \frac{|\mathrm{GL}_3^{\epsilon}(q)|}{|\mathrm{GL}_2^{\epsilon}(q)||\mathrm{GL}_1^{\epsilon}(q)|} < 2(q+1)q^4, \;\; |x^G| \geqslant \frac{|\mathrm{O}_7(q)|}{|\mathrm{O}_4^-(q)||\mathrm{O}_3(q)|} > \frac{1}{4}q^{12}$$

and we deduce that $f(x, H) < .549$. Now assume x is semisimple and r is odd. Then a calculation with the adjoint representation reveals that $\nu(x) \neq 5 - \delta_{3,p}$ and thus $|x^G| > \frac{1}{2}(q+1)^{-1}q^{19-4\delta_{3,p}}$. For instance, if $p \neq 3$ then $C_{\bar{G}}(x)$ is not of type GL_4 (since x is centralized by a triality graph automorphism, see [5, 3.55(iv)]) nor $\mathrm{SO}_2 \times \mathrm{GL}_3$ (since $\nu(x) \neq 5$) and thus

$$|x^G| \geqslant |\mathrm{O}_8^+(q) : \mathrm{O}_4^+(q)\mathrm{GU}_2(q)| > \frac{1}{2}(q+1)^{-1}q^{19}$$

as claimed. The desired result now follows since $|x^G \cap H| \leqslant |\mathrm{PGL}_3^{\epsilon}(q)| < q^8$.

To complete the proof, let us suppose $x \in H - \mathrm{PGL}(V)$ is an element of prime order r. We begin by assuming x is a field automorphism, in which case $q = q_0^r$. If r is odd then $x^G \cap H \subseteq \mathrm{PGL}_3^{\epsilon}(q)x$ and x induces a field automorphism on $\mathrm{PGL}_3^{\epsilon}(q)$. Therefore [5, 3.43, 3.48] imply that

$$|x^G \cap H| \leqslant |\mathrm{PGL}_3^{\epsilon}(q) : \mathrm{PGL}_3^{\epsilon}(q^{1/r})| < 2q^{8\left(1-\frac{1}{r}\right)}, \;\; |x^G| > \frac{1}{4}q^{\frac{1}{2}n(n-1)\left(1-\frac{1}{r}\right)}$$

and the desired result follows. Similarly, if $r = 2$ and $p \neq 3$ then $\epsilon = +$ (since $q_0^2 \equiv 1\,(3)$) and the bounds

$$|x^G \cap H| \leqslant |\mathrm{PGL}_3(q) : \mathrm{PGL}_3(q^{1/2})| + |\mathrm{PGL}_3(q) : \mathrm{PGU}_3(q^{1/2})| < 4q^4 \tag{15}$$

and $|x^G| > \frac{1}{4}q^{14}$ are always sufficient. Alternatively, if $(r, p) = (2, 3)$ then $n = 7$ and again we get $|x^G \cap H| < 4q^4$ if $\epsilon = +$; if $\epsilon = -$ then $|x^G \cap H| \leqslant |\mathrm{PGU}_3(q) : \mathrm{PGO}_3(q)| < 2q^5$ and in both cases the bound $|x^G| > \frac{1}{4}q^{21/2}$ is good enough. If $n = 8$ and x is an involutory graph-field automorphism then $\epsilon = +$ (since $q = q_0^2$ and $p \neq 3$) and the bounds (15) and $|x^G| > \frac{1}{4}q^{14}$ are always sufficient.

Finally, let us assume $n = 8$ and x is a triality automorphism. If x is a graph-field automorphism then $q = q_0^3$, $|x^G| > \frac{1}{4}q^{56/3}$ and the trivial bound $|x^G \cap H| < |H| < 6\log_2 q.q^8$ suffices. If x is a triality graph automorphism then $x^G \cap H \subseteq \mathrm{PGL}_3^{\epsilon}(q) \times \langle\tau\rangle$, where τ is a non-G_2 triality graph automorphism which centralizes $\mathrm{PGL}_3^{\epsilon}(q)$ (see [5, 3.47]). Applying [5, 3.14, 3.48] we deduce that

$$|x^G \cap H| \leqslant 2i_3(\mathrm{PGL}_3^{\epsilon}(q)) + 2 \leqslant 4(q+1)q^5, \;\; |x^G| > \frac{1}{8}q^{14}$$

and the desired result follows.

Case ii. *The irreducible embedding* ($\mathscr{D}10$)
Here ρ is the restriction of a spin representation for an orthogonal group $\mathrm{P}\Omega_8^+(q_0^2)$ which contains $\mathrm{P}\Omega_8^-(q_0)$. Observe that $H \cap \mathrm{PGL}(V) \leqslant \mathrm{PGO}_8^-(q_0) = \widetilde{H}$ and note that we may assume G is without triality (see [5, 3.3]). Now, if $x \in H - \mathrm{PGL}(V)$ then [5, 3.50] implies that $x^G \cap H \subseteq \widehat{H}x$, where $\widehat{H} = \mathrm{Inndiag}(\mathrm{P}\Omega_8^-(q_0))$. In particular, if x is a field automorphism of odd prime order r then $q_0 = q_1^r$ and applying [5, 3.43] we deduce that

$$|x^G \cap H| < 2q_0^{28\left(1-\frac{1}{r}\right)}, \ \ |x^G| > \frac{1}{4}q_0^{56\left(1-\frac{1}{r}\right)}$$

and the result follows. If x is an involutory field automorphism then x induces an involutory graph automorphism on $\widehat{H}$ and therefore [5, 3.14] implies that

$$|x^G \cap H| \leqslant i_2(\mathrm{Aut}(\mathrm{P}\Omega_8^-(q_0))) < 2(q_0+1)q_0^{15}.$$

In this case it is easy to check that the bound

$$|x^G| \geqslant 2^{2(\delta_{2,p}-1)}|\mathrm{O}_8^+(q_0^2) : \mathrm{O}_8^+(q_0)| = 2^{2(\delta_{2,p}-1)}q_0^{12}(q_0^2+1)(q_0^4+1)^2(q_0^6+1)$$

is sufficient unless $q_0 = 3$. Here $i_2(\mathrm{Aut}(\mathrm{P}\Omega_8^-(3))) = 60504111$ and the previous bound is in fact sufficient. The argument for an involutory graph-field automorphism is entirely similar.

Assume for the remainder that $x \in H \cap \mathrm{PGL}(V)$ is an element of prime order r. If $r = p = 2$ then applying [5, 3.22, 3.55(ii)] we obtain the following results.

$\mathrm{O}_8^-(q_0)$-class of x	b_1	a_2	c_2	b_3	c_4
$\mathrm{O}_8^+(q_0^2)$-class of $\rho(x)$	b_1	a_2	a_4	b_3	c_4
$f(x,H) <$	.507	.504	.503	.479	.500

For example, if $\rho(x)$ is $\mathrm{O}_8^+(q_0^2)$-conjugate to b_1 then using the proof of [5, 3.22] we deduce that $|x^G \cap H| = q_0^3(q_0^4+1)$, $|x^G| = q_0^6(q_0^8-1)$ and thus $f(x,H) < .507$ as claimed. Similarly, if $r = p > 2$ then we derive the bounds listed in the next table. Here the symbol † (resp. ‡) indicates the additional condition $p \geqslant 5$ (resp. $p \geqslant 7$).

$\mathrm{PGO}_8^-(q_0)$-class of x	$[J_2^2, I_4]$	$[J_3, I_5]$	$[J_3, J_2^2, I_1]$	$[J_3^2, I_2]$	$[J_5, I_3]^\dagger$	$[J_7, I_1]^\ddagger$
$\mathrm{PGO}_8^+(q_0^2)$-class of $\rho(x)$	$[J_2^2, I_4]$	$[J_2^4]$	$[J_3, J_2^2, I_1]$	$[J_3^2, I_2]$	$[J_4^2]$	$[J_7, I_1]$
$f(x,H) <$	.517	.514	.521	.522	.506	.508

If $r = 2$ and p is odd then we may apply [5, 3.37, 3.55(iii)]. For instance, if $\rho(x)$ is $\bar{G}$-conjugate to $[-iI_4, iI_4]$ then

$$|x^G \cap H| \leqslant \frac{|\mathrm{O}_8^-(q_0)|}{|\mathrm{O}_6^+(q_0)||\mathrm{O}_2^-(q_0)|} + \frac{|\mathrm{O}_8^-(q_0)|}{|\mathrm{O}_6^-(q_0)||\mathrm{O}_2^+(q_0)|} < 4q_0^{12}, \ \ |x^G| > \frac{1}{4}(q_0^2+1)^{-1}q_0^{26}$$

and we conclude that $f(x,H) < .586$ for all $q_0 \geqslant 3$.

Finally, let us assume $r \neq p$ and r is odd. Let $i_0 \geqslant 1$ be minimal such that r divides $q_0^{i_0} - 1$ and let $\hat{x} \in \mathrm{O}_8^-(q_0)$ be the lift of x to an element of order r. Let l_0 denote the dimension of the 1-eigenspace of $\hat{x}$ on the natural $\mathrm{O}_8^-(q_0)$-module and observe that [5, 3.29] implies that $l_0 \geqslant 2$ if $i_0 \neq 8$. Further, using [5, 3.55(iv)] we can easily identify the possibilities for $C_{\bar{G}}(x)$ and $C_{\bar{G}}(\rho(x))$, where $\bar{G} = \mathrm{SO}_8(K)$ and K is the algebraic closure of $\mathbb{F}_{q_0}$. The desired result quickly follows.

Case iii. *The inclusions* ($\mathscr{D}7$)*,* ($\mathscr{D}8$) *and* ($\mathscr{D}15$)
In ($\mathscr{D}7$), V is a section of $\bigwedge^2 V_6$ and $H \cap \mathrm{PGL}(V) \leqslant \mathrm{PGSp}_6(q)$. If $x \in H - \mathrm{PGL}(V)$ has prime order then [5, 3.49] gives $|x^G| > \frac{1}{4}q^{39}$ and one checks that the trivial bound

$$|x^G \cap H| < |\mathrm{Aut}(H_0)| < \log_2 q.q^{21} \tag{16}$$

is sufficient. Now assume $x \in H \cap \mathrm{PGL}(V)$ has prime order r. If $r = p$ then we claim that $\nu(x) \geqslant 4$ if $\lambda' = (2, 1^4)$, otherwise $\nu(x) \geqslant 6$, where $\lambda' \vdash 6$ denotes the associated partition of $x \in H_0$. Now, if $\lambda' = (2, 1^4)$ then an easy calculation with the module $\bigwedge^2 V_6$ reveals that $\rho(x)$ acts on V with Jordan form $[J_2^4, I_{n-8}]$, where $n = \dim V$. If $\lambda' \neq (2, 1^4)$ and $p = 2$ then a similar calculation gives $\nu(x) = 6$; if p is odd and $y \in H_0$ is unipotent with associated partition $\lambda' = (2^2, 1^2)$ then $\rho(y)$ is conjugate to $[J_3, J_2^4, I_{n-11}]$ and the claim follows. In particular, if $\nu(x) = 4$ then $|x^G \cap H| < q^6$, $|x^G| > \frac{1}{4}q^{32}$ and the result follows; if $\nu(x) \neq 4$ then our calculation with y implies that

$$|x^G| \geqslant \frac{1}{2}\frac{|\mathrm{O}_{13}(q)|}{|\mathrm{Sp}_4(q)||\mathrm{O}_2^-(q)||\mathrm{O}_1(q)|q^{25}} > \frac{1}{8}(q+1)^{-1}q^{43}$$

and we find that (16) is sufficient. Finally, let us assume $x \in H \cap \mathrm{PGL}(V)$ has order $r \neq p$. Then $\nu(x) \geqslant 6 - \delta_{3,p}$ and the desired result follows via [5, 3.36] and (16).

Now consider the embedding labelled ($\mathscr{D}$8). If $x \in H - \mathrm{PGL}(V)$ then [5, 3.49] gives $|x^G| > \frac{1}{4}q^{105/2}$ and the trivial bound

$$|x^G \cap H| < |\mathrm{Aut}(H_0)| < \log_3 q.q^{21} \tag{17}$$

is always sufficient. Now assume $x \in H \cap \mathrm{PGL}(V) \leqslant \mathrm{PGSp}_6(q)$ has prime order r. If $r \neq p$ then Theorem 2.4 implies that $\nu(x) \geqslant 4$ since there is no semisimple element $y \in G$ with $\nu(y) = 3$. Therefore $|x^G| > \frac{1}{2}q^{40}$ (minimal if $x = [-I_4, I_{10}]$) and the result follows via (17). Finally, suppose $r = p$. A well-known theorem of Steinberg states that the number of unipotent elements in a finite group of Lie type over $\mathbb{F}_q$ of the form $\bar{H}_\sigma$ is precisely $q^{2|\Phi^+(\bar{H})|}$, where $\Phi^+(\bar{H})$ is the set of positive roots of $\bar{H}$. Therefore $|x^G \cap H| < q^{18}$ and the desired result follows since $|x^G| > \frac{1}{4}q^{36}$ (minimal if x has associated partition $\lambda = (2^3, 1^8)$).

Finally, let us consider ($\mathscr{D}$15). If τ is an involutory graph automorphism of the algebraic group E_6 then $M(\lambda_1)^\tau = M(\lambda_6)$ and so we need only consider $M(\lambda_1)$. If $x \in H - \mathrm{PGL}(V)$ has prime order then [5, 3.49] gives $|x^G| > \frac{1}{2}q^{348}$ and the result follows since

$$|x^G \cap H| < |\mathrm{Aut}(H_0)| < 2\log_2 q.q^{78}. \tag{18}$$

Now assume $x \in H \cap \mathrm{PGL}(V)$ has prime order r. We claim that $\nu(x) \geqslant 7 - \delta_{r,p}$. This is immediate from [20, Table 5] if $r = p$ so assume x is semisimple. Then x lies in a maximal torus $T_6 < E_6$ (as algebraic groups defined over the algebraic closure of $\mathbb{F}_q$) and therefore some conjugate of x lies in a maximal rank subgroup $A_1A_5 < E_6$ (algebraic groups). The claim follows since [24, 2.3] gives

$$V \downarrow A_1A_5 = (V_2 \otimes V_6) \oplus (0 \otimes (\textstyle\bigwedge^2 V_6)^*),$$

where V_2 (resp. V_6) denotes the natural module for A_1 (resp. A_5) and 0 is the trivial 1-dimensional module for A_1. Therefore [5, 3.38] implies that $|x^G| > \frac{1}{2}q^{250}$ and (18) is always sufficient.

Case iv. *The remaining cases in Table 2.10*
These pose few problems. In a number of cases we can calculate directly with the corresponding representation and improve the lower bound on $\nu(x)$ given in Theorem 2.4. Indeed, there are several such calculations in [7, §7] and many of those results are useful here. For example, consider ($\mathscr{D}$13) and ($\mathscr{D}$14). If $x \in H \cap \mathrm{PGL}(V)$ has prime order then the proof of [7, 7.4] gives $\nu(x) \geqslant 6$ and thus [5, 3.38] implies that $|x^G| > \frac{1}{2}q^{107}$. Clearly, the same bound holds if $x \in H - \mathrm{PGL}(V)$ (see [5, 3.49]) and the desired result follows since $|x^G \cap H| < |\mathrm{Aut}(H_0)| < 2\log_2 q.q^{52}$. The other cases are just as easy.

This completes the proof of Lemma 2.20. □

Case 2. *H_0 is not a simple group of Lie type in defining characteristic*

Recall that we may assume $n < N$, where $N = N(G_0)$ is given in Table 2.9. We begin by assuming $H_0 = \mathrm{PSL}_2(l)$, where l is coprime to q. The various possibilities for G_0 are listed in [13, Table 2] and for brevity we shall only give details for the particular case $G_0 = \mathrm{PSp}_{(l-1)/2}(q)$, where $\mathbb{F}_q = \mathbb{F}_p[\sqrt{l}]$ and $l \equiv 1\,(4)$, the other cases are very similar. Here the hypothesis $6 \leqslant n \leqslant 62$ implies that $13 \leqslant l \leqslant 125$ and applying Theorem 2.4 and [5, 3.38] we deduce that

$$|x^G \cap H| < \log_3 l.l(l^2-1), \;\; |x^G| > \frac{1}{8}(q+1)^{-1}q^{\frac{1}{2}\alpha(l-1)-\alpha^2+1},$$

where $\alpha = 4$ if $l \geqslant 73$, otherwise $\alpha = 3$. These bounds are sufficient with the exception of the cases $(l,q) \in \{(25,2),(17,2),(13,4),(13,3)\}$. Here the desired conclusion is easily obtained. For instance, in each case the corresponding Brauer character is listed in [16] and we can compute $f(x,H)$ precisely when x is semisimple.

For the remainder let us assume $H_0 \neq \mathrm{PSL}_2(l)$. In [14], Hiss and Malle list all the absolutely irreducible representations of degree at most 250 of quasisimple finite groups, excluding groups of Lie type in their defining characteristic. Frobenius-Schur indicators are also recorded and information is given which allows one to calculate the smallest field over which each representation can be written. We make extensive use of these results. To illustrate our approach, let us assume $G_0 = \mathrm{PSp}_n(q)$ and observe that

$$|x^G \cap H| < |\mathrm{Aut}(H_0)|, \;\; |x^G| > \frac{1}{8}(q+1)^{-1}q^{\alpha(n-\alpha)+1}, \tag{19}$$

where $\alpha = 4$ if $n \geqslant 36$ and $\alpha = 3$ otherwise. Since we are free to assume $n \leqslant 62$, close inspection of [14, Table 2] reveals that $|\mathrm{Aut}(H_0)| \leqslant |\mathrm{Sp}_6(5)|$ and thus (19) is sufficient for all $n \geqslant 36$. For $n \leqslant 34$ we consider in turn each pair (H_0, n) listed in [14, Table 2] with Frobenius-Schur indicator -1 and apply the above bounds (19). Excluding any inclusions which belong to one of the collections $\mathscr{A}$, $\mathscr{B}$ or $\mathscr{C}$, we find that we are left to deal with the following cases:

$$(H_0, G_0) \in \{(G_2(4), \mathrm{PSp}_{12}(3)), (HS, \mathrm{Sp}_{20}(2)), (Suz, \mathrm{PSp}_{12}(3)), (Co_3, \mathrm{Sp}_{22}(2))\}.$$

Similarly, if $G_0 = \mathrm{PSL}^{\epsilon}_n(q)$ and $n \leqslant 63$ then using [14, Table 2] we deduce that $|\mathrm{Aut}(H_0)| \leqslant |\mathrm{Sp}_8(3)|$ and applying Theorem 2.4 and [5, 3.38] we reduce to the case $n \leqslant 20$. Further scrutiny of [14, Table 2] reveals that we are left to deal with the single case $(H_0, G_0) = (J_3, \mathrm{PSU}_9(2))$. We do likewise when G_0 is orthogonal, noting that $|x^G \cap H| < |Co_1|$ since $n \leqslant 63$. The cases which remain to be considered are listed in Table 2.11 (for $G_0 = \mathrm{P}\Omega_8^+(q)$ we have excluded any subgroups which are not maximal - see [18]). In the table, the † symbol in the row for ($\mathscr{D}$22) denotes the additional condition $(p,\epsilon) \in \{(2,-),(3,+)\}$.

Lemma 2.21. *The conclusion to Theorem 1.1 holds for the embeddings in Table 2.11.*

Proof. Let $x \in H$ be an element of prime order r. We claim that (10) holds. If $x \in H - \mathrm{PGL}(V)$, which is only possible in cases ($\mathscr{D}$20) and ($\mathscr{D}$23), then the claim follows via [5, 3.49] since $|x^G \cap H| \leqslant i_r(\mathrm{Aut}(H_0) - H_0)$. Let us assume for the remainder that $x \in H \cap \mathrm{PGL}(V)$.

If $r = p$ then we bound $|x^G|$ by applying Theorem 2.4 and [5, 3.22] (note that Theorem 2.4 implies that $\nu(x) \geqslant 4$ if G_0 is orthogonal and p is odd). Since $|x^G \cap H| \leqslant i_r(H \cap \mathrm{PGL}(V))$, one can check that the subsequent upper bound on $f(x,H)$ is sufficient with the single exception of ($\mathscr{D}$26). Here a more accurate lower bound for $|x^G|$ suffices. Indeed, [8] gives $|x^G \cap H| \leqslant i_3(Suz) = 151236800$ and we conclude that $f(x,H) < .582 < 7/12$ since

$$|x^G| \geqslant \frac{|\mathrm{Sp}_{12}(3)|}{|\mathrm{Sp}_6(3)||\mathrm{O}_3(3)|3^{24}} = \frac{1}{2}3^2(3^2+1)(3^4+1)(3^{10}-1)(3^{12}-1)$$

(minimal if x has associated partition $\lambda = (2^3, 1^6)$).

Now suppose $r \neq p$. Let χ be the Brauer character of the corresponding representation $\rho : \widehat{H}_0 \to \mathrm{GL}(V)$ and note that χ is listed in [16] for each of the embeddings ($\mathscr{D}$16)-($\mathscr{D}$26). Therefore, in these cases we can compute $f(x,H)$ precisely and easily deduce that (10) holds.

	H_0	G_0
($\mathscr{D}$16)	A_{10}	$\Omega^+_{16}(2)$
($\mathscr{D}$17)	$\mathrm{SL}_3(3)$	$\Omega^-_{12}(2)$
($\mathscr{D}$18)	$G_2(3)$	$\Omega^\epsilon_{14}(2)$
($\mathscr{D}$19)	$G_2(4)$	$\mathrm{PSp}_{12}(3)$
($\mathscr{D}$20)	$Sz(8)$	$\mathrm{P}\Omega^+_8(5)$
($\mathscr{D}$21)	M_{11}	$\Omega^-_{10}(2)$
($\mathscr{D}$22)	M_{12}	$\mathrm{P}\Omega^\epsilon_{10}(p)^\dagger$
($\mathscr{D}$23)	J_3	$\mathrm{PSU}_9(2)$
($\mathscr{D}$24)	HS	$\mathrm{Sp}_{20}(2)$
($\mathscr{D}$25)	M^cL	$\Omega^\epsilon_{22}(2)$
($\mathscr{D}$26)	Suz	$\mathrm{PSp}_{12}(3)$
($\mathscr{D}$27)	Co_1	$\mathrm{P}\Omega^\epsilon_{24}(3)$
($\mathscr{D}$28)	Co_2	$\Omega^+_{22}(2)$
($\mathscr{D}$29)	Co_3	$\mathrm{Sp}_{22}(2)$

Table 2.11: The collection $\mathscr{D}$, II

In the three remaining cases we have $H = Co_i$, for $i \in \{1,2,3\}$. If $x \in H$ is non-trivial then [12, Table 1] indicates that H can be generated by five conjugates of x, whence $\nu(x) \geqslant n/5$ and we conclude that $\nu(x) \geqslant 5$ for all non-trivial elements $x \in H$. The desired result now follows as before, using [5, 3.38] and the upper bound $|x^G \cap H| \leqslant i_r(H)$. □

This completes the proof of Proposition 2.19. □

Proposition 2.22. *If $H \in \mathscr{S}$ and $\dim V < 6$ then the conclusion to Theorem 1.1 holds.*

Proof. We begin by assuming H_0 is a simple group of Lie type in defining characteristic. First consider the case $H_0 = \mathrm{PSL}_2(q')$, where $q' = p^e$. Here we may assume $G_0 \in \{\Omega_5(q), \mathrm{PSp}_4(q)'\}$ since $n = 2$ if $G_0 = \Omega_4^-(q)$ (see Remark 1.2). Also recall that there exists an integer $i \geqslant 1$ such that $q' = q^i$ and $\dim V = l^i$, where $l \geqslant 2$ is the dimension of an irreducible $K\widehat{H}_0$-module (see the proof of Proposition 2.19). Clearly, we may assume $i = 1$ and $p \geqslant 5$. Now, if $x \in H - \mathrm{PGL}(V)$ has prime order r then $q = q_0^r$ and x acts on G_0 as a field automorphism. Applying [5, 3.43, 3.48] we see that

$$|x^G \cap H| \leqslant \frac{|\mathrm{PGL}_2(q)|}{|\mathrm{PGL}_2(q^{1/r})|} < 2q^{3(1-\frac{1}{r})},\ \ |x^G| > \frac{1}{4}q^{10(1-\frac{1}{r})}$$

and the result follows. Now assume $x \in H \cap \mathrm{PGL}(V)$ has prime order r. If $r = p$ then

$$|x^G \cap H| \leqslant \frac{|\mathrm{GL}_2(q)|}{|\mathrm{GL}_1(q)|q} = q^2 - 1,\ \ |x^G| \geqslant \frac{|\mathrm{Sp}_4(q)|}{2|\mathrm{Sp}_2(q)|q^3} = \frac{1}{2}(q^4 - 1)$$

and we conclude that $f(x,H) < .554$ for all $q \geqslant 5$. If $r = 2 < p$ then $f(x,H) < .565$ since $|x^G \cap H| \leqslant q^2$ and $|x^G| \geqslant \frac{1}{2}q^2(q^2-1)$. Finally, if $r \neq p$ and r is odd then

$$|x^G \cap H| \leqslant 2\log_5 q.\frac{|\mathrm{GL}_2(q)|}{|\mathrm{GL}_1(q)|^2} = 2\log_5 q.q(q+1),\ \ |x^G| \geqslant \frac{|\mathrm{Sp}_4(q)|}{|\mathrm{GU}_2(q)|} = q^3(q-1)(q^2+1)$$

and again the desired conclusion follows.

Now assume $H_0 \neq \mathrm{PSL}_2(p^e)$. Studying the tables in [25] (or [19, 5.4.13]) we find that we need only consider the irreducible inclusion $H_0 = Sz(q) < \mathrm{Sp}_4(q) = G_0$, where $q = 2^f$ and $f = 2m+1 \geqslant 3$. Here $H_0 = C_{G_0}(\psi)$, where ψ is an involutory graph-field automorphism of G_0 (see [5, 3.44]). Let us begin by assuming $x \in H - \mathrm{PGL}(V)$ has prime order r. If x is a field automorphism then r must divide f and the result follows since

$$|x^G \cap H| \leqslant |Sz(q) : Sz(q^{1/r})| < 2q^{5(1-\frac{1}{r})},\ \ |x^G| \geqslant |\mathrm{Sp}_4(q) : \mathrm{Sp}_4(q^{1/r})| > \frac{1}{2}q^{10(1-\frac{1}{r})}.$$

	H_0	G_0	
$(\mathscr{E}1)$	A_5	$\mathrm{PSp}_4(p)$	$p \geqslant 5$
$(\mathscr{E}2)$	A_6	$\mathrm{PSL}_3(4)$	
$(\mathscr{E}3)$	A_6	$\mathrm{PSU}_3(5)$	
$(\mathscr{E}4)$	A_6	$\mathrm{PSp}_4(p)$	$p \geqslant 5$
$(\mathscr{E}5)$	A_7	$\mathrm{PSU}_3(5)$	
$(\mathscr{E}6)$	A_7	$\mathrm{SL}_4(2)$	
$(\mathscr{E}7)$	A_7	$\mathrm{PSp}_4(7)$	
$(\mathscr{E}8)$	A_7	$\mathrm{PSL}_4^{\epsilon}(p)$	$p \neq 2, 7^{\dagger}$
$(\mathscr{E}9)$	$\mathrm{PSL}_2(11)$	$\mathrm{PSL}_5^{\epsilon}(p)$	$p \neq 11^{\ddagger}$
$(\mathscr{E}10)$	$\mathrm{SL}_3(2)$	$\mathrm{PSL}_3^{\epsilon}(p)$	$p \neq 2, 7^{\dagger}$
$(\mathscr{E}11)$	$\mathrm{SL}_3(2)$	$\mathrm{PSL}_4^{\epsilon}(p)$	$p \neq 2, 7^{\dagger}$
$(\mathscr{E}12)$	$\mathrm{PSL}_3(4)$	$\mathrm{PSU}_4(3)$	
$(\mathscr{E}13)$	$\mathrm{SU}_4(2)$	$\mathrm{PSL}_4^{\epsilon}(p)$	$p \equiv \epsilon\,(3)$, $p \geqslant 5$
$(\mathscr{E}14)$	$\mathrm{SU}_4(2)$	$\mathrm{SU}_5(5)$	
$(\mathscr{E}15)$	M_{11}	$\mathrm{SL}_5(3)$	

Table 2.12: The collection $\mathscr{E}$

On the other hand, if x is an involutory graph-field automorphism then we may assume x centralizes H_0 and we deduce that $f(x,H) < .600$ since

$$|x^G \cap H| \leqslant i_2(H_0) + 1 = (q-1)(q^2+1) + 1, \ \ |x^G| \geqslant |\mathrm{Sp}_4(q) : Sz(q)| = q^2(q+1)(q^2-1).$$

Now assume $x \in H \cap \mathrm{PGL}(V)$ has prime order r. If $r = 2$ then [5, 3.52] implies that x is $\mathrm{Sp}_4(q)$-conjugate to c_2 and the subsequent bounds $|x^G \cap H| = (q-1)(q^2+1)$ and $|x^G| = (q^2-1)(q^4-1)$ are always sufficient. If r is odd then $r \geqslant 5$ and $\nu(x) = 3$ (see [5, 3.52]), whence

$$|x^G| \geqslant \frac{|\mathrm{Sp}_4(q)|}{|\mathrm{GU}_1(q)|^2} = q^4(q-1)^2(q^2+1)$$

and thus $f(x,H) < .695$ for all $q \geqslant 8$ since $|x^G \cap H| < |\mathrm{Aut}(Sz(q))| = \log_2 q.q^2(q-1)(q^2+1)$.

For the remainder let us assume H_0 is not a group of Lie type in defining characteristic. In view of Proposition 2.5, we may also assume that the embedding $H < G$ is not in the collection $\mathscr{A}$. Then close inspection of [13, Table 2], [14, Table 2] and [8, 16] reveals that we are left to deal with the irreducible inclusions listed in Table 2.12. Here the symbol † (resp. ‡) signifies that $\epsilon = +$ if and only if $p \equiv 1, 2, 4\,(7)$ (resp. $p \equiv 1, 3, 4, 5, 9\,(11)$).

Lemma 2.23. *The conclusion to Theorem 1.1 holds for the embeddings in Table 2.12.*

Proof. As usual, let V denote the natural G_0-module and let χ be the Brauer character corresponding to each of the irreducible inclusions in Table 2.12. Observe that χ is given in [8, 16]. Let $x \in H$ be an element of prime order r.

Case 1. $x \in H \cap \mathrm{PGL}(V)$, $r \neq p$

Here we can use χ to compute $f(x,H)$ precisely. For example, consider the embedding labelled $(\mathscr{E}6)$. From the 2-modular Brauer character table for A_7 (see [16, p.13]) we derive the following results. Here $\zeta = 1$ if G contains an involutory graph automorphism of G_0, otherwise $\zeta = 0$.

A_7-class of x	$\mathrm{SL}_4(2)$-class of x	$\lvert x^G \cap H\rvert$	$\lvert x^G\rvert$	$f(x,H) <$	
$3A$	$3A$	70	112	$.901^*$	
$3B$	$3B$	280	1120	$.803^*$	
$5A$	$5A$	504	1344	$.864^*$	
$(7A, 7B)$	$(7A, 7B)$	$2^{\zeta}.360$	$2^{\zeta}.2880$	$.739$	$(.760^*)$

Note that the elements $7A$ and $7B$ in A_7 are conjugate in G if and only if $\zeta = 1$; if they are conjugate then $f(x,H) < .760^*$, otherwise $f(x,H) < .739$. As usual, the asterisks in the final column indicate that this case is an exception to the main statement of Theorem 1.1 and it is therefore included in Table 1.1. Similarly, for ($\mathscr{E}12$) we deduce that either $f(x,H) < 3/4$ or x is G_0-conjugate to $2B$ and $f(x,H) = (\log 120)/(\log 540) \approx .761^*$. Again, this exceptional case appears in Table 1.1. In each of the remaining cases in Table 2.12, the reader can check that $f(x,H) < 1/2 + 1/n$, where n is defined as in Remark 1.2.

Case 2. $x \in H \cap \mathrm{PGL}(V)$, $r = p$
Let us begin with the embedding labelled ($\mathscr{E}2$). Now A_6 is a maximal subgroup of $\mathrm{PSL}_3(4)$ (see [8] for example) and therefore $H \cap \mathrm{PGL}(V) = A_6$. Since there is a unique class of involutions in both A_6 and $\mathrm{PSL}_3(4)$, we deduce that $|x^G \cap H| = 45$, $|x^G| = 315$ and thus $f(x,H) < .662$. In each of the remaining cases, [17] implies that $x \in G$ is not a long root element and we use this fact to obtain subsequent lower bounds for $|x^G|$. Now $|x^G \cap H| \leqslant i_r(H \cap \mathrm{PGL}(V))$ and one can check that these bounds imply that $f(x,H) < 1/2 + 1/n$ with the exception of the cases ($\mathscr{E}6$) and ($\mathscr{E}12$). For example, consider ($\mathscr{E}8$). Here $H \cap \mathrm{PGL}(V) = A_7$, so $p \in \{3,5\}$ and $G_0 = \mathrm{PSU}_4(p)$ (see Table 2.12). Now $i_3(A_7) = 350$, $i_5(A_7) = 504$ and

$$|x^G| \geqslant \frac{1}{2}\frac{|\mathrm{GU}_4(p)|}{|\mathrm{GU}_2(p)|p^4} = \frac{1}{2}p(p^3+1)(p^4-1)$$

since x is not a long root element. We conclude that $f(x,H) < .722$ if $p = 3$ and $f(x,H) < .511$ if $p = 5$. The other cases are similar. The embeddings ($\mathscr{E}6$) and ($\mathscr{E}12$) can be analysed using GAP [10]: for ($\mathscr{E}6$) we deduce that $f(x,H) = (\log 105)/(\log 210) \approx .871^*$ and $f(x,H) = (\log 2240)/(\log 40320) \approx .728$ for ($\mathscr{E}12$).

Case 3. $x \in H - \mathrm{PGL}(V)$
Here $r = 2$ and therefore $|x^G \cap H| \leqslant i_2(\mathrm{Aut}(H_0) - H_0)$. An accurate lower bound for $|x^{G_0}|$ is easy to compute (see [5, 3.48]) and these bounds imply that $f(x,H) < 1/2 + 1/n$ with the exception of the following cases

$$(\mathscr{E}2),\ (\mathscr{E}6),\ (\mathscr{E}8),\ (\mathscr{E}12),\ (\mathscr{E}13).$$

For instance, in ($\mathscr{E}9$) we have $(H_0, G_0) = (\mathrm{PSL}_2(11), \mathrm{PSL}_5^{\epsilon}(p))$ and thus $f(x,H) < .440$ for all $p \geqslant 2$ since

$$|x^G \cap H| \leqslant i_2(\mathrm{PGL}_2(11) - \mathrm{PSL}_2(11)) = 66, \ \ |x^G| \geqslant \frac{|\mathrm{PSL}_5(p)|}{|\mathrm{Sp}_4(p)|} = \frac{p^6(p^3-1)(p^5-1)}{(5,p-1)},$$

where $(5,p-1)$ denotes the highest common factor of 5 and $p-1$. For the five exceptional cases we claim that the following bounds hold:

	($\mathscr{E}2$)	($\mathscr{E}6$)	($\mathscr{E}8$)	($\mathscr{E}12$)	($\mathscr{E}13$)
$f(x,H) <$	.711	.914*	.630	.761*	.445

The bounds in cases ($\mathscr{E}2$), ($\mathscr{E}6$) and ($\mathscr{E}12$) are easily checked using GAP [10]. (For ($\mathscr{E}12$) we have $f(x,H) < 3/4$ unless x lies in the G_0-class $2B$ in which case $f(x,H) = (\log 120)/(\log 540) \approx .761^*$.) Now consider ($\mathscr{E}8$). Here $i_2(S_7 - A_7) = 126$ and we immediately reduce to the case $p = 3$ since

$$|x^G| \geqslant \frac{|\mathrm{PSL}_4^{\epsilon}(p)|}{|\mathrm{Sp}_4(p)|} = (4, p-\epsilon)^{-1}p^2(p^3+1)$$

(note that Table 2.12 states that $\epsilon = -$ if $p = 5$). The case $p = 3$ can be analysed using GAP and the desired result quickly follows. Finally, let us consider ($\mathscr{E}13$). Here $H_0 = \mathrm{SU}_4(2)$ and $G_0 = \mathrm{PSL}_4^{\epsilon}(p)$, where $p \equiv \epsilon\,(3)$ and $p \geqslant 5$. Now x induces an involutory graph automorphism on both H_0 and G_0 and we claim that the centralizers $C_{H_0}(x)$ and $C_{G_0}(x)$ are of the same type (see [5, 3.47] for a description of the possible types). To see this, let τ be a symplectic-type

	G_0	type of H	conditions
(i)	$\mathrm{Sp}_4(q)'$	$\mathrm{O}_2^{\epsilon}(q) \wr S_2$	$p=2$
(ii)	$\mathrm{Sp}_4(q)'$	$\mathrm{O}_2^{-}(q^2).2$	$p=2$
(iii)	$\mathrm{P\Omega}_8^{+}(q)$	$\mathrm{GL}_3^{\epsilon}(q) \times \mathrm{GL}_1^{\epsilon}(q)$	$q \geqslant 3$ if $\epsilon = +$
(iv)	$\mathrm{P\Omega}_8^{+}(q)$	$\mathrm{O}_2^{-}(q^2) \times \mathrm{O}_2^{-}(q^2)$	
(v)	$\mathrm{P\Omega}_8^{+}(q)$	$\mathrm{O}_1(q) \wr S_8$	$q = p > 2$
(vi)	$\mathrm{P\Omega}_8^{+}(q)$	$G_2(q)$	

Table 3.1: The $\mathscr{N}$ collection

graph automorphism of H_0 and first note that $C_{G_0}(\tau)$ is symplectic since $\mathrm{Sp}_4(2) \not< \mathrm{PSO}_4^{\epsilon'}(p)$. In the $2B$-class of H_0 there is an involution h such that $[h,\tau]=1$ and $C_{H_0}(h\tau)$ is non-symplectic; moreover h is $\bar{G}$-conjugate to $[-iI_2, iI_2]$ (see [16, p.62]) and therefore $C_{G_0}(h\tau)$ is orthogonal and the claim follows. Therefore we have $f(x,H) < .445$ if x is symplectic since $|x^G \cap H| = 36$ and $|x^G| \geqslant 3150$; if x is orthogonal then $|x^G \cap H| = 540$, $|x^G| \geqslant 1890000$ and $f(x,H) < .436$. □

This completes the proof of Proposition 2.22. □

3 Proof of Theorem 1.1: $H \in \mathscr{N}$

To complete the proof of Theorem 1.1, let us assume H is a subgroup in the collection $\mathscr{N}$ (see [5, §3.1]). Recall that $\mathscr{N}$ is empty unless one of the following holds:

(a) $G_0 = \mathrm{Sp}_4(q)'$, $p=2$ and G contains graph-field automorphisms;
(b) $G_0 = \mathrm{P\Omega}_8^{+}(q)$ and G contains triality automorphisms.

The subgroups contained in the collection $\mathscr{N}$ are listed in Table 3.1 (see [5, 3.3]). Here the *type of H* gives an approximate group-theoretic structure for $H \cap \mathrm{PGL}(V)$.

3.1 Symplectic groups in dimension four

Proposition 3.1. *The conclusion to Theorem 1.1 holds in case (i) of Table 3.1.*

Proof. Let V denote the natural $\mathrm{Sp}_4(q)$-module and let $x \in H$ be an element of prime order r. Note that we may assume $q > 2$ since $n = 2$ if $G_0 = \mathrm{Sp}_4(2)'$ (see Remark 1.2). We start by assuming $x \in H \cap \mathrm{PGL}(V) = \widetilde{H} = \mathrm{O}_2^{\epsilon}(q) \wr S_2$. If $r = 2$ then applying [5, 3.52] we easily derive the following results:

$\mathrm{Sp}_4(q)$-class of x	$\lvert x^G \cap H\rvert$	$\lvert x^G\rvert$	$f(x,H) <$
b_1, a_2	$4(q-\epsilon)$	$2(q^4-1)$	.481
c_2	$(q-\epsilon)^2$	$(q^2-1)(q^4-1)$	.391

Similarly, if r is odd then r divides $q - \epsilon$ and the bounds

$$|x^G \cap H| \leqslant 16\log_2 q, \quad |x^G| \geqslant 2|\mathrm{Sp}_4(q) : \mathrm{GU}_2(q)| = 2q^3(q-1)(q^2+1)$$

are sufficient. Now let us assume $x \in H - \mathrm{PGL}(V)$, in which case [5, 3.50] implies that $x^G \cap H \subseteq \widetilde{H}x$. If x is a field automorphism of prime order r then $q = q_0^r$ and the bounds

$$|x^G \cap H| \leqslant |\widetilde{H}x| \leqslant 8(q+1)^2, \quad |x^G| = |\mathrm{Sp}_4(q) : \mathrm{Sp}_4(q^{1/r})| > \frac{1}{2}q^{10(1-\frac{1}{r})}$$

are sufficient unless $(r,q) = (2,4)$; here we calculate that $f(x,H) < .735$ since $|\widetilde{H}x| \leqslant 200$ and $|x^G| = 1360$. Finally, let us assume x is an involutory graph-field automorphism. Then $\log_2 q$ is odd and the result follows since

$$|x^G \cap H| \leqslant |\widetilde{H}x| \leqslant 8(q+1)^2, \quad |x^G| = |\mathrm{Sp}_4(q) : Sz(q)| = q^2(q+1)(q^2-1).$$

□

Proposition 3.2. *The conclusion to Theorem 1.1 holds in case (ii) of Table 3.1.*

Proof. Again, we may assume $q > 2$. Suppose $x \in H \cap \mathrm{PGL}(V) = \widetilde{H} = \mathrm{O}_2^-(q^2).2$ has prime order r. If r is odd then r divides q^2+1 and thus $|x^G| > \frac{1}{2}q^8$ and the trivial bound $|x^G \cap H| \leqslant |\widetilde{H}| = 4(q^2+1)$ is always sufficient. Now assume $r = 2$. Then x is G-conjugate to c_2 and the desired result follows since $|x^G \cap H| = q^2+1$ and $|x^G| = (q^2-1)(q^4-1)$. Finally, if $x \in H - \mathrm{PGL}(V)$ then $x^G \cap H \subseteq \widetilde{H}x$ and the bounds

$$|x^G \cap H| \leqslant |\widetilde{H}x| = 4(q^2+1), \;\; |x^G| \geqslant |\mathrm{Sp}_4(q) : Sz(q)| = q^2(q+1)(q^2-1)$$

are always sufficient. □

3.2 Orthogonal groups in dimension eight

For the remainder we shall adopt the following notation.

Notation. Let $G_0 = \mathrm{P}\Omega_8^+(q)$, where $q = p^f$ and p is prime. Let $\bar{G} = \mathrm{PSO}_8(K)$, where K denotes the algebraic closure of $\mathbb{F}_q$, and let σ be a Frobenius morphism of $\bar{G}$ such that $\bar{G}_\sigma$ is almost simple with socle G_0 and natural module V over $\mathbb{F}_q$. Then G denotes an almost simple group which has socle G_0 and contains triality automorphisms.

Proposition 3.3. *The conclusion to Theorem 1.1 holds in case (iii) of Table 3.1.*

Proof. Let $H \leqslant G$ be a subgroup of type $\mathrm{GL}_3^\epsilon(q) \times \mathrm{GL}_1^\epsilon(q)$ and define

$$B = \frac{\mathrm{GL}_3^\epsilon(q) \times \mathrm{GL}_1^\epsilon(q)}{(2, q-1)}.$$

If q is even then $H \cap \mathrm{PGL}(V) \leqslant B.\langle \psi_1, \psi_2 \rangle = B.2^2$, where ψ_1 acts on B by sending (x_1, x_2) to (x_1^γ, x_2) and γ is the familiar inverse-transpose graph automorphism of $\mathrm{GL}_3^\epsilon(q)$, while ψ_2 sends (x_1, x_2) to (x_1, x_2^{-1}). Similarly, if q is odd then

$$H \cap \mathrm{PGL}(V) \leqslant (B.\langle \delta \rangle).\langle \psi_1, \psi_2 \rangle = (B.2).2^2,$$

where $\delta \in \bar{G}_\sigma - \mathrm{PSO}_8^+(q)$ is an involution. We claim that $f(x, H) < 5/8$ for all elements $x \in G$ of prime order.

Case 1. $x \in H \cap \mathrm{PGL}(V)$

Let $x \in H \cap \mathrm{PGL}(V)$ be an element of prime order r and note that each $y \in x^G \cap B$ lifts to an element $\hat{y} = (\hat{y}_1, \hat{y}_2) \in \widehat{B}$, where $\widehat{B} = \mathrm{GL}_3^\epsilon(q) \times \mathrm{GL}_1^\epsilon(q)$ and

$$|y^B| = |\hat{y}^{\widehat{B}}| = |\hat{y}_1^{\mathrm{GL}_3^\epsilon(q)}||\hat{y}_2^{\mathrm{GL}_1^\epsilon(q)}|$$

(see [5, 3.11]). First assume $r = p > 2$. Then $x^G \cap H \subseteq B$ and $\lambda \in \{(2^2, 1^4), (3^2, 1^2)\}$, where λ denotes the associated partition of x. If $\lambda = (2^2, 1^4)$ then

$$|x^G \cap H| \leqslant \frac{|\mathrm{GL}_3^\epsilon(q)|}{|\mathrm{GL}_1^\epsilon(q)|^2 q^3} < 2q^4, \;\; |x^G| \geqslant \frac{|\mathrm{O}_8^+(q)|}{|\mathrm{O}_4^+(q)||\mathrm{Sp}_2(q)|q^9} > \frac{1}{2}q^{10} \tag{20}$$

and the desired result follows. The case $\lambda = (3^2, 1^2)$ is very similar. Next assume $r = p = 2$. If $x^G \cap H \subseteq B$ then x is G-conjugate to a_2 and appealing to [5, 3.22, 3.55(ii)] we see that the bounds in (20) are valid and the result follows. Alternatively, if $x^G \cap (H - B)$ is non-empty then $x^G \cap B = \emptyset$ and there are at most three possibilities for x up to G-conjugacy. If x is a c_4-involution then the bounds

$$|x^G \cap H| \leqslant |\mathrm{GL}_3^\epsilon(q) : \Omega_3(q)|.|\mathrm{GL}_1^\epsilon(q)| = q^2(q-\epsilon)^2(q^3-\epsilon), \;\; |x^G| > \frac{1}{2}q^{16}$$

are always sufficient. Similarly, if x is conjugate to b_3 then $|x^G| > \frac{3}{2}q^{15}$ and the desired result follows since

$$|x^G \cap H| \leqslant 3\left(\frac{|\mathrm{GL}_3^{\epsilon}(q)|}{|\Omega_3(q)|} + \frac{|\mathrm{GL}_3^{\epsilon}(q)|}{|\mathrm{GL}_1^{\epsilon}(q)|^2q^3}|\mathrm{GL}_1^{\epsilon}(q)|\right) = 3(q^3-\epsilon)(q^2(q-\epsilon)+q^2-1).$$

The argument for a b_1-involution is very similar.

Now assume $r \neq p$, beginning with the case $r=2$. If $x^G \cap H \subseteq B$ then x is conjugate to $[-I_2, I_6]$ and applying [5, 3.55(iii)] we deduce that

$$|x^G \cap H| \leqslant (2\alpha+1)(|\mathrm{GL}_3^{\epsilon}(q) : \mathrm{GL}_2^{\epsilon}(q)\mathrm{GL}_1^{\epsilon}(q)|+1) < (2\alpha+1)(2q^4+1)$$

and $|x^G| > \frac{3}{4}(q+1)^{-1}q^{13}$, where $\alpha = 1$ if $q \equiv \epsilon\,(4)$, otherwise $\alpha = 0$. The reader can check that these bounds are always sufficient. The case where $x^G \cap (H-B)$ is non-empty is just as easy. For example, if x is conjugate to $[-I_4, I_4]$ then $|x^G| > \frac{1}{8}q^{16}$ and the desired result follows since

$$|x^G \cap H| \leqslant 2\frac{|\mathrm{GL}_3^{\epsilon}(q)|}{|\mathrm{GL}_2^{\epsilon}(q)||\mathrm{GL}_1^{\epsilon}(q)|} + \frac{|\mathrm{GL}_3^{\epsilon}(q)|}{|\mathrm{SO}_3(q)|}.|\mathrm{GL}_1^{\epsilon}(q)| < 4q^4 + q^2(q+1)^2(q^3+1).$$

Now assume $r > 2$. Then $x^G \cap H \subseteq B$ and x lifts to a unique element $\hat{x} \in \Omega_8^+(q)$ of order r. Write $\mathcal{E}_x$ for the multiset of eigenvalues of $\hat{x}$ on the natural $\Omega_8^+(q)$-module, and let $i \geqslant 1$ be minimal such that r divides $q^i - 1$. Also, define the integer $c = c(i,\epsilon)$ as in the statement of [5, 3.33] and observe that $c \in \{1,2,3\}$ (note that $c = i$ if $\epsilon = +$). If $c = 2$ then x is $\bar{G}$-conjugate to $[I_4, \omega I_2, \omega^{-1}I_2]$, where $\omega \in K$ is a primitive r^{th} root of unity, and [5, 3.55(iv)] implies that x and x^{τ} are $\bar{G}_{\sigma}$-conjugate for any triality graph automorphism τ. Therefore

$$|x^G \cap H| \leqslant \log_2 q.\left(\frac{|\mathrm{GL}_3^{\epsilon}(q)|}{|\mathrm{GL}_1(q^2)||\mathrm{GL}_1^{\epsilon}(q)|}\right) \leqslant \log_2 q.q^3(q^3+1),$$

$$|x^G| \geqslant \frac{|\mathrm{O}_8^+(q)|}{|\mathrm{O}_4^+(q)||\mathrm{GU}_2(q)|} > \frac{1}{2}(q+1)^{-1}q^{19}$$

and the result follows. The case $c=3$ is similar so assume $c=1$. We claim that

$$|x^G \cap H| < 3\log_2 q.2^6.2^{1+\epsilon}q^6, \quad |x^G| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{3}{2}(1-\epsilon)+1} q^{\dim x^{\bar{G}}}. \tag{21}$$

The bound on $|x^G|$ follows immediately from [5, 3.30] and it is clear that $|y^B| < 2^{1+\epsilon}q^6$ for all $y \in x^G \cap H$. It remains to show that there are at most $3\log_2 q.2^6$ distinct B-classes in $x^G \cap H$. Here the term $3\log_2 q$ accounts for the effect of field and triality graph automorphisms on $\mathcal{E}_x$. Let l be the dimension of the 1-eigenspace of $\hat{x}$ on the natural $\Omega_8^+(q)$-module and suppose $y = (y_1, y_2)$ is an element of $x^G \cap B$ such that $\mathcal{E}_y \cup \mathcal{E}_y^{-1} = \mathcal{E}_x$. Evidently, there are at most $2^{4-l/2}$ distinct possibilties for $\mathcal{E}_y = \mathcal{E}_{y_1} \cup \mathcal{E}_{y_2}$, and for each of these, there are at most four choices for $\mathcal{E}_{y_2}$. We conclude that there at most $2^{6-l/2} \leqslant 2^6$ choices for y up to B-conjugacy and (21) follows.

Let us now apply the bounds in (21), beginning with the case $\epsilon = +$. If $\dim x^{\bar{G}} \geqslant 18$ then it remains to deal with the case $(r,q) = (3,4)$, where x is $\bar{G}$-conjugate to $[I_4, \omega I_2, \omega^2 I_2]$ or $[I_2, \omega I_3, \omega^2 I_3]$ and $\omega \in K$ is a primitive cube root of unity. In the latter case we have

$$|x^G \cap H| \leqslant 6|\mathrm{GL}_3(4) : \mathrm{GL}_2(4)\mathrm{GL}_1(4)| + 2|\mathrm{GL}_3(4) : \mathrm{GL}_1(4)^3| + 2 = 15458$$

and we deduce that $f(x,H) < .378$ since $|x^G| \geqslant |\mathrm{O}_8^+(4) : \mathrm{O}_2^+(4)\mathrm{GL}_3(4)|$. Similarly, we calculate that $f(x,H) < .358$ if $x = [I_4, \omega I_2, \omega^2 I_2]$. If $\dim x^{\bar{G}} < 18$ then we may assume $x = [I_6, \mu, \mu^{-1}]$ and thus $\dim x^{\bar{G}} = 12$. If τ is a triality graph automorphism then x^{τ} is conjugate to $[\mu I_4, \mu^{-1}I_4]$, whence $|x^G| > \frac{3}{2}q^{12}$ and the desired result follows since

$$|x^G \cap H| \leqslant \log_2 q.(6|\mathrm{GL}_3(q) : \mathrm{GL}_2(q)\mathrm{GL}_1(q)| + 6) < \log_2 q.(12q^4+6).$$

The case $\epsilon = -$ is very similar.

Case 2. $x \in H - \mathrm{PGL}(V)$
Let us begin by assuming x is a field automorphism of prime order r, in which case $q = q_0^r$ and [5, 3.50] implies that $x^G \cap H \subseteq \widetilde{H}x$, where $\widetilde{H} = H \cap \mathrm{PGL}(V)$. If we assume $r > 2$ then the bounds

$$|x^G \cap H| \leqslant |\widetilde{H}x| < 4(q+1)^2q^8, \;\; |x^G| > \frac{1}{4}q^{28(1-\frac{1}{r})}$$

(see [5, 3.48]) are always sufficient. If $r = 2$ then applying [5, 3.14] we deduce that

$$|x^G \cap H| \leqslant (q-\epsilon)^2 i_2(\mathrm{Aut}(\mathrm{PSL}_3^{\epsilon}(q))) < 2(q+1)^3q^4, \;\; |x^G| > \frac{1}{4}q^{14}$$

and again the result follows. The same bounds are valid if x is an involutory graph-field automorphism. If x is a triality graph-field automorphism then $q = q_0^3$, $|x^G| > \frac{1}{4}q^{56/3}$ (see [5, 3.48]) and we find that the trivial bound $|x^G \cap H| < |H| < 3\log_2 q.4(q+1)^2q^8$ is sufficient unless $q = 8$. In this case we conclude that $f(x,H) < .620$ since

$$|x^G \cap H| < |H| \leqslant 2^{\zeta+1}9|\mathrm{GU}_3(8)||\mathrm{GU}_1(8)|, \;\; |x^G| = 2^{\zeta}|\Omega_8^+(8) : {}^3D_4(2)|$$

where $\zeta = 1$ if G contains an involutory graph automorphism, otherwise $\zeta = 0$.

Finally, let us assume x is a triality graph automorphism. We claim that

$$|x^G \cap H| \leqslant 3(q-\epsilon)^2 i_3(\mathrm{PGL}_3^{\epsilon}(q)). \tag{22}$$

To see this, first observe that there exists an element $b \in B$ of order three such that $C_{G_0}(bx) = G_2(q)$ and $\mathrm{SL}_3^{\epsilon}(q) \leqslant C_B(bx)$. Of course, if $Z = Z(B)$ and $\widetilde{B} := B/Z \;\cong \mathrm{PGL}_3^{\epsilon}(q)$ then

$$|x^G \cap H| \leqslant i_3(B.\langle x\rangle) \leqslant |Z|.i_3(\widetilde{B}.\langle x\rangle) \leqslant (q-\epsilon)^2.i_3(\widetilde{B}.\langle x\rangle)$$

and (22) follows since $\widetilde{B}.\langle x\rangle \;\cong (\mathrm{PSL}_3^{\epsilon}(q) \times \langle \tilde{b}x\rangle).(3, q-\epsilon)$, where $\tilde{b}$ is the image of b in $\widetilde{B}$. Now, if x is a non-G_2 triality then

$$|x^G| \geqslant \frac{|\mathrm{P}\Omega_8^+(q)|}{|\mathrm{SL}_2(q)|q^5} = 2^{2(\delta_{2,p}-1)}q^6(q^4-1)^2(q^6-1)$$

and we find that (22) is always sufficient since $i_3(\mathrm{PGL}_3^{\epsilon}(q))$ is given as follows:

	$q \equiv 0\,(3)$	$q \equiv \epsilon\,(3)$	$q \equiv -\epsilon\,(3)$
$i_3(\mathrm{PGL}_3^{\epsilon}(q))$	q^6-1	$q^6+2q^4+3\epsilon q^3+2q^2$	$q^6-\epsilon q^3$

Now assume x is a G_2-type triality. Then

$$|x^G| \geqslant |\mathrm{P}\Omega_8^+(q) : G_2(q)| \geqslant 2^{2(\delta_{2,p}-1)}q^6(q^4-1)^2 \tag{23}$$

and (22) is only sufficient if $q > 13$. To deal with the remaining cases we need a more accurate upper bound for $|x^G \cap H|$. We claim that

$$|x^G \cap H| \leqslant 2^{\zeta}(q+1)^2 \tag{24}$$

for all values of q, where $\zeta = 1$ if G contains an involutory graph automorphism, otherwise $\zeta = 0$. To see this, let $\{x_{\alpha}(t) : \alpha \in \Phi,\; t \in K\}$ be a set of Chevalley generators for the algebraic group $\mathrm{SO}_8(K)$, where Φ is a root system of type D_4. Let $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subset \Phi$ be a set of simple roots, where α_2 corresponds to the middle node of the associated Dynkin diagram D_4. Write $\alpha_0 = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$ for the highest root and consider the subgroup

$$\bar{J} := \langle U_{\pm\alpha_2}, U_{\pm\alpha_0}, h_{\alpha_1}(t), h_{\alpha_3}(u) : t, u \in K^*\rangle = \mathrm{GL}_3(K) \times \mathrm{GL}_1(K) \leqslant \mathrm{SO}_8(K),$$

where $U_{\pm\alpha} = \langle x_{\alpha}(t), x_{-\alpha}(u) : t, u \in K\rangle$ and $h_{\alpha}(t) = x_{\alpha}(t)x_{-\alpha}(-t^{-1})x_{\alpha}(t-1)x_{-\alpha}(1)x_{\alpha}(-1)$. Let τ be a G_2-type triality graph automorphism of $\mathrm{SO}_8(K)$ which centralizes $\langle U_{\pm\alpha_2}, U_{\pm\alpha_0}\rangle =$

$\mathrm{SL}_3(K) \leqslant \bar{J}$ and sends $h_{\alpha_1}(t)$ to $h_{\alpha_3}(t)$ and $h_{\alpha_3}(t)$ to $h_{\alpha_4}(t) = h_{\alpha_3}(t^{-1})h_{\alpha_2}(t^{-2})h_{\alpha_1}(t^{-1})h_{\alpha_0}(t) \in \bar{J}$. From the well-known Chevalley relations (see [11, 1.12.1] for example) we can determine the elements of order three in the coset $\bar{J}\tau$. Furthermore, if $j\tau \in \bar{J}\tau$ has order three then we can identify $C_{\mathrm{SO}_8(K)}(j\tau)$ by calculating $\dim C_{\mathscr{L}(\mathrm{SO}_8(K))}(j\tau)$, where $\mathscr{L}(\mathrm{SO}_8(K))$ is the Lie algebra of $\mathrm{SO}_8(K)$. In this way we deduce that $j\tau \in \bar{J}\tau$ is a G_2-type triality automorphism if and only if $j \in Z(\bar{J})$, i.e.

$$j = h_{\alpha_0}(c)h_{\alpha_2}(c^{-2})h_{\alpha_1}(a)h_{\alpha_3}(b)$$

where a and b are arbitrary non-zero elements of K and $c^3 = (ab)^{-1}$. The claim follows immediately and it is easy to check that the bounds (23) and (24) are always sufficient. □

Proposition 3.4. *The conclusion to Theorem 1.1 holds in case (iv) of Table 3.1.*

Proof. Here $H = N_G(H_0)$, where $H_0 = N_{G_0}(L)$ and L is a Sylow l-subgroup of G_0 for an odd prime l which divides q^2+1. According to [18, 3.3.1] we have

$$H_0 \cong (D_{\frac{2}{d}(q^2+1)} \times D_{\frac{2}{d}(q^2+1)}).2^2 \leqslant (\Omega_4^-(q) \times \Omega_4^-(q)).2^2,$$

where $d = (2, q-1)$. We claim that every involution in $H \cap \mathrm{PGL}(V)$ lies in $\mathrm{Inndiag}(G_0)$. Suppose $z \in H \cap \mathrm{PGL}(V)$ is an involution which does not lie in $\mathrm{Inndiag}(G_0)$. Then z must centralize the direct product $K_0 = \frac{1}{d}(q^2+1) \times \frac{1}{d}(q^2+1) \leqslant H_0$, but this is not possible since each of the direct factors in K_0 acts irreducibly on a 4-space and therefore K_0 is self-centralizing.

Let $x \in H \cap \mathrm{PGL}(V)$ be an element of prime order r. If r is odd then Lagrange's Theorem implies that r divides q^2+1, so $|x^G| > \frac{1}{2}(q+1)^{-1}q^{21}$ and the trivial bound

$$|x^G \cap H| \leqslant |H \cap \mathrm{PGL}(V)| \leqslant 32(q^2+1)^2$$

is always sufficient. If $r = 2$ then $|x^G| > \frac{3}{4}(q+1)^{-1}q^{13}$ (since $x \in \mathrm{Inndiag}(G_0)$) and the previous bound is sufficient unless $q = 2$. Here the desired result is easily obtained using GAP [10].

Now assume $x \in H - \mathrm{PGL}(V)$ has prime order. If x is not a triality graph automorphism then $q \geqslant 4$, [5, 3.48] implies that $|x^G| > \frac{1}{4}q^{14}$ and it is easy to check that the trivial bound

$$|x^G \cap H| < |H| \leqslant 96(q^2+1)^2 \tag{25}$$

is always sufficient. Finally, assume x is a triality graph automorphism. If $q \in \{2,3\}$ then using GAP [10] we deduce that $x^G \cap H$ is empty if x is a G_2-type triality, while $|x^G \cap H| \leqslant 4^{\delta_{3,q}}200$ if x is a non-G_2 triality. If $q \geqslant 5$ then the bounds $|x^G| > \frac{1}{8}q^{14}$ and (25) are always sufficient. □

Proposition 3.5. *The conclusion to Theorem 1.1 holds in case (v) of Table 3.1.*

Proof. Here $q = p$ is odd and $H = N_G(P)$, where $P \leqslant G_0$ is a group of order 8 which centralizes a non-degenerate 1-decomposition $\mathscr{D}$ of the natural G_0-module V (see [18, 3.4.2]). Then [18, 3.4.2(ii)] gives $H_0 \cong [2^9].\mathrm{SL}_3(2)$, where $[2^9]$ denotes a group of order 2^9, and

$$H \cap \mathrm{PGL}(V) \leqslant N_{\mathrm{PGO}_8^+(q)}(\mathscr{D}) = 2^7.S_8 = \widetilde{H},$$

where $\widetilde{H}$ is a $\mathscr{C}_2$-subgroup of type $\mathrm{O}_1(q) \wr S_8$. According to [18, Table I], the maximality of H in G implies that $G \cap \bar{G}_\sigma = G_0$, whence $|G : G_0| \leqslant 6$ and $|H| \leqslant 6.2^9|\mathrm{SL}_3(2)|$.

First assume $x \in H \cap \mathrm{PGL}(V)$ is an element of odd prime order r. Then Lagrange's Theorem implies that $r \in \{3,7\}$ and from [5, 3.55] we see that there are the following possibilities for x (up to $\bar{G}_\sigma$-conjugacy), where $\omega \in K$ is a primitive r^{th} root of unity.

	$p \neq r$	$p = r$
$r = 3$	$[I_4, \omega I_2, \omega^2 I_2]$	$[J_3^2, I_2]$
$r = 7$	$[I_2, \omega, \ldots, \omega^6]$	$[J_7, I_1]$

The result now follows from [6, 2.10] since $|x^G \cap H| \leqslant |x^{\widetilde{G}} \cap \widetilde{H}|$, where $\widetilde{G} = \mathrm{PGO}_8^+(q)$. Now assume $r = 2$. If x is conjugate to $[-I_2, I_6]$ then applying [5, 3.55(iii)] and the proof of [6, 2.10] we calculate that $f(x,H) < .602$ for all $q \geqslant 3$ since

$$|x^G \cap H| \leqslant \binom{8}{2} + \frac{8!}{4!} + \frac{8!}{2!4!} + \frac{8!}{6!}6 = 2884, \;\; |x^G| \geqslant 3\frac{|\mathrm{SO}_8^+(q)|}{|\mathrm{GU}_4(q)|2} = \frac{3}{2}q^6(q-1)(q^2+1)(q^3-1).$$

Similarly, if $x = [-I_3, I_5]$ then $f(x,H) < .591$ since

$$|x^G \cap H| \leqslant 3\left[\binom{8}{3} + \frac{8!}{6!}\binom{6}{2} + \frac{8!}{4!2!}4 + \frac{8!}{3!2!}\right] = 22848, \;\; |x^G| \geqslant \frac{3}{2}q^7(q^4-1)(q^4+q^2+1)$$

(see [6, (48)]). The case $x = [-I_1, I_7]$ is very similar.

Case 2. $x \in H - \mathrm{PGL}(V)$
Here x is a triality graph automorphism and the bounds

$$|x^G \cap H| < |H| = 2^{\zeta}3.2^9|\mathrm{SL}_3(2)| = 2^{\zeta}.258048, \;\; |x^G| \geqslant 2^{\zeta}|\mathrm{P\Omega}_8^+(q) : G_2(q)| = 2^{\zeta-2}q^6(q^4-1)^2$$

are sufficient for all $q \geqslant 5$, where $\zeta = 1$ if G contains an involutory graph automorphism, otherwise $\zeta = 0$. Finally, if $q = 3$ then using GAP [10] we calculate that $|x^G \cap H| \leqslant 128$ if x is a G_2-type triality, while $|x^G \cap H| \leqslant 7168$ if x is a non-G_2 triality. The desired result quickly follows. □

Proposition 3.6. *The conclusion to Theorem 1.1 holds in case (vi) of Table 3.1.*

Proof. Following [18], we say that a subgroup $H_0 \leqslant G_0$ is a *G_2-group* if it is isomorphic to $G_2(q)$. According to [18, 3.1.1(i)], such a subgroup fixes a 1-dimensional non-singular subspace U of the natural G_0-module V and we may identify H_0 with the image of the composition

$$G_2(q) \xrightarrow{\rho} \mathrm{Stab}_{G_0}(U) \hookrightarrow G_0,$$

where ρ is the irreducible embedding labelled $(\mathscr{C}4)$ in Table 2.3. In particular, our earlier work in Lemma 2.13 applies. We also note that $H_0 = C_{G_0}(\tau)$ for a suitably chosen triality graph automorphism τ.

Let H be a subgroup of G such that $H \cap G_0$ is a G_2-group and observe that

$$H \cap \mathrm{PGL}(V) = \begin{cases} G_2(q) \times \langle\gamma\rangle & \text{if } G \text{ contains an involutory graph automorphism} \\ G_2(q) & \text{otherwise,} \end{cases}$$

where γ is an involution such that $\nu(\gamma) = 1$ with respect to V. We claim that $f(x,H) < 5/8$ for all prime order elements $x \in G$. If $x \in H \cap \mathrm{PGL}(V)$ then the claim quickly follows from the proof of Lemma 2.13 and we leave the reader to make the necessary minor adjustments. For the remainder, let us assume $x \in H - \mathrm{PGL}(V)$.

If x is a field automorphism of prime order r then $q = q_0^r$ and the bounds

$$|x^G \cap H| \leqslant 2|G_2(q) : G_2(q^{1/r})| < 4q^{14\left(1-\frac{1}{r}\right)}, \;\; |x^G| > \frac{1}{4}q^{28\left(1-\frac{1}{r}\right)}$$

are always sufficient. The same bounds are valid (with $r = 2$) if x is an involutory graph-field automorphism. Next fix a triality graph automorphism τ such that $C_{G_0}(\tau) = G_2(q)$. If x is a triality graph-field automorphism then $q = q_0^3$ and without loss we may assume $x = \tau\phi$, where ϕ is a field automorphism of order 3 and $[\tau, \phi] = 1$. Then $x^G \cap H \subseteq G_2(q)\phi \times \langle\tau\rangle$ and the result follows via [5, 3.43, 3.48] since $|x^G \cap H| < 4q^{28/3}$ and $|x^G| > \frac{1}{4}q^{56/3}$.

Finally, let us assume x is a triality graph automorphism, in which case $x^G \cap H \subseteq G_2(q) \times \langle\tau\rangle$. If x is a non-G_2 triality then using [21, 1.3(ii)] we deduce that

$$|x^G \cap H| \leqslant 2^{\zeta}i_3(G_2(q)) < 2^{1+\zeta}(q+1)q^9, \;\; |x^G| \geqslant 2^{\zeta}\frac{|\mathrm{P\Omega}_8^+(q)|}{|\mathrm{PGU}_3(q)|} = 2^{\zeta}\frac{q^9(q^3-1)(q^4-1)^2}{(2,q-1)^2},$$

where $\zeta = 1$ if G contains an involutory graph automorphism, otherwise $\zeta = 0$. These bounds are sufficient for all $q \geqslant 2$. On the other hand, if x is a G_2-type triality then

$$|x^G \cap H| = 2^{\zeta}|\{h\tau : \; h \in G_2(q), \; h^3 = 1, \; C_{G_0}(h\tau) = G_2(q)\}|$$

and

$$|x^G| \geqslant 2^{\zeta}\frac{|\mathrm{P}\Omega_8^+(q)|}{|G_2(q)|} = 2^{\zeta}\frac{q^6(q^4-1)^2}{(2,q-1)^2}. \tag{26}$$

If $p \equiv \epsilon\,(3)$ then there are exactly two distinct classes of elements of order three in $G_2(q)$, with representatives x_1 and x_2 where

$$|x_1^{G_2(q)}| = \frac{|G_2(q)|}{|\mathrm{SL}_3^{\epsilon}(q)|} = q^3(q^3+\epsilon), \;\; |x_2^{G_2(q)}| = \frac{|G_2(q)|}{|\mathrm{GL}_2^{\epsilon}(q)|} = q^5(q+\epsilon)(q^4+q^2+1)$$

and we deduce that x_1 (resp. x_2) is $\bar{G}$-conjugate to $[I_2, \omega I_3, \omega^2 I_3]$ (resp. $[I_4, \omega I_2, \omega^2 I_2]$) since

$$V \downarrow A_2 = V_3 \oplus V_3^* \oplus 0 \oplus 0,$$

where $A_2 < G_2$ (algebraic groups) is generated by long root subgroups and V_3 and 0 denote the natural and trivial A_2-modules respectively. From [11, p.215] it follows that

$$|\{h\tau : \; h \in G_2(q), \; h^3 = 1, \; C_{G_0}(h\tau) = G_2(q)\}| = |x_1^{G_2(q)}| + 1 = q^3(q^3+\epsilon) + 1$$

and thus $f(x,H) < 5/8$ as required. Now assume $p = 3$ and suppose $C_{G_0}(h\tau) = G_2(q)$, where $h \in G_2(q)$ is an element of order three. In the notation of [20], h lies in one of the unipotent classes A_1, $\tilde{A}_1$, $\tilde{A}_1^{(3)}$, $G_2(a_1)$ of the algebraic group G_2. In fact, arguing as in the proof of [21, 6.3] we deduce that h must lie in the class $\tilde{A}_1$ and thus $|x^G \cap H| = 2^{\zeta}q^6$. As before, the desired result follows via (26). □

References

[1] M. Aschbacher, *On the maximal subgroups of the finite classical groups*, Invent. Math. **76** (1984), 469–514.

[2] M. Aschbacher and G.M. Seitz, *Involutions in Chevalley groups over fields of even order*, Nagoya Math. J. **63** (1976), 1–91.

[3] N. Bourbaki, *Groupes et Algebrès de Lie (Chapters 4,5 and 6)*, Hermann, Paris, 1968.

[4] T.C. Burness, *Fixed point ratios in actions of finite classical groups,* I, to appear.

[5] ———, *Fixed point ratios in actions of finite classical groups,* II, to appear.

[6] ———, *Fixed point ratios in actions of finite classical groups,* III, to appear.

[7] ———, *Fixed point spaces in actions of classical algebraic groups*, J. Group Theory **7** (2004), 311–346.

[8] J. Conway, R. Curtis, S. Norton, R. Parker, and R. Wilson, *Atlas of Finite Groups*, Oxford University Press, 1985.

[9] H. Enomoto, *The conjugacy classes of Chevalley groups of type* (G_2) *over finite fields of characteristic 2 or 3*, J. Fac. Sci. Univ. Tokyo **16** (1970), 497–512.

[10] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2004.

[11] D. Gorenstein, R. Lyons, and R. Solomon, *The Classification of the Finite Simple Groups, Number 3*, Mathematical Surveys and Monographs, vol. 40, Amer. Math. Soc., 1998.

[12] R. Guralnick and J. Saxl, *Generation of finite almost simple groups by conjugates*, J. Algebra **268** (2003), 519–571.

[13] G. Hiss and G. Malle, *Low dimensional representations of quasi-simple groups*, LMS J. Comput. Math. **4** (2001), 22–63.

[14] ——, *Corrigenda: Low dimensional representations of quasi-simple groups*, LMS J. Comput. Math. **5** (2002), 95–126.

[15] J.E. Humphreys, *Conjugacy Classes in Semisimple Algebraic Groups*, AMS Math. Surveys and Monographs, vol. 43, 1995.

[16] C. Jansen, K. Lux, R. Parker, and R. Wilson, *An Atlas of Brauer Characters*, LMS Monographs, no. 11, Oxford University Press, 1995.

[17] W. Kantor, *Subgroups of classical groups generated by long root elements*, Trans. Amer. Math. Soc. **248** (1979), 347–379.

[18] P.B. Kleidman, *The maximal subgroups of the finite 8-dimensional orthogonal groups* $\mathrm{P}\Omega_8^+(q)$ *and of their automorphism groups*, J. Algebra **110** (1987), 173–242.

[19] P.B. Kleidman and M.W. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, London Math. Soc. Lecture Note Series, vol. 129, Cambridge University Press, 1990.

[20] R. Lawther, *Jordan block sizes of unipotent elements in exceptional algebraic groups*, Comm. Algebra **23** (1995), 4125–4156.

[21] R. Lawther, M.W. Liebeck, and G.M. Seitz, *Fixed point ratios in actions of finite exceptional groups of Lie type*, Pacific J. Math. **205** (2002), 393–464.

[22] M.W. Liebeck, *On the orders of maximal subgroups of the finite classical groups*, Proc. London Math. Soc. **50** (1985), 426–446.

[23] M.W. Liebeck and J. Saxl, *Minimal degrees of primitive permutation groups, with an application to monodromy groups of coverings of Riemann surfaces*, Proc. London Math. Soc. **63** (1991), 266–314.

[24] M.W. Liebeck and G.M. Seitz, *Reductive subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc. **580** (1996).

[25] F. Lübeck, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. **4** (2001), 135–169.

[26] N. Spaltenstein, *Caractères unipotents de* ${}^3D_4(\mathbb{F}_q)$, Comment. Math. Helv. **57** (1982), 676–691.