Fixed point ratios in actions of finite classical groups, III

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Abstract

This is the third in a series of four papers on fixed point ratios in non-subspace actions of finite classical groups. Our main result states that if G is a finite almost simple classical group and Ω is a non-subspace G-set then either $\mathrm{fpr}(x) \lesssim |x^G|^{-\frac{1}{2}}$ for all elements $x \in G$ of prime order, or (G, Ω) is one of a small number of known exceptions. In this paper we consider the case where G_ω is contained in one of the Aschbacher families $\mathscr{C}_2$ or $\mathscr{C}_3$.

1 Introduction

Let G be a finite almost simple classical group over $\mathbb{F}_q$, with socle G_0 and natural module V. Recall from [3] that a subgroup H of G is said to be a *non-subspace subgroup* if $H \cap G_0$ is contained in a maximal subgroup of G_0 which acts irreducibly on V. A transitive action of G on a set Ω is a *non-subspace action* if the point stabilizer G_ω is a non-subspace subgroup of G. Our main result, which we shall refer to as Theorem 1, states that if Ω is a faithful, transitive, non-subspace G-set then

$$\mathrm{fpr}(x) < |x^G|^{-\frac{1}{2}+\frac{1}{n}+\iota}$$

for all elements $x \in G$ of prime order, where either $\iota = 0$ or (G_0, Ω, ι) belongs to a short list of known exceptions (see [3, Table 1]). Here $\mathrm{fpr}(x)$ denotes the *fixed point ratio of* x, i.e. the proportion of points in Ω which are fixed by x. In almost all cases $n = \dim V$ (see Remark 1.2).

In order to prove this theorem, we may assume G acts primitively on Ω and therefore apply Aschbacher's well-known result on the subgroup structure of finite classical groups. In [1], eight collections of subgroups of G are defined, labelled $\mathscr{C}_i$ for $1 \leqslant i \leqslant 8$, and it is shown that if H is a maximal subgroup of G not containing G_0 then either H is contained in one of the $\mathscr{C}_i$ collections, or it belongs to a family $\mathscr{S}$ of almost simple groups which act irreducibly on V (a small additional collection $\mathscr{N}$ arises when G_0 is $\mathrm{Sp}_4(q)'$ (q even) or $\mathrm{P\Omega}_8^+(q)$). A detailed description of these subgroup collections is given in [9] (also see [4, §3.1]).

This is the third in a series of four papers. In the introductory note [3] we provided some background and motivation, stated our main results and described two further applications of Theorem 1 to the study of primitive permutation groups. In [4] we established Theorem 1 when the stabilizer G_ω is a non-subspace subgroup contained in a member of one of the collections $\mathscr{C}_i$, where $4 \leqslant i \leqslant 8$. In this paper we assume G_ω belongs to $\mathscr{C}_2$ or $\mathscr{C}_3$. Roughly speaking, the subgroups in $\mathscr{C}_2$ are the stabilizers of decompositions $V = \bigoplus_i V_i$, where $\dim V_i = m$, while the stabilizers of extension fields of $\mathbb{F}_q$ of prime index comprise $\mathscr{C}_3$. Again, we refer the reader to [9, §§4.2-3] for further details. We complete the proof of Theorem 1 in [5] when we consider the subgroups in the remaining collections $\mathscr{S}$ and $\mathscr{N}$.

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G_0	type of H	ι
$\mathrm{PSp}_n(q)$	$\mathrm{Sp}_{n/2}(q) \wr S_2$	$1/n$
$\mathrm{PSp}_n(q)$	$\mathrm{Sp}_{n/2}(q^2)$	$1/(n+2)$
$\mathrm{P\Omega}^{\epsilon}_n(q)$	$\mathrm{GL}^{\epsilon'}_{n/2}(q)$	$1/(n-2)$
$\mathrm{SU}_4(2)$	$\mathrm{GU}_1(2) \wr S_4$	.010
$\Omega^+_8(2)$	$\mathrm{O}^-_4(2) \wr S_2$	.001
$\mathrm{SL}_4(2)$	$\mathrm{GL}_2(4)$	.020

Table 1.1: The exceptional cases with $\iota > 0$

Theorem 1.1. *Let G be a finite almost simple classical group acting transitively and faithfully on a set Ω with point stabilizer $G_\omega \leqslant H$, where H is a maximal non-subspace subgroup of G in one of the Aschbacher collections $\mathscr{C}_2$ or $\mathscr{C}_3$. Then*

$$\mathrm{fpr}(x) < |x^G|^{-\frac{1}{2}+\frac{1}{n}+\iota}$$

for all elements $x \in G$ of prime order, where $\iota = 0$ or (G_0, H, ι) is listed in Table 1.1, where G_0 denotes the socle of G.

Remark 1.2. The integer $n = n(G)$ in the statement of Theorem 1.1 is defined as follows: if $G_0 \in \{\mathrm{Sp}_4(2)', \mathrm{SL}_3(2)\}$ then $n = 2$, otherwise n is defined to be the minimal degree of a non-trivial irreducible $K\widehat{G}_0$-module, where $\widehat{G}_0$ is a covering group of G_0 and K is the algebraic closure of $\mathbb{F}_q$. The *type of H* referred to in Table 1.1 provides an approximate group-theoretic structure for $H \cap \mathrm{PGL}(V)$.

Notation. We follow [9] in our notation for classical groups. In particular, we write $\mathrm{PSL}^{\epsilon}_n(q)$ for $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q)$ if $\epsilon = +$ and $-$ respectively. Other notation and terminology is consistent with the earlier papers [3] and [4]. In particular, if $H \leqslant G$ and $x \in G$ then we define

$$f(x,H) := \frac{\log |x^G \cap H|}{\log |x^G|}$$

and thus Theorem 1 states that $f(x,H) < 1/2 + 1/n + \iota$ when H is a non-subspace subgroup and x has prime order (see [4, (1)]). We label representatives for conjugacy classes of unipotent involutions in symplectic and orthogonal groups as in [2] and our terminology for graph automorphisms is explained in [4, 3.47]. The *associated partition* of an arbitrary unipotent element $x \in \mathrm{PGL}(V)$ is the partition of the integer $\dim V$ which corresponds to the Jordan normal form of x on V (see [4, §3.3]). Its semisimple analogue, the *associated σ-tuple*, is defined in [4, 3.27]. In addition, for $x \in \mathrm{PGL}(V)$ we define $\nu(x)$ to be the codimension of the largest eigenspace of x on V (see [4, 3.16]). For any $r \in \mathbb{N}$ and subset $S \subseteq X$ of a finite group X we write $i_r(S)$ for the number of elements of order r in S.

2 Proof of Theorem 1.1: $H \in \mathscr{C}_2$

The subgroups in $\mathscr{C}_2$ are the stabilizers of m-decompositions $V = V_1 \oplus \cdots \oplus V_t$ of the natural G_0-module, where $t \geqslant 2$ and $\dim V_i = m$. The particular cases we shall consider in this section are listed in Table 2.1; in the last column we record some necessary conditions for the existence and maximality of H in G (see [9, p.100 and Tables 3.5.A-H]). For convenience, we postpone the analysis of totally singular $n/2$-decompositions to the next section (see cases (ii) and (vii)-(ix) in Table 3.1).

	G_0	type of H	conditions
(i)	$\mathrm{PSL}_n^{\epsilon}(q)$	$\mathrm{GL}_{n/t}^{\epsilon}(q) \wr S_t$	$(n,q) \neq (2t,2)$, $q>3$ if $(n,\epsilon)=(t,+)$
(ii)	$\mathrm{PSp}_n(q)$	$\mathrm{Sp}_{n/t}(q) \wr S_t$	$q>2$ if $n=2t$
(iii)	$\mathrm{P\Omega}_n^{\epsilon}(q)$	$\mathrm{O}_1(q) \wr S_n$	$q=p \geqslant 3$, $\epsilon=-$ if and only if $n \equiv 2\,(4)$ and $q \equiv 3\,(4)$
(iv)	$\mathrm{P\Omega}_n^{\epsilon}(q)$	$\mathrm{O}_{n/t}^{\epsilon'}(q) \wr S_t$	$n \geqslant 2t$, q odd if n/t odd, $(n/t,q) \neq (3,3)$;
			if $\epsilon=+$: $q>4$ if $(\epsilon',n/t)=(+,2)$, $(\epsilon',n/t,q) \neq (+,4,2)$,
			$(\epsilon')^t=+$ if n/t even;
			if $\epsilon=-$: $\epsilon' \in \{\circ,-\}$, t is odd if n/t even
(v)	$\mathrm{P\Omega}_n^{\epsilon}(q)$	$\mathrm{O}_{n/2}(q) \wr S_2$	$n/2$ odd, $q \equiv 2+\epsilon\,(4)$

Table 2.1: The collection $\mathscr{C}_2$

2.1 Preliminary results

Let $\bar{G}$ be a simple classical algebraic group over an algebraically closed field K of characteristic $p \geqslant 0$, with natural module $\bar{V}$ of dimension n. Let $\bar{H} \in \mathscr{C}_2$ be a maximal closed subgroup of $\bar{G}$, say $\bar{H}$ is the stabilizer in $\bar{G}$ of a decomposition $\bar{V} = V_1 \oplus \cdots \oplus V_t$ and assume that each V_i is non-degenerate if $\bar{G}$ is a symplectic or orthogonal group. If $x \in \bar{G}$ has prime order then [6, Theorem 1] states that

$$\dim(x^{\bar{G}} \cap \bar{H}) \leqslant \left(\frac{1}{2}+\delta\right)\dim x^{\bar{G}},$$

where $\delta = 1/n$ if $\bar{G} = \mathrm{Sp}_n(K)$ and $t=2$, otherwise $\delta = 0$ (note that the entry '$1/(2n+2)$' appearing in the final column of [6, Table 1] should be '$1/2n$'). In fact, as the next proposition demonstrates, much better bounds hold when $t>2$.

Proposition 2.1. *Let $\bar{H} \in \mathscr{C}_2$ be the stabilizer in $\bar{G}$ of a decomposition $\bar{V} = V_1 \oplus \cdots \oplus V_t$ and assume that each V_i is non-degenerate if $\bar{G}$ is a symplectic or orthogonal group. Then*

$$\dim(x^{\bar{G}} \cap \bar{H}) \leqslant \left(\frac{1}{t}+\zeta\right)\dim x^{\bar{G}}$$

for all elements $x \in \bar{G}$ of prime order, where $\zeta = 0$ unless $\bar{G}$ is symplectic, when $\zeta = (1+\alpha)/(n+2\alpha)$ and $\alpha = 1-\delta_{2,t}$.

Proof. Let us start by assuming $\bar{G} = \mathrm{PSp}_n(K)$, so $\bar{H} = (\mathrm{Sp}_m(K) \wr S_t) \cap \bar{G}$, where $m = n/t$. Let $x \in \bar{G}$ be an element of prime order r and note that $x^{\bar{G}} \cap \bar{H}$ is a finite union of $\bar{B}$-classes, where $\bar{B} = (\mathrm{Sp}_m(K)^t) \cap \bar{G} = \bar{H}^0$. In particular, replacing x by a suitable $\bar{G}$-conjugate, we may assume that $\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}}$.

First suppose x is semisimple and r is odd, in which case x is the image (modulo scalars) of an element $\hat{x} = (x_1,\ldots,x_t)\pi \in \mathrm{Sp}_m(K)^t\pi$, where $\pi \in S_t$ has cycle-shape $(r^h, 1^{t-hr})$ for some $h \geqslant 0$. We claim that

$$\dim(x^{\bar{G}} \cap \bar{H}) \leqslant \left(\frac{1}{t}+\frac{1+\alpha}{n+2}\right)\dim x^{\bar{G}}. \tag{1}$$

Now, if π induces the permutation $\prod_{i=1}^{h}((i-1)r+1\ldots ir)$ on coordinates then the proof of [10, 4.5] implies that $\hat{x}$ is $\mathrm{Sp}_m(K)^t$-conjugate to $(I_m,\ldots,I_m,x_{hr+1},\ldots,x_t)\pi$, where $x_i^r = 1$ for all $i > hr$, and thus

$$\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}} = h(r-1)\dim \mathrm{Sp}_m + \sum_{i=hr+1}^{t} \dim x_i^{\mathrm{Sp}_m}.$$

Let $\omega \in K$ be a primitive r^{th} root of unity and suppose $\hat{x}$ admits the eigenvalue ω^i with multiplicity l_i on the natural $\mathrm{Sp}_n(K)$-module, where $0 \leqslant i \leqslant r-1$. We claim that there exist

tr rational numbers $\{l_{ij} : 0 \leqslant i \leqslant r-1,\ 1 \leqslant j \leqslant t\}$ such that $\sum_j l_{ij} = l_i$ and

$$\dim(x^{\bar{G}} \cap \bar{H}) = \frac{1}{2}nm + \frac{1}{2}n - \frac{1}{2}l_0 - \frac{1}{2}\sum_{i=0}^{r-1}\left(\sum_{j=1}^{t} l_{ij}^2\right). \qquad (2)$$

If $t = hr$ then $l_i = mh$ for each i, whence

$$\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}} = h(r-1)\dim \mathrm{Sp}_m = \frac{1}{2}n(m+1)\left(1-\frac{1}{r}\right)$$

and (2) holds if we set $l_{ij} = m/r$ for all i,j. If $t - hr = f > 0$ we may assume that x fixes each subspace in the set $\{V_j : 1 \leqslant j \leqslant f\}$. For $1 \leqslant j \leqslant f$, let y_{ij} be the multiplicity of ω^i as an eigenvalue of x in its action on V_j, so

$$\dim(x^{\bar{G}} \cap \bar{H}) = \frac{1}{2}(t-h)(m^2+m) - \frac{1}{2}\sum_{j=1}^{f}\left(y_{0j} + \sum_{i=0}^{r-1} y_{ij}^2\right)$$

and (2) follows if we define $l_{ij} = y_{ij}$ for $1 \leqslant j \leqslant f$ and set $l_{ij} = m/r$ for $j > f$. Applying (2) we deduce that

$$\dim(x^{\bar{G}} \cap \bar{H}) \leqslant \frac{1}{2}nm + \frac{1}{2}n - \frac{1}{2}l_0 - \frac{1}{2t}\sum_{i=0}^{r-1} l_i^2 = \frac{1}{t}\dim x^{\bar{G}} + \frac{1}{2}(n-l_0)\left(1-\frac{1}{t}\right)$$

and (1) follows since

$$\dim x^{\bar{G}} \geqslant \frac{1}{4}(n+2)(n-l_0) \geqslant \frac{(n+2)(n-l_0)}{2(1+\alpha)}\left(1-\frac{1}{t}\right).$$

Next assume x is a semisimple involution and suppose $C_{\bar{G}}(x)^0 = \mathrm{GL}_{n/2}$, in which case $\dim x^{\bar{G}} = \frac{1}{4}n(n+2)$. If $x \in \bar{B}\pi$ and π induces the permutation $(12)\ldots(2h-1\,2h)$ on the coordinates then x lifts to an element $\hat{x} = (x_1,\ldots,x_t)\pi \in \mathrm{Sp}_m(K) \wr S_t$ of order 4 and the proof of [10, 4.5] implies that $\hat{x}$ is $\mathrm{Sp}_m(K)^t$-conjugate to $(-I_m, I_m, \ldots, -I_m, I_m, x_{2h+1}, \ldots, x_t)\pi$, where $x_j = z = [-iI_{m/2}, iI_{m/2}]$ for all $j > 2h$ (here $i \in K$ satisfies $i^2 = -1$). In particular, the hypothesis $\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}}$ implies that $x \in \bar{B}$ since $2\dim z^{\mathrm{Sp}_m} > \dim \mathrm{Sp}_m$. Therefore

$$\dim(x^{\bar{G}} \cap \bar{H}) = \frac{1}{4}nm + \frac{1}{2}n = \left(\frac{1}{t} + \frac{1+\alpha}{n+2}\right)\dim x^{\bar{G}} - n\left(\frac{\alpha}{4} + \frac{1}{2t} - \frac{1}{4}\right)$$

and (1) follows. Next suppose x is $\bar{G}$-conjugate to $[-I_l, I_{n-l}]$, where $2 \leqslant l \leqslant n/2$ is even. Then $\dim x^{\bar{G}} = l(n-l)$ and the hypothesis $\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}}$ implies that $x \in \bar{B}\pi$, where $\pi \in S_t$ has cycle-shape $(2^a, 1^f)$ for $a = \lfloor l/m \rfloor$. If $f = 0$ then x is $\bar{G}$-conjugate to $[-I_{n/2}, I_{n/2}]$ and thus

$$\dim(x^{\bar{G}} \cap \bar{H}) = \frac{t}{2}\dim \mathrm{Sp}_m = \left(\frac{1}{t} + \frac{1}{n}\right)\dim x^{\bar{G}}.$$

On the other hand, if $f > 0$ then we may assume that x fixes the subspaces $\{V_1, \ldots, V_f\}$ and that the restriction of x to such a subspace V_j is Sp_m-conjugate to $[-I_{l_j}, I_{n-l_j}]$ for some even integer $l_j \geqslant 0$. Then

$$\dim(x^{\bar{G}} \cap \bar{H}) = a\dim \mathrm{Sp}_m + \sum_{j=1}^{f} l_j(m-l_j) \leqslant \frac{1}{2}am(m+1) + m(l-ma) - \frac{1}{f}(l-ma)^2$$

and we conclude that

$$\dim(x^{\bar{G}} \cap \bar{H}) \leqslant ml - \frac{l^2}{t} + \frac{1}{2}ma \leqslant ml - \frac{l^2}{t} + \frac{1}{2}l \leqslant \left(\frac{1}{t} + \frac{1}{n}\right)\dim x^{\bar{G}}.$$

Now assume x has order $r = p > 2$ and associated partition $(r^{a_r}, \ldots, 1^{a_1}) \vdash n$. (Recall that λ is the partition of $\dim V$ corresponding to the Jordan normal form of x on V - see [4, §3.3]). In analogy with the semisimple case, we can find tr rational numbers $\{a_{ij}\}$ such that $\sum_j a_{ij} = a_i$ for each $1 \leqslant i \leqslant r$ and

$$\dim(x^{\bar{G}} \cap \bar{H}) = \frac{1}{2}nm + \frac{1}{2}n - \frac{1}{2}\sum_{j=1}^{t}\left(\sum_{i=1}^{r}\left(\sum_{k=i}^{r} a_{kj}\right)^2\right) - \frac{1}{2}\sum_{i \text{ odd}} a_i$$

(see [6, 2.3]). This implies that

$$\begin{aligned}\dim(x^{\bar{G}} \cap \bar{H}) &\leqslant \frac{1}{2}nm + \frac{1}{2}n - \frac{1}{2t}\sum_{i=1}^{m}\left(\sum_{k=i}^{r} a_k\right)^2 - \frac{1}{2}\sum_{i \text{ odd}} a_i \\ &= \frac{1}{t}\dim x^{\bar{G}} + \frac{1}{2}\left(1 - \frac{1}{t}\right)\left(n - \sum_{i \text{ odd}} a_i\right)\end{aligned}$$

and (1) follows since $n = \sum_{i=1}^{r} ia_i$ and thus

$$\dim x^{\bar{G}} \geqslant \frac{n+2}{2(1+\alpha)}\left(n - \sum_{i \text{ odd}} a_i\right)\left(1 - \frac{1}{t}\right).$$

Finally, let us assume $r = p = 2$ (we adopt the standard Aschbacher-Seitz [2] notation for representatives of the classes of unipotent involutions in $\bar{G}$). If x is $\bar{G}$-conjugate to b_l or c_l (according to the parity of l) then [6, 2.3(iv)] gives $\dim x^{\bar{G}} = l(n-l+1)$ and the hypothesis $\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}}$ implies that $x \in \bar{B}$. In particular, if x acts on V_j with associated partition $(2^{l_j}, 1^{m-2l_j})$ then

$$\dim(x^{\bar{G}} \cap \bar{H}) \leqslant \sum_{j=1}^{t}((m+1)l_j - l_j^2) \leqslant (m+1)l - \frac{l^2}{t} = \frac{1}{t}\dim x^{\bar{G}} + l\left(1 - \frac{1}{t}\right)$$

and (1) quickly follows. On the other hand, if x is $\bar{G}$-conjugate to a_l then $\dim x^{\bar{G}} = l(n-l)$ and from the definition of an a-type involution (see [2]) it is clear that the restriction of each $y \in x^{\bar{G}} \cap \bar{H}$ to a fixed subspace V_j is Sp_m-conjugate to a_{l_j} for some even integer $l_j \geqslant 0$ (where we set $a_0 = I_m$). Therefore, if $y \in x^{\bar{G}} \cap \bar{B}$ then

$$\dim y^{\bar{B}} = \sum_j l_j\left(\frac{n}{t} - l_j\right) = \frac{nl}{t} - \sum_j l_j^2 \leqslant \frac{1}{t}\dim x^{\bar{G}}.$$

Now, if $x \in \mathrm{Sp}_m \times \mathrm{Sp}_m = \bar{J}$ is Sp_{2m}-conjugate to a_m then

$$\dim x^{\bar{J}} = 2\dim a_{m/2}^{\mathrm{Sp}_m} = \frac{1}{2}m^2 < \dim \mathrm{Sp}_m$$

and so the hypothesis $\dim(x^{\bar{G}} \cap \bar{H}) = \dim x^{\bar{B}}$ implies that $x \in \bar{B}\pi$, where $\pi \in S_t$ has cycle-shape $(2^a, 1^f)$ and $a = \lfloor l/m \rfloor$. In the usual manner we conclude that

$$\dim(x^{\bar{G}} \cap \bar{H}) \leqslant \left(\frac{1}{t} + \frac{1}{n}\right)\dim x^{\bar{G}}.$$

The argument for linear and orthogonal groups is very similar and left to the reader. □

Remark 2.2. The conclusion to Proposition 2.1 holds for arbitrary unipotent elements if $p = 0$.

Following [4, §3.2], if X is a subset of a finite group and $r \in \mathbb{N}$ then we write $i_r(X)$ for the number of elements of order r in X. The following result is an easy exercise.

Lemma 2.3. *Let r be a prime and let $i_{r,k}(S_t)$ be the number of permutations in S_t with cycle shape $(r^k, 1^{t-rk})$, where S_t is the symmetric group on t letters. Then*

$$i_{r,k}(S_t) = \frac{t!}{k!(t-kr)!r^k} \text{ and } i_r(S_t) = \sum_{k=1}^{\lfloor t/r \rfloor} i_{r,k}(S_t).$$

In §2.4 we will need the following technical result on orthogonal groups.

Lemma 2.4. *If q is odd and $l \geqslant 1$ then the following hold for all m.*

(i) $|\mathrm{O}^+_{2l}(q) : \mathrm{O}^+_{2m}(q)\mathrm{O}^+_{2l-2m}(q)| + |\mathrm{O}^+_{2l}(q) : \mathrm{O}^-_{2m}(q)\mathrm{O}^-_{2l-2m}(q)| < 2q^{2m(2l-2m)}$;

(ii) $|\mathrm{O}^-_{2l}(q) : \mathrm{O}^+_{2m}(q)\mathrm{O}^-_{2l-2m}(q)| + |\mathrm{O}^+_{2l}(q) : \mathrm{O}^-_{2m}(q)\mathrm{O}^+_{2l-2m}(q)| < 2q^{2m(2l-2m)}$;

(iii) $|\mathrm{O}_{2l+1}(q) : \mathrm{O}^+_{2m}(q)\mathrm{O}_{2l+1-2m}(q)| + |\mathrm{O}_{2l+1}(q) : \mathrm{O}^-_{2m}(q)\mathrm{O}_{2l+1-2m}(q)| < 2q^{2m(2l+1-2m)}$;

(iv) $|\mathrm{O}_{2l+1}(q) : \mathrm{O}^+_{2l}(q)\mathrm{O}_1(q)| + |\mathrm{O}_{2l+1}(q) : \mathrm{O}^-_{2l}(q)\mathrm{O}_1(q)| = q^{2l}$;

(v) $|\mathrm{O}^+_{2l}(q) : \mathrm{O}_{2m+1}(q)\mathrm{O}_{2l-2m-1}(q)| < |\mathrm{O}^-_{2l}(q) : \mathrm{O}_{2m+1}(q)\mathrm{O}_{2l-2m-1}(q)| < q^{(2m+1)(2l-2m-1)}$.

Proof. First consider (i). Without loss we may assume $m \geqslant l/2$ and thus

$$|\mathrm{O}^+_{2l}(q) : \mathrm{O}^+_{2m}(q)\mathrm{O}^+_{2l-2m}(q)| + |\mathrm{O}^+_{2l}(q) : \mathrm{O}^-_{2m}(q)\mathrm{O}^-_{2l-2m}(q)| = \frac{q^{2m(l-m)}\prod_{i=1}^{l-m}(q^{2m+2i}-1)}{\prod_{i=1}^{l-m}(q^{2i}-1)}.$$

The result now follows from [4, 3.8] and the other statements are derived in a similar fashion. □

Recall that in order to prove Theorem 1.1 it suffices to show that

$$f(x,H) := \frac{\log|x^G \cap H|}{\log|x^G|} < \frac{1}{2} + \frac{1}{n} + \iota$$

for all elements $x \in G$ of prime order, where ι is defined as in the statement of Theorem 1.1. We start with the case $G_0 = \mathrm{PSL}^\epsilon_n(q)$.

2.2 Proof of Theorem 1.1: Case (i) of Table 2.1

Let σ be a Frobenius morphism of $\bar{G} = \mathrm{PSL}_n(K)$ such that $\bar{G}_\sigma$ has socle $G_0 = \mathrm{PSL}^\epsilon_n(q)$ and natural module V, where K is the algebraic closure of $\mathbb{F}_q$ and $q = p^f$ for a prime p. Let $\bar{B}$ denote the image of $\mathrm{GL}_{n/t}(K)^t$ in $\mathrm{PSL}_n(K)$ and observe that

$$H \cap \mathrm{PGL}(V) \leqslant \left([(q-\epsilon)^{t-1}].\mathrm{PGL}^\epsilon_{\frac{n}{t}}(q)^t\right).S_t = B.S_t = \widetilde{H},$$

where B is the image of $\mathrm{GL}^\epsilon_{n/t}(q)^t$ in $\mathrm{PGL}^\epsilon_n(q)$ and $[(q-\epsilon)^{t-1}]$ is a group of order $(q-\epsilon)^{t-1}$. For the reader's convenience we partition the proof into three parts: in Proposition 2.5 we assume $x \in H \cap \mathrm{PGL}(V)$ is semisimple, in Proposition 2.6 we consider unipotent elements in $H \cap \mathrm{PGL}(V)$ and then in Proposition 2.7 we deal with the outer automorphisms in $H - \mathrm{PGL}(V)$.

Proposition 2.5. *The conclusion to Theorem 1.1 holds in case (i) of Table 2.1 for semisimple elements of prime order in $H \cap \mathrm{PGL}(V)$.*

Proof. Let $x \in H \cap \mathrm{PGL}(V)$ be a semisimple element of prime order r. We partition the proof into several cases, where Case i.j.k is a subcase of Case i.j which itself is a subcase of Case i.

Case 1. $x^G \cap H \subseteq B$

Let $E = C_{\bar{G}}(x)$ and let $H^1(\sigma, E/E^0)$ denote the set of σ-equivalence classes with respect to the induced action of σ on the finite group E/E^0 (see [4, 3.5]).

Case 1.1. $r > 2$, $|H^1(\sigma, E/E^0)| = 1$
Let $i \geqslant 1$ be minimal such that r divides $q^i - 1$ and define the integer $c = c(i, \epsilon)$ as in the statement of [4, 3.33] (so $c = i$ if $\epsilon = +$). According to [4, 3.35], the hypothesis $|H^1(\sigma, E/E^0)| = 1$ is equivalent to assuming that E is connected if $c = 1$. Furthermore, [4, 3.11] implies that each $y \in x^G \cap H$ lifts to an element $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_t) \in \widehat{B}$ of order r, where $\widehat{B} = \mathrm{GL}^{\epsilon}_{n/t}(q)^t$ and

$$|y^B| = |\hat{y}^{\widehat{B}}| = \prod_j |\hat{y}_j^{\mathrm{GL}^{\epsilon}_{n/t}(q)}|.$$

Define the integers l and d as in [4, 3.32] and note that the hypothesis $x^G \cap H \subseteq B$ implies that $n \geqslant \max(tc, l + dc)$. Since $|x^{\bar{G}_\sigma}| = |x^{G_0}|$ (see [8, 4.2.2(j)]) we deduce that

$$|x^G| \geqslant |x^{\bar{G}_\sigma}| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{d\alpha} q^{\dim x^{\bar{G}}} \tag{3}$$

(see [4, 3.30]) where

$$\alpha = \begin{cases} 1 & \text{if } \epsilon = - \text{ and } i \equiv 2\,(4) \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

and [4, 3.33] gives

$$\dim x^{\bar{G}} \geqslant n^2 - l^2 - \frac{1}{c}(n - l - c(d-1))^2 - c(d-1). \tag{5}$$

Case 1.1.1. $c > 1$
Let $\mu = (l, a_1, \ldots, a_k)$ denote the associated σ-tuple of x (see [4, 3.27]) and note that d is the number of terms a_j in μ which are non-zero. Write $\mathcal{E}_x$ for the multiset of eigenvalues of $\hat{x} \in \mathrm{GL}^{\epsilon}_n(q)$, where $\hat{x}$ is the unique lift of x of order r. We claim that

$$|x^G \cap H| < 2\log_2 q . 2^{td(1-\alpha)}\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} (d+1)^{\frac{n}{c}} q^{\frac{1}{t}\dim x^{\bar{G}}}. \tag{6}$$

To see this, first observe that Proposition 2.1 and [4, 3.30] imply that

$$|y^B| = |\hat{y}^{\widehat{B}}| < 2^{td(1-\alpha)}\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}}$$

for all $y \in x^G \cap H$ and so it remains to show that the number of B-classes in $x^G \cap H$ is at most $2\log_2 q.(d+1)^{n/c}$. The term $2\log_2 q$ accounts for the effect of field and graph automorphisms of G_0 on $\mathcal{E}_x$ and so we need to show that $M \leqslant (d+1)^{n/c}$, where M is the number of distinct ways the non-trivial σ-orbits in $\mathcal{E}_x$ can be distributed among the t direct factors in $\widehat{B}$. Now, if $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_t) \in \widehat{B}$ and $n/t \equiv j\,(c)$ then it follows that $l_u \equiv j\,(c)$ for each $1 \leqslant u \leqslant t$, where l_u denotes the multiplicity of 1 in the eigenvalue set $\mathcal{E}_{\hat{x}_u}$. Therefore

$$M \leqslant \binom{\frac{n-tj}{c}}{\frac{l-tj}{c}\; a_{k_1}\; \ldots\; a_{k_d}},$$

where $a_{k_v} > 0$ for all $1 \leqslant v \leqslant d$, and the multinomial theorem implies that

$$M \leqslant (d+1)^{\frac{n-tj}{c}} \leqslant (d+1)^{\frac{n}{c}} \tag{7}$$

as required.

If we assume $t \geqslant 3$ then one can check that the bounds (3), (5) and (6) are sufficient to imply that $f(x, H) < 1/2 + 1/n$ unless $\epsilon = +$ and $(t, i, q) = (3, 2, 2)$. Here $(r, d) = (3, 1)$ and it remains to deal with the cases $(n, l) \in \{(12, 10), (9, 7), (6, 4), (6, 0)\}$. If $(n, l) = (6, 4)$ then x is

$\bar{G}$-conjugate to $[I_4, \omega, \omega^2]$, where $\omega \in K$ is a primitive cube root of unity, and we calculate that $f(x,H) < .141$ since

$$|x^G \cap H| \leqslant 3|\mathrm{GL}_2(2) : \mathrm{GL}_1(4)| = 6, \;\; |x^G| \geqslant |\mathrm{GL}_6(2) : \mathrm{GL}_4(2)\mathrm{GL}_1(4)| = 333312.$$

The other cases are similar. For $t=2$ we require greater accuracy. We claim that

$$|x^G \cap H| < 2\log_2 q.\left(\frac{n-l}{cd}+1\right)^{d(t-1)} 2^{td(1-\alpha)} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}} \tag{8}$$

for all $t \geqslant 2$, where α is defined as in (4). Arguing as before, it suffices to show that

$$M \leqslant \left(\frac{n-l}{cd}+1\right)^{d(t-1)},$$

where M is defined as above. If we assume $a_{k_1}, \ldots, a_{k_d}$ are non-zero then it is clear that $\mathcal{E}_{\hat{x}_1}$ is determined by a choice of d-tuple $(b_1, \ldots, b_d)$, where $0 \leqslant b_j \leqslant a_{k_j}$ for each $1 \leqslant j \leqslant d$. If N denotes the number of such d-tuples then $M \leqslant N^{t-1}$ since $\mathcal{E}_{\hat{x}_t}$ is uniquely determined once $\mathcal{E}_{\hat{x}_1}, \ldots, \mathcal{E}_{\hat{x}_{t-1}}$ have been chosen. Now

$$N = \prod_{j=1}^{d} (a_{k_j}+1) \leqslant \left(\frac{\sum_j a_{k_j}}{d}+1\right)^d = \left(\frac{n-l}{cd}+1\right)^d \tag{9}$$

and (8) follows. If we assume $t=2$ then the bounds (3), (5) and (8) are always sufficient if $d \geqslant 3$; if $d=2$ then we are left to deal with a handful of cases with $\epsilon = +$ and the desired result quickly follows through direct calculation. For example, if $(n,l,i,q) = (8,2,3,2)$ then $r=7$, $\nu(x) = 6$ (i.e. 6 is the codimension of the largest eigenspace of x on the natural $\bar{G}$-module; see [4, 3.16] for the formal definition of $\nu(x)$) and we conclude that $f(x,H) < .445$ since

$$|x^G \cap H| \leqslant 2|\mathrm{GL}_4(2) : \mathrm{GL}_1(2)\mathrm{GL}_1(2^3)|^2, \;\; |x^G| \geqslant |\mathrm{GL}_8(2) : \mathrm{GL}_2(2)\mathrm{GL}_1(2^3)^2|.$$

Now assume $(t,d,\epsilon) = (2,1,+)$. We claim that

$$|x^G \cap H| < 2\log_2 q.2^2 \left(\frac{q^2+1}{q^2-1}\right) q^{\frac{1}{2}\dim x^{\bar{G}}}. \tag{10}$$

Without loss of generality, suppose $a_1 > 0$. If $l=0$ then $a_1 = n/i$ and [4, 3.30] implies that

$$|x^G \cap H| \leqslant 2\log_2 q.|\mathrm{GL}_{n/2}(q) : \mathrm{GL}_{n/2i}(q^i)|^2 < 2\log_2 q.q^{\frac{1}{2}\dim x^{\bar{G}}}$$

so let us assume $l>0$. For all possible integers j in the range $0 \leqslant j \leqslant a_1$, choose $z_j = (y_1, y_2) \in x^G \cap H$ so that $\mathcal{E}_{y_1}$ contains precisely j copies of the non-trivial σ-orbit Ω_1 (recall that a_1 is defined to be the multiplicity of Ω_1 in $\mathcal{E}_x$; see [4, §3.4] for the definition of a σ-orbit). Then $|x^G \cap H| \leqslant 2\log_2 q. \sum_j |z_j^B|$, where

$$|z_j^B| = \frac{|\mathrm{GL}_{n/2}(q)|}{|\mathrm{GL}_{n/2-ji}(q)||\mathrm{GL}_j(q^i)|}.\frac{|\mathrm{GL}_{n/2}(q)|}{|\mathrm{GL}_{l+ji-n/2}(q)||\mathrm{GL}_{a_1-j}(q^i)|}$$

and thus

$$|x^G \cap H| < 2\log_2 q.2^2 \sum_j q^{\dim z_j^{\bar{B}}},$$

where

$$\dim z_j^{\bar{B}} = -2i(i+1)j^2 + 2(n-l)(i+1)j + nl - l^2 - \frac{1}{i}(n-l)^2.$$

If $(n-l)/i$ is even then $\dim z_j^{\bar{B}} \leqslant \dim z_{(n-l)/2i}^{\bar{B}} = \frac{1}{2}\dim x^{\bar{G}}$ and (10) follows since

$$\sum_j q^{\dim z_j^{\bar{B}}} \leqslant 2\left(1+q^2+\cdots+q^{\frac{1}{2}\dim x^{\bar{G}}-2}\right) + q^{\frac{1}{2}\dim x^{\bar{G}}} \leqslant \left(\frac{q^2+1}{q^2-1}\right) q^{\frac{1}{2}\dim x^{\bar{G}}}.$$

Similarly, if $(n-l)/i$ is odd then

$$\sum_j q^{\dim z_j^{\bar{B}}} \leqslant 2\left(1+q^2+\cdots+q^{\frac{1}{2}\dim x^{\bar{G}}-\frac{1}{2}i(i+1)}\right)$$

and again the claim follows. With minor adjustments, the same argument applies when $(t,d,\epsilon)=(2,1,-)$ and it is easy to see that (10) holds. Furthermore, if $\epsilon=-$ then the bounds (3), (5) and (10) are sufficient unless $(n,l,i,q)=(4,0,1,4)$, where direct calculation yields $f(x,H)<.529$. If $(\epsilon,l)=(+,0)$ then the same bounds are almost always sufficient and the few cases which remain are easily dealt with. Finally, if $\epsilon=+$ and $l>0$ then we quickly reduce to the case $(n,i,q)=(l+2,2,2)$, where x is $\bar{G}$-conjugate to $[I_{n-2},\omega,\omega^2]$ and $\omega\in K$ is a primitive cube root of unity. Here the reader can check that the bounds

$$|x^G\cap H|\leqslant 2\left(\frac{|\mathrm{GL}_{n/2}(2)|}{|\mathrm{GL}_{n/2-2}(2)||\mathrm{GL}_1(2^2)|}\right)=\frac{1}{3}2^{n-2}(2^{\frac{n}{2}-1}-1)(2^{\frac{n}{2}}-1)$$

and

$$|x^G|\geqslant\frac{|\mathrm{GL}_n(2)|}{|\mathrm{GL}_{n-2}(2)||\mathrm{GL}_1(2^2)|}=\frac{1}{3}2^{2n-3}(2^{n-1}-1)(2^n-1)$$

are always sufficient.

Case 1.1.2. $c=1$
Here $l>0$ and $d+l\leqslant n\leqslant(d+1)l$ (see [4, 3.32(i)]). If $n=t\geqslant 3$ then $|x^B|=1$ and arguing as before (see (6)) we deduce that $|x^G\cap H|\leqslant 2\log_2 q.(d+1)^n$. Then (3) and (5) are sufficient unless $(\epsilon,q)=(-,2)$. Here $r=3$, $d\in\{1,2\}$ and the desired result quickly follows through direct calculation. For the remainder we will assume $n\geqslant 2t$.

First consider the case $n=l+d$. If $d=1$ then one can check that the bounds

$$|x^G\cap H|\leqslant 2\log_2 q.t\left(\frac{|\mathrm{GL}^\epsilon_{n/t}(q)|}{|\mathrm{GL}^\epsilon_{n/t-1}(q)||\mathrm{GL}^\epsilon_1(q)|}\right)=2t\log_2 q.\left(\frac{q^{n/t-1}(q^{n/t}-\epsilon^{n/t})}{q-\epsilon}\right),$$

$$|x^G|\geqslant\frac{|\mathrm{GL}^\epsilon_n(q)|}{|\mathrm{GL}^\epsilon_{n-1}(q)||\mathrm{GL}^\epsilon_1(q)|}=\frac{q^{n-1}(q^n-\epsilon^n)}{q-\epsilon}$$

are always sufficient (note that $\epsilon=-$ if $q=2$ since $r|(q-\epsilon)$). For $d\geqslant 2$ we claim that

$$|x^G\cap H|<2\log_2 q.2^{\frac{d}{2}(1+\epsilon)}t^d q^{\frac{1}{t}\dim x^{\bar{G}}-d\left(1-\frac{1}{t}\right)}. \tag{11}$$

To see this, let $\hat{y}=(\hat{y}_1,\ldots,\hat{y}_t)\in\widehat{B}$ be a lift of $y\in x^G\cap H$ of order r so that the 1-eigenspace of $\hat{y}$ has dimension l. Let l_k denote the multiplicity of 1 in $\mathcal{E}_{\hat{y}_k}$, so $\sum_k l_k=l$ and $|y^B|<2^{\frac{d}{2}(1+\epsilon)}q^{\dim y^{\bar{B}}}$, where

$$\dim y^{\bar{B}}=\frac{n^2}{t}-\sum_{k=1}^t l_k^2-n+l\leqslant\frac{n^2}{t}-\frac{l^2}{t}-n+l=\frac{1}{t}\dim x^{\bar{G}}-d\left(1-\frac{1}{t}\right).$$

Then (11) follows since there at most t^d distinct ways to distribute the d distinct eigenvalues $\lambda_i\neq 1$ among the t direct factors. Now, $\dim x^{\bar{G}}=2ld+d^2-d$ and applying (3) we find that (11) is sufficient unless $(\epsilon,t,d,q)=(-,2,2,2)$. Here $n\geqslant 6$ (see Table 2.1) and x is $\bar{G}$-conjugate to $[I_{n-2},\omega,\omega^2]$, where $\omega\in K$ is a primitive cube root of unity. Moreover,

$$|x^G\cap H|\leqslant 2\frac{|\mathrm{GU}_{n/2}(2)|}{|\mathrm{GU}_{n/2-2}(2)||\mathrm{GU}_1(2)|^2}+2\left(\frac{|\mathrm{GU}_{n/2}(2)|}{|\mathrm{GU}_{n/2-1}(2)||\mathrm{GU}_1(2)|}\right)^2<10.2^{2n-6}$$

and the desired result follows since $|x^G|>\frac{1}{9}2^{4n-5}$.

Now suppose $n>l+d$ and $t\geqslant 3$. Then (6) gives

$$|x^G\cap H|<2\log_2 q.2^{\frac{1}{2}td(1+\epsilon)}(d+1)^n q^{\frac{1}{t}\dim x^{\bar{G}}}$$

and we find that (3) and (5) are sufficient if $\epsilon = +$ unless (n, l, d) is one of a handful of cases with $(t, q) = (3, 4)$. These are all easily dealt with. Similarly, if $\epsilon = -$, $t \geqslant 3$ and $q \geqslant 4$ then the same bounds are always sufficient. Now assume $(\epsilon, q) = (-, 2)$ and $t \geqslant 3$. If $d = 1$ then $x = [I_l, \lambda I_{n-l}]$ for some λ and we observe that

$$|x^G \cap H| < 2\binom{t+n-l-1}{n-l} 2^{\frac{1}{t}\dim x^{\bar{G}}},$$

where $\dim x^{\bar{G}} = 2l(n-l)$ (the binomial coefficient can be interpreted combinatorially as the number of ways the $n-l$ eigenvalues λ can be distributed among the t direct factors). Since $l+1 < n \leqslant 2l$, it is easy to check that this bound with (3) is always sufficient whenever $t \geqslant 3$. Finally, if $d = 2$ then (8) gives

$$|x^G \cap H| < 2\left(\frac{n-l}{2}+1\right)^{2(t-1)} 2^{\frac{1}{t}\dim x^{\bar{G}}}$$

and (3) and (5) are sufficient unless $(n, t, l) = (6, 3, 3)$, where more accurate bounds yield $f(x, H) < .312$.

Next assume $n > l + d$ and $t = 2$. Here we require a greater degree of accuracy. If $d = 1$ then $\dim x^{\bar{G}} = 2l(n-l)$ and an earlier argument (see (10)) implies that

$$|x^G \cap H| < 2\log_2 q.2^{1+\epsilon}\left(\frac{q^2+1}{q^2-1}\right) q^{l(n-l)}.$$

If $\epsilon = +$ then this bound with (3) is sufficient unless $(n, l, q) = (4, 2, 4)$; if $\epsilon = -$ then we are left to deal with the case $(n, l, q) = (6, 4, 2)$. In both cases, the desired result is easily obtained through direct calculation. Next assume $d = 2$, so x is $\bar{G}$-conjugate to $[I_l, \alpha I_a, \beta I_{n-l-a}]$ for distinct $\alpha, \beta \in K$. We claim that

$$|x^G \cap H| < 2\log_2 q.\left(\frac{q^2+1}{q^2-1}\right)^2 2^{2(1+\epsilon)} q^{\frac{1}{2}\dim x^{\bar{G}}}. \tag{12}$$

To see this, first observe that $|x^G \cap H| \leqslant 2\log_2 q.\sum_{j,k}|\hat{x}_{jk}^{\widehat{B}}|$, where $\hat{x}_{jk} = (\hat{y}_1, \hat{y}_2) \in \widehat{B}$ and $\hat{y}_1 = [I_j, \alpha I_k, \beta I_{n/2-j-k}]$ up to $\mathrm{GL}_{n/2}(K)$-conjugacy. Next fix j and note that

$$|\hat{x}_{jk}^{\widehat{B}}| = \frac{|\mathrm{GL}_{n/2}^{\epsilon}(q)|}{|\mathrm{GL}_j^{\epsilon}(q)||\mathrm{GL}_k^{\epsilon}(q)||\mathrm{GL}_{n/2-j-k}^{\epsilon}(q)|}.\frac{|\mathrm{GL}_{n/2}^{\epsilon}(q)|}{|\mathrm{GL}_{l-j}^{\epsilon}(q)||\mathrm{GL}_{a-k}^{\epsilon}(q)||\mathrm{GL}_{n/2-l+j-a+k}^{\epsilon}(q)|}.$$

Then $\sum_k |\hat{x}_{jk}^{\widehat{B}}| < 2^{2(1+\epsilon)}\sum_k q^{f(j,k)}$, where $f(j,k) := \dim x_{jk}^{\bar{B}}$, and (12) quickly follows. If $\epsilon = +$ then the bounds (3), (5) and (12) are sufficient unless $(n, l, q) = (6, 3, 4)$, where direct calculation yields $f(x, H) < .499$. If $\epsilon = -$ then the same bounds are sufficient unless $q = 2$ and $n = l + 3$; here $\dim x^{\bar{G}} = 6n - 14$ and the desired result follows via (3) since $|x^G \cap H| < 2\log_2 q.2q^{3n-14}(1 + q^4 + q^6)$. Finally, let us assume $n > l + d$, $t = 2$ and $d \geqslant 3$, in which case $q \geqslant 8$ if $\epsilon = +$ and $q \geqslant 4$ if $\epsilon = -$. Arguing as before (see (8)) we deduce that

$$|x^G \cap H| < 2\log_2 q.\left(\frac{n-l}{d}+1\right)^d 2^{d(1+\epsilon)} q^{\frac{1}{2}\dim x^{\bar{G}}}$$

and the desired conclusion follows via (3) and (5).

Case 1.2. $r > 2$, $|H^1(\sigma, E/E^0)| = r$

According to [4, 3.34, 3.35], the hypotheses imply that $c = 1$, r divides n and $E = C_{\bar{G}}(x)$ is non-connected. Furthermore, $\dim x^{\bar{G}} = n^2(1 - 1/r)$ and the hypothesis $x^G \cap H \subseteq B$ implies that r does not divide $t!$, whence $r \geqslant 5$ if $t \geqslant 3$. In addition, the proof of [4, 3.35] implies that x lifts to an element $\hat{x} \in \mathrm{GL}_n^{\epsilon}(q)$ which is $\mathrm{GL}_n^{\epsilon}(q)$-conjugate to

$$\begin{pmatrix} & \lambda^j I_{n/r} \\ I_{n-n/r} & \end{pmatrix} \tag{13}$$

for some unique integer $0 \leqslant j \leqslant r-1$, where $Z(\mathrm{GL}_n^{\epsilon}(q)) = \langle \lambda I_n \rangle$. If $j=0$ then $\hat{x}$ is $\mathrm{GL}_n^{\epsilon}(q)$-conjugate to the diagonal matrix $[I_{\frac{n}{r}}, \omega I_{\frac{n}{r}}, \ldots, \omega^{r-1} I_{\frac{n}{r}}] \in \mathrm{GL}_n^{\epsilon}(q)$, where ω is a primitive r^{th} root of unity. Further, [4, 3.35] gives

$$|x^G| \geqslant \frac{1}{r}|\mathrm{GL}_n^{\epsilon}(q) : \mathrm{GL}_{n/r}^{\epsilon}(q)^r| > \frac{1}{2r}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(r-1)(1-\epsilon)} q^{n^2\left(1-\frac{1}{r}\right)} \tag{14}$$

and we claim that

$$|x^G \cap H| < \left(\frac{n}{r}+1\right)^{(r-1)(t-1)} 2^{\frac{t}{2}(r-1)(1+\epsilon)} q^{\frac{1}{t}n^2\left(1-\frac{1}{r}\right)}.$$

To see this, first observe that each $y \in x^G \cap H$ lifts to an element $\hat{y} \in \widehat{B}$ of order r. Then appealing to Proposition 2.1 we deduce that

$$|y^B| \leqslant |\hat{y}^{\widehat{B}}| < 2^{\frac{t}{2}(r-1)(1+\epsilon)} q^{\frac{1}{t}n^2\left(1-\frac{1}{r}\right)}$$

and the claim follows since there are at most $(n/r+1)^{(r-1)(t-1)}$ distinct ways to partition the eigenvalue set $\mathcal{E}_{\hat{x}}$ into t subsets (see (9) for example). These bounds are sufficient unless $(n,t,r) = (6,2,3)$ and $q = 3+\epsilon$, where direct calculation yields $f(x,H) < .537$.

Finally, if $1 \leqslant j \leqslant r-1$ then [4, 3.35] implies that

$$|x^G| \geqslant \frac{|\mathrm{GL}_n^{\epsilon}(q)|}{|\mathrm{GL}_{n/r}^{\epsilon}(q^r)|r} > \frac{1}{2r} q^{n^2\left(1-\frac{1}{r}\right)} \tag{15}$$

and applying [4, 3.51] we deduce that

$$|x^G \cap H| \leqslant \sum_{j=1}^{r-1} |\hat{z}_j^{\mathrm{GL}_{\frac{n}{t}}^{\epsilon}(q)}|^t = (r-1)\left(\frac{|\mathrm{GL}_{n/t}^{\epsilon}(q)|}{|\mathrm{GL}_{n/tr}^{\epsilon}(q^r)|}\right)^t < (r-1).2^{\frac{t}{2}(1+\epsilon)}\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}n^2\left(1-\frac{1}{r}\right)},$$

where

$$\hat{z}_j = \begin{pmatrix} & \lambda^j I_{n/tr} \\ I_{n/t-n/tr} & \end{pmatrix} \in \mathrm{GL}_{n/tr}^{\epsilon}(q). \tag{16}$$

These bounds are always sufficient.

Case 1.3. $r=2$

Write $s = \nu(x)$ for the codimension of the largest eigenspace of x on the natural $\bar{G}$-module (see [4, 3.16]) and note that the hypothesis $x^G \cap H \subseteq B$ implies that $s < n/t$. In particular, $C_{\bar{G}}(x)$ is connected and each $y \in x^G \cap H$ lifts to an involution $\hat{y} \in \widehat{B}$. Now, $\dim x^{\bar{G}} = 2s(n-s)$ and applying Proposition 2.1 we deduce that

$$|x^G \cap H| < \binom{t+s-1}{s} 2^t q^{\frac{2s}{t}(n-s)}, \quad |x^G| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)} q^{2s(n-s)}.$$

If $t=2$ then one can check that these bounds are sufficient unless $(s,q) = (1,3)$ or $(n,s,q) \in \{(6,2,3),(4,1,5)\}$. If $(s,q) = (1,3)$ then the bounds

$$|x^G \cap H| \leqslant 2\left(\frac{|\mathrm{GL}_{n/2}^{\epsilon}(3)|}{|\mathrm{GL}_{n/2-1}^{\epsilon}(3)||\mathrm{GL}_1^{\epsilon}(3)|}\right) = 2\left(\frac{3^{n/2-1}(3^{n/2}-\epsilon^{n/2})}{3-\epsilon}\right),$$

$$|x^G| \geqslant \frac{|\mathrm{GL}_n^{\epsilon}(3)|}{|\mathrm{GL}_{n-1}^{\epsilon}(3)||\mathrm{GL}_1^{\epsilon}(3)|} = \frac{3^{n-1}(3^n-\epsilon^n)}{3-\epsilon}$$

are always sufficient; the remaining two cases are easily dealt with through direct calculation. If $t \geqslant 3$ and $n \geqslant 2t$ then it remains to deal with the case $(n,t,s,q) = (6,3,1,3)$ where more accurate bounds yield $f(x,H) < .315$.

Case 2. $x^G \cap (H-B) \neq \emptyset$
Here $x^G \cap B\pi \neq \emptyset$ for a non-trivial permutation $\pi \in S_t$, where π has order r and cycle-shape $(r^{h(\pi)}, 1^{t-h(\pi)r})$. Define

$$h = \max\{h(\pi) \,:\, \pi \in S_t \text{ and } x^G \cap B\pi \neq \emptyset\} \tag{17}$$

and fix $\pi \in S_t$ such that $h(\pi) = h$. Without loss, we may assume that π fixes V_j for all $j \geqslant hr+1$ in the decomposition $V = V_1 \oplus \cdots \oplus V_t$. If $|H^1(\sigma, E/E^0)| = 1$, where $E = C_{\bar{G}}(x)$, and $y \in B\rho$ is G-conjugate to x, where $\rho \in S_t$ has cycle-shape $(r^k, 1^{t-kr})$, then [4, 3.11] implies that y lifts to an element $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_t)\rho$ in $\mathrm{GL}_n^{\epsilon}(q)$ of order r and the proof of [10, 4.5] reveals that $\hat{y}$ is $\widehat{B}$-conjugate to $(I_{n/t}, \ldots, I_{n/t}, \hat{y}_{kr+1}, \ldots, \hat{y}_t)\rho$. Therefore

$$|y^B| \leqslant |\hat{y}^{\widehat{B}}| = |\mathrm{GL}_{n/t}^{\epsilon}(q)|^{k(r-1)} \prod_{j>kr} |\hat{y}_j^{\mathrm{GL}_{n/t}^{\epsilon}(q)}| \tag{18}$$

and we deduce that

$$\dim x^{\bar{G}} \geqslant \dim \pi^{\bar{G}} = n^2 h(r-1)\frac{1}{t}\left(2 - \frac{hr}{t}\right). \tag{19}$$

Case 2.1. $r > 2$, $|H^1(\sigma, E/E^0)| = 1$, $c > 1$
Define the integers i, c and α as in Case 1 and observe that

$$|x^G| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\alpha(r-1)} q^{\dim x^{\bar{G}}}. \tag{20}$$

If $n = t$ then the hypothesis $c > 1$ implies that $x^G \cap B\rho \neq \emptyset$ if and only if $\rho \in S_t$ has cycle-shape $(r^h, 1^{t-hr})$, whence $\dim x^{\bar{G}} = nh(r-1)(2 - hr/n)$ and applying (18) and Lemma 2.3 we deduce that

$$|x^G \cap H| = |x^G \cap (H-B)| < 2\log_2 q . \left(\frac{n!}{h!(n-hr)!r^h}\right)(q-\epsilon)^{h(r-1)}. \tag{21}$$

This bound with (20) is sufficient unless $(n, h, r, q, \epsilon) = (5, 1, 5, 2, -)$, where direct calculation yields $f(x, H) < .552$.

Now assume $n \geqslant 2t$. We claim that

$$|x^G \cap H| < 2\log_2 q . 2\left(\frac{t^r}{r}\right)^h \left(\frac{r-1}{c}+1\right)^{\frac{n}{c}} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} 2^{\frac{t}{c}(r-1)(1-\alpha)} q^{\frac{1}{t}\dim x^{\bar{G}}}. \tag{22}$$

To see this, first observe that $|x^G \cap H| \leqslant \sum_{k=0}^{h} |\rho_k^{S_t}| M_k N_k$, where $\rho_k \in S_t$ has cycle-shape $(r^k, 1^{t-rk})$, M_k is the size of the largest B-class in $x^G \cap B\rho_k$ and N_k is the number of distinct B-classes in $x^G \cap B\rho_k$. Applying Proposition 2.1 and [4, 3.9] we deduce that

$$M_k < M_k' = \left(\frac{q+1}{q}\right)^{\frac{1}{2}(t-rk)(1-\epsilon)} 2^{\frac{1}{c}(t-rk)(r-1)(1-\alpha)} q^{\frac{1}{t}\dim x^{\bar{G}}}$$

and arguing as before (see (7)) we have

$$N_k \leqslant N_k' = 2\log_2 q . \left(\frac{r-1}{c}+1\right)^{\frac{1}{c}\left(n - \frac{nrk}{t}\right)}.$$

Therefore

$$|x^G \cap H| < \sum_{k=0}^{h} \frac{t!}{k!(t-rk)!r^k} M_0' N_0' < \sum_{k=0}^{h} \left(\frac{t^r}{r}\right)^k M_0' N_0' < 2\left(\frac{t^r}{r}\right)^h M_0' N_0'$$

and (22) follows. It is easy to check that the bounds (19), (20) and (22) are always sufficient (note that we may assume $n \geqslant 3t$ if $(\epsilon, q) = (+, 2)$ - see Table 2.1).

Case 2.2. $r > 2$, $|H^1(\sigma, E/E^0)| = 1$, $c = 1$
Here $d = r-1$ and $r \leqslant n \leqslant rl$ since the σ-orbit of each r^{th} root of unity in K is a singleton set. Also note that $t > hr$ (if $t = hr$ then $C_{\bar{G}}(x)$ is non-connected and [4, 3.35] implies that $|H^1(\sigma, E/E^0)| = r$). First suppose $n = t$. Then (19) and (20) hold and appealing to (21) and (22) we deduce that

$$|x^G \cap H| < 2\log_2 q.2\left(\frac{t^r}{r}\right)^h (q-\epsilon)^{h(r-1)} r^t.$$

If $(\epsilon, q) \neq (-,2)$ then these bounds are almost always sufficient; the remaining cases are easily dealt with through direct calculation. If $(\epsilon, q) = (-,2)$ then $r = 3$,

$$|x^G \cap H| \leqslant 2\sum_{k=0}^{h}\left[\left(\frac{n!}{k!(n-3k)!3^k}\right)3^{2k}.3^{n-3k}\right] < 4.3^{n-2h}n^{3h}$$

and we deduce that (19) and (20) are sufficient for all $h \geqslant 3$. If $h = 2$ then we are left to deal with the cases $n \in \{7,8,9\}$ with which we can calculate directly. Finally, if $h = 1$ then the maximality of h implies that x is $\bar{G}$-conjugate to $[I_l, \omega I_{n-l-1}, \omega^2]$ and thus

$$|x^G \cap H| \leqslant 2\left(\frac{n!}{l!(n-l-1)!} + \frac{n!}{(n-3)!3}3^2\binom{n-3}{l-1}\right), \quad |x^G| > \frac{2}{9}2^{2nl+4n-2l^2-6l-6}.$$

These bounds are always sufficient if $n \geqslant 6$; if $n = 5$ then direct calculation gives $f(x,H) < .599$; we get $f(x,H) < .718$ if $n = 4$.

Finally, if $n \geqslant 2t$ then (19) and (20) hold, and (22) is valid with $c = 1$. The reader can check that these bounds are sufficient.

Case 2.3. $r > 2$, $|H^1(\sigma, E/E^0)| = r$
First assume x lifts to an element $\hat{x} \in \mathrm{GL}_n^\epsilon(q)$ of order r. Then (14) holds and appealing to Proposition 2.1 and the proof of [10, 4.5] we deduce that

$$\begin{aligned}|x^G \cap H| &< \sum_{k=0}^{\lfloor t/r \rfloor}\left[\frac{t!}{k!(t-rk)!r^k}\left(\frac{q+1}{q}\right)^{\frac{1}{2}k(r-1)(1-\epsilon)} r^{n\left(1-\frac{rk}{t}\right)}2^{\frac{1}{2}(r-1)(t-rk)(1+\epsilon)}\right]q^{\frac{1}{t}n^2\left(1-\frac{1}{r}\right)} \\ &< 2\left(\frac{t^r}{r}\right)^{\frac{t}{r}}\left(\frac{q+1}{q}\right)^{\frac{t}{2r}(r-1)(1-\epsilon)} r^n 2^{\frac{t}{2}(r-1)(1+\epsilon)}q^{\frac{1}{t}n^2\left(1-\frac{1}{r}\right)}.\end{aligned}$$

If $\epsilon = +$ and $n > t$ then these bounds are sufficient unless $(n,t,r,q) = (6,3,3,4)$, where direct calculation yields $f(x,H) < .356$. Similarly, if $\epsilon = -$ and $n > t$ then we are left to deal with the case $(t,r,q) = (3,3,2)$ for $n \leqslant 12$. Here the more accurate bounds

$$|x^G \cap H| \leqslant \frac{n!}{(n/3)!^3}2^{\frac{2}{9}n^2} + 2|\mathrm{GU}_{n/3}(2)|^2, \quad |x^G| \geqslant \frac{|\mathrm{GU}_n(2)|}{|\mathrm{GU}_{n/3}(2)|^3 3}$$

are always sufficient. If $n = t$ then it remains to deal with a handful of cases (n,r). In these cases the desired result is easily obtained by evaluating the bounds

$$|x^G \cap H| \leqslant \sum_{k=0}^{\lfloor t/r \rfloor}\left[\frac{t!}{k!(t-rk)!r^k}(q-\epsilon)^{k(r-1)}\frac{(n-rk)!}{(n/r-k)!^r}\right], \quad |x^G| \geqslant \frac{1}{r}|\mathrm{GL}_n^\epsilon(q) : \mathrm{GL}_{\frac{n}{r}}^\epsilon(q)^r|.$$

Now assume x lifts to an element $\hat{x} \in \mathrm{GL}_n^\epsilon(q)$ as in (13), with $j \geqslant 1$. Write $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_t)\rho$, where $\hat{x}_i \in \mathrm{GL}_{n/t}^\epsilon(q)$ and $\rho \in S_t$ has cycle-shape $(r^k, 1^{t-rk})$. Assume for now that tr divides n, so if ρ induces the permutation $\prod_{i=1}^k((i-1)r+1\ldots ir)$ on the coordinates and $i > kr$ then $\hat{x}_i$ is $\mathrm{GL}_{n/t}^\epsilon(q)$-conjugate to $\hat{z}_j$, where $\hat{z}_j$ is given in (16). Since $\hat{x}^r = \lambda^j I_n$ it follows that

$$\hat{x}_1\ldots\hat{x}_r = \hat{x}_{r+1}\ldots\hat{x}_{2r} = \ldots = \hat{x}_{(k-1)r+1}\ldots\hat{x}_{kr} = \lambda^j I_{n/t}$$

and using the proof of [10, 4.5] we deduce that $\hat{x}$ is $\widehat{B}$-conjugate to $b\rho$, where

$$b = (I_{n/t}, \ldots, I_{n/t}, \lambda^j I_{n/t}, \ldots, I_{n/t}, \ldots, I_{n/t}, \lambda^j I_{n/t}, \hat{x}_{kr+1}, \ldots, \hat{x}_t) \in \widehat{B}.$$

Therefore

$$\begin{aligned} |x^G \cap H| &\leqslant (r-1) \sum_{k=0}^{\lfloor t/r \rfloor} \left[\frac{t!}{k!(t-kr)!r^k} |\mathrm{GL}^{\epsilon}_{n/t}(q)|^{k(r-1)} \left(\frac{|\mathrm{GL}^{\epsilon}_{n/t}(q)|}{|\mathrm{GL}^{\epsilon}_{n/tr}(q^r)|} \right)^{t-kr} \right] \\ &< (r-1).2 \left(\frac{t^r}{r} \right)^{\frac{t}{r}} \left(\frac{q+1}{q} \right)^{\frac{t}{2}(1-\epsilon)} 2^{\frac{t}{2}(1+\epsilon)} q^{\frac{1}{t}n^2\left(1-\frac{1}{r}\right)} \end{aligned} \tag{23}$$

and one can check that (15) is always sufficient. Finally, let us assume n is not divisible by tr. Then r divides t and $x^G \cap B\rho$ is non-empty if and only if ρ has cycle-shape $(r^{t/r})$. Therefore

$$\begin{aligned} |x^G \cap H| &\leqslant (r-1) \frac{t!}{(t/r)! r^{t/r}} |\mathrm{GL}^{\epsilon}_{n/t}(q)|^{\frac{t}{r}(r-1)} \\ &< (r-1) \frac{t!}{(t/r)! r^{t/r}} \left(\frac{q+1}{q} \right)^{\frac{t}{2r}(r-1)(1-\epsilon)} q^{\frac{n^2}{tr}(r-1)} \end{aligned} \tag{24}$$

and the result now follows by applying (15).

Case 2.4. $r = 2$
Write $s = \nu(x)$ and observe that $s = nh/t + j$ for some integer $0 \leqslant j < n/t$. Let us first assume $s < n/2$. Then $C_{\bar{G}}(x)$ is connected, $t > 2h$ and

$$|x^G| > \frac{1}{2} \left(\frac{q}{q+1} \right)^{\frac{1}{2}(1-\epsilon)} q^{\dim x^{\bar{G}}}, \tag{25}$$

where $\dim x^{\bar{G}} = 2s(n-s)$ (see [4, Table 3.8]). Arguing as before we deduce that

$$|x^G \cap H| < 2 \left(\frac{t^2}{2} \right)^h 2^{\frac{t}{2}(1+\epsilon)} 2^n q^{\frac{1}{t}\dim x^{\bar{G}}} \tag{26}$$

and if we assume $n \geqslant 2t$ and $h \geqslant 2$ then the bounds (19), (25) and (26) are always sufficient. If $h = 1$ and $s = n/t + j$ then

$$|x^G \cap H| < \left(2^{\frac{1}{2}(1+\epsilon)} \binom{t}{2} \binom{t-3+j}{j} + \binom{n/t+j+t-1}{n/t+j} \right) 2^{\frac{1}{2}(t-2)(1+\epsilon)} q^{\frac{2s}{t}(n-s)}$$

and (25) is sufficient if $n \geqslant 2t$. If $n = t$ then the maximality of h implies that $s = h$, whence

$$|x^G \cap H| \leqslant \sum_{k=0}^{h} \left[\frac{n!}{k!(n-2k)!2^k} \binom{n-2k}{h-k} (q-\epsilon)^k \right] < \binom{n}{h} + \frac{n!}{(n-2h)!} (q-\epsilon)^h \tag{27}$$

and the desired result follows via (25).

For the remainder, let us assume $s = n/2$. Here $C_{\bar{G}}(x)$ is non-connected and

$$|x^G| > \frac{1}{4} \left(\frac{q}{q+1} \right)^{\frac{1}{2}(1-\epsilon)} q^{\frac{1}{2}n^2}. \tag{28}$$

Let us first assume $C_G(x)$ is of type $\mathrm{GL}^{\epsilon}_{n/2}(q)^2$, so x lifts to an involution $\hat{x} \in \mathrm{GL}^{\epsilon}_n(q)$. If $n = t$ then (27) holds (with $h = n/2$) and this bound with (28) is always sufficient. If $n \geqslant 2t$ then (26) holds (with $h = \lfloor t/2 \rfloor$) and if we assume $t \geqslant 3$ then it remains to deal with four cases with which we can calculate directly. If $t = 2$ then

$$|x^G \cap (H - B)| \leqslant |\mathrm{GL}^{\epsilon}_{n/2}(q)| \leqslant (q+1) q^{\frac{1}{4}n^2 - 1},$$

$$|x^G\cap B| \leqslant \sum_{l=0}^{n/2}\left[\left(\frac{|\mathrm{GL}^{\epsilon}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon}_{l}(q)||\mathrm{GL}^{\epsilon}_{n/2-l}(q)|}\right)^2\right] < 4\sum_{l=0}^{n/2} q^{2l(n-2l)} < 4\left(\frac{q^2+1}{q^2-1}\right)q^{\frac{1}{4}n^2}$$

and (28) is sufficient unless $(n,q)=(4,3)$, where direct calculation yields $f(x,H)<.617$. Finally, let us assume $C_G(x)$ is of type $\mathrm{GL}_{n/2}(q^2)$. If n/t is even then arguing as before (see (23)) we deduce that

$$\begin{aligned}|x^G\cap H| &\leqslant \sum_{k=0}^{\lfloor t/2\rfloor}\left[\frac{t!}{k!(t-2k)!2^k}|\mathrm{GL}^{\epsilon}_{n/t}(q)|^k\left(\frac{|\mathrm{GL}^{\epsilon}_{n/t}(q)|}{|\mathrm{GL}_{n/2t}(q^2)|}\right)^{t-2k}\right]\\ &< 2\left(\frac{t^2}{2}\right)^{\frac{t}{2}}\left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)}2^t q^{\frac{n^2}{2t}}\end{aligned}$$

and (28) is sufficient unless $(n,t)=(4,2)$ or $(\epsilon,n,t)=(-,6,3)$. If $(n,t)=(4,2)$ then

$$|x^G\cap H| \leqslant |\mathrm{GL}^{\epsilon}_2(q):\mathrm{GL}_1(q^2)|^2+|\mathrm{GL}^{\epsilon}_2(q)| = q^2(q-\epsilon)^2+q(q-\epsilon)(q^2-1),$$

$$|x^G| \geqslant \frac{1}{2}|\mathrm{GL}^{\epsilon}_4(q):\mathrm{GL}_2(q^2)| = \frac{1}{2}q^4(q-\epsilon)(q^3-\epsilon)$$

and we conclude that $f(x,H)<.651$ for all $q\geqslant 3$. In the same way we calculate that $f(x,H)<.550$ if $(\epsilon,n,t)=(-,6,3)$. Finally, if n/t is odd then t is even, (24) holds (setting $r=2$) and one can check that (28) is always sufficient. □

Proposition 2.6. *The conclusion to Theorem 1.1 holds in case (i) of Table 2.1 for unipotent elements of prime order in $H\cap \mathrm{PGL}(V)$.*

Proof. Let $x\in H\cap\mathrm{PGL}(V)$ be an element of order p, with associated partition $\lambda\vdash n$ (see [4, §3.3]). Write $\widehat{B}=\mathrm{GL}^{\epsilon}_{n/t}(q)^t$ and define h as in (17) (setting $r=p$), so $h=0$ if and only if $x^G\cap H\subseteq B$. Fix $\pi\in S_t$ with cycle-shape $(r^h,1^{t-hr})$ and assume π induces the permutation $\prod_{i=1}^{h}((i-1)p+1,\ldots,ip)$ on coordinates. If $y\in B\pi$ is G-conjugate to x then [4, 3.11] implies that y lifts to a unique element $\hat{y}=(\hat{y}_1,\ldots,\hat{y}_t)\pi\in\widehat{B}\pi$ of order p which is $\widehat{B}$-conjugate to $(I_{n/t},\ldots,I_{n/t},\hat{y}_{hp+1},\ldots,\hat{y}_t)\pi$. Furthermore, (18) holds and

$$\lambda=(p^{\frac{nh}{t}+b_p},(p-1)^{b_{p-1}},\ldots,1^{b_1}), \tag{29}$$

where the restriction of y to $V_{hp+1}\oplus\cdots\oplus V_t$ has associated partition $\lambda'=(p^{b_p},\ldots,1^{b_1})\vdash n(t-hp)/t$. We follow the approach of Proposition 2.5, partitioning the proof into a number of cases, where Case i.j is a subcase of Case i.

Case 1. $x^G\cap H\subseteq B$
We begin by considering two special cases; the general case is handled in Case 1.3.

Case 1.1. $\lambda=(k^{n/k})$
Here $2\leqslant k\leqslant p$ and k divides n/t since $x^G\cap H\subseteq B$. Furthermore, the hypotheses imply that p does not divide t if $k=p$. Applying Proposition 2.1 and [4, 3.18, 3.20(i)] we deduce that

$$|x^G\cap H| < \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)}q^{\frac{1}{t}\dim x^{\bar{G}}},\;\; |x^G| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)}q^{\dim x^{\bar{G}}-1}$$

and [6, 2.4] implies that $\dim x^{\bar{G}}\geqslant \frac{1}{2}n^2$ (minimal if $k=2$). The result follows.

Case 1.2. $\lambda=(2^j,1^{n-2j})$
We may assume that $j<n/2$ and thus [4, 3.20(i)] implies that

$$|x^G| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)}q^{\dim x^{\bar{G}}}, \tag{30}$$

where $\dim x^{\bar{G}} = 2j(n-j)$. We also note that the prime order hypothesis on x implies that λ must have this form if $p=2$, in which case $j < n/t$ since $x^G \cap H \subseteq B$. If $j=1$ then

$$|x^G \cap H| < t.2^{\frac{1}{2}(1+\delta_{2,q})(1+\epsilon)} q^{2\left(\frac{n}{t}-1\right)}$$

and (30) is always sufficient if $t \geqslant 3$. If $(t,j)=(2,1)$ then the bounds

$$|x^G \cap H| \leqslant 2(q-\epsilon)^{-1}(q^{n/2-1}-\epsilon)(q^{n/2}-1), \quad |x^G| \geqslant (q-\epsilon)^{-1}(q^{n-1}-1)(q^n-\epsilon)$$

are sufficient. Now assume $j \geqslant 2$. Applying (18), Proposition 2.1 and [4, 3.18] we deduce that

$$|x^G \cap H| < \binom{t+j-1}{j} 2^{\frac{t}{2}(1+\delta_{2,q})(1+\epsilon)} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{2j}{t}(n-j)}$$

since each B-class in $x^G \cap B$ is determined by a distribution of the j J_2-blocks among the t direct factors in B. For $t \geqslant 3$, this bound with (30) is sufficient unless $(n,t,j,q)=(9,3,2,2)$, where direct calculation yields $f(x,H) < .354$. Now assume $t=2$. Arguing as in the proof of Proposition 2.5 (see (10)) we deduce that

$$|x^G \cap H| < 2^{(1+\delta_{2,q})(1+\epsilon)} \left(\frac{q+1}{q}\right)^{1-\epsilon} \left(\frac{q^2+1}{q^2-1}\right) q^{j(n-j)}. \tag{31}$$

First assume $(q,\epsilon)=(2,+)$. Then $|x^G| > 2^{2j(n-j)-1}$ and if we assume $j \geqslant 4$ then (31) is sufficient unless $(n,j)=(10,4)$, where direct calculation yields $f(x,H) < .514$. If $j=3$ then there are at most two essentially distinct ways to write $(2^3,1^{n-6})$ as a sum of two partitions of $n/2$ and we deduce that $|x^G \cap H| < 16.2^{3n-10} + 8a.2^{3n-18}$, where $a=1$ if $n \geqslant 12$, otherwise $a=0$. The result now follows since $|x^G| > 2^{6n-19}$; the case $j=2$ is very similar. Finally, if $t=2$ and $(q,\epsilon) \neq (2,+)$ then the bounds (30) and (31) are sufficient unless $(\epsilon,n,j,q)=(-,6,2,2)$. Here direct calculation yields $f(x,H) < .399$.

Case 1.3. *General λ*
Write $\lambda = (m^{a_m}, \ldots, 2^{a_2}, 1^l) \vdash n$, where $m = n/t$. In view of Case 1.2, we may assume $p > 2$. Let $d \geqslant 1$ be the number of non-zero terms a_j in λ and observe that

$$|x^G| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)(d+1)} q^{\dim x^{\bar{G}}-1}. \tag{32}$$

If $d=1$ and $a_k > 0$ then we may assume $k > 2$ and $l > 0$. Then Proposition 2.1 implies that

$$|x^G \cap H| < \binom{t-1+(n-l)/k}{(n-l)/k} 2^t q^{\frac{1}{t}\dim x^{\bar{G}}},$$

where $n \geqslant \max(l+k, tk)$ and $\dim x^{\bar{G}} = (n^2-l^2)(1-1/k)$, and we find that (32) is sufficient unless $(n,t,l,k,q)=(6,2,3,3,3)$, where direct calculation yields $f(x,H) < .370$. Now assume $d \geqslant 2$. We claim that

$$n \geqslant \max(t(d+1), \frac{1}{2}d^2 + \frac{3}{2}d + l) \tag{33}$$

and

$$\dim x^{\bar{G}} \geqslant \frac{1}{2}n^2 + \frac{1}{2}(d^2-d)n - \frac{1}{8}d^4 - \frac{1}{12}d^3 + \frac{3}{8}d^2 - \frac{1}{6}d - \frac{1}{2}l^2. \tag{34}$$

To see this, suppose $\{r_1, \ldots, r_d\}$ is the set of indices with $a_{r_k} > 0$, where $r_i > r_{i+1}$ for each i. Since $x^G \cap H \subseteq B$ we have $n/t \geqslant r_1 \geqslant d+1$ and (33) follows since

$$n = l + \sum_{j=1}^{d} r_j a_{r_j} \geqslant l + \sum_{j=2}^{d+1} j = l + \frac{1}{2}(d+1)(d+2) - 1.$$

The lower bound on $\dim x^{\bar{G}}$ follows from [6, 2.3, 2.4]. For example, if $\alpha = \frac{1}{2}(2n-2l-d^2-3d+4)$ is even and l and d are fixed then $(d+1, d, \ldots, 3, 2^{\alpha/2}, 1^l) \vdash n$ is the least possible partition of n with respect to the familiar dominance ordering on partitions and the result follows via [6, 2.3, 2.4]. Next we claim that

$$|x^G \cap H| < 2^{\frac{1}{2}td(1+\epsilon)} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} \left(\frac{n/2 - d^2/4 + d/4 - l/2 - 1}{d} + 1\right)^{d(t-1)} q^{\frac{1}{t}\dim x^{\bar{G}}}. \quad (35)$$

In view of (18), Proposition 2.1 and [4, 3.18] it is sufficient to show that the number N of B-classes in $x^G \cap B$ satisfies $N \leqslant Y^{d(t-1)}$, where

$$Y = \frac{n/2 - d^2/4 + d/4 - l/2 - 1}{d} + 1.$$

Such a B-class is determined by a choice of t partitions $\lambda_i \vdash n/t$ with $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_t$ and clearly λ_t is uniquely determined once $\lambda_1, \ldots, \lambda_{t-1}$ have been chosen, whence $N \leqslant M^{t-1}$, where M is the number of choices for λ_1. If $r_1 > \ldots > r_d \geqslant 2$ are the indices with $a_{r_k} > 0$ then λ_1 is uniquely determined by a choice of d-tuple $(x_1, \ldots, x_d)$, where $0 \leqslant x_j \leqslant a_{r_j}$ for each j. Of course, if M' denotes the number of all such d-tuples then

$$M \leqslant M' = \prod_{j=1}^{d} (a_{r_j} + 1) \leqslant \left(\frac{\sum_j a_j}{d} + 1\right)^d$$

and thus $M \leqslant Y^d$ since $\sum_j a_j$ is maximal when a_2 is as large as possible.

Calculating, we find that the bounds (32), (34) and (35) are sufficient unless $(\epsilon, t, d, q) = (+, 2, 2, 3)$ and $(n, l) \in \{(8,3), (8,1), (6,1)\}$. These cases are easily settled via direct calculation.

Case 2. $x^G \cap (H - B) \neq \emptyset$

Define $h > 0$ as in (17). Referring to (29), we observe that $\dim x^{\bar{G}}$ is minimal if $b_j = 0$ for all $j > 0$ and thus (19) holds (with $r = p$). Also note that [4, 3.20(i)] implies that $|x^{\bar{G}_\sigma}| = |x^{G_0}|$.

Case 2.1. $n = t$

Here $\lambda = (p^h, 1^{n-hp})$ and we deduce that

$$|x^G \cap H| = |x^G \cap (H - B)| \leqslant \frac{n!}{h!(n-hp)!p^h}(q-\epsilon)^{h(p-1)}$$

and (30) holds since x^G meets $B\pi$ if and only if π has cycle-shape $(p^h, 1^{n-hp})$. If $\epsilon = +$ then we may assume $q \geqslant 4$ (see Table 2.1) and the above bounds with (19) are always sufficient, also if $\epsilon = -$ and $q > 2$. Finally, if $(\epsilon, q) = (-, 2)$ then we are left to deal with the following cases:

(n,h)	$\lvert x^G \cap H\rvert$	$\lvert x^G\rvert$	$f(x,H) <$
$(5,1)$	30	165	.667
$(4,1)$	18	45	.760*

These results are obtained through direct calculation. The asterisk appearing in the last row indicates that the case $(n,t,h,q) = (4,4,1,2)$ is an exception to the main statement of Theorem 1.1 and is therefore recorded in Table 1.1.

Case 2.2. $n \geqslant 2t$, $p = 2$

Let us begin by assuming $h = 1$. Here $\lambda = (2^{n/t+j}, 1^{n-2n/t-2j})$ for some $0 \leqslant j \leqslant n(1/2 - 1/t)$ and arguing as before we deduce that

$$|x^G \cap B| < \binom{n/t + j + t - 1}{n/t + j} 2^{\frac{t}{2}(1+\delta_{2,q})(1+\epsilon)} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}}$$

and

$$|x^G \cap (H-B)| < \binom{t}{2}\binom{t-3+j}{j} 2^{\frac{1}{2}(t-2)(1+\delta_{2,q})(1+\epsilon)} \left(\frac{q+1}{q}\right)^{\frac{1}{2}(t-1)(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}},$$

where $\dim x^{\bar{G}} = 2(n/t+j)(n-n/t-j)$. If $t \geqslant 3$ then the result follows via (30); if $t=2$ then

$$|x^G \cap (H-B)| \leqslant |\mathrm{GL}^{\epsilon}_{n/2}(q)| < \left(\frac{q+1}{q}\right)^{\frac{1}{2}(1-\epsilon)} q^{\frac{1}{4}n^2}, \;\; |x^G| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(1-\epsilon)} q^{\frac{1}{2}n^2}$$

and either $x^G \cap B$ is empty or $n \equiv 0\,(4)$ and

$$|x^G \cap B| < \left(\frac{|\mathrm{GL}^{\epsilon}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon}_{n/4}(q)|q^{n^2/16}}\right)^2 < \left(\frac{q+1}{q}\right)^{1-\epsilon} q^{\frac{1}{4}n^2}.$$

These bounds are always sufficient. Now assume $h>1$. Arguing as in the proof of Proposition 2.5 (see (22)) we deduce that

$$|x^G \cap H| < 2\left(\frac{t^2}{2}\right)^h 2^{\frac{n}{2}} 2^{\frac{t}{2}(1+\delta_{2,q})(1+\epsilon)} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}}.$$

(Note that if $\rho \in S_t$ has cycle-shape $(2^{nk/t}, 1^{n-2nk/t})$ then the number of B-classes in $x^G \cap B\rho$ is at most N, where N is the number of distinct distributions of $n(h-k)/t$ J_2-blocks among $t-2k$ direct factors. This accounts for the $2^{n/2}$ factor since $N \leqslant 2^{n/2-nk/t}$.) The reader can check that this bound with (30) and (19) is sufficient unless $(n,q)=(2t,2)$ and (n,h,ϵ) is one of a handful of possibilities, each of which is easily dealt with.

Case 2.3. $n \geqslant 2t$, $p>2$
Here

$$|x^G| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(p-1)(1-\epsilon)} q^{\dim x^{\bar{G}}}$$

and in the usual manner we deduce that

$$|x^G \cap H| < 2\left(\frac{t^p}{p}\right)^h p^{n+\frac{n}{t}h(1-p)} 2^{\frac{t}{2}(p-1)(1+\epsilon)} \left(\frac{q+1}{q}\right)^{\frac{t}{2}(1-\epsilon)} q^{\frac{1}{t}\dim x^{\bar{G}}}. \tag{36}$$

Applying the lower bound on $\dim x^{\bar{G}}$ given in (19) we find that these bounds are sufficient unless $(n,t,h,q,\epsilon) = (6,3,1,3,+)$. In this case direct calculation yields $f(x,H) < .321$. □

Proposition 2.7. *The conclusion to Theorem 1.1 holds in case (i) of Table 2.1 for elements of prime order in $H - \mathrm{PGL}(V)$.*

Proof. Let us start by assuming $x \in G$ is a field automorphism of prime order r, in which case r is odd if $\epsilon = -$ (see [4, 3.42]). Then $q = q_0^r$ and

$$|x^G| \geqslant \frac{|\mathrm{PSL}^{\epsilon}_n(q)|}{|\mathrm{PGL}^{\epsilon}_n(q^{1/r})|} > \frac{1}{2}(q+1)^{-1} q^{(n^2-1)(1-\frac{1}{r})}. \tag{37}$$

Also, [4, 3.50] implies that $x^G \cap H \subseteq \widetilde{H}x$, where $\widetilde{H} = B.S_t$, and applying [4, 3.43] we deduce that

$$|x^G \cap H| \leqslant (q-\epsilon)^{-1} \sum_{j=0}^{\lfloor t/r \rfloor} \left[|\rho_j^{S_t}| |\mathrm{GL}^{\epsilon}_{n/t}(q)|^{j(r-1)} \left(\frac{|\mathrm{GL}^{\epsilon}_{n/t}(q)|}{|\mathrm{PGL}^{\epsilon}_{n/t}(q^{1/r})|}\right)^{t-jr} \right], \tag{38}$$

where $\rho_j \in S_t$ has cycle-shape $(r^j, 1^{t-jr})$. In particular, if $n = t$ then

$$|x^G \cap H| \leqslant (q-\epsilon)^{t-1}(i_r(S_t)+1) \leqslant (q-\epsilon)^{t-1}t!$$

(where $i_r(S_t)$ denotes the number of elements of order r in S_t) and one can check that (37) is sufficient unless $(r,\epsilon) = (2,+)$ and $t \leqslant 4$. If $(r,t) = (2,4)$ then $|x^G \cap H| \leqslant 10(q-1)^3$ since $i_2(S_4) = 9$ and the result follows via (37); if $(r,t) = (2,3)$ then (38) gives $|x^G \cap H| \leqslant q^2+q-2$ and the bound $|x^G| > \frac{1}{6}q^4$ is always sufficient. Now assume $n \geqslant 2t$. Then (38) implies that

$$|x^G \cap H| < (q-\epsilon)^{t-1}t!2^t q^{\left(\frac{n^2}{t}-t\right)\left(1-\frac{1}{r}\right)}$$

and we are left to deal with the case $(n,t,r,\epsilon) = (4,2,2,+)$ for $q \in \{4,9,16\}$. Here the result is easily derived through direct calculation. For example, if $q = 4$ then $f(x,H) < .546$ since

$$|x^G \cap H| \leqslant \frac{1}{3}\left(\frac{|\mathrm{GL}_2(4)|}{|\mathrm{PGL}_2(2)|}\right)^2 + |\mathrm{PGL}_2(4)| = 360, \quad |x^G| \geqslant \frac{|\mathrm{SL}_4(4)|}{|\mathrm{PGL}_4(2)|} = 48960.$$

The argument for involutory graph-field automorphisms is very similar.

Finally, let us assume x is an involutory graph automorphism and assume for now that $n \geqslant 3t$. Then x permutes the t direct factors in B, inducing an involutory graph automorphism on any factor which is fixed. Recall from [4] that x is said to be a *symplectic type* graph automorphism if $C_{G_0}(x)$ has socle $\mathrm{PSp}_n(q)$, otherwise x is *non-symplectic* (see [4, 3.47]). Note that

$$|x^G| > \frac{1}{2}(q+1)^{-1}q^{\frac{1}{2}(n^2+\alpha n-2)}, \tag{39}$$

where $\alpha = 1$ if x is non-symplectic, otherwise $\alpha = -1$ (see [4, 3.48]). We claim that the following two conditions hold:

(I) If $C_{G_0}(x)$ is symplectic then x induces a symplectic-type graph automorphism on each factor in B which is fixed;

(II) If $C_{G_0}(x)$ is non-symplectic and $p \neq 2$ then x induces a non-symplectic graph automorphism on each fixed factor in B; if $p = 2$ then at least one factor must be fixed and acted on as a non-symplectic graph automorphism.

An easy way to see this is to view the algebraic group $\mathrm{GL}_n(K)$ (where K is the algebraic closure of $\mathbb{F}_q$) as the stabilizer in $\mathrm{Sp}_{2n}(K)$ of a maximal totally singular subspace of the natural $\mathrm{Sp}_{2n}(K)$-module $\bar{V}$ and calculate the action of x on $\bar{V}$. Then $\nu(x) = n$ (with respect to $\bar{V}$) and it is easy to see that $C_{G_0}(x)$ is symplectic if and only if n is even and x is $\mathrm{Sp}_{2n}(K)$-conjugate to $[-I_n, I_n]$ or a_n according to the parity of p. Set $\delta = +$ if x is non-symplectic, otherwise $\delta = -$. Suppose x permutes the t factors in B with cycle-shape $(2^j, 1^{t-2j})$ and induces a non-symplectic graph automorphism on precisely $0 \leqslant k \leqslant t-2j$ of the fixed factors. Then x is $\mathrm{Sp}_{2n}(K)$-conjugate to the block-diagonal matrix $[X_\delta^j, Y^k, Z^{t-2j-k}] \in \mathrm{Sp}_{2n}(K)$, where the elements $X_\delta \in \mathrm{Sp}_{4n/t}(K)$ and $Y, Z \in \mathrm{Sp}_{2n/t}(K)$ are given as follows up to conjugacy (here $i \in K$ satisfies $i^2 = -1$ and we adopt the notation of [2] for unipotent involutions in symplectic groups):

	$p \neq 2$	$p = 2$
X_+	$[-iI_{2n/t}, iI_{2n/t}]$	$a_{2n/t}$
X_-	$[-I_{2n/t}, I_{2n/t}]$	$a_{2n/t}$
Y	$[-iI_{n/t}, iI_{n/t}]$	$b_{n/t}$ or $c_{n/t}$
Z	$[-I_{n/t}, I_{n/t}]$	$a_{n/t}$

The conditions (I) and (II) follow immediately.

If $n = 2t$ then x induces a non-trivial automorphism on each fixed direct factor $\mathrm{GL}_2^\epsilon(q)$ in $\widehat{B} = \mathrm{GL}_2^\epsilon(q)^t$ which restricts to an inner automorphism i_x of $\mathrm{SL}_2(q)$. In analogy with the case $n \geqslant 3t$, we say that x induces a *symplectic-type* automorphism on a fixed factor if and only if

i_x centralizes $\mathrm{SL}_2(q)$, otherwise the action of x on the fixed factor is said to be *non-symplectic*. With this terminology, it is easy to see that conditions (I) and (II) are valid if we delete each occurrence of the term 'graph'. Finally, if $n = t$ then x acts by inversion on each fixed factor and it is easy to see that x does not fix any factors if $C_{G_0}(x)$ is symplectic, while at least one factor is fixed if $C_{G_0}(x)$ is non-symplectic and $p = 2$.

Let us begin by assuming $C_{G_0}(x)$ is symplectic, so n is even. Now, if n/t is odd then t is even and our above comments imply that x permutes the t factors with cycle-shape $(2^{t/2})$. Therefore

$$|x^G \cap H| \leqslant (q-\epsilon)^{-1}\frac{t!}{(t/2)!2^{t/2}}|\mathrm{GL}^{\epsilon}_{n/t}(q)|^{\frac{t}{2}} \leqslant \frac{t!}{(t/2)!2^{t/2}}(q-\epsilon)^{\frac{t}{2}-1}q^{\frac{n^2}{2t}-\frac{t}{2}}$$

and (39) is sufficient unless $q = 2$ and $(n,t) \in \{(6,2),(4,4)\}$. These cases are easily settled through direct calculation. Next assume $C_{G_0}(x)$ is symplectic and n/t is even. In view of condition (I) we deduce that

$$|x^G \cap H| \leqslant (q-\epsilon)^{-1}\sum_{j=0}^{\lfloor t/2\rfloor}\left[|\rho_j^{S_t}||\mathrm{GL}^{\epsilon}_{n/t}(q)|^j\left(\frac{|\mathrm{GL}^{\epsilon}_{n/t}(q)|}{|\mathrm{Sp}_{n/t}(q)|}\right)^{t-2j}\right] < (q-\epsilon)^{t-1}t!2^tq^{\frac{n^2}{2t}-\frac{t}{2}}$$

and thus (39) is sufficient if $t \geqslant 3$. If $t = 2$ then

$$|x^G \cap H| \leqslant (q-\epsilon)\left(\frac{|\mathrm{PGL}^{\epsilon}_{n/2}(q)|}{|\mathrm{Sp}_{n/2}(q)|}\right)^2 + |\mathrm{PGL}^{\epsilon}_{n/2}(q)| < q^{\frac{1}{4}n^2-\frac{1}{2}n-2}\left(4(q-\epsilon)+q^{\frac{1}{2}n+1}\right)$$

and if we assume $n \geqslant 8$ then (39) is sufficient unless $(n,q,\epsilon) = (8,2,-)$, where a direct calculation yields $f(x,H) < .439$. Finally, if $(n,t) = (4,2)$ then $q > 2$ (see Table 2.1) and the bounds $|x^G \cap H| \leqslant (q-\epsilon) + |\mathrm{PGL}_2(q)|$ and $|x^G| \geqslant (2,q-\epsilon)^{-1}q^2(q^3-\epsilon)$ (see [9, 4.5.6, 4.8.2]) are always sufficient.

Now assume $C_{G_0}(x)$ is non-symplectic. If $n = t$ then

$$|x^G \cap H| \leqslant (q-\epsilon)^{-1}\sum_{j=0}^{\alpha}\left[|\rho_j^{S_t}|(q-\epsilon)^{t-j}\right] \leqslant (q-\epsilon)^{t-1}(i_2(S_t)+1) \leqslant t!(q-\epsilon)^{t-1},$$

where $\alpha = t/2-1$ if $(t,p) = 2$, otherwise $\alpha = \lfloor t/2\rfloor$. Now if $\epsilon = +$ then (39) is sufficient unless $t = 3$ and $q < 5$; if $\epsilon = -$ then the same bounds are sufficient if $t \geqslant 14$ or if $q \geqslant 8$. For the cases which remain, the desired result can be obtained by applying (39) with the more accurate upper bound $|x^G \cap H| \leqslant (q-\epsilon)^{t-1}(i_2(S_t)+1)$. Now assume $n \geqslant 2t$. If $p = 2$ and n/t is even then (II) implies that

$$|x^G \cap H| \leqslant (q-\epsilon)^{-1}\sum_{j=0}^{\alpha}\left[|\rho_j^{S_t}||\mathrm{GL}^{\epsilon}_{n/t}(q)|^jg(j)\right],$$

where

$$g(j) = \sum_{k=1}^{t-2j}\left[\binom{t-2j}{k}\left(\frac{|\mathrm{GL}^{\epsilon}_{n/t}(q)|}{|\mathrm{Sp}_{n/t-2}(q)|q^{n/t-1}}\right)^k\left(\frac{|\mathrm{GL}^{\epsilon}_{n/t}(q)|}{|\mathrm{Sp}_{n/t}(q)|}\right)^{t-2j-k}\right]$$

and α is defined as before. Therefore

$$|x^G \cap H| < (q-\epsilon)^{t-1}t!2^tq^{\frac{n^2}{2t}+\frac{n}{2}-t}$$

and it is easy to see that this bound also holds if $\mathrm{hcf}(n/t,p,2) = 1$. If we assume $t \geqslant 3$ then the desired result follows via (39). Next assume $t = 2$ and $n \geqslant 6$. If p is odd then

$$|x^G \cap H| \leqslant (q-\epsilon)\left(\frac{|\mathrm{PGL}^{\epsilon}_{n/2}(q)|}{|\mathrm{SO}^{\epsilon'}_{n/2}(q)|}\right)^2 + |\mathrm{PGL}^{\epsilon}_{n/2}(q)| < q^{\frac{1}{4}n^2-1}\left(4(q-\epsilon)q^{\frac{1}{2}n-1}+1\right)$$

and (39) is sufficient unless $(n,q,\epsilon)=(6,3,-)$, where direct calculation yields $f(x,H)<.586$. Similarly, if $p=2$ then

$$|x^G\cap H|<4(q-\epsilon)q^{\frac{1}{4}n^2-2}(q^{\frac{1}{2}n}+2\beta),$$

where $\beta=1$ if $n/2$ is even, otherwise $\beta=0$. Applying (39), we see that we are left to deal with a handful of cases which are easily dealt with by deriving more accurate bounds. Finally, let us assume $(n,t)=(4,2)$. If $p\neq 2$ then

$$|x^G\cap H|\leqslant (q-\epsilon)\left(\frac{|\mathrm{PGL}_2(q)|}{|\mathrm{PGO}_2^+(q)|}+\frac{|\mathrm{PGL}_2(q)|}{|\mathrm{PGO}_2^-(q)|}\right)^2+|\mathrm{PGL}_2(q)|=q^4(q-\epsilon)+q(q^2-1)$$

and the desired result follows since

$$|x^G|\geqslant\frac{|\mathrm{PSL}_4^\epsilon(q)|}{|\mathrm{SO}_4^-(q)|}=(4,q-\epsilon)^{-1}q^4(q^2-1)(q^3-\epsilon).$$

Similarly, if $p=2$ then $|x^G\cap H|\leqslant(q^4-1)(q-\epsilon)$, $|x^G|\geqslant q^2(q^3-\epsilon)(q^4-1)$ and again the result follows. □

2.3 Proof of Theorem 1.1: Case (ii) of Table 2.1

Let σ be a Frobenius morphism of $\bar{G}=\mathrm{PSp}_n(K)$ such that $\bar{G}_\sigma$ has socle $G_0=\mathrm{PSp}_n(q)$. Here $\iota=1/n$ when $t=2$ and so we may assume $n\geqslant 8$ if $t=2$. Observe that

$$H\cap\mathrm{PGL}(V)\leqslant\left(\left((2,q-1)^{t-1}.\mathrm{PSp}_{\frac{n}{t}}(q)^t\right).(2,q-1)\right).S_t=B.S_t,$$

where B is the image of $\mathrm{GSp}_{n/t}(q)^t$ in $\mathrm{PGSp}_n(q)=\bar{G}_\sigma$. If q is odd then $B=\widetilde{B}.\langle\delta\rangle$, where $\widetilde{B}$ is the image of $\mathrm{Sp}_{n/t}(q)^t$ in $\mathrm{PSp}_n(q)$ and δ is an involutory diagonal automorphism of $\mathrm{PSp}_n(q)$.

Let $x\in H\cap\mathrm{PGL}(V)$ be an element of prime order r and suppose $y\in B\rho$ is G-conjugate to x, where $\rho\in S_t$ has cycle-shape $(r^k,1^{t-kr})$ for some $k\geqslant 0$. Assume y fixes each subspace V_j with $j>kr$ in the decomposition $V=V_1\oplus\cdots\oplus V_t$. If r is odd then [4, 3.11] implies that y lifts to a unique element $\hat{y}=(\hat{y}_1,\ldots,\hat{y}_t)\pi\in\mathrm{Sp}_n(q)$ of order r. Furthermore, the proof of [10, 4.5] implies that $\hat{y}$ is $\widehat{B}$-conjugate to $(I_{n/t},\ldots,I_{n/t},\hat{y}_{kr+1},\ldots,\hat{y}_t)\pi$, where $\widehat{B}=\mathrm{Sp}_{n/t}(q)^t$ and $\hat{y}_j\in\mathrm{Sp}_{n/t}(q)$ satisfies $\hat{y}_j^r=I_{n/t}$ for all $j>kr$. Therefore

$$|y^B|\leqslant|\hat{y}^{\widehat{B}}|=|\mathrm{Sp}_{n/t}(q)|^{k(r-1)}\prod_{j>kr}|\hat{y}_j^{\mathrm{Sp}_{n/t}(q)}| \tag{40}$$

and it is easy to see that the same bound holds if $r=2$ and $C_{\bar{G}}(x)$ is connected. We also note that if $p=2$ then each involution $\rho\in S_t$ with cycle-shape $(2^k,1^{t-2k})$ is G-conjugate to $a_{nk/t}$, where we label representatives of G-classes of involutions as in [2]. Throughout, we define the integer h as in (17).

The case $x\in H-\mathrm{PGL}(V)$ is straightforward. Here $x\in G$ is a field automorphism of prime order r, so $q=q_0^r$ and [4, 3.48] states that $|x^G|>\frac{1}{4}q^{(n^2+n)(1-1/r)/2}$. Furthermore, we have

$$|x^G\cap H|\leqslant\sum_{j=0}^{\lfloor t/r\rfloor}\left[|\rho_j^{S_t}||\mathrm{Sp}_{n/t}(q)|^{j(r-1)}\left(\frac{|\mathrm{Sp}_{n/t}(q)|}{|\mathrm{Sp}_{n/t}(q^{1/r})|}\right)^{t-jr}\right]<2^tt!q^{\frac{1}{2}\left(\frac{n^2}{t}+n\right)\left(1-\frac{1}{r}\right)},$$

where $\rho_j\in S_t$ has cycle-shape $(r^j,1^{t-jr})$, and these bounds are sufficient unless $(n,t,r,q)=(6,3,2,4)$, where direct calculation yields $f(x,H)<.535$.

Proposition 2.8. *The conclusion to Theorem 1.1 holds in case (ii) of Table 2.1 for semisimple elements of prime order in $H\cap\mathrm{PGL}(V)$.*

Proof. Let $x \in H \cap \mathrm{PGL}(V)$ be a semisimple element of prime order $r \neq p$. We prove the proposition in two parts, starting with the case $x^G \cap H \subseteq B$.

Case 1. $x^G \cap H \subseteq B$
Let us start by assuming $r > 2$. Let $i \geqslant 1$ be minimal such that $r|(q^i - 1)$ and let $\mu = (l, a_1, \ldots, a_k)$ denote the associated σ-tuple of x, where $k = (r-1)/i$ (see [4, 3.27]). Let d denote the number of non-zero terms a_j in μ and note that d is even if i is odd. From the proof of Proposition 2.1 we have

$$\dim x^{\bar{B}} \leqslant \frac{1}{t} \dim x^{\bar{G}} + \frac{1}{2}(n-l)\left(1 - \frac{1}{t}\right)$$

and [4, 3.30] implies that

$$|x^G| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{d(2-e)} q^{\dim x^{\bar{G}}}, \tag{41}$$

where $e = 2$ if i is odd, otherwise $e = 1$. Appealing to (40) and arguing as in the proof of Proposition 2.5 (see (8)) we deduce that

$$|x^G \cap H| < \log_2 q. \left(\frac{n-l}{di} + 1\right)^{\frac{d}{e}(t-1)} 2^{\frac{1}{2}(e-1)dt} q^{\frac{1}{t}\dim x^{\bar{G}} + \frac{1}{2}(n-l)\left(1-\frac{1}{t}\right)}.$$

Now $n \geqslant \max(l + di, eti)$ and [4, 3.33] gives a lower bound for $\dim x^{\bar{G}}$. If $i \geqslant 3$ then one can check that these bounds are always sufficient; if $i < 3$ then we are left to deal with a handful of cases and the desired result is easily obtained by computing more accurate bounds.

Now assume $r = 2$. Write $s = \nu(x)$ and observe that the hypothesis $x^G \cap H \subseteq B$ implies that $s < n/t$. In particular, s is even and x lifts to an involution $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_t) \in \widehat{B}$. If $\nu(\hat{x}_i) = s_i$ (with respect to the natural $\mathrm{Sp}_{n/t}(q)$-module) then

$$\dim x^{\bar{B}} = \sum_{i=1}^{t} s_i\left(\frac{n}{t} - s_i\right) = \frac{ns}{t} - \sum_{i=1}^{t} s_i^2 \leqslant \frac{1}{t}s(n-s) = \frac{1}{t}\dim x^{\bar{G}}$$

and thus

$$|x^G \cap H| < \binom{t + s/2 - 1}{s/2} 2^t q^{\frac{1}{t}s(n-s)}, \quad |x^G| > \frac{1}{2}q^{s(n-s)}.$$

It is easy to check that these bounds are sufficient for all $t \geqslant 3$. Finally, if $t = 2$ then

$$\begin{aligned}
|x^G \cap H| &\leqslant \sum_{j=0}^{s/2}\left[\frac{|\mathrm{Sp}_{n/2}(q)|}{|\mathrm{Sp}_{2j}(q)||\mathrm{Sp}_{n/2-2j}(q)|}.\frac{|\mathrm{Sp}_{n/2}(q)|}{|\mathrm{Sp}_{s-2j}(q)||\mathrm{Sp}_{n/2-s+2j}(q)|}\right] \\
&< 4\sum_{j=0}^{s/2} q^{\frac{1}{2}(ns+8sj-2s^2-16j^2)} < 4\left(\frac{q^2+1}{q^2-1}\right) q^{\frac{1}{2}s(n-s)}
\end{aligned}$$

and the bound $|x^G| > \frac{1}{2}q^{s(n-s)}$ is always sufficient.

Case 2. $x^G \cap (H - B) \neq \emptyset$
Write $x = b\pi$, where $b \in B$ and $h(\pi) = h > 0$. If $C_{\bar{G}}(x)$ is connected then x lifts to an element $\hat{x} = (\hat{x}_1, \ldots, \hat{x}_t)\pi \in \mathrm{Sp}_n(q)$ of order r which is $\widehat{B}$-conjugate to $(I_{n/t}, \ldots, I_{n/t}, \hat{x}_{hr+1}, \ldots, \hat{x}_t)\pi$ and it is easy to see that $\dim x^{\bar{G}}$ is minimal if $\hat{x}_j = I_{n/t}$ for each j, i.e.

$$\dim x^{\bar{G}} \geqslant \dim \pi^{\bar{G}} = \begin{cases} \frac{nh}{t}\left(n - \frac{nh}{t}\right) & \text{if } r = 2 \\ \frac{nh}{2t}(r-1)(2n+1-nhr/t) & \text{otherwise.} \end{cases} \tag{42}$$

Case 2.1. $r > 2$
Let $i \geqslant 1$ be minimal such that $r|(q^i - 1)$ and define e as in Case 1. Then [4, 3.30] gives

$$|x^G| > \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(2-e)(r-1)} q^{\dim x^{\bar{G}}} \tag{43}$$

and we claim that

$$|x^G \cap H| < \log_2 q.2\left(\frac{t^r}{r}\right)^h 2^{\frac{1}{2}(e-1)(r-1)t}\left(\frac{1}{2}(r+1)\right)^{\frac{n}{2}} q^{\left(\frac{1}{t}+\frac{2}{n+2}\right)\dim x^{\bar{G}}}. \tag{44}$$

If $y \in x^G \cap H$ then using (40) and the proof of Proposition 2.1 we deduce that

$$|y^B| < 2^{\frac{1}{2}(e-1)(r-1)t} q^{\left(\frac{1}{t}+\frac{2}{n+2}\right)\dim x^{\bar{G}}}.$$

The claim now follows because the number of distinct B-classes in $x^G \cap H$ is at most

$$\log_2 q.\sum_{k=0}^{h}\left[\frac{t!}{k!(t-kr)!r^k}\left(\frac{r-1}{ei}+1\right)^{\frac{1}{ei}\left(n-\frac{nrk}{t}\right)}\right] < \log_2 q.2\left(\frac{t^r}{r}\right)^h\left(\frac{1}{2}(r+1)\right)^{\frac{n}{2}}$$

(see (7) and (22) for example). The reader can check that the bounds (42), (43) and (44) are sufficient with the exception of a small number of cases when $h = 1$ and $r \in \{3, 5\}$. Here the desired result quickly follows through direct calculation.

Case 2.2. $r = 2$
We begin by assuming $C_{\bar{G}}(x)$ is connected, so $t > 2h$ since $\nu(x) < n/2$. Appealing to the proof of Proposition 2.1 we deduce that

$$|x^G \cap H| < 2\left(\frac{t^2}{2}\right)^h 2^{\frac{n}{2}+t} q^{\left(\frac{1}{t}+\frac{1}{n}\right)\dim x^{\bar{G}}}$$

(see (44)) where $\dim x^{\bar{G}} \geqslant (nh/t)(n - nh/t)$. Now $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ and the reader can check that these bounds are always sufficient if $n > 2t$ and $h \geqslant 2$; if $h = 1$ and $n > 2t$ then we are left to deal with the case $(n, t) = (12, 3)$ for $q \in \{3, 5\}$, where direct calculation yields $f(x, H) < .343$. If $n = 2t$ then the maximality of h implies that $\nu(x) = 2h$ and the result follows since $|x^G| > \frac{1}{2}q^{4h(t-h)}$ and

$$|x^G \cap H| \leqslant \sum_{k=0}^{h}\left[\frac{t!}{k!(t-2k)!2^k}\binom{t-2k}{h-k}|\mathrm{Sp}_2(q)|^k\right] < t^{2h}q^h(q^2-1)^h.$$

Now assume $C_{\bar{G}}(x)$ is non-connected. There are four cases to consider. If $C_G(x)$ is of type $\mathrm{Sp}_{n/2}(q)^2$ then $|x^G| > \frac{1}{4}q^{n^2/4}$ and our earlier arguments apply since each $y \in x^G \cap B$ lifts to an involution in $\widehat{B}$. We leave the details to the reader. Next assume $C_G(x)$ is of type $\mathrm{Sp}_{n/2}(q^2)$. If $n/2t$ is odd then t is even and the intersection $x^G \cap B\rho$ is non-empty if and only if $\rho \in S_t$ has cycle-shape $(2^{t/2})$. Therefore

$$|x^G \cap H| \leqslant \frac{t!}{(t/2)!2^{t/2}}\frac{1}{2}|\mathrm{Sp}_{n/t}(q)|^{\frac{t}{2}} < \frac{t!}{(t/2)!2^{t/2+1}}q^{\frac{n^2}{4t}+\frac{n}{4}}$$

and the desired result follows since $|x^G| > \frac{1}{4}q^{n^2/4}$. Alternatively, if $n/2t$ is even then

$$|x^G \cap H| \leqslant \sum_{k=0}^{\lfloor t/2\rfloor}\left[\frac{t!}{k!(t-2k)!2^k}|\mathrm{Sp}_{n/t}(q)|^k\left(\frac{|\mathrm{Sp}_{n/t}(q)|}{|\mathrm{Sp}_{n/2t}(q^2)|}\right)^{t-2k}\right] < 2\left(\frac{t^2}{2}\right)^{\lfloor\frac{t}{2}\rfloor} q^{\frac{n^2}{4t}+\frac{n}{4}}$$

and one can check that the bound $|x^G| > \frac{1}{4}q^{n^2/4}$ is sufficient unless $(n,t,q) = (8,2,3)$, where direct calculation yields $f(x,H) < .618$. Finally, if $C_G(x)$ is of type $\mathrm{GL}^{\epsilon}_{n/2}(q)$ then

$$|x^G \cap H| \leqslant \sum_{k=0}^{\lfloor t/2 \rfloor} \left[\frac{t!}{k!(t-2k)!2^k} |\mathrm{Sp}_{n/t}(q)|^k \left(\frac{|\mathrm{Sp}_{n/t}(q)|}{|\mathrm{GL}^{\epsilon}_{n/2t}(q)|} \right)^{t-2k} \right] < 2 \left(\frac{t^2}{2} \right)^{\lfloor \frac{t}{2} \rfloor} 2^t q^{\frac{n^2}{4t} + \frac{n}{2}},$$

$$|x^G| = \frac{|\mathrm{Sp}_n(q)|}{|\mathrm{GL}^{\epsilon}_{n/2}(q)|2} > \frac{1}{4} \left(\frac{q}{q+1} \right) q^{\frac{1}{4}n(n+2)}$$

and we are left to deal with a handful of cases which are easily settled. For example, if $t = 3$ then the above bounds are sufficient unless $n = 6$, where we calculate that $f(x,H) < .609$ since

$$|x^G \cap H| \leqslant \left(\frac{|\mathrm{Sp}_2(q)|}{|\mathrm{GL}^{\epsilon}_1(q)|} \right)^3 + 3|\mathrm{Sp}_2(q)| \frac{|\mathrm{Sp}_2(q)|}{|\mathrm{GL}^{\epsilon}_1(q)|}, \;\; |x^G| = \frac{|\mathrm{Sp}_6(q)|}{|\mathrm{GL}^{\epsilon}_3(q)|2}.$$

□

Proposition 2.9. *The conclusion to Theorem 1.1 holds in case (ii) of Table 2.1 for unipotent elements of prime order in $H \cap \mathrm{PGL}(V)$.*

Proof. Let $x \in H \cap \mathrm{PGL}(V)$ be a unipotent element of order p, with associated partition $\lambda \vdash n$. Note that any odd parts in λ must occur with an even multiplicity (see [4, §3.3]).

Case 1. $x^G \cap H \subseteq B$, $p > 2$
According to the proof of Proposition 2.1 we have

$$\dim x^{\bar{B}} \leqslant \frac{1}{t} \dim x^{\bar{G}} + \frac{1}{2}(n-e) \left(1 - \frac{1}{t} \right), \tag{45}$$

where e is the number of odd parts in λ. If $\lambda = (k^{n/k})$, for some $k \geqslant 2$, then the hypothesis $x^G \cap H \subseteq B$ implies that $k \leqslant n/t$ and applying (40) and (45) we deduce that

$$|x^G \cap H| < 2^t q^{\frac{1}{t}\dim x^{\bar{G}} + \frac{1}{2}(1-\frac{1}{t})n}, \;\; |x^G| > \frac{1}{4}(q+1)^{-1} q^{\dim x^{\bar{G}}+1},$$

where $\dim x^{\bar{G}} \geqslant \frac{1}{4}n(n+2)$. These bounds are sufficient unless $(n,t) = (6,3)$, where direct calculation yields $f(x,H) < .501$. Next assume $\lambda = (2^j, 1^{n-2j})$ for some $1 \leqslant j < n/2$, so $\dim x^{\bar{G}} = j(n-j+1)$. If $j = 1$ then the desired result follows from the bounds $|x^G \cap H| < t.q^{n/t}$ and $|x^G| > \frac{1}{4}q^n$. Now assume $j \geqslant 2$. If $n = 2t$ then $t \geqslant 3$ and the bounds $|x^G \cap H| < \binom{t}{j} q^{2j}$ and $|x^G| > \frac{1}{4}(q+1)^{-1} q^{j(2t-j+1)+1}$ are always sufficient so assume for the remainder that $n \geqslant 4t$. If $j = 2$ then $|x^G \cap H| < \binom{t}{2} q^{2n/t} + 2tq^{2n/t-2}$, $|x^G| > \frac{1}{4}(q+1)^{-1} q^{2n-1}$ and the result follows. If $j \geqslant 3$ then the bounds

$$|x^G \cap H| < \binom{t+j-1}{j} 2^t q^{\frac{1}{t}\dim x^{\bar{G}} + j(1-\frac{1}{t})}, \;\; |x^G| > \frac{1}{4} q^{\dim x^{\bar{G}}}$$

are sufficient unless $(n,t,j,q) = (8,2,3,3)$, where more accurate bounds yield $f(x,H) < .423$.

Now assume $\lambda = (m^{a_m}, \ldots, 2^{a_2}, 1^l) \vdash n$, where $m = n/t \geqslant 2$, and let $d \geqslant 1$ denote the number of non-zero terms a_j. (Note that the prime order hypothesis implies that $m \leqslant p$.) The case $d = 1$ is straightforward so let us assume $d \geqslant 2$. Then arguing as in the proof of Proposition 2.6 (see (34)) we deduce that

$$\dim x^{\bar{G}} \geqslant \frac{1}{4}n^2 + \frac{1}{4}(d^2-d+2)n - \frac{1}{16}d^4 - \frac{1}{24}d^3 + \frac{3}{16}d^2 - \frac{1}{3}d - \frac{1}{4}l^2 - \frac{1}{2}l$$

and $n \geqslant \max(t(4d+2)/3, l+2d+2d^2/3-2/3)$ since odd parts in λ must occur with an even multiplicity. Furthermore, using (45) we deduce that

$$|x^G \cap H| < 2^{td}\left(\frac{n/2-d^2/4+d/4-l/2-1}{d}+1\right)^{d(t-1)} q^{\frac{1}{t}\dim x^{\bar{G}}+\frac{1}{2}(n-l)\left(1-\frac{1}{t}\right)}$$

(see (35)). Now [4, 3.18] implies that

$$|x^G| > \left(\frac{1}{2}\right)^{d+1}\left(\frac{q}{q+1}\right)^{d} q^{\dim x^{\bar{G}}}$$

and one can check that these bounds are sufficient with the exception of a small number of cases when $(t,q)=(2,3)$. These remaining cases are easily dealt with by computing more accurate bounds. For instance, if $(n,l)=(12,0)$ then $\lambda=(3^2,2^3)$ and we deduce that $f(x,H)<.547$ since $|x^G \cap H| < 2.3^{26}$ and $|x^G| > \frac{1}{4}3^{50}$.

Case 2. $x^G \cap H \subseteq B$, $p=2$

Let us begin by assuming x is G-conjugate to a_l for some even integer l. Then the hypothesis $x^G \cap H \subseteq B$ implies that $l<n/t$ since every element of order two in S_t is an a-type involution. If $(\ ,\ )$ is a non-degenerate G-invariant symmetric bilinear form on V then $(vx,v)=0$ for all $v \in V$ and thus if $y=(y_1,\ldots,y_t) \in x^G \cap B$ then each non-trivial y_i must be an a-type involution, hence $n \geqslant 4t$. Now, if $l=2$ then the bounds $|x^G \cap H| < 2tq^{2n/t-4}$ and $|x^G| > \frac{1}{2}q^{2n-4}$ are always sufficient. If $l \geqslant 4$ then using Proposition 2.1 we deduce that

$$|x^G \cap H| < \binom{t+l/2-1}{l/2}2^t q^{\frac{1}{t}l(n-l)},\ \ |x^G| > \frac{1}{2}q^{l(n-l)}$$

and the reader can check that these bounds are always sufficient.

Assume for the remainder that x is G-conjugate to either b_l or c_l, the precise type depending on the parity of l. Then the hypothesis $x^G \cap H \subseteq B$ implies that $l \leqslant n/t$ and we note that if $y=(y_1,\ldots,y_t) \in x^G \cap B$ then at least one y_j is a b- or c-type involution. Now, if $n=2t$ then each non-trivial y_i must be $\mathrm{Sp}_2(q)$-conjugate to b_1 and the subsequent bounds $|x^G \cap H| < \binom{t}{l}q^{2l}$ and $|x^G| > \frac{1}{2}q^{l(2t-l+1)}$ are always sufficient. Assume for the remainder that $n \geqslant 4t$. If $l=1$ then $|x^G \cap H| < tq^{n/t}$, $|x^G| > \frac{1}{2}q^n$ and the result follows. Similarly, if $l=2$ then the bounds $|x^G \cap H| < \binom{t}{2}q^{2n/t}+2tq^{2n/t-2}$ and $|x^G| > \frac{1}{2}q^{2n-2}$ are always sufficient. Now assume $l \geqslant 3$. Using the proof of Proposition 2.1 we deduce that

$$|x^G \cap H| < \binom{t+l-1}{l}2^{2t}q^{\frac{1}{t}\dim x^{\bar{G}}+(1-\frac{1}{t})l},\ \ |x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}},$$

where $\dim x^{\bar{G}} = l(n-l+1)$. (Note that the number of B-classes in $x^G \cap B$ is determined by the number of ways l J_2-blocks can be distributed among the t direct factors, together with one of two choices (either a- or c-type) for each factor which is assigned an even number of J_2-blocks. This number is at most $\binom{t+l-1}{l}2^t$.) If we assume $t \geqslant 3$ then these bounds are sufficient with the exception of a handful of cases with which we can calculate directly. For $t=2$ we require more accurate bounds. Let N_1 (resp. N_2) denote the number of elements $(y_1,y_2) \in x^G \cap B$ such that one (resp. neither) of the y_i is an a-type involution. Then $|x^G \cap H| = N_1+N_2$ and we claim that

$$N_1 < 2^3\left(\frac{q^2+1}{q^2-1}\right)q^{\frac{1}{2}\dim x^{\bar{G}}},\ \ N_2 < 2^2\left(\frac{q^2+1}{q^2-1}\right)q^{\frac{1}{2}(\dim x^{\bar{G}}+l)}. \tag{46}$$

First consider N_1. For all possible even integers $j \geqslant 0$, choose $x_j=(y_1,y_2) \in x^G \cap B$ such that y_1 is $\mathrm{Sp}_{n/2}(q)$-conjugate to a_j (set $a_0 = I_{n/2}$). Then using [4, 3.22] we calculate that

$$N_1 = 2\sum_j |x_j^B| < 2^3\sum_j q^{j(n/2-j)+(l-j)(n/2-l+j+1)} = 2^3\sum_j q^{f(j)}.$$

Evidently $\max_{j\in\mathbb{Z}} f(j) \leqslant f(l/2) = \frac{1}{2}\dim x^{\bar{G}}$ and the claim for N_1 now follows since $f(j)$ is even and $|f(j+1)-f(j)| \geqslant 2$ for all j. Similar reasoning justifies the upper bound for N_2. Now $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ and if we assume $l \geqslant 3$ then the reader can check that the upper bound on $|x^G \cap H|$ derived from (46) is always sufficient if $q \geqslant 4$. Similarly, if $q=2$ and $l \geqslant 4$ then it remains to deal with the case $(n,l) = (8,4)$, where direct calculation gives $f(x,H) < .590$. Finally, if $(l,q) = (3,2)$ then using the proof of [4, 3.22] we calculate that

$$|x^G \cap H| \leqslant 2|b_1^{\mathrm{Sp}_{n/2}(2)}|\left(|a_2^{\mathrm{Sp}_{n/2}(2)}| + |c_2^{\mathrm{Sp}_{n/2}(2)}|\right) = \frac{8}{3}(2^{\frac{n}{2}-2}-1)(2^{\frac{n}{2}}-1)^2,$$

$$|x^G| \geqslant \frac{4}{3}(2^{n-4}-1)(2^{n-2}-1)(2^n-1)$$

and the reader can check that these bounds are always sufficient.

Case 3. $x^G \cap (H-B) \neq \emptyset$

Define $h>0$ as in (17) and fix $\pi \in S_t$ such that $h(\pi) = h$. It is easy to see that (42) holds (on substituting $r=p$). First assume $p>2$. Here $|x^G|$ is minimal if $\lambda = (p^{nh/t}, 1^{n-nhp/t}) \vdash n$ and thus

$$|x^G| > \frac{1}{2}q^{\frac{nh}{2t}(p-1)(2n+1-nhp/t)}. \tag{47}$$

If $n=2t$ then the maximality of h implies that $\lambda = (p^{2h}, 1^{n-2hp})$, so

$$|x^G \cap H| = |x^G \cap (H-B)| \leqslant \frac{t!}{h!(t-hp)!p^h}(q(q^2-1))^{h(p-1)}$$

and the result follows via (47). Now assume $n \geqslant 4t$. Using Proposition 2.1 and arguing as in the proof of Proposition 2.6 (see (36)) we deduce that

$$|x^G \cap H| < 2\left(\frac{t^p}{p}\right)^h p^{n+\frac{n}{t}h(1-p)}2^{pt}q^{\left(\frac{1}{t}+\frac{2}{n+2}\right)\dim x^{\bar{G}}}.$$

Applying the lower bound on $\dim x^{\bar{G}}$ given in (42), we calculate that this bound with (47) is sufficient unless $(h,p) = (1,3)$ and (n,t) is one of a handful of cases. As usual, these exceptional cases are easily settled through direct calculation.

Next assume $p=2$ and x is an a-type involution, i.e. x is G-conjugate to $a_{nh/t+j}$, where $0 \leqslant j < n/t$ is even. If $t=2$ then $(h,j) = (1,0)$ and

$$|x^G \cap (H-B)| \leqslant |\mathrm{Sp}_{n/2}(q)| < q^{\frac{1}{8}n(n+2)}, \quad |x^G| > \frac{1}{2}q^{\frac{1}{4}n^2}.$$

Clearly, either $x^G \cap B$ is empty or $n \equiv 0\,(8)$ and $|x^G \cap B| < 4q^{n^2/8}$ since each $y \in x^G \cap B$ must act on both V_1 and V_2 as an $a_{n/4}$ involution. We leave the reader to check that these bounds are always sufficient. Now assume $t \geqslant 3$. Evidently, each B-class in $x^G \cap B\pi$ is determined by a choice of elements $x_{2h+1}, \ldots, x_t$ in $\mathrm{Sp}_{n/t}(q)$ (up to conjugacy) such that each non-trivial x_k is conjugate to a_{l_k} for some even integer l_k and $\sum_k l_k = j$. If $n=2t$ then $q \geqslant 4$ (see Table 2.1), $j=0$ and the result follows since

$$|x^G \cap H| = |x^G \cap (H-B)| < \left(\frac{t!}{h!(t-2h)!2^h}\right)q^{3h}, \quad |x^G| > \frac{1}{2}q^{4h(t-h)}.$$

Now assume $n \geqslant 4t$. Then [4, 3.22] gives $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ and using the proof of Proposition 2.1 we deduce that

$$|x^G \cap H| < 2\left(\frac{t^2}{2}\right)^h 2^{\frac{n}{4}+t}q^{\left(\frac{1}{t}+\frac{1}{n}\right)\dim x^{\bar{G}}}.$$

(Note that if $x^G \cap B\rho$ is non-empty then the number of B-classes in $x^G \cap B\rho$ is at most $2^{n/4}$.) Applying (42) (with $r=p=2$) we find that the above bounds are always sufficient if $h \geqslant 2$.

If $h=1$ then $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$, where $\dim x^{\bar{G}} = (n/t+j)(n-n/t-j)$, and the desired result follows since the proof of Proposition 2.1 implies that

$$|x^G \cap B| < \binom{t+n/2t+j/2-1}{n/2t+j/2} 2^t q^{\frac{1}{t}\dim x^{\bar{G}}}$$

and

$$|x^G \cap (H-B)| < \binom{t}{2}\binom{t-3+j/2}{j/2} 2^{t-2} q^{\left(\frac{1}{t}+\frac{1}{n}\right)\dim x^{\bar{G}}}.$$

Finally, let us assume $p=2$ and x is a b- or c-type involution. In this case, x is conjugate to b_l or c_l, where $l = nh/t+j$ and $1 \leqslant j \leqslant n/t$. In particular, we note that $t > 2h$. If $n=2t$ then

$$|x^G \cap H| < \sum_{k=0}^{h}\left[|\rho_k^{S_t}|\binom{t-2k}{2h+j-2k} q^{4h+2j-k}\right] < \left(\binom{t}{2h+j} + \frac{t!}{(t-2h-j)!j!}\right) q^{4h+2j},$$

where $\rho_k \in S_t$ has cycle-shape $(2^k, 1^{t-2k})$, and the result follows since $|x^G| > \frac{1}{2}q^{(2h+j)(2t-2h-j+1)}$. Now assume $n \geqslant 4t$. Arguing as before we deduce that

$$|x^G \cap H| < 2\binom{t^2}{2}^h 2^{\frac{n}{2}+t} 2^t q^{\left(\frac{1}{t}+\frac{1}{n}\right)\dim x^{\bar{G}}}, \quad |x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}},$$

where $\dim x^{\bar{G}} \geqslant (nh/t+1)(n-nh/t)$, and the result follows if $h \geqslant 2$. If $h=1$ then

$$|x^G \cap B| < 2^t \binom{t+n/t+j-1}{n/t+j} 2^t q^{\left(\frac{1}{t}+\frac{2}{n+2}\right)\dim x^{\bar{G}}}$$

and

$$|x^G \cap (H-B)| < 2^{t-2}\binom{t}{2}\binom{t-3+j}{j} 2^{t-2} q^{\left(\frac{1}{t}+\frac{2}{n+2}\right)\dim x^{\bar{G}}},$$

where $\dim x^{\bar{G}} \geqslant (nh/t+j)(n-nh/t-j+1)$. Now $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ and we find that these bounds are sufficient with the exception of a handful of cases (n,t,j) with which we can calculate directly. For example, if $(n,t,j) = (12,3,2)$ then x is G-conjugate to c_6, so $|x^G| > \frac{1}{2}q^{42}$. If $y = (y_1, y_2, y_3) \in x^G \cap B$ then at least one y_i is $\mathrm{Sp}_4(q)$-conjugate to c_2 and [4, 3.22] implies that $|x^G \cap B| < 2^3(q^{18}+3q^{16}+3q^{14})$. If $\pi = (12) \in S_3$ and $z \in B\pi$ then z is B-conjugate to $[I_4, I_4, c_2]\pi$ and $|x^G \cap (H-B)| < 3.2q^{16}$. We conclude that $f(x,H) < .539$ for all $q \geqslant 2$. ☐

2.4 Proof of Theorem 1.1: Cases (iii), (iv) and (v) of Table 2.1

Fix a Frobenius morphism σ of $\bar{G} = \mathrm{PSO}_n(K)$ such that $\bar{G}_\sigma$ has socle $\mathrm{P\Omega}_n^\epsilon(q)$. Let $(\triangle)$ denote the hypothesis "$(n,\epsilon) = (8,+)$ *and G contains triality automorphisms*", and note that if $(\triangle)$ holds then we may assume that H is of type $\mathrm{O}_4^+(q) \wr S_2$ or $\mathrm{O}_2^{\epsilon'}(q) \wr S_4$ (see [4, 3.3]).

Proposition 2.10. *The conclusion to Theorem 1.1 holds in case (iii) of Table 2.1.*

Proof. Here $q = p \geqslant 3$, $n \geqslant 7$ and $H \leqslant 2^{n-1}.S_n = B.S_n \leqslant \mathrm{PGL}(V)$. Let $x \in H$ be an element of prime order r and note that $x^G \cap (H-B)$ is non-empty. If r is odd then $x^G \cap B\pi$ is non-empty if and only if $\pi \in S_t$ has cycle-shape $(r^h, 1^{n-hr})$ for a uniquely determined integer $h \geqslant 1$. Therefore

$$|x^G \cap H| = |x^G \cap (H-B)| \leqslant \left(\frac{n!}{h!(n-hr)!r^h}\right) 2^{h(r-1)}$$

and

$$|x^G| > \begin{cases} \frac{1}{2}\left(\frac{q}{q+1}\right)^{\frac{1}{2}(r-1)} q^{\dim x^{\bar{G}}} & \text{if } r \neq p \\ \frac{1}{8}\left(\frac{q}{q+1}\right)^{2} q^{\dim x^{\bar{G}}} & \text{if } r = p, \end{cases}$$

where $\dim x^{\bar{G}} = \frac{1}{2}(r-1)(2nh-h^2r-h)$. These bounds are sufficient unless $(n,r,q)=(7,3,3)$, where direct calculation yields $f(x,H)<.590$.

Now assume $r=2$. Define $h \geqslant 1$ as in (17) and observe that the maximality of h implies that $\nu(x)=h$. If $n=2h$ and $C_G(x)$ is of type $\mathrm{GL}^{\epsilon}_{n/2}(q)$ then the result follows since

$$|x^G \cap H| = |x^G \cap (H-B)| \leqslant \frac{n!}{(n/2)!}, \quad |x^G| > \frac{1}{4}(q+1)^{-1}q^{\frac{1}{4}n(n-2)+1}.$$

For the remainder, we may assume that x lifts to an involution in $\mathrm{O}_1(q) \wr S_n$ and thus

$$|x^G \cap H| \leqslant \sum_{j=0}^{h}\left[\frac{n!}{j!(n-2j)!2^j}|\mathrm{O}_1(q)|^j\binom{n-2j}{h-j}\right] \leqslant \frac{n!}{(n-2h)!}. \tag{48}$$

If $n=2h$ then $|x^G| > \frac{1}{8}q^{n^2/4}$ and we are left to deal with the case $(n,q)=(8,3)$. Here (48) gives $|x^G \cap H| \leqslant 14630$ and we conclude that $f(x,H)<.619<5/8$ since $|x^G| > \frac{1}{8}3^{16}$. On the other hand, if $n>2h$ then $|x^G| > \frac{1}{4}(q+1)^{-1}q^{h(n-h)+1}$ and if $h \geqslant 2$ then (48) is sufficient with the exception of a handful of cases which we can deal with by computing more accurate bounds. For instance, if $(n,h,q)=(7,2,3)$ then $H \cap G_0 \cong 2^6.A_7$ (see [9, 4.2.15]) and therefore $f(x,H)<.609$ if $x \in G_0$ since

$$|x^G \cap H| \leqslant \binom{7}{2} + \frac{7!}{2!3!} = 441, \quad |x^G| = |\mathrm{O}_7(3) : \mathrm{O}_5(3)\mathrm{O}_2^-(3)| = 22113.$$

Similarly, we calculate that $f(x,H)<.500$ if $x \notin G_0$. Finally, if $h=1$ then $|x^G| > \frac{1}{4}q^{n-1}$, $|x^G \cap H| \leqslant n + \binom{n}{2}|\mathrm{O}_1(q)| = n^2$ and we are left to deal with the cases $(n,q)=(8,3)$ and $(7,3)$. These cases are easily settled through direct calculation. □

Proposition 2.11. *The conclusion to Theorem 1.1 holds for cases (iv) and (v) of Table 2.1.*

Proof. We deal with both cases simultaneously. Let $\bar{B} = \mathrm{PSO}_{n/t}(K)^t$ and observe that

$$H \cap \mathrm{PGL}(V) \leqslant \left(\left((2,q-1)^{t-1}.\mathrm{PO}^{\epsilon'}_{\frac{n}{t}}(q)^t\right).(2,n/t,q-1)\right).S_t = B.S_t,$$

where B is the image of $\mathrm{GO}^{\epsilon'}_{n/t}(q)^t$ in $\mathrm{PGO}^{\epsilon}_n(q)$ and $(2,n/t,q-1)$ is a cyclic group of order $\mathrm{hcf}(2,n/t,q-1)$. Let $x \in H \cap \mathrm{PGL}(V)$ be an element of prime order r and suppose $y \in x^G \cap B\rho$, where $\rho \in S_t$ has cycle-shape $(r^k,1^{t-kr})$ for some $k \geqslant 0$ and y fixes each subspace V_j with $j>kr$ in the decomposition $V = V_1 \oplus \cdots \oplus V_t$. If we assume $\nu(x)<n/2$ if $r=2<p$ then y lifts to an element $\hat{y} = (\hat{y}_1,\ldots,\hat{y}_t)\rho \in \widehat{B}\rho$ of order r which is $\widehat{B}$-conjugate to $(I_{n/t},\ldots,I_{n/t},\hat{y}_{kr+1},\ldots,\hat{y}_t)\rho$, where $\widehat{B} = \mathrm{O}^{\epsilon'}_{n/t}(q)^t$ and

$$|y^B| \leqslant |\hat{y}^{\widehat{B}}| = |\mathrm{O}^{\epsilon'}_{n/t}(q)|^{k(r-1)}\prod_{j>kr}|\hat{y}_j^{\mathrm{O}^{\epsilon'}_{n/t}(q)}|. \tag{49}$$

We note that if $p=2$ then every element of order two in S_t acts on V as an a-type involution.

Case 1. $x \in H \cap \mathrm{PGL}(V)$

Here we can argue as in the proof of Proposition 2.9 and for brevity we only provide details in the case where x is a semisimple involution.

Assume for now that ($\triangle$) does not hold. Write $s=\nu(x)$, define h as in (17) and let us start by assuming $s<n/2$ and $h=0$, so $x^G \cap H \subseteq B$ and $\dim x^{\bar{G}} = s(n-s)$. If $s=1$ then applying Lemma 2.4 we deduce that $|x^G \cap H| < t(q+1)q^{n/t-2}$, $|x^G| > \frac{1}{4}q^{n-1}$ and we are left to deal with the case $(t,q)=(2,3)$. Here $\epsilon=+$ (see Table 2.1) and if we assume $n \equiv 0\,(4)$ then the bounds

$$|x^G \cap H| \leqslant 2|\mathrm{O}^-_{\frac{n}{2}}(3) : \mathrm{O}_{\frac{n}{2}-1}(3)|, \quad |x^G| \geqslant \frac{1}{2}|\mathrm{O}^+_n(3) : \mathrm{O}_{n-1}(3)|$$

are sufficient; the case $n \equiv 2\,(4)$ is similar. If $s=2$ then the desired result follows since

$$|x^G \cap H| < \binom{t}{2}\left(\frac{q+1}{q}\right)^2 q^{2\left(\frac{n}{t}-1\right)} + t.2q^{2\frac{n}{t}-4}, \quad |x^G| > \frac{1}{4}(q+1)^{-1}q^{2n-3}.$$

Now assume $s \geqslant 3$. Applying Proposition 2.1 and Lemma 2.4 we see that

$$|x^G \cap H| < \binom{t+s-1}{s} 2^t q^{\frac{1}{t}s(n-s)}, \quad |x^G| > \frac{1}{4}q^{s(n-s)}$$

and it is easy to check that these bounds are always sufficient if $t \geqslant 3$. If $t=2$ then

$$|x^G \cap H| < 2^2\left(\frac{q^2+1}{q^2-1}\right) q^{\frac{1}{2}s(n-s)} \tag{50}$$

and if we assume $s \geqslant 3$ then the bound $|x^G| > \frac{1}{4}q^{s(n-s)}$ is sufficient unless $s=q=3$ and $n \in \{8,10\}$. These cases are easily dealt with through direct calculation.

Now assume $s < n/2$ and $h > 0$. Then $t > 2h$,

$$|x^G| > \frac{1}{4}(q+1)^{-1}q^{s(n-s)+1} \tag{51}$$

and applying Proposition 2.1, Lemma 2.4 and (49) we deduce that

$$|x^G \cap H| < 2\left(\frac{t^2}{2}\right)^h 2^{t+n} q^{\frac{1}{t}s(n-s)}. \tag{52}$$

If we assume $h \geqslant 3$ then these bounds are sufficient unless $(n,t,h,q) = (14,7,3,3)$. Here the hypothesis $s < n/2$ implies that $s=6$ and direct calculation yields $f(x,H) < .260$. If $h=2$ then we are left to deal with a handful of cases with $n=2t$; here the maximality of h implies that $s \in \{4,5\}$ and the desired result follows by applying (51) and a more accurate upper bound for $|x^G \cap H|$. For instance, if $s=4$ then

$$|x^G \cap H| \leqslant \binom{t}{4}(q+1)^4 + 12\binom{t}{4}(q+1)^3 + \left(12\binom{t}{4}+3\binom{t}{3}\right)(q+1)^2 + 6\binom{t}{3}(q+1) + \binom{t}{2}$$

and the result follows via (51). Now assume $h=1$. If $n=2t$ then $s \in \{2,3\}$. If $s=2$ then $|x^G \cap H| \leqslant \binom{t}{2}(q+1)^2 + 2\binom{t}{2}(q+1) + t$ and (51) is good enough. The case $s=3$ is similar. If $n > 2t$ then (51) is always sufficient since

$$|x^G \cap H| < \left(2\binom{t+s-1}{s} + \binom{t}{2}\binom{t+j-3}{j}\right) 2^{t-1} q^{\frac{1}{t}s(n-s)}.$$

Next assume $s = n/2$. If $C_G(x)$ is of type $\mathrm{O}^{\epsilon''}_{n/2}(q) \times \mathrm{O}^{\epsilon'''}_{n/2}(q)$ then our earlier work applies since each $y \in x^G \cap B\rho$ lifts to an involution $\hat{y} = (\hat{y}_1,\ldots,\hat{y}_t)\rho \in \widehat{B}\rho$. In particular, if $t=2$ then appealing to (49) and (50) we deduce that

$$|x^G \cap B| < 2^2\left(\frac{q^2+1}{q^2-1}\right) q^{\frac{1}{8}n^2}, \quad |x^G \cap (H-B)| \leqslant \frac{1}{2}|\mathrm{O}^{\epsilon'}_{n/2}(q)| < q^{\frac{1}{8}n(n-2)}$$

and the bound $|x^G| > \frac{1}{8}q^{n^2/4}$ is sufficient unless $(n,q) \in \{(10,3),(8,3)\}$; these cases are easily resolved. Now assume $t \geqslant 3$. Applying Proposition 2.1, Lemma 2.4 and (49) we deduce that

$$|x^G \cap H| < 2\left(\frac{t^2}{2}\right)^{\lfloor \frac{t}{2} \rfloor} 2^{t+n} q^{\frac{n^2}{4t}}$$

(see (52)) and if we assume $n > 2t$ then the bound $|x^G| > \frac{1}{8}q^{n^2/4}$ is sufficient unless $(n,t,q) = (12,3,3)$, where direct calculation yields $f(x,H) < .370$. Finally, suppose $n=2t$. If $t \geqslant 5$ then the previous bounds are sufficient with the exception of the cases

$$(t,q) \in \{(8,3),(7,3),(6,3),(5,5),(5,3)\}.$$

Here we apply the more accurate upper bound

$$|x^G \cap H| \leqslant \sum_{k=0}^{t'/2} \left[\frac{t!}{k!(t-2k)!}(q-\epsilon')^{k+\beta} \sum_{l=0}^{t'/2-k} \left(\binom{t'-2k}{2l}(q-\epsilon')^{2l} \binom{t'-2k-2l}{t'/2-k-l} \right) \right], \tag{53}$$

where $t' = t - \beta$, $t \equiv \beta\,(2)$ and the desired result quickly follows. For instance, if $(t, q) = (5, 3)$ then (53) gives $|x^G \cap H| \leqslant 11416$ and thus $f(x, H) < .348$ since $|x^G| \geqslant |\mathrm{O}_{10}^{+}(3) : \mathrm{O}_5(3)^2|$. Finally, if $(n, t) = (8, 4)$ then (53) implies that

$$|x^G \cap H| \leqslant (q-\epsilon')^4 + 12(q-\epsilon')^3 + 24(q-\epsilon')^2 + 24(q-\epsilon') + 6$$

and we conclude that $f(x, H) < .456$ since $|x^G| \geqslant \frac{1}{2}|\mathrm{O}_8^{+}(q) : \mathrm{O}_4^{-}(q)^2|$.

Next assume $C_G(x)$ is of type $\mathrm{O}_{n/2}^{\epsilon''}(q^2)$, where $\epsilon'' = \epsilon$ if $n/2$ is even. Then $|x^G| > \frac{1}{8}q^{n^2/4}$ and if we assume n/t is odd then

$$|x^G \cap H| \leqslant \frac{t!}{(t/2)!2^{t/2}} \frac{1}{2}|\mathrm{O}_{n/t}^{\epsilon'}(q)|^{\frac{t}{2}} < \frac{t!}{(t/2)!2} q^{\frac{n^2}{4t} - \frac{n}{4}}$$

since t is even and $x^G \cap B\rho$ is non-empty if and only if $\rho \in S_t$ has cycle-shape $(2^{t/2})$. These bounds are always sufficient. On the other hand, if n/t is even then

$$|x^G \cap H| \leqslant \sum_{k=0}^{\lfloor t/2 \rfloor} \left[\frac{t!}{k!(t-2k)!2^k} |\mathrm{O}_{n/t}^{\epsilon'}(q)|^k \left(\frac{|\mathrm{O}_{n/t}^{\epsilon'}(q)|}{|\mathrm{O}_{n/2t}^{\zeta}(q^2)|} \right)^{t-2k} \right] < 2\left(\frac{t^2}{2}\right)^{\lfloor \frac{t}{2} \rfloor} \left(\frac{q+1}{q}\right)^t 2^t q^{\frac{n^2}{4t}},$$

where $\zeta = \epsilon'$ if $n/2t$ is even. If we assume $t \geqslant 3$ then this bound with $|x^G| > \frac{1}{8}q^{n^2/4}$ is sufficient unless $(n, t, q) \in \{(10, 5, 3), (8, 4, 3)\}$. In these cases the desired result is easily obtained through direct calculation. If $t = 2$ then the more accurate bounds

$$|x^G \cap H| \leqslant \left(\frac{|\mathrm{O}_{n/2}^{\epsilon'}(q)|}{|\mathrm{O}_{n/4}^{\zeta}(q^2)|} \right)^2 + \frac{1}{2}|\mathrm{O}_{n/2}^{\epsilon'}(q)| < (q^{\frac{n}{4}}+1)q^{\frac{1}{8}n(n-2)}, \;\; |x^G| > \frac{1}{8}q^{\frac{1}{4}n^2}$$

are sufficient unless $(n, q) = (8, 3)$, where direct calculation yields $f(x, H) < .541$.

Finally, suppose $C_G(x)$ is of type $\mathrm{GL}_{n/2}^{\epsilon''}(q)$, in which case

$$|x^G| > \frac{1}{4}(q+1)^{-1}q^{\frac{1}{4}n(n-2)+1}. \tag{54}$$

Assume for now that n/t is even and let the symbol (†) represent the following conditions on ϵ'' and n/t with respect to ϵ':

ϵ'	conditions
$+$	$\epsilon'' = +$ if $n/t \equiv 2\,(4)$
$-$	$\epsilon'' = -$ and $n/t \equiv 2\,(4)$

If (†) holds then using [8, Table 4.5.1] we deduce that

$$|x^G \cap H| \leqslant \sum_{k=0}^{\lfloor t/2 \rfloor} \left[\frac{t!}{k!(t-2k)!2^k} |\mathrm{O}_{n/t}^{\epsilon'}(q)|^k \left(\frac{|\mathrm{O}_{n/t}^{\epsilon'}(q)|}{|\mathrm{GL}_{n/2t}^{\epsilon''}(q)|} \right)^{t-2k} \right] < 2\left(\frac{t^2}{2}\right)^{\lfloor \frac{t}{2} \rfloor} 2^{2t} q^{\frac{n^2}{4t} - \frac{n}{4}} \tag{55}$$

and if we assume $t \geqslant 3$ then one can check that (54) is sufficient unless $(n, t) = (8, 4)$ or $(n, t, q) \in \{(12, 6, 3), (10, 5, 3)\}$. If $(n, t) = (8, 4)$ then the result follows via (54) since (55) gives

$$|x^G \cap H| \leqslant 2^4 + 2^3\binom{4}{2}(q+1) + 3.2^2(q+1)^2;$$

the other two cases are similar. If $t=2$ then $\epsilon=+$ (see Table 2.1), $n\equiv 0\,(4)$ and the bounds

$$|x^G\cap H|\leqslant \left(\frac{|\mathrm{O}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon''}_{n/4}(q)|}\right)^2+\frac{1}{2}|\mathrm{O}^{\epsilon'}_{n/2}(q)|<(2^4+q^{\frac{n}{4}})q^{\frac{1}{8}n(n-4)}$$

and (54) are sufficient unless $n=8$ and $q\in\{3,5\}$. Here $x^G\cap B$ is empty if $\epsilon'=-$ (see [4, Table 3.8]) and direct calculation yields $f(x,H)<.546$.

Now assume that either n/t is odd or (†) does not hold if n/t is even. Then t is even and $x^G\cap B\rho$ is non-empty if and only if $\rho\in S_t$ has cycle-shape $(2^{t/2})$. Therefore

$$|x^G\cap H|\leqslant \frac{t!}{(t/2)!2^{t/2}}\frac{1}{2}|\mathrm{O}^{\epsilon'}_{n/t}(q)|^{\frac{t}{2}}<\frac{t!}{(t/2)!}\frac{1}{2}\left(\frac{q+1}{q}\right)^{\frac{t}{2}}q^{\frac{1}{4t}n(n-t)}$$

and the reader can check that this bound with (54) is always sufficient.

To complete the proof, let us assume (△) holds and recall that we may assume H is of type $\mathrm{O}^+_4(q)\wr S_2$ or $\mathrm{O}^{\epsilon'}_2(q)\wr S_4$. In view of [4, 3.55(iii)], we can also assume that $C_G(x)$ is not of type $\mathrm{O}^+_4(q)^2$. If $\nu(x)=1$ then $x^G\cap H\subseteq B$, $|x^G|\geqslant\frac{3}{2}q^3(q^4-1)$ and the bounds

$$|x^G\cap H|\leqslant\left\{\begin{array}{ll}6q(q^2-1) & \text{if } t=2\\ 12(q+1) & \text{if } t=4\end{array}\right.$$

are always sufficient. Similarly, if $\nu(x)=3$ and $t=2$ then $x^G\cap H\subseteq B$ and the bounds

$$|x^G\cap H|\leqslant 12\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}_3(q)||\mathrm{O}_1(q)|}\left(1+\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}^+_2(q)|^2}+\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}^-_2(q)|^2}\right)<24q^7+12q^3$$

and $|x^G|>\frac{3}{4}q^{15}$ are sufficient unless $q=3$, where direct calculation yields $f(x,H)<.559$. If $t=4$ then $|x^G|>\frac{3}{4}q^{15}$ and the bounds

$$|x^G\cap B|\leqslant 12(q-\epsilon')^3+36(q-\epsilon'),\;\; |x^G\cap(H-B)|\leqslant 3\binom{4}{2}|\mathrm{O}^{\epsilon'}_2(q)|2(q-\epsilon')=72(q-\epsilon')^2$$

are always sufficient.

Next assume $C_G(x)$ is of type $\mathrm{GL}^{\epsilon''}_4(q)$. If $t=2$ then the bounds

$$\begin{aligned}|x^G\cap H|&\leqslant 2\left(\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}^+_2(q)|^2}+\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}^-_2(q)|^2}\right)+\left(\frac{2|\mathrm{O}^+_4(q)|}{|\mathrm{O}_3(q)||\mathrm{O}_1(q)|}\right)^2+\left(\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{GL}^{\epsilon''}_2(q)|}\right)^2+\frac{1}{2}|\mathrm{O}^+_4(q)|\\ &=2q^6+4q^2(q+\epsilon'')^2-2q^4+4q^2\end{aligned}$$

(see [4, 3.55(iii)]) and

$$|x^G|\geqslant 3\frac{|\mathrm{SO}^+_8(q)|}{|\mathrm{GL}^{\epsilon''}_4(q)|2}=\frac{3}{2}q^6(q+\epsilon'')(q^2+1)(q^3+\epsilon'') \tag{56}$$

are always sufficient. Similarly, if $t=4$ then

$$|x^G\cap B|\leqslant 4+\binom{4}{2}(q-\epsilon')^2+|\mathrm{O}^{\epsilon'}_2(q):\mathrm{GL}^{\epsilon'}_1(q)|^4=6(q-\epsilon')^2+20,$$

$$|x^G\cap(H-B)|\leqslant\binom{4}{2}|\mathrm{O}^{\epsilon'}_2(q)|\left(1+|\mathrm{O}^{\epsilon'}_2(q):\mathrm{GL}^{\epsilon'}_1(q)|^2\right)+\frac{3}{2}|\mathrm{O}^{\epsilon'}_2(q)|^2=60(q-\epsilon')+6(q-\epsilon')^2$$

and the desired result follows via (56). Finally, let us assume x is conjugate to $[-I_4,I_4]$. If $t=2$ then the bounds

$$\begin{aligned}|x^G\cap H|&\leqslant\left(\frac{2|\mathrm{O}^+_4(q)|}{|\mathrm{O}_3(q)||\mathrm{O}_1(q)|}\right)^2+\left(\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}^+_2(q)|^2}+\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}^-_2(q)|^2}\right)^2+2\left(\frac{|\mathrm{O}^+_4(q)|}{|\mathrm{O}^+_2(q^2)|}\right)^2+1+\frac{3}{2}|\mathrm{O}^+_4(q)|\\ &=4q^2(q^2-1)^2+(q^4+q^2)^2+2q^4(q^2-1)^2+1\end{aligned}$$

and

$$|x^G| \geqslant 3\frac{|\mathrm{SO}_8^+(q)|}{|\mathrm{SO}_4^-(q)|^2 4} = \frac{3}{4}q^8(q^2-1)(q^6-1)$$

are always sufficient. The case $t = 4$ is very similar.

Case 2. $x \in H - \mathrm{PGL}(V)$
If $x \in G$ is a field automorphism of prime order r then $q = q_0^r$ and [4, 3.48] gives

$$|x^G| > \frac{1}{4}q^{\frac{1}{2}(n^2-n)(1-\frac{1}{r})}. \tag{57}$$

Now, if r is odd then

$$|x^G \cap H| \leqslant \sum_{j=0}^{\lfloor t/r \rfloor}\left[\frac{t!}{j!(t-jr)!r^j}|\mathrm{O}_{n/t}^{\epsilon'}(q)|^{j(r-1)}\left(\frac{|\mathrm{O}_{n/t}^{\epsilon'}(q)|}{|\mathrm{O}_{n/t}^{\epsilon'}(q^{1/r})|}\right)^{t-jr}\right] < 2^t t! q^{\frac{1}{2}\left(\frac{n^2}{t}-n\right)(1-\frac{1}{r})} \tag{58}$$

and the desired result follows via (57). Next assume $q = q_0^2$ and x is an involutory field or graph-field automorphism, so $\epsilon \neq -$ and (57) holds (with $r = 2$). If n/t is odd then (58) is valid (with $r = 2$) and we find that (57) is always sufficient. If $\epsilon' = +$ then

$$|x^G \cap H| \leqslant \sum_{j=0}^{\lfloor t/2 \rfloor}\left[\frac{t!}{j!(t-2j)!2^j}|\mathrm{O}_{n/t}^{+}(q)|^{j}\left(\frac{|\mathrm{O}_{n/t}^{+}(q)|}{|\mathrm{O}_{n/t}^{+}(q^{1/2})|}+\frac{|\mathrm{O}_{n/t}^{+}(q)|}{|\mathrm{O}_{n/t}^{-}(q^{1/2})|}\right)^{t-2j}\right] < 2^{2t} t! q^{\frac{n^2}{4t}-\frac{n}{4}}$$

and (57) is sufficient unless $(n,t,q) = (8,2,4)$ (note that $(n,t,q) \neq (8,4,4)$ - see Table 2.1). Here direct calculation yields $f(x,H) < .530$. Finally, if $\epsilon' = -$ then t is even (see Table 2.1) and $x^G \cap B\rho$ is non-empty if and only if $\rho \in S_t$ has cycle-shape $(2^{t/2})$. Therefore

$$|x^G \cap H| \leqslant \frac{t!}{(t/2)!2^{t/2}}|\mathrm{O}_{n/t}^{-}(q)|^{\frac{t}{2}} < \frac{t!}{(t/2)!}\left(\frac{q+1}{q}\right)^{\frac{t}{2}} q^{\frac{n^2}{4t}-\frac{n}{4}}$$

and (57) is always sufficient.

Now assume $(\triangle)$ holds. If x is a triality graph-field automorphism then $q = q_0^3$ and [4, 3.48] gives $|x^G| > \frac{1}{4}q^{56/3}$. If $t = 4$ then the trivial bound

$$|x^G \cap H| < |H| \leqslant 3\log_2 q.4!2^4(q+1)^4 \tag{59}$$

is always sufficient. On the other hand, if $t = 2$ then we may assume $\epsilon' = +$. Since $\Omega_4^+(q) \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ (central product) we deduce that

$$|x^G \cap H| \leqslant \frac{4!}{3}|\mathrm{SL}_2(q)|^2\frac{|\mathrm{SL}_2(q)|}{|\mathrm{SL}_2(q^{1/3})|} < 16q^8$$

and the desired result follows since $|x^G| > \frac{1}{4}q^{56/3}$.

Finally, let us assume x is a triality graph automorphism, and assume for now that $t = 4$. If x is a non-G_2 triality (see [4, 3.47]) then $|x^G| > \frac{1}{8}q^{20}$ (see [4, Table 3.10]) and we find that (59) is sufficient for all $q \geqslant 4$. If $q = 3$ then a calculation using GAP [7] yields $f(x,H) < .405$. Similarly, if $q = 2$ then $\epsilon' = -$ (see Table 2.1) and using GAP we deduce that $f(x,H) < .555$. If x is a G_2-type triality then $|x^G| > \frac{1}{8}q^{14}$ and (59) is sufficient for all $q \geqslant 9$. The cases $5 \leqslant q \leqslant 8$ are easily settled. For example, if $q = 5$ then $|H \cap G_0| \leqslant 62208$ (see [9, 4.2.11]), $|x^{G_0}| = 1521000000$ and thus

$$f(x,H) \leqslant \frac{\log(24.62208)}{\log(8.1521000000)} < .613.$$

Now assume $q < 5$, in which case $\epsilon' = -$ (see Table 2.1). Here we compute the following results using GAP [7, 11]:

q	$\lvert x^G \cap H\rvert \leqslant$	$\lvert x^G\rvert \geqslant$	$f(x,H) <$
4	800	266342400	.345
3	512	1166400	.447
2	288	14400	.592

For the remainder we may assume $t = 2$, so $\epsilon' = +$ and $q \geqslant 3$ (see Table 2.1). As previously remarked, there is an isomorphism $\Omega_4^+(q) \cong \mathrm{SL}_2(q) \circ \mathrm{SL}_2(q)$ and it is helpful to consider the corresponding situation for algebraic groups. Here $A_1^4.S_4 \leqslant D_4.S_3$ and a triality graph automorphism τ acts as a 3-cycle on the A_1-factors and centralizes the remaining factor only if $C_{D_4}(\tau) = G_2$; either τ centralizes this factor or it acts on it as an inner automorphism of order three. Therefore, if x is a G_2-type triality then

$$|x^G \cap H| \leqslant \frac{4!}{3}|\mathrm{SL}_2(q)|^2 < 8q^6, \;\; |x^G| > \frac{1}{8}q^{14}$$

and we are left to deal with the case $q = 3$, where direct calculation yields $f(x, H) < .604$. Likewise, if x is a non-G_2 triality then

$$|x^G \cap H| \leqslant \frac{4!}{3}|\mathrm{SL}_2(q)|^2 \frac{|\mathrm{SL}_2(q)|}{q-1} < 16q^8, \;\; |x^G| > \frac{1}{8}q^{20}$$

and we conclude that $f(x, H) < .582$ for all $q \geqslant 3$. □

3 Proof of Theorem 1.1: $H \in \mathscr{C}_3$

The subgroups in $\mathscr{C}_3$ arise from field extensions of prime degree k, where k divides the dimension n of the natural G_0-module V. As advertised in §2, we also deal with the $\mathscr{C}_2$-subgroups of unitary, symplectic and orthogonal groups which stabilize a totally singular $n/2$-decomposition of V. The cases we shall consider in this section are listed in Table 3.1 (see [9, Tables 4.2.A, 4.3.A]).

	G_0	type of H	conditions
(i)	$\mathrm{PSL}_n^\epsilon(q)$	$\mathrm{GL}_{n/k}^\epsilon(q^k)$	k odd if $\epsilon = -$
(ii)	$\mathrm{PSU}_n(q)$	$\mathrm{GL}_{n/2}(q^2)$	n even
(iii)	$\mathrm{PSp}_n(q)$	$\mathrm{Sp}_{n/k}(q^k)$	n/k even
(iv)	$\mathrm{P\Omega}_n^\epsilon(q)$	$\mathrm{O}_{n/k}^\epsilon(q^k)$	$n/k \geqslant 4$ even
(v)	$\Omega_n(q)$	$\mathrm{O}_{n/k}(q^k)$	nkq odd, $n/k \geqslant 3$
(vi)	$\mathrm{P\Omega}_n^\epsilon(q)$	$\mathrm{O}_{n/2}(q^2)$	$n/2$ odd, $q \equiv -\epsilon\,(4)$
(vii)	$\mathrm{PSp}_n(q)$	$\mathrm{GL}_{n/2}^\epsilon(q).2$	q odd
(viii)	$\mathrm{P\Omega}_n^+(q)$	$\mathrm{GL}_{n/2}^{\epsilon'}(q).2$	$n \equiv 0\,(4)$ if $\epsilon' = -$
(ix)	$\mathrm{P\Omega}_n^-(q)$	$\mathrm{GU}_{n/2}(q).2$	$n \equiv 2\,(4)$

Table 3.1: The collection $\mathscr{C}_3$

Proposition 3.1. *The conclusion to Theorem 1.1 holds in cases (i) and (ii) of Table 3.1.*

Proof. We may assume $n \geqslant 3$. Let $\bar{G} = \mathrm{PSL}_n(K)$, $\bar{B} = \mathrm{PSL}_{n/k}(K)$ and let σ be a Frobenius morphism of $\bar{G}$ such that $\bar{G}_\sigma$ has socle $\mathrm{PSL}_n^\epsilon(q)$. Let V denote the natural G_0-module. We only give details for case (i) of Table 3.1; a very similar argument applies in case (ii) and the reader can easily make the necessary minor adjustments. We partition the proof into a number of separate cases, where Case i.j is a subcase of Case i.

Case 1. $x \in H \cap \mathrm{PGL}(V)$
According to [9, (4.3.10)] we have

$$H \cap \mathrm{PGL}(V) \leqslant \left(\left(\frac{q^k - \epsilon}{q - \epsilon}\right).\mathrm{PGL}_{\frac{n}{k}}^\epsilon(q^k)\right).\langle\phi\rangle = B.k,$$

where ϕ acts on $\mathrm{PGL}_{n/k}^\epsilon(q^k)$ as a field automorphism of order k and B is the image of $\mathrm{GL}_{n/k}^\epsilon(q^k)$ in $\mathrm{PGL}_n^\epsilon(q)$. Let $x \in H \cap \mathrm{PGL}(V)$ be an element of prime order r and write $B = \widehat{B}/Z$, where

$\widehat{B} = \mathrm{GL}^{\epsilon}_{n/k}(q^k)$ and $Z = q - \epsilon$ (i.e. Z is a cyclic group of order $q-\epsilon$). If $x \in B$ then the proof of [4, 3.11] implies that either there exists an element $\hat{x} \in \widehat{B}$ of order r such that $|x^B| = |\hat{x}^{\widehat{B}}|$, or $r|(q-\epsilon)$ and $C_{\bar{G}}(x)$ is non-connected. Set $s = \nu(x)$ with respect to V and note that

$$|x^G \cap H| \leqslant |H \cap \mathrm{PGL}(V)| < 2kq^{\frac{n^2}{k}-1}. \tag{60}$$

Case 1.1. $k \geqslant 5$
If $C_{\bar{G}}(x)$ is non-connected then r divides n and the bounds (14) and (60) are sufficient for all $k \geqslant 5$. Now assume $C_{\bar{G}}(x)$ is connected. If $s \geqslant n/2$ then [4, 3.38] implies that $|x^G| > \frac{1}{2}(q+1)^{-n}q^{(n^2+2n-2)/2}$ and (60) is sufficient unless $(n,k,q) = (5,5,2)$. Here r is odd and thus [4, 3.36] implies that $|x^G| > (1/2)(2/3)^4 2^{15}$ since we are assuming $s \geqslant 3$. We conclude that $f(x,H) < .625$ since $|x^G \cap H| \leqslant |H \cap \mathrm{PGL}(V)| \leqslant 155$.

Next suppose $s < n/2$. If $x^G \cap (H - B) \neq \emptyset$ then $r = k$ and

$$x = \begin{cases} [I_{n/k}, \omega I_{n/k}, \ldots, \omega^{k-1} I_{n/k}] & \text{if } p \neq k \\ [J_k^{n/k}] & \text{if } p = k \end{cases} \tag{61}$$

(up to $\bar{G}$-conjugacy) where $\omega \in K$ is a primitive k^{th} root of unity and J_k is a standard Jordan block of size k. Therefore $s = n(1-1/k) \geqslant n/2$, and so the hypothesis $s < n/2$ implies that $x^G \cap H \subseteq B$. Consequently, we may define $s_0 = \nu(x)$ with respect to the action of x on the natural B-module and we note that the hypothesis $s < n/2$ implies that $s_0 > 0$. Therefore $s \geqslant k$ and $n \geqslant 3k$ since $s \geqslant ks_0$ (see the proof of [10, 4.2]). If x is unipotent then $|x^G| > \frac{1}{2}(q+1)^{-1}q^{2s(n-s)+1}$ and appealing to [4, 3.15, 3.24, 3.38] we deduce that

$$\begin{aligned} |x^G \cap H| &< \left(\frac{q^k - \epsilon}{q - \epsilon}\right).2\left(\frac{q^k}{q^k - 1}\right)^{\frac{s}{k}} q^{\frac{1}{k}(2ns - s^2 - sk)}.\sum_{s_0=1}^{\lfloor s/k \rfloor} k_{s_0,p,u}(\mathrm{PGL}^{\epsilon}_{\frac{n}{k}}(q^k)) \\ &< 4\left(\frac{q^k}{q^k - 1}\right)^{\frac{s}{k}+1} q^{\frac{1}{k}(2ns - s^2 + k^2 - k)}, \end{aligned}$$

where $k_{s_0,p,u}(\mathrm{PGL}^{\epsilon}_{\frac{n}{k}}(q^k))$ denotes the number of distinct classes in $\mathrm{PGL}^{\epsilon}_{\frac{n}{k}}(q^k)$ of elements y of order p such that $\nu(y) = s_0$. These bounds also hold if x is semisimple (see [4, 3.40]) and the desired result follows since $5 \leqslant k \leqslant s \leqslant \frac{1}{2}(n-1)$.

Case 1.2. $k < 5$, $x^G \cap (H - B) \neq \emptyset$
Here $r = k$ and (61) holds (up to $\bar{G}$-conjugacy). If $k = 3$ then the desired result quickly follows via (60). Now assume $k = 2$, so $\epsilon = +$ (see Table 3.1). Applying [4, 3.43] we deduce that

$$|x^G \cap (H - B)| \leqslant (q+1)|\phi^{\mathrm{PGL}_{n/2}(q^2)}| < 2(q+1)q^{\frac{1}{4}n^2 - 1}. \tag{62}$$

If $p = 2$ then $|x^G| > \frac{1}{2}q^{n^2/2}$ and the desired result follows via (62) if $n \equiv 2\,(4)$ since $x^G \cap B$ is empty; if $n \equiv 0\,(4)$ then any element in $x^G \cap B$ is $\bar{B}$-conjugate to $[J_2^{n/4}]$, whence $|x^G \cap B| < q^{n^2/4}$ and we are left to deal with the case $(n,q) = (4,2)$, where direct calculation yields $f(x,H) < .712$. Now assume $p \neq 2$. If $n \equiv 2\,(4)$ then either $x^G \cap B$ is empty or $C_G(x)$ is of type $\mathrm{GL}_{n/2}(q^2)$ and $x^G \cap B = \{z\}$, where z is the unique central involution in B. In this case, the desired result follows via (62) since $|x^G| > \frac{1}{4}q^{n^2/2}$. On the other hand, if $n \equiv 0\,(4)$ and $C_G(x)$ is of type $\mathrm{GL}_{n/2}(q)^2$ then $x^G \cap B = y_1^B$; if $C_G(x)$ is of type $\mathrm{GL}_{n/2}(q^2)$ then $x^G \cap B = \{z\} \cup y_2^B$, where

$$y_1 = \begin{pmatrix} & I_{n/4} \\ I_{n/4} & \end{pmatrix}, \quad y_2 = \begin{pmatrix} & \omega^{q+1} I_{n/4} \\ I_{n/4} & \end{pmatrix}$$

and $\mathbb{F}^*_{q^2} = \langle \omega \rangle$. In either case we deduce that

$$|x^G \cap B| \leqslant 1 + \frac{1}{2}|\mathrm{GL}_{n/2}(q^2) : \mathrm{GL}_{n/4}(q^2)^2| < q^{\frac{1}{4}n^2}, \quad |x^G| > \frac{1}{4}q^{\frac{1}{2}n^2}$$

and (62) is sufficient unless $(n,q)=(4,3)$, where direct calculation gives $f(x,H)<.661$.

Case 1.3. $k<5$, $x^G\cap H\subseteq B$, $r=p$

If $x\in B$ has associated partition $\lambda'=(m^{a_m},\ldots,1^{a_1})\vdash m$, where $m=n/k$, then it is clear that the Jordan form of x on V corresponds to the partition $\lambda=(m^{ka_m},\ldots,1^{ka_1})\vdash n$. In particular, the corresponding $\bar{B}$- and $\bar{G}$-classes are uniquely determined by λ and [6, 2.3] implies that $\dim x^{\bar{G}}=k^2\dim x^{\bar{B}}$. Therefore

$$|x^G\cap H|\leqslant |x^{\mathrm{PGL}^{\epsilon}_{n/k}(q^k)}|<2^tq^{\frac{1}{k}\dim x^{\bar{G}}},\ \ |x^G|>\frac{1}{2}\left(\frac{q}{q+1}\right)^t q^{\dim x^{\bar{G}}-1} \tag{63}$$

where t denotes the number of non-zero terms a_j in λ. If $t=1$ then $n\geqslant 2k$, [6, 2.4] implies that $\dim x^{\bar{G}}\geqslant \frac{1}{2}n^2$ and one can check that the bounds in (63) are sufficient unless $(n,k,q)=(4,2,3)$, where direct calculation yields $f(x,H)<.545$ (note that $x^G\cap(H-B)$ is non-empty if $(n,k,q)=(4,2,2)$). Now assume $t\geqslant 2$. Here $n\geqslant\frac{1}{2}kt(t+1)$ and [4, 3.25] implies that

$$\dim x^{\bar{G}}\geqslant k^2\left(\frac{n}{k}(t^2-t)-\frac{1}{4}t^4+\frac{1}{6}t^3+\frac{1}{4}t^2-\frac{1}{6}t\right).$$

Therefore (63) is sufficient unless $(k,t)=(2,2)$ and $q\leqslant 3$; here the bounds $|x^G\cap H|<2q^{\frac{1}{2}\dim x^{\bar{G}}}$ and $|x^G|>\frac{1}{2}q^{\dim x^{\bar{G}}}$ are always sufficient since $\dim x^{\bar{G}}\geqslant 4n-8$.

Case 1.4. $k<5$, $x^G\cap H\subseteq B$, $r\neq p$

Suppose $r=2$. If $C_{\bar{G}}(x)$ is connected then $|x^G\cap H|<2q^{\frac{1}{k}\dim x^{\bar{G}}}$, $|x^G|>\frac{1}{2}(q+1)^{-1}q^{\dim x^{\bar{G}}+1}$ and the result follows since $\dim x^{\bar{G}}\geqslant 2k(n-k)$. If $C_{\bar{G}}(x)$ is non-connected then the hypotheses imply that $k=3$ and $n\equiv 0\,(6)$ and the subsequent bounds $|x^G\cap H|<2q^{n^2/6}$ and $|x^G|>\frac{1}{4}(q+1)^{-1}q^{n^2/2+1}$ are always sufficient.

Now assume $r>2$ and suppose $C_{\bar{G}}(x)$ is connected. Let $i\geqslant 1$ be minimal such that $r|(q^i-1)$ and $i_0\geqslant 1$ minimal such that $r|(q^{ki_0}-1)$. Observe that

$$i_0=\left\{\begin{array}{ll} i/k & \text{if } k \text{ divides } i\\ i & \text{otherwise.}\end{array}\right. \tag{64}$$

Define the integers l and d as in [4, 3.32] and define $c=c(i,\epsilon)$ as in the statement of [4, 3.33].

Suppose k does not divide i. Then $i=i_0$ and σ- and σ^k-orbits coincide (see [4, 3.26]). In particular, if $c>1$ and $x\in G$ has associated σ-tuple $\mu=(l,a_1,\ldots,a_t)$ then each non-zero term in μ must be a multiple of k. Indeed, x acts on the natural B-module with associated σ^k-tuple $\mu'=(l/k,a_1/k,\ldots,a_t/k)$ and thus $\dim x^{\bar{G}}=k^2\dim x^{\bar{B}}$. Now

$$|x^G|>\frac{1}{2}\left(\frac{q}{q+1}\right)^{\alpha d}q^{\dim x^{\bar{G}}} \tag{65}$$

and we deduce that $|x^G\cap H|<2\log_2 q.2^{d(1-\alpha)+\alpha}q^{\frac{1}{k}\dim x^{\bar{G}}}$ where α is defined in (4) and

$$\dim x^{\bar{G}}\geqslant n^2-l^2-\frac{1}{c}(n-l-kc(d-1))^2-ck^2(d-1).$$

The same bounds hold if $c=1$ and the result follows since $n\geqslant l+kdc$.

Now assume k does divide i, so $i_0=i/k$ and each non-trivial σ-orbit is a union of k distinct σ^k-orbits. In particular, $c>1$ and we may assume x has associated σ-tuple $\mu=(l,a_1,\ldots,a_t)$. For $k\leqslant 3$ we claim that

$$|x^G\cap H|<2\log_2 q.2^{kd(1-\alpha)+\alpha}\left(\frac{q^k}{q^k-1}\right)^d q^{\frac{1}{k}\dim x^{\bar{G}}}, \tag{66}$$

where α is defined as before. Applying (65) and the lower bound on $\dim x^{\bar{G}}$ given in [4, 3.33], we find that (66) is sufficient with the exception of a handful of cases for which we can calculate more accurate bounds. For example, if $(k,i,q) = (2,2,2)$ and $n = l+2$ then the reader can check that the bounds

$$|x^G \cap H| \leqslant 2\frac{|\mathrm{GL}_{n/2}(4)|}{|\mathrm{GL}_{n/2-1}(4)||\mathrm{GL}_1(4)|},\;\; |x^G| \geqslant \frac{|\mathrm{GL}_n(2)|}{|\mathrm{GL}_{n-2}(2)||\mathrm{GL}_1(2^2)|}$$

are sufficient for all $n \geqslant 4$. It remains to justify (66).

Proof of (66).
Modulo field and graph automorphisms, each B-class in $x^G \cap B$ is determined by a tk-tuple of the form

$$(b_{11}, \ldots, b_{1k}, b_{21}, \ldots, b_{2k}, \ldots, b_{tk}),$$

where the b_{ij} are non-negative integers such that $\sum_j b_{ij} = a_i$ for each $1 \leqslant i \leqslant t$. Let $\mathscr{B}$ denote the set of all such tk-tuples and for each $b \in \mathscr{B}$, let $x_b \in B$ represent the corresponding B-class in $x^G \cap B$ and fix $\hat{x}_b \in \widehat{B}$ of order r such that $|x_b^B| = |\hat{x}_b^{\widehat{B}}|$. Accounting for the possible effect of field and graph automorphisms, it follows that

$$|x^G \cap H| \leqslant 2\log_2 q. \sum_{b \in \mathscr{B}} |\hat{x}_b^{\widehat{B}}|.$$

Assume for now that $k=3$, so $|\hat{x}_b^{\widehat{B}}| < 2^{3d(1-\alpha)+\alpha} q^{3\dim x_b^{\bar{B}}}$ for all $b \in \mathscr{B}$. Let a be the number of terms a_j in μ which are not divisible by 3, so if

$$\Sigma := \sum_{b \in \mathscr{B}} q^{3\dim x_b^{\bar{B}}} < 3^a \left(\frac{q^3}{q^3-1}\right)^d q^{3\zeta} \tag{67}$$

holds, where $\zeta = \max_{b \in \mathscr{B}} \dim x_b^{\bar{B}} = \frac{1}{9}\dim x^{\bar{G}} - \frac{2}{9}ac$, then

$$|x^G \cap H| < 2\log_2 q. 2^{3d(1-\alpha)+\alpha} \left(\frac{3}{q^{2c/3}}\right)^a \left(\frac{q^3}{q^3-1}\right)^d q^{\frac{1}{3}\dim x^{\bar{G}}}$$

and (66) follows. To establish (67) we argue by induction on d. Without loss of generality we may assume that $a_1 > 0$.

If $d=1$ then for $0 \leqslant j \leqslant a_1$ define

$$\Sigma_j = \sum_{b \in \mathscr{B}_j} q^{3\dim x_b^{\bar{B}}},\;\; \zeta_j = \max_{b \in \mathscr{B}_j} \dim x_b^{\bar{B}},$$

where a tuple $b \in \mathscr{B}$ lies in the subset $\mathscr{B}_j$ if and only if $b_{11} = j$. Clearly $\Sigma = \Sigma_0 + \cdots + \Sigma_{a_1}$. Next fix j and observe that if $b \in \mathscr{B}_j$ with $b_{12} = v$ then $\dim x_b^{\bar{B}} = f(v) = c_1v^2 + c_2v + c_3$ for some constants c_i, with $c_1 < 0$. Now $f(v)$ is even (it is the dimension of a $\bar{B}$-class) and thus

$$\Sigma_j < 2\left(1 + (q^3)^2 + (q^3)^4 + \cdots + (q^3)^{\zeta_j - 2}\right) + \eta q^{3\zeta_j} < \eta\left(\frac{q^6+2-\eta}{q^6-1}\right) q^{3\zeta_j},$$

where $\eta = 2$ if $a_1 - j$ is odd, otherwise $\eta = 1$. Now, if $a = 0$ then $\zeta = \zeta_j$ if and only if $j = \frac{1}{3}a_1$ and it follows that

$$\Sigma < 2^2\left(\frac{q^6}{q^6-1}\right)\left(1 + (q^3)^2 + (q^3)^4 + \cdots + (q^3)^{\zeta-2}\right) + \left(\frac{q^6+1}{q^6-1}\right) q^{3\zeta} < \left(\frac{q^3}{q^3-1}\right) q^{3\zeta}$$

since $\frac{2}{3}a_1$ is even. Similarly, if $a = 1$ then

$$\begin{aligned}\Sigma &< 2^2\left(\frac{q^6}{q^6-1}\right)\left(1 + (q^3)^2 + (q^3)^4 + \cdots + (q^3)^{\zeta-2}\right) + \left(2\left(\frac{q^6}{q^6-1}\right) + \frac{q^6+1}{q^6-1}\right) q^{3\zeta} \\ &< 3\left(\frac{q^3}{q^3-1}\right) q^{3\zeta}\end{aligned}$$

and we conclude that (67) holds when $d = 1$.

Now assume $d > 1$. For $0 \leqslant j \leqslant a_1$ define $\mathscr{B}_j$, Σ_j and ζ_j as before. If $a_1 \equiv 0\,(3)$ then the inductive hypothesis implies that

$$\Sigma_j < 3^a \left(\frac{q^3}{q^3-1}\right)^{d-1} q^{3\zeta_j},$$

and since $\Sigma = \Sigma_0 + \cdots + \Sigma_{a_1}$ we conclude that

$$\Sigma < 3^a \left(\frac{q^3}{q^3-1}\right)^{d-1} \left(2(1+(q^3)^2+\cdots+(q^3)^{\zeta-2}) + q^{3\zeta}\right) < 3^a \left(\frac{q^3}{q^3-1}\right)^{d} q^{3\zeta}.$$

The case $a_1 \not\equiv 0\,(3)$ is very similar. This establishes (67) and thus (66) holds when $k = 3$. The argument when $k = 2$ is similar (and shorter).

Finally, let us assume r is odd and $C_{\bar{G}}(x)$ is non-connected. Then the hypothesis $x^G \cap H \subseteq B$ implies that $r \neq k$, so rk divides n. If $k = 3$ then the bounds (14) and (60) are sufficient, so assume $k = 2$, in which case $\epsilon = +$. Define $i \geqslant 1$ as before and observe that our earlier arguments apply if $i > 1$. For example, if i is even then $|x^G| > \frac{1}{2}q^{n^2(1-1/r)}$ and (66) implies that

$$|x^G \cap H| < 2^{r-1}\left(\frac{q^2}{q^2-1}\right)^{\frac{1}{2}(r-1)} q^{\frac{1}{2}n^2(1-\frac{1}{r})}$$

since $d \leqslant \frac{1}{2}(r-1)$ and $\mathcal{E}_x$, the multiset of eigenvalues of $\hat{x} \in \mathrm{GL}_n(q)$, is fixed by all automorphisms of G_0. One can check that this bound with (14) is always sufficient. If $i = 1$ and x is $\bar{G}_\sigma$-conjugate to x_0 (in the notation of [4, 3.35]) then $x^G \cap H = x^H$ and the result follows via (14) since

$$|x^G \cap H| \leqslant \frac{|\mathrm{GL}_{n/2}(q^2)|}{|\mathrm{GL}_{n/2r}(q^2)|^r r} < 2^{r-1} q^{\frac{1}{2}n^2(1-\frac{1}{r})}.$$

On the other hand, if x is not $\bar{G}_\sigma$-conjugate to x_0 then

$$|x^G \cap H| \leqslant (r-1)\frac{|\mathrm{GL}_{n/2}(q^2)|}{|\mathrm{GL}_{n/2r}(q^{2r})|r} < 2q^{\frac{1}{2}n^2(1-\frac{1}{r})}$$

and the result follows once again via (14).

Case 2. $x \in H - \mathrm{PGL}(V)$
Let us begin by assuming $x \in G$ is a field automorphism of prime order r, so $q = q_0^r$ and $r \neq k$ since every element of order k in $H \cap \mathrm{P\Gamma L}(V)$ lies in $B.\langle\phi\rangle \leqslant \mathrm{PGL}(V)$, where $\mathrm{P\Gamma L}(V)$ is the projective general semilinear group on V. Applying [4, 3.15, 3.43] we deduce that

$$|x^G \cap H| \leqslant \left(\frac{q^k-\epsilon}{q-\epsilon}\right)\frac{|\mathrm{PGL}^\epsilon_{n/k}(q^k)|}{|\mathrm{PGL}^\epsilon_{n/k}(q^{k/r})|} < 2\left(\frac{q^k-1}{q-1}\right) q^{k\left(\frac{n^2}{k^2}-1\right)\left(1-\frac{1}{r}\right)}$$

and (37) is sufficient unless $(n,k,r) = (3,3,2)$. Here $\epsilon = +$ and we conclude that $f(x,H) < .812$ for all $q \geqslant 4$ since $|x^G \cap H| \leqslant q^2+q+1$ and $|x^G| > \frac{1}{6}q^4$. Similar reasoning applies when $x \in G$ is an involutory graph-field automorphism. (Note that k is odd since every involution in H lies in $\mathrm{PGL}(V).\langle\gamma\rangle$ if $k = 2$, where γ is an involutory graph automorphism.)

To complete the proof, let us assume that $x \in G$ is an involutory graph automorphism. We begin by assuming $n \geqslant 3k$ and k is odd. Then $x^G \cap H \subseteq Bx$ and x induces an involutory graph automorphism on B such that $C_B(x)$ and $C_{G_0}(x)$ are of the same type. In particular, if n is even and $C_{G_0}(x)$ is symplectic then

$$|x^G \cap H| \leqslant \left(\frac{q^k-\epsilon}{q-\epsilon}\right)\frac{|\mathrm{PGL}^\epsilon_{n/k}(q^k)|}{|\mathrm{Sp}_{n/k}(q^k)|} < 2\left(\frac{q^k-1}{q-1}\right) q^{\frac{n^2}{2k}-\frac{n}{2}-k}$$

and the result follows via (39). The non-symplectic case is very similar. Now assume k is odd and $n < 3k$. If $n = k$ then the bounds $|x^G \cap H| \leqslant (q-1)^{-1}(q^n - 1)$ and $|x^G| > \frac{1}{2}(q+1)^{-1}q^{(n^2+n-2)/2}$ are always sufficient. If $n = 2k$ then x induces an automorphism on $\widehat{B} = \mathrm{GL}_2^{\epsilon}(q^k)$ which restricts to an inner automorphism i_x of $\mathrm{SL}_2(q^k)$; if i_x is non-trivial then $C_{G_0}(x)$ is non-symplectic and (39) is sufficient since

$$|x^G \cap H| \leqslant \left(\frac{q^k - \epsilon}{q - \epsilon}\right)\left(\frac{|\mathrm{PGL}_2(q^k)|}{|\mathrm{PGO}_2^+(q^k)|} + \frac{|\mathrm{PGL}_2(q^k)|}{|\mathrm{PGO}_2^-(q^k)|}\right) \leqslant \left(\frac{q^k - 1}{q - 1}\right) q^{2k};$$

if i_x centralizes $\mathrm{SL}_2(q^k)$ then $C_{G_0}(x)$ is symplectic and again the result follows via (39) since $|x^G \cap H| \leqslant (q-1)^{-1}(q^k - 1)$.

Next assume $k = 2$ and $n \neq 4$. Then $\epsilon = +$ (see Table 3.1) and we observe that $C_{G_0}(x)$ is non-symplectic if $n \equiv 2\,(4)$. Therefore, for any n, we have $x^G \cap H \subseteq Bx \cup Bx\phi$ where x acts on B as an involutory graph automorphism such that $C_B(x)$ and $C_{G_0}(x)$ are of the same type and $x\phi$ induces an involutory graph-field automorphism on B. Now, if $n \equiv 0\,(4)$ and $C_{G_0}(x)$ is symplectic then using [4, 3.43] we deduce that

$$|x^G \cap H| \leqslant (q+1)\frac{|\mathrm{PGL}_{n/2}(q^2)|}{|\mathrm{Sp}_{n/2}(q^2)|} + \frac{|\mathrm{PGL}_{n/2}(q^2)|}{|\mathrm{PGU}_{n/2}(q)|} < (2q^{\frac{n}{2}+1} + q + 1)q^{\frac{1}{4}n^2 - \frac{n}{2} - 2}$$

and $|x^G| > \frac{1}{2}q^{(n^2-n-4)/2}$. If we assume $n \geqslant 8$ then these bounds are sufficient unless $(n, q) = (8, 2)$, where $B = \mathrm{GL}_4(4)$ and direct calculation yields $f(x, H) < .573$. The non-symplectic case is similar. Finally, if $n = 4$ and $C_{G_0}(x)$ is non-symplectic then $f(x, H) < .699$ for all $q \geqslant 2$ since

$$|x^G \cap H| \leqslant (q+1)\left(\frac{|\mathrm{PGL}_2(q^2)|}{|\mathrm{PGO}_2^+(q^2)|} + \frac{|\mathrm{PGL}_2(q^2)|}{|\mathrm{PGO}_2^-(q^2)|}\right) + \frac{|\mathrm{PGL}_2(q^2)|}{|\mathrm{PGU}_2(q)|} = q(q^4 + q^3 + q^2 + 1)$$

and $|x^G| \geqslant (4, q-1)^{-1}q^4(q^2 - 1)(q^3 - 1)$. On the other hand, if $C_{G_0}(x)$ is symplectic then

$$|x^G \cap H| \leqslant (q+1) + \frac{|\mathrm{PGL}_2(q^2)|}{|\mathrm{PGU}_2(q)|} = q^3 + 2q + 1, \quad |x^G| \geqslant (4, q-1)^{-1}q^2(q^3 - 1)$$

and it follows that $f(x, H) < 3/4$ unless $q = 2$, where $f(x, H) = (\log 13)/(\log 28) \approx .770^*$; this exceptional case is recorded in Table 1.1. $\square$

Proposition 3.2. *The conclusion to Theorem 1.1 holds in cases (iii)-(vi) of Table 3.1.*

Proof. These cases are all very similar and we only give details for (iii). Here the statement of Theorem 1.1 gives $\iota = 1/(n+2)$ if $k = 2$. Define $\bar{G} = \mathrm{PSp}_n(K)$, $\bar{B} = \mathrm{PSp}_{n/k}(K)$ and let σ be a Frobenius morphism of $\bar{G}$ such that $\bar{G}_\sigma$ has socle $G_0 = \mathrm{PSp}_n(q)$. According to [4, 3.3], if $(n, p) = (4, 2)$ then we may assume G does not contain any graph-field automorphisms. (Similarly, in cases (iv) and (vi) we may assume that G does not contain a triality automorphism if $G_0 = \mathrm{P}\Omega_8^+(q)$ - see [4, 3.3].)

According to [9, p.116] we have $H \cap \mathrm{PGL}(V) \leqslant B.\langle\phi\rangle = \widetilde{H}$, where ϕ acts on B as a field automorphism of order k and

$$B \cong \begin{cases} \langle z\rangle \times \mathrm{PSp}_{n/2}(q^2) & \text{if } k = 2 \text{ and } p \neq 2 \\ \mathrm{PGSp}_{n/k}(q^k) & \text{otherwise.} \end{cases}$$

Here $z \in \bar{G}_\sigma - G_0$ is an involution with $C_G(z)$ of type $\mathrm{Sp}_{n/2}(q^2)$.

Now, if $x \in H - \mathrm{PGL}(V)$ has prime order r then x is a field automorphism, so $q = q_0^r$, $r \neq k$ and the result follows since [4, 3.43, 3.48] imply that

$$|x^G \cap H| \leqslant |\mathrm{Sp}_{n/k}(q^k) : \mathrm{Sp}_{n/k}(q^{k/r})| < 2q^{\frac{n}{2k}(n+k)(1-\frac{1}{r})}, \quad |x^G| > \frac{1}{4}q^{\frac{1}{2}n(n+1)(1-\frac{1}{r})}.$$

For the remainder, let us assume $x \in H \cap \mathrm{PGL}(V)$ of prime order r. Arguing as in the proof of the previous proposition, applying [4, 3.24, 3.38, 3.40], we quickly reduce to the case $k < 5$.

Case 1. $k < 5$, $x^G \cap (H - B) \neq \emptyset$
Here $r = k$ and applying [4, 3.43] we deduce that

$$|x^G \cap (H - B)| \leqslant (k-1)|\mathrm{Sp}_{n/k}(q^k) : \mathrm{Sp}_{n/k}(q)| < 2(k-1)q^{\frac{n}{2k}(k-1)\left(\frac{n}{k}+1\right)}. \tag{68}$$

Let us start by assuming $k = 3$. If $p = 3$ then x has associated partition $\lambda = (3^{n/3})$ (see (61)) and thus $|x^G| > \frac{1}{2}q^{n(n+1)/3}$. Furthermore, if $x^G \cap B$ is non-empty then $n \equiv 0\,(18)$ and $|x^G \cap B| < q^{n(n+3)/9}$ since each $y \in x^G \cap B$ is $\bar{B}$-conjugate to $[J_3^{n/9}]$. These bounds with (68) are always sufficient. The case $k = 3$ with $p \neq 3$ is very similar. Now assume $(k,p) = (2,2)$. Then $|x^G| > \frac{1}{2}q^{n^2/4}$ since x is $\bar{G}$-conjugate to $a_{n/2}$. Furthermore, $x^G \cap B$ is either empty or $n \equiv 0\,(8)$ and $|x^G \cap B| < 2q^{n^2/8}$ since each $y \in x^G \cap B$ is $\bar{B}$-conjugate to $a_{n/4}$ (see (69) below). If we assume $n \geqslant 8$ then (68) is sufficient unless $(n,q) = (8,2)$, where direct calculation yields $f(x,H) < .668$. If $n = 4$ then $|x^G \cap H| = |x^G \cap (H-B)| = q(q^2+1)$, $|x^G| = q^4 - 1$ and thus $f(x,H) < .851$ for all $q \geqslant 2$.

Finally, let us assume $k = 2$ and p is odd. Here $C_{\bar{G}}(x)$ is non-connected and there are four cases to consider. If $C_G(x)$ is of type $\mathrm{Sp}_{n/2}(q^2)$ then $|x^G| > \frac{1}{4}q^{n^2/4}$, $x^G \cap B = \{z\}$ and we find that (68) is always sufficient if $n \geqslant 8$. If $n = 4$ then $f(x,H) < .774$ for all $q \geqslant 3$ since

$$|x^G \cap H| \leqslant 1 + \frac{1}{2}|\mathrm{Sp}_2(q^2) : \mathrm{Sp}_2(q)| = 1 + \frac{1}{2}q(q^2+1), \;\; |x^G| = \frac{1}{2}q^2(q^2-1)$$

The case where $C_G(x)$ is of type $\mathrm{Sp}_{n/2}(q)^2$ is similar. Now assume $C_G(x)$ is of type $\mathrm{GL}^{\epsilon}_{n/2}(q)$. If $q \equiv \epsilon\,(4)$ then $x^G \cap B = (1,t)^B$, where $t \in \mathrm{PSp}_{n/2}(q^2)$ is an involution with centralizer of type $\mathrm{GL}_{n/4}(q^2)$ (note that $t \in \mathrm{PSp}_{n/2}(q^2)$ since $q^2 \equiv 1\,(4)$ - see [8, Table 4.5.1]). Therefore

$$|x^G \cap B| \leqslant \frac{|\mathrm{Sp}_{n/2}(q^2)|}{|\mathrm{GL}_{n/4}(q^2)|2} < q^{\frac{1}{8}n(n+4)}, \;\; |x^G| > \frac{1}{4}(q+1)^{-1}q^{\frac{1}{4}(n^2+2n+4)}$$

and (68) is sufficient unless $(n,q) = (4,3)$, where direct calculation gives $f(x,H) < .732$. Similarly, if $q \equiv -\epsilon\,(4)$ then $x^G \cap B = (z,t)^B$ so the previous bounds hold and again it remains to deal with the case $(n,q) = (4,3)$. This time direct calculation yields $f(x,h) < .651$.

Case 2. $k < 5$, $x^G \cap H \subseteq B$, $r = p$
First assume $p = 2$ and observe that the natural embedding $\mathrm{Sp}_{n/k}(q^k) \hookrightarrow \mathrm{Sp}_n(q)$ maps involution class representatives as follows:

$$a_l \mapsto a_{kl}, \;\; c_l \mapsto c_{kl}, \;\; b_l \mapsto \left\{ \begin{array}{ll} b_{kl} & \text{if } k \text{ is odd} \\ c_{kl} & \text{if } k = 2. \end{array} \right. \tag{69}$$

If x is G-conjugate to a_{kl} then $\dim x^{\bar{G}} = k^2 \dim x^{\bar{B}}$, [4, 3.22] implies that $|x^G \cap H| < 2q^{\dim x^{\bar{B}}}$, $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ and the desired result follows since $\dim x^{\bar{G}} \geqslant 2n-4$ and $n \geqslant 4k$. Similarly, if x is G-conjugate to b_{kl} or c_{kl} then the bounds $|x^G \cap H| < 2q^{l(n-kl+k)}$ and $|x^G| > \frac{1}{2}q^{kl(n-kl+1)}$ are sufficient unless $(k,n,l,q) = (2,4,1,2)$. Here direct calculation yields $f(x,H) < .712$ since $|x^G \cap H| = 15$ and $|x^G| = 45$.

Now assume p is odd. Let $\lambda = (m^{ka_m}, \ldots, 1^{ka_1}) \vdash n$ be the associated partition of x, where $m = n/k$. We claim that

$$|x^G \cap H| < 2^t q^{\left(\frac{1}{2}+\frac{\delta_{2,k}}{n+2}\right)\dim x^{\bar{G}}}, \;\; |x^G| > \left(\frac{1}{2}\right)^{t+1}\left(\frac{q}{q+1}\right)^{t\delta_{2,k}} q^{\dim x^{\bar{G}}}, \tag{70}$$

where t is the number of non-zero terms a_j in λ. In view of [4, 3.18], it is sufficient to show that

$$\dim x^{\bar{B}} \leqslant \left(\frac{1}{2k} + \frac{\delta_{2,k}}{2n+4}\right)\dim x^{\bar{G}}. \tag{71}$$

Let $d = \sum_{i \text{ odd}} a_i$ and observe that $\dim x^{\bar{G}} = k^2 \dim x^{\bar{B}} - \frac{1}{2}(k-1)(n-kd)$. If $k=3$ then [6, 2.3, 2.4] imply that $\dim x^{\bar{G}} \geqslant n^2/4 + n/2 - 9d^2/4 + 9d/2$ (minimal if $\lambda = (2^{n/2-3d/2}, 1^{3d})$) and (71) quickly follows since $n \geqslant 3d+6$. Similar reasoning applies when $k=2$ (note that we have equality in (71) if $\lambda = (2^{n/2})$).

Let us now apply (70). If $t=1$ then $\dim x^{\bar{G}} \geqslant \frac{1}{4}n(n+2)$ and we are left to deal with the case $(n,k,q) = (4,2,3)$, where direct calculation yields $f(x,H) < .800$. Now assume $t \geqslant 2$. We claim that

$$\dim x^{\bar{G}} \geqslant g(n,t) = \left\{ \begin{array}{ll} \frac{3}{2}nt(t-1) - \frac{9}{8}t^4 + \frac{3}{4}t^3 + \frac{15}{8}t^2 - \frac{3}{4}t - \frac{3}{4} & \text{if } k=3 \\ (t^2-t)n - \frac{1}{2}t^4 + \frac{1}{3}t^3 + t^2 - \frac{1}{3}t - \frac{1}{2} & \text{if } k=2. \end{array} \right. \tag{72}$$

If $k=2$ then [6, 2.4] implies that $\dim x^{\bar{G}} \geqslant \dim y^{\bar{G}}$, where $y \in \bar{G}$ is unipotent with associated partition $(t^2, \ldots, 2^2, 1^{n-t^2-t+2}) \vdash n$, and (72) follows from [6, 2.3]. Now assume $k=3$ and define

$$f(\rho) = \frac{1}{2}n^2 + \frac{1}{2}n - \sum_{i<j} i a_i a_j - \frac{1}{2}\sum_i i a_i^2 - \frac{1}{2}\sum_{i \text{ odd}} a_i,$$

where $\rho = (n^{a_n}, \ldots, 1^{a_1}) \vdash n$ is an arbitrary partition of n. Note that $f(\rho) = \dim y^{\bar{G}}$ if ρ is the associated partition of a unipotent element $y \in \bar{G}$, and $g(n,t) = f(\rho')$ where

$$\rho' = (t^3, (t-1)^3, \ldots, 2^3, 1^{n-\frac{3}{2}t^2-\frac{3}{2}t+3}) \vdash n.$$

The claim now follows by arguing as in the proof of [4, 3.25].

If $k=3$ then $n \geqslant \frac{3}{2}t(t+1)$ and (72) implies that the bounds in (70) are sufficient unless $(t,q) = (2,3)$. Here $n \geqslant 12$ and (70) is good enough since $\dim x^{\bar{G}} \geqslant 3n-6$ (minimal if $\lambda = (2^3, 1^{n-6})$). Now assume $k=2$. Then $n \geqslant t(t+1)$ and if we assume $t \geqslant 3$ then (70) (with (72)) is sufficient unless $(t,q) = (3,3)$. In this case $n \geqslant 20$ (since a_1 and a_3 must be positive multiples of 4) and the result follows via (70) since (72) gives $\dim x^{\bar{G}} \geqslant 6n-24$. Finally, if $t=2$ and $\lambda \neq (2^2, 1^{n-4})$ then $\dim x^{\bar{G}} \geqslant 4n-12$ (minimal if $\lambda = (2^4, 1^{n-8})$) and (70) is sufficient. If $\lambda = (2^2, 1^{n-4})$ then $|x^G \cap H| < q^n$, $|x^G| > \frac{1}{4}(q+1)^{-1}q^{2n-1}$ and the desired result follows.

Case 3. $k<5$, $x^G \cap H \subseteq B$, $r \neq p$

If $r=2$ then the hypothesis $x^G \cap H \subseteq B$ implies that x is $\bar{G}$-conjugate to $[-I_{2ka}, I_{n-2ka}]$ for some $1 \leqslant a < n/4k$ and the subsequent bounds $|x^G \cap H| < 2q^{\frac{1}{k}\dim x^{\bar{G}}}$ and $|x^G| > \frac{1}{2}q^{\dim x^{\bar{G}}}$ are always sufficient since $\dim x^{\bar{G}} \geqslant 2k(n-2k)$. Now assume $r>2$. Define the integers i and i_0 as in the proof of the previous proposition and observe that (64) holds. Let $\mu = (l, a_1, \ldots, a_t)$ denote the associated σ-tuple of $x \in G$ and let d denote the number of non-zero parts a_j in μ. Note that (41) holds and that d is even if i is odd.

If k does not divide i then $i = i_0$ and thus σ- and σ^k-orbits coincide. Therefore each term in μ is divisible by k and we calculate that $k^2 \dim x^{\bar{B}} = \dim x^{\bar{G}} + \frac{1}{2}(n-l)(k-1)$, whence

$$|x^G \cap H| < \log_2 q . 2^{\frac{d}{2}(e-1)} q^{\frac{1}{k}\dim x^{\bar{G}} + \frac{1}{2k}(n-l)(k-1)},$$

where $e=2$ if i is odd, $e=1$ if i is even, and

$$\dim x^{\bar{G}} \geqslant \frac{1}{2}(n^2 + n - l^2 - l - \frac{1}{ei}(n-l-ki(d-e))^2 - k^2 i(d-e)). \tag{73}$$

The result follows via (41). Now assume k divides i, so $i_0 = i/k$ and each non-trivial σ-orbit is a union of k distinct σ^k-orbits. If $k=2$ and $i \equiv 2\,(4)$ then each term in μ must be even and we deduce that

$$|x^G \cap H| < \log_2 q . 2^d q^{\frac{1}{2}\dim x^{\bar{G}} + \frac{1}{4}(n-l)},$$

where (73) holds and $n \geqslant l + 2di$. Then (41) is sufficient unless $(n,l,i,d) = (4,0,2,1)$ and $q \in \{2,4\}$, where direct calculation yields $f(x,H) < .813$. Now assume $(k, i \bmod 4) \neq (2,2)$. Then arguing as in the proof of Proposition 3.1 (in particular, the proof of (66)) we deduce that

$$|x^G \cap H| < \log_2 q . 2^{\frac{1}{2}kd(e-1)} \left(\frac{q^k}{q^k-1}\right)^{\frac{d}{e}} q^{\frac{1}{k}\dim x^{\bar{G}} + \frac{1}{2k}(n-l)(k-1)}$$

and the desired result follows via (41) and the lower bound on $\dim x^{\bar{G}}$ given in [4, 3.33]. $\square$

Proposition 3.3. *The conclusion to Theorem 1.1 holds in cases (vii)-(ix) of Table 3.1.*

Proof. All three cases are very similar and we only give details for (viii) and (ix), which we deal with simultaneously; say H is of type $\mathrm{GL}^{\epsilon'}_{n/2}(q).2$. Define $\bar{G} = \mathrm{PSO}_n(K)$, $\bar{B} = \mathrm{PSL}_{n/2}(K)$, where $n \geqslant 8$, and let σ be a Frobenius morphism of $\bar{G}$ such that $\bar{G}_\sigma$ has socle $G_0 = \mathrm{P}\Omega^{\epsilon}_n(q)$. In addition, let σ' be a Frobenius morphism of $\bar{B}$ such that $\bar{B}_{\sigma'} \cong \mathrm{PGL}^{\epsilon'}_{n/2}(q)$. Recall from the statement of Theorem 1.1 that $\iota = 1/(n-2)$ and note that we may assume G is without triality if $(n,\epsilon) = (8,+)$ (see [4, 3.3]). Also observe that $H \cap \mathrm{PGL}(V) \leqslant \widetilde{H}$, where $\widetilde{H} = C_{\widetilde{G}}(z)$ for a suitable involution $z \in \bar{G}_\sigma$ if p is odd, and $\widetilde{H} = \mathrm{GL}^{\epsilon'}_{n/2}(q).\langle\psi\rangle = B.2$ if $p = 2$ where ψ induces an involutory graph automorphism on B (see [9, 4.2.7, 4.3.18] for example).

If $x \in H - \mathrm{PGL}(V)$ has odd prime order r then x is a field automorphism, $q = q_0^r$ and [4, 3.48] states that $|x^G| > \frac{1}{4}q^{n(n-1)(1-1/r)/2}$. Moreover, [4, 3.15, 3.38] imply that

$$|x^G \cap H| \leqslant \frac{1}{2}(q-\epsilon')\frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{PGL}^{\epsilon'}_{n/2}(q^{1/r})|} < (q+1)q^{\frac{1}{4}(n^2-4)\left(1-\frac{1}{r}\right)}$$

and the result follows. If x is an involution then $q = q_0^2$ and $\epsilon' = +$ since every involution in H lies in $\mathrm{PGL}(V)$ if $\epsilon' = -$. Again, applying [4, 3.15, 3.38] we deduce that

$$|x^G \cap H| \leqslant \frac{1}{2}(q-1)\left(\frac{|\mathrm{PGL}_{n/2}(q)|}{|\mathrm{PGL}_{n/2}(q^{1/2})|} + \frac{|\mathrm{PGL}_{n/2}(q)|}{|\mathrm{PGU}_{n/2}(q^{1/2})|}\right) < 2(q-1)q^{\frac{1}{8}n^2-\frac{1}{2}}$$

and the bound $|x^G| > \frac{1}{4}q^{n(n-1)/4}$ is always sufficient. Now assume $x \in H \cap \mathrm{PGL}(V)$. For the reader's convenience, we partition the proof into three cases.

Case 1. $r = p$

Let us start by assuming $p = 2$. If $x^G \cap H \subseteq B$ then x is $\bar{G}$-conjugate to a_{2l} for some $1 \leqslant l < n/4$ and the desired result follows since $|x^G \cap H| < 2q^{l(n-2l)}$ and $|x^G| > \frac{1}{2}q^{2l(n-2l-1)}$. Now assume $x^G \cap (H-B) \neq \emptyset$, so $\nu(x) = n/2$. If $n \equiv 0\,(4)$ and x is $\widetilde{G}$-conjugate to $a_{n/2}$ then [4, 3.22] implies that $|x^G \cap B| < 2q^{n^2/8}$, $|x^G| > \frac{1}{2}q^{n(n-2)/4}$ and we deduce that

$$|x^G \cap (H-B)| \leqslant |\mathrm{GL}^{\epsilon'}_{n/2}(q) : \mathrm{Sp}_{n/2}(q)| < 2(q+1)q^{\frac{1}{8}(n^2-2n-8)}$$

since each $y \in x^G \cap (H-B)$ acts on B as a symplectic-type graph automorphism. These bounds are sufficient unless $(n,q) = (8,2)$, where direct calculation yields $f(x,H) < .721$. On the other hand, if x is G-conjugate to $b_{n/2}$ or $c_{n/2}$ (according to the parity of $n/2$) then $|x^G| > \frac{1}{2}q^{n^2/4}$ and $|x^G \cap H| < 2(q+1)q^{(n^2+2n-8)/8}$ since $x^G \cap B$ is empty and each $y \in x^G \cap (H-B)$ induces a non-symplectic graph automorphism on B. These bounds are always sufficient.

Now assume $p > 2$. Let $\lambda = (m^{2a_m}, \ldots, 1^{2a_1}) \vdash n$ denote the associated partition of $x \in G$, where $m = n/2$, and let t denote the number of non-zero terms a_j. Then applying [6, Theorem 1] and [4, 3.21] we deduce that

$$|x^G \cap H| < 2^t q^{\left(\frac{1}{2}+\frac{1}{n-2}\right)\dim x^{\bar{G}}},\ \ |x^G| > \left(\frac{1}{2}\right)^{t+1}\left(\frac{q}{q+1}\right)^t q^{\dim x^{\bar{G}}}. \tag{74}$$

If $t = 1$ then [6, 2.4] implies that $\dim x^{\bar{G}} \geqslant \frac{1}{4}n(n-2)$ and thus (74) is sufficient unless $(n,q) = (8,3)$, where direct calculation yields $f(x,H) < .707$. Now assume $t \geqslant 2$ and observe that $n \geqslant t(t+1)$ and $\dim x^{\bar{G}} \geqslant n(t^2-t) - t^4/2 + t^3/3 - t/3$ (minimal if $\lambda = (t^2, \ldots, 2^2, 1^{n-t^2-t+2})$). If $t \geqslant 3$ then these bounds imply that (74) is sufficient unless $(t,q) = (3,3)$ and $12 \leqslant n \leqslant 16$. These cases are easily settled through direct calculation. Now assume $t = 2$ and set $d = \sum_{i \text{ odd}} a_i$. If $d = 0$ then there exists a non-zero a_j with $j \geqslant 4$, hence $p \geqslant 5$, $\dim x^{\bar{G}} \geqslant n^2/4 + 3n/2 - 8$

(minimal if $\lambda = (4^2, 2^{n/2-4})$) and it is easy to check that (74) is always sufficient. Now assume $d > 0$ and observe that $n \equiv 0\,(4)$ if and only if d is even. Applying [6, 2.3, 2.4] we deduce that

$$\dim x^{\bar{B}} = \frac{1}{2}\dim x^{\bar{G}} + \frac{1}{4}n - \frac{1}{2}d,\;\; \dim x^{\bar{G}} \geqslant \frac{1}{4}n^2 - \frac{1}{2}n - d^2 + d$$

(minimal if $\lambda = (2^{n/2-d}, 1^{2d})$). In particular, if $d = 1$ then $n \geqslant 10$ and the bounds

$$|x^G \cap H| < 2q^{\frac{1}{2}\dim x^{\bar{G}} + \frac{1}{4}n - \frac{1}{2}},\;\; |x^G| > \frac{1}{4}(q+1)^{-1}q^{\dim x^{\bar{G}}+1}$$

are always sufficient. Finally, if $d \geqslant 2$ then $n \geqslant 2d+4$ and the desired result follows since $|x^G \cap H| < 2q^{\frac{1}{4}(2\dim x^{\bar{G}} + n - 2d)}$ and $|x^G| > \frac{1}{4}q^{\dim x^{\bar{G}}}$.

Case 2. $r \neq p$, $r = 2$
If $\nu(x) < n/2$ then x is $\bar{G}$-conjugate to $[-I_{2a}, I_{n-2a}]$ for some $1 \leqslant a < n/4$ and the bounds $|x^G \cap H| < 2q^{a(n-2a)}$ and $|x^G| > \frac{1}{4}(q+1)^{-1}q^{2a(n-2a)+1}$ are sufficient without exception. Now assume $\nu(x) = n/2$. If $n \equiv 2\,(4)$ and x is an involutory graph automorphism of G_0 then $|x^G| > \frac{1}{4}q^{n^2/4}$ and the result follows since

$$|x^G \cap H| \leqslant \frac{1}{2}(q-\epsilon')\frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{SO}_{n/2}(q)|} < (q+1)q^{\frac{1}{8}(n^2+2n-8)}.$$

On the other hand, if $n \equiv 2\,(4)$ and $C_G(x)$ is of type $\mathrm{GL}^{\epsilon''}_{n/2}(q)$ then (54) holds and $\epsilon = \epsilon' = \epsilon''$ (see Table 3.1 and [4, Table 3.8]). Moreover, we have

$$|x^G \cap H| \leqslant \sum_{j=0}^{\frac{1}{4}(n-2)} \frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon'}_{j}(q)||\mathrm{GL}^{\epsilon'}_{n/2-j}(q)|} < 2\left(\frac{q^2}{q^2-1}\right)q^{\frac{1}{8}n^2-\frac{1}{2}}$$

and the desired result follows. Now assume $n \equiv 0\,(4)$, so $\epsilon = +$ (see Table 3.1). If $C_G(x)$ is of type $\mathrm{O}^{+}_{n/2}(q^2)$ or $\mathrm{O}^{\epsilon''}_{n/2}(q)^2$ then

$$\begin{aligned}|x^G \cap H| &\leqslant \frac{1}{2}(q-\epsilon')\left(\frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{PGO}^{+}_{n/2}(q)|} + \frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{PGO}^{-}_{n/2}(q)|}\right) + \frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon'}_{n/4}(q)|^2 2} + \frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}_{n/4}(q^2)|2}\\ &< ((q+1)q^{\frac{n}{4}-1}+2)q^{\frac{1}{8}n^2}\end{aligned}$$

and the result follows since $|x^G| > \frac{1}{8}q^{n^2/4}$. Finally, let us assume $n \equiv 0\,(4)$ and $C_G(x)$ is of type $\mathrm{GL}^{\epsilon''}_{n/2}(q)$. If $\epsilon' = \epsilon''$ then

$$\begin{aligned}|x^G \cap H| &\leqslant \frac{1}{2}(q-\epsilon')\frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{Sp}_{n/2}(q)|} + \frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon'}_{n/4}(q)|^2 2} + \sum_{j=0}^{\frac{1}{4}n-1}\frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}^{\epsilon'}_{j}(q)||\mathrm{GL}^{\epsilon'}_{n/2-j}(q)|}\\ &< (q^{\frac{n}{4}+1}+q+1)q^{\frac{1}{8}n^2-\frac{1}{4}n-1} + 2(q^2-1)^{-1}q^{\frac{1}{8}n^2}\end{aligned}$$

and (54) is sufficient unless $(n,q) = (8,3)$, where direct calculation yields $f(x,H) < .681$. Similarly, if $\epsilon' = -\epsilon''$ then

$$|x^G \cap H| \leqslant \frac{1}{2}(q-\epsilon')\frac{|\mathrm{PGL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{Sp}_{n/2}(q)|} + \frac{|\mathrm{GL}^{\epsilon'}_{n/2}(q)|}{|\mathrm{GL}_{n/4}(q^2)|2} < (q^{\frac{n}{4}+1}+q+1)q^{\frac{1}{8}n^2-\frac{1}{4}n-1}$$

and (54) is always sufficient.

Case 3. $r \neq p$, $r > 2$
Since r is odd, each $y \in x^G \cap H$ lifts to an element $\hat{y} \in \mathrm{GL}^{\epsilon'}_{n/2}(q)$ of order r. Let $i \geqslant 1$ be minimal

such that $r|(q^i-1)$, let $\mu=(l,a_1,\ldots,a_t)$ denote the associated σ-tuple of x and let d denote the number of non-zero terms a_j in μ, so d is even if i is odd. Define the integer $c=c(i,\epsilon')$ as in the statement of [4, 3.33].

We begin by assuming c is even. Then each non-zero term in μ is even and

$$\dim x^{\bar{B}}=\frac{1}{4}(n^2-l^2-c\sum_j a_j^2)=\frac{1}{2}\dim x^{\bar{G}}+\frac{1}{4}(n-l).$$

In particular, we deduce that

$$|x^G\cap H|<\log_2 q.2^{\frac{d}{e}}\left(\frac{q+1}{q}\right)^{\frac{1}{2}(1-\epsilon')}q^{\frac{1}{2}\dim x^{\bar{G}}+\frac{1}{4}(n-l)}$$

and

$$|x^G|>\frac{1}{2}\left(\frac{q}{q+1}\right)^{d(2-e)+1}q^{\dim x^{\bar{G}}}, \tag{75}$$

where

$$\dim x^{\bar{G}}\geqslant\frac{1}{2}(n^2-n-l^2+l-\frac{1}{ei}(n-l-2i(d-e))^2-4i(d-e))$$

and $e=2$ if i is odd, otherwise $e=1$. Now $n\geqslant l+2di$ and these bounds are sufficient with the exception of a handful of cases with which we can calculate directly.

Now assume c is odd. Then $\dim x^{\bar{B}}\leqslant\frac{1}{2}\dim x^{\bar{G}}+\frac{1}{4}(n-l)$ and we claim that

$$|x^G\cap H|<\log_2 q.2^{\frac{d}{2}(1+\epsilon')}\left(\frac{q+1}{q}\right)^{\frac{1}{2}(1-\epsilon')}\left(\frac{q}{q-1}\right)^{\frac{d}{e}}q^{\frac{1}{2}\dim x^{\bar{G}}+\frac{1}{4}(n-l)}. \tag{76}$$

To see this, suppose $\epsilon'=+$, in which case i is odd and d is even. Then modulo field automorphisms, each B-class in $x^G\cap B$ is determined by a choice of s-tuple $(b_1,\ldots,b_s)$, where each $b_j\leqslant a_j$ is a non-negative integer and $s=(r-1)/2c=t/2$. Let $\mathscr{B}$ denote the set of all such s-tuples and for each $b\in\mathscr{B}$ let $x_b\in B$ represent the B-class corresponding to b. Then

$$|x^G\cap H|\leqslant\log_2 q.\sum_{b\in\mathscr{B}}|\hat{x}_b^{\mathrm{GL}_{n/2}(q)}|,$$

where $\hat{x}_b\in\mathrm{GL}_{n/2}(q)$ has order r and $|x_b^B|=|\hat{x}_b^{\mathrm{GL}_{n/2}(q)}|$, and thus (76) holds if

$$\Sigma:=\sum_{b\in\mathscr{B}}q^{\dim x_b^{\bar{B}}}\leqslant\left(\frac{q}{q-1}\right)^{\frac{d}{2}}q^{\frac{1}{2}\dim x^{\bar{G}}+\frac{1}{4}(n-l)}.$$

If $a\geqslant 0$ is the number of terms a_j in μ which are odd then

$$\alpha:=\max_{b\in\mathscr{B}}\dim x_b^{\bar{B}}=\frac{1}{2}\dim x^{\bar{G}}+\frac{1}{4}(n-l)-\frac{1}{4}ai$$

and so it suffices to show that

$$\Sigma\leqslant 2^{\frac{a}{2}}\left(\frac{q}{q-1}\right)^{\frac{d}{2}}q^{\alpha}.$$

We now proceed by induction on d. The argument is similar to the proof of (66) and we leave the details to the reader. The case $\epsilon'=-$ is very similar.

Now $n\geqslant l+di$ and if we apply (76), together with (75) and the lower bound on $\dim x^{\bar{G}}$ given in [4, 3.33], we find that we are left to deal with a handful of exceptional cases. For example, if $\epsilon'=+$ then it remains to deal with the cases $(n,q)\in\{(10,4),(8,8),(8,7),(8,4)\}$ for $(i,l,d)=(1,0,2)$. These are easily settled. For instance, if $(n,q)=(10,4)$ then $r=3$ and $f(x,H)<.631$ since

$$|x^G\cap H|\leqslant 2+2\frac{|\mathrm{GL}_5(4)|}{|\mathrm{GL}_4(4)||\mathrm{GL}_1(4)|}+2\frac{|\mathrm{GL}_5(4)|}{|\mathrm{GL}_3(4)||\mathrm{GL}_2(4)|},\quad |x^G|\geqslant|\Omega_{10}^+(4):\mathrm{GL}_5(4)|.$$

□

References

[1] M. Aschbacher, *On the maximal subgroups of the finite classical groups*, Invent. Math. **76** (1984), 469–514.

[2] M. Aschbacher and G.M. Seitz, *Involutions in Chevalley groups over fields of even order*, Nagoya Math. J. **63** (1976), 1–91.

[3] T.C. Burness, *Fixed point ratios in actions of finite classical groups,* I, to appear.

[4] ———, *Fixed point ratios in actions of finite classical groups,* II, to appear.

[5] ———, *Fixed point ratios in actions of finite classical groups,* IV, to appear.

[6] ———, *Fixed point spaces in actions of classical algebraic groups*, J. Group Theory **7** (2004), 311–346.

[7] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2004.

[8] D. Gorenstein, R. Lyons, and R. Solomon, *The Classification of the Finite Simple Groups, Number 3*, Mathematical Surveys and Monographs, vol. 40, Amer. Math. Soc., 1998.

[9] P.B. Kleidman and M.W. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, London Math. Soc. Lecture Note Series, vol. 129, Cambridge University Press, 1990.

[10] M.W. Liebeck and A. Shalev, *Simple groups, permutation groups, and probability*, J. Amer. Math. Soc. **12** (1999), 497–520.

[11] F. Lübeck, private communication, 2005.