# Fixed point ratios in actions of finite classical groups, I

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**Abstract**

This is the first in a series of four papers on fixed point ratios in actions of finite classical groups. Our main result states that if $G$ is a finite almost simple classical group and $\Omega$ is a non-subspace $G$-set then either $\mathrm{fpr}(x) \lesssim |x^G|^{-\frac{1}{2}}$ for all elements $x \in G$ of prime order, or $(G, \Omega)$ is one of a small number of known exceptions. In this introductory note we present our results and describe an application to the study of minimal bases for primitive permutation groups. A further application concerning monodromy groups of covers of Riemann surfaces is also outlined. The proof of the main theorem appears in three subsequent papers [3], [4] and [5].

## Introduction

If a group $G$ acts on a set $\Omega$ then we define $C_\Omega(x)$ to be the set of points in $\Omega$ which are fixed by a given element $x \in G$. If $G$ and $\Omega$ are finite then we define the *fixed point ratio of* $x$, which we denote by $\mathrm{fpr}(x)$, to be the proportion of points in $\Omega$ fixed by $x$, i.e. $\mathrm{fpr}(x) = |C_\Omega(x)|/|\Omega|$. If $G$ acts transitively on $\Omega$ then it is easy to see that

$$\mathrm{fpr}(x) = \frac{|x^G \cap H|}{|x^G|} \tag{1}$$

where $H$ is the stabilizer in $G$ of some element $\omega \in \Omega$. In this way the analysis of such ratios is reduced to a study of conjugacy classes and their intersections with subgroups of $G$.

Fixed point ratios have been extensively studied since the days of Jordan in the nineteenth century and this work has found a wide range of applications. In recent years, a number of papers have appeared where bounds on fixed point ratios for actions of simple groups are obtained and then applied to a number of different problems, see [11, 12, 13, 14, 17, 20] for example. More specifically, in [19] Liebeck and Saxl prove that if $G$ is a finite almost simple group of Lie type over $\mathbb{F}_q$ acting faithfully and transitively on a set $\Omega$ then either

$$\mathrm{fpr}(x) \leqslant \frac{4}{3q} \tag{2}$$

for all non-identity elements $x \in G$, or $(G, \Omega, x)$ belongs to a short list of known exceptions which involves certain 'small' classical groups of (Lie) rank less than four.

In studying actions of classical groups, it is natural to distinguish between those actions which permute subspaces of the natural module and those which do not. In particular, the stabilizers of subspaces tend to be large subgroups (parabolic subgroups for example) and therefore (1) suggests that such actions give rise to relatively large fixed point ratios. In this series of papers

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we obtain upper bounds on fixed point ratios for finite almost simple classical groups in *non-subspace actions*; our main result is Theorem 1 below. If $G$ is such a group with socle $G_0$, a classical group with natural module $V$ over a field of prime characteristic $p$, then roughly speaking we say that a subgroup $H$ of $G$ is a *non-subspace subgroup* if $H \cap G_0$ is irreducible on $V$. This notion was introduced by Liebeck and Shalev in [20].

**Definition 1.** A subgroup $H$ of $G$ is a *subspace subgroup* if for each maximal subgroup $M$ of $G_0$ containing $H \cap G_0$ one of the following holds:

(a) $M$ is the stabilizer in $G_0$ of a proper non-zero subspace $U$ of $V$, where $U$ is totally singular, non-degenerate, or, if $G_0$ is orthogonal and $p = 2$, a non-singular 1-space ($U$ can be any subspace if $G_0 = \mathrm{PSL}(V)$);

(b) $M = \mathrm{O}^{\pm}_{2m}(q)$ if $(G_0, p) = (\mathrm{Sp}_{2m}(q)', 2)$.

A transitive action of $G$ on a set $\Omega$ is a *subspace action* if the point stabilizer $G_\omega$ of an element $\omega \in \Omega$ is a subspace subgroup of $G$; non-subspace subgroups and actions are defined accordingly.

The main theorem on the subgroup structure of finite classical groups is due to Aschbacher. In [1], eight collections of subgroups of $G$ are defined, labelled $\mathscr{C}_i$ for $1 \leqslant i \leqslant 8$, and in general it is shown that if $H$ is a maximal subgroup of $G$ not containing $G_0$ then either $H$ is contained in $\mathscr{C}(G) := \bigcup_i \mathscr{C}_i$ or it belongs to a family $\mathscr{S}$ of almost simple groups which act irreducibly on the natural $G$-module $V$ (see [16] for a detailed description of these subgroup collections). Due to the existence of certain outer automorphisms, a small additional collection $\mathscr{N}$ arises when $G_0$ is $\mathrm{Sp}_4(q)'$ ($q$ even) or $\mathrm{P}\Omega^+_8(q)$ (see [3, Table 3.1] and [5, §3]). Roughly speaking, a maximal subgroup is non-subspace unless it is a member of the collection $\mathscr{C}_1$ or is a particular example of a subgroup in $\mathscr{C}_8$, where we label the $\mathscr{C}_i$ collections as in [16].

The statement of Theorem 1 involves a natural number $n$ which we associate to each finite classical group $G$. In general, $n$ is simply the dimension of the natural $G$-module. However, in some small-dimensional cases we have a choice of natural $G$-module due to the existence of exceptional isomorphisms between certain finite simple classical groups. The list of isomorphisms between finite simple classical groups which act naturally on vector spaces of different dimensions is as follows

$$\mathrm{P}\Omega^{\epsilon}_6(q) \cong \mathrm{PSL}^{\epsilon}_4(q),\ \Omega_5(q) \cong \mathrm{PSp}_4(q),\ \Omega^-_4(q) \cong \mathrm{PSL}_2(q^2),\ \Omega_3(q) \cong \mathrm{PSL}_2(q),$$
$$\mathrm{Sp}_4(2)' \cong \mathrm{PSL}_2(9),\ \mathrm{SL}_3(2) \cong \mathrm{PSL}_2(7)$$

and subsequently we make the following definition.

**Definition 2.** Let $G$ be a finite almost simple classical group over $\mathbb{F}_q$ with socle $G_0$ and let $K$ denote the algebraic closure of $\mathbb{F}_q$. If $G_0 \in \{\mathrm{Sp}_4(2)', \mathrm{SL}_3(2)\}$ then set $n = n(G) = 2$, in all other cases we define $n = n(G)$ to be the minimal degree of a non-trivial irreducible $K\widehat{G}_0$-module, where $\widehat{G}_0$ is a covering group of $G_0$.

A major motivation for this work comes from an existence theorem of Liebeck and Shalev [20, Theorem ($\star$)]. This result states that there exists an absolute constant $\epsilon > 0$ so that if $G$ is any finite almost simple classical group in a non-subspace action then

$$\mathrm{fpr}(x) < |x^G|^{-\epsilon} \tag{3}$$

for all elements $x \in G$ of prime order. Evidently, for non-subspace actions of classical groups, this result is a significant improvement on the Liebeck-Saxl upper bound (2) and it finds a wide range of striking applications in [20]. In particular, the result lies at the heart of their proof of the Cameron-Kantor base conjecture and also their attack on the Guralnick-Thompson genus conjecture (see §§1.2-3). However, [20, Theorem ($\star$)] is strictly an existence result and no information on $\epsilon$ is given. For future applications, it is very desirable to obtain an explicit value for $\epsilon$ in (3). The central aim of this series of papers is to show that (3) holds with $\epsilon \approx 1/2$.

| $G_0$ | type of $H$ | $\iota$ |
|---|---|---|
| $\mathrm{PSL}_n^{\epsilon}(q)$ | $\mathrm{Sp}_n(q)$ | $1/n$ |
| $\mathrm{PSp}_n(q)$ | $\mathrm{Sp}_{n/2}(q) \wr S_2$ | $1/n$ |
| $\mathrm{PSp}_n(q)$ | $\mathrm{Sp}_{n/2}(q^2)$ | $1/(n+2)$ |
| $\mathrm{P\Omega}_n^{\epsilon}(q)$ | $\mathrm{GL}_{n/2}^{\epsilon'}(q)$ | $1/(n-2)$ |
| $\mathrm{SU}_4(2)$ | $\mathrm{GU}_1(2) \wr S_4$ | .010 |
| $\Omega_8^+(2)$ | $\mathrm{O}_4^-(2) \wr S_2$ | .001 |
| $\mathrm{SL}_4(2)$ | $\mathrm{GL}_2(4)$ | .020 |
| $\mathrm{PSp}_4(3)$ | $2^4.\mathrm{O}_4^-(2)$ | .086 |
| $\mathrm{P\Omega}_8^+(q)$ | $\Omega_7(q)$ | .219 |
| $\Omega_7(q)$ | $G_2(q)$ | .108 |
| $\Omega_{10}^-(2)$ | $A_{12}$ | .087 |
| $\mathrm{Sp}_8(2)$ | $A_{10}$ | .062 |
| $\Omega_8^+(2)$ | $A_9$ | .124 |
| $\mathrm{P\Omega}_8^+(3)$ | $\Omega_8^+(2)$ | .081 |
| $\Omega_7(3)$ | $\mathrm{Sp}_6(2)$ | .065 |
| $\mathrm{PSU}_6(2)$ | $\mathrm{PSU}_4(3)$ | .076 |
| $\mathrm{Sp}_6(2)$ | $\mathrm{SU}_3(3)$ | .054 |
| $\mathrm{PSU}_4(3)$ | $\mathrm{PSL}_3(4)$ | .011 |
| $\mathrm{SL}_4(2)$ | $A_7$ | .164 |

Table 1: The exceptional cases with $\iota > 0$

**Theorem 1.** *Let $G$ be a finite almost simple classical group acting transitively and faithfully on a set $\Omega$ with point stabilizer $G_\omega \leqslant H$, where $H$ is a maximal non-subspace subgroup of $G$. Then*

$$\mathrm{fpr}(x) < |x^G|^{-\frac{1}{2}+\frac{1}{n}+\iota}$$

*for all elements $x \in G$ of prime order, where $n$ is given in Definition 2 and either $\iota = 0$ or $(G_0, H, \iota)$ is listed in Table 1, where $G_0$ denotes the socle of $G$.*

**Remark 1.** In Table 1 we refer to the *type of $H$*. If $H$ lies in one of the $\mathscr{C}_i$ collections then the type provides an approximate group-theoretic structure for $H \cap \mathrm{PGL}(V)$. This notation is consistent with [16, §4]. In the remaining cases, the type refers to the socle of the almost simple group $H \cap G_0$.

As an immediate corollary, we obtain the following result.

**Corollary 1.** *If $n > 10$ and $x \in G$ has prime order then $\mathrm{fpr}(x) < |x^G|^{-\frac{1}{2}+\frac{1}{n}+\iota}$, where either $\iota = 0$ or $(G_0, H, \iota)$ is one of the following:*

| $G_0$ | type of $H$ | $\iota$ |
|---|---|---|
| $\mathrm{PSL}_n^{\epsilon}(q)$ | $\mathrm{Sp}_n(q)$ | $1/n$ |
| $\mathrm{PSp}_n(q)$ | $\mathrm{Sp}_{n/2}(q) \wr S_2$ | $1/n$ |
| $\mathrm{PSp}_n(q)$ | $\mathrm{Sp}_{n/2}(q^2)$ | $1/(n+2)$ |
| $\mathrm{P\Omega}_n^{\epsilon}(q)$ | $\mathrm{GL}_{n/2}^{\epsilon'}(q)$ | $1/(n-2)$ |

The next result follows from Theorem 1 and Lemma 1.1 below. Here the *untwisted Lie rank of $G_0$* simply refers to the rank of the ambient simple algebraic group.

**Corollary 2.** *Let $G$ be a finite almost simple classical group over $\mathbb{F}_q$ with socle $G_0$ and natural module $V$, where $\dim V \geqslant 7$. If $l$ denotes the untwisted Lie rank of $G_0$ then*

$$\mathrm{fpr}(x) < 2q^{-\frac{l(l-1)}{l+1}}$$

*for all non-identity elements $x \in G$ and all non-subspace actions of $G$.*

This work constitutes the second stage of a two-part programme. In [6] we studied non-subspace actions of simple classical algebraic groups. In this context, the natural analogue of the fixed point ratio is the difference in dimensions $\dim C_{\Omega}(x) - \dim \Omega$. The main result of [6] can be stated as follows.

**Theorem ([6, Theorem 1]).** *Let $\bar{G}$ be a simple classical algebraic group over an algebraically closed field and let $\Omega$ be a primitive non-subspace $\bar{G}$-variety. If $x \in \bar{G}$ is a non-scalar element of prime order then*

$$\dim \Omega - \dim C_{\Omega}(x) \geqslant \left(\frac{1}{2} - \delta\right) \dim x^{\bar{G}},$$

*where $\delta = 0$, or $(\bar{G}, \Omega, \delta)$ is one of a small number of known exceptions. In the exceptional cases, $\delta$ tends to 0 as $\mathrm{rank}(\bar{G}) \to \infty$.*

This theorem says that the algebraic group analogue of (3) holds with $\epsilon \approx 1/2$. Our aim is to use this information to establish a similar result for the corresponding finite groups, passing from algebraic to finite groups by taking the set of fixed points of a Frobenius morphism. This is similar to the approach taken by Lawther, Liebeck and Seitz in their work on fixed point spaces in actions of simple groups of exceptional type [17, 18].

**Layout.** This work is organised as follows. In this introductory note we state our main results and describe an application to the study of minimal bases for primitive permutation groups. We also explain how Theorem 1 may be useful in efforts to classify primitive monodromy groups of covers of Riemann surfaces. The proof of Theorem 1 appears in the three subsequent papers [3], [4] and [5]. In [3] we collect a number of results from the literature and we show that the conclusion to Theorem 1 holds when $H$ is a maximal non-subspace subgroup in one of the collections $\mathscr{C}_i$, where $4 \leqslant i \leqslant 8$. In [4] we show that the same conclusion holds when $H$ lies in $\mathscr{C}_2$ or $\mathscr{C}_3$. Finally, in [5] we complete the proof of Theorem 1 by dealing with the almost simple subgroups in the $\mathscr{S}$ collection and the small additional set of subgroups $\mathscr{N}$.

**Notation and terminology.** A finite group $G$ is *almost simple* if $G_0 \trianglelefteqslant G \leqslant \mathrm{Aut}(G_0)$ for some non-abelian simple group $G_0$, the socle of $G$. We say that $G$ is classical if $G_0$ is. Associated to each finite simple classical group $G_0$ is a *natural module* $V$ over a finite field $\mathbb{F}$ of characteristic $p$. Strictly speaking, in most cases $V$ is not a module for $G_0$, but a module for some covering group. This slight abuse of terminology is harmless. Following [16] we define $\mathbb{F} = \mathbb{F}_{q^u}$, where $u = 2$ if $G_0$ is unitary, otherwise $u = 1$. Here $\mathbb{F}_{q^u} = \mathrm{GF}(q^u)$ is the field with $q^u$ elements and the term *finite classical group over* $\mathbb{F}_q$ will refer to any finite classical group whose natural module is defined over $\mathbb{F}$. Our notation for such groups is standard (see [16, Table 2.1.B] for example). For example, $\mathrm{PSL}_n^{\epsilon}(q)$ denotes $\mathrm{PSL}_n(q)$ if $\epsilon = +$ and $\mathrm{PSU}_n(q)$ if $\epsilon = -$. We note that there exists an isomorphism $\Omega_{2m+1}(q) \cong \mathrm{Sp}_{2m}(q)$ when $q$ is even and thus we only refer explicitly to odd-dimensional orthogonal groups when the characteristic of the underlying field is odd.

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# 1 Applications of Theorem 1

Here we prove Corollary 2 and describe how Theorem 1 can be applied to the study of bases in primitive actions of finite classical groups. We also explain how Theorem 1 may be useful in ongoing efforts to classify the primitive monodromy groups of covers of Riemann surfaces whose socle is not a product of alternating groups. In particular, Theorem 1 applies to monodromy groups of arbitrary genus.

## 1.1 Fixed point ratios

In order to prove Corollary 2 we first establish a lower bound on $|x^G|$ in terms of the field $\mathbb{F}_q$ and the untwisted Lie rank $l$. In [6, p.314] we remarked that if $\bar{G}$ is a simple classical algebraic group of rank $r$ then $\dim x^{\bar{G}} \geqslant 2r$ for all non-identity elements $x \in \bar{G}$. The following result can be thought of as a finite analogue. In the proof, there are numerous references to the preliminary results in [3, §3].

**Lemma 1.1.** *Let $G$ be a finite almost simple classical group over $\mathbb{F}_q$ with socle $G_0$ and let $l$ denote the untwisted Lie rank of $G_0$. If $l \geqslant f(G_0)$, where $f$ is defined as follows,*

| $G_0$ | $\mathrm{PSL}^{\epsilon}_{l+1}(q)$ | $\mathrm{PSp}_{2l}(q)$ | $\mathrm{P\Omega}^{\epsilon}_{2l}(q)$ | $\Omega_{2l+1}(q)$ |
|---|---|---|---|---|
| $f(G_0)$ | 4 | 2 | 4 | 3 |

*then $|x^G| > \frac{1}{4}q^{2l-\alpha}$ for all non-identity elements $x \in G$, where $\alpha = 1$ if $G_0 = \mathrm{P\Omega}^{\epsilon}_{2l}(q)$, otherwise $\alpha = 0$.*

*Proof.* First observe that $C_G(x) \leqslant C_G(x^m)$ for all $m \geqslant 1$ and so we may assume $x$ has prime order. Suppose $x \in \mathrm{PGL}(V)$ has prime order $r$, where $V$ denotes the natural $G_0$-module. Let $\hat{x}$ be a pre-image of $x$ in $\mathrm{GL}(V)$ and define

$$\nu(x) = \min\{\dim[\bar{V}, \lambda\hat{x}] : \lambda \in K^*\}, \tag{4}$$

where $K$ is the algebraic closure of $\mathbb{F}_q$ and $\bar{V} = V \otimes K$. (Note that $\nu(x)$ is equal to the codimension of the largest eigenspace of $\hat{x}$ on $\bar{V}$.) If $r = p$ (where $q = p^f$) and $(G_0, p) \neq (\mathrm{P\Omega}^{\epsilon}_{2l}(q), 2)$ then the proof of [3, 3.22] implies that $|x^G|$ is minimal when $x$ is a long root element and the result quickly follows. For example, if $G_0 = \mathrm{PSp}_{2l}(q)$ and $q$ is odd then [3, Table 3.4] indicates that $|x^{\widetilde{G}}| > \frac{1}{2}(q+1)^{-1}q^{s(2l-s)+1}$, where $\widetilde{G}$ is the group of inner-diagonal automorphisms of $G_0$ and $s = \nu(x)$. In particular, if $s \geqslant 2$ then $|x^G| > \frac{1}{2}(q+1)^{-1}q^{4l-3}$ and the claim follows; if $s = 1$ then $x$ is a long root element and via [3, 3.18, 3.20] we conclude that $|x^G| > \frac{1}{2}(q+1)^{-1}q^{2l}$. Similarly, if $(G_0, r, p) = (\mathrm{P\Omega}^{\epsilon}_{2l}(q), 2, 2)$ then the proof of [3, 3.22] yields $|x^G| \geqslant |y^G| > \frac{1}{2}q^{2l-1}$ where $y \in \mathrm{O}^{\epsilon}_{2l}(q)$ is a transvection (i.e. $y$ is conjugate to $b_1$ in the terminology of [2]).

The case $r \neq p$ is just as easy. For instance, if $G_0 = \mathrm{PSp}_{2l}(q)$ then from [3, Tables 3.7, 3.8] we deduce that $|x^G| > \frac{1}{4}q^{4(l-1)}$ (minimal if $r = 2$ and $x$ is conjugate to $[-I_2, I_{2l-2}]$) and the result follows since $l \geqslant 2$. Finally, let us assume $x \in G - \mathrm{PGL}(V)$. If $G_0 = \mathrm{PSL}^{\epsilon}_{l+1}(q)$ then [3, 3.49] gives $|x^G| > \frac{1}{4}q^{(l^2+l-4)/2}$ and the desired conclusion follows since $l \geqslant 4$. The other cases are very similar and we leave them to the reader. □

**Proof of Corollary 2.**
First observe that the hypothesis $n \geqslant 7$ implies that $l \geqslant f(G_0)$, where $f$ is the function defined in the statement of Lemma 1.1. In particular, the conclusion to Lemma 1.1 holds and Theorem 1 applies since $\mathrm{fpr}(x) \leqslant \mathrm{fpr}(x^m)$ for all $m \geqslant 1$.

Let us begin by considering the relevant exceptional cases listed in Table 1, excluding the first four cases for now. Here it is easy to check that the corollary holds. For example, if $G_0 = \Omega^+_8(2)$

then we obtain the following results (see [4, 2.11] and [5, 2.5, 2.7]):

| type of $H$ | $\mathrm{O}_4^-(2) \wr S_2$ | $A_9$ | $\mathrm{Sp}_6(2)$ |
|---|---|---|---|
| $\mathrm{fpr}(x) \leqslant$ | $1/6$ | $3/10$ | $3/10$ |
| $G_0$-class of $x$ | $2F$ | $2F$ | $3B$ |

The entries in the second row of the table provide sharp upper bounds for $\max_{x \in G^{\#}} \mathrm{fpr}(x)$, where $G^{\#}$ denotes the set of non-identity elements in $G$. In the last row we record a specific $G_0$-class which contains an element $x$ which realizes this maximal fixed point ratio (here we adopt the standard notation of [10]). The conclusion to Corollary 2 follows immediately since $(l, q) = (4, 2)$. The other cases are just as easily verified. For example, if $G_0 = \mathrm{P}\Omega_8^+(3)$ and $H$ is of type $\Omega_7(3)$ then [5, (3)] implies that

$$\max_{x \in G^{\#}} \mathrm{fpr}(x) \leqslant \frac{32760}{262080} = \frac{1}{8} < 2.3^{-\frac{3}{2}}.$$

Similarly, if $G_0 = \mathrm{Sp}_8(2)$ and $H$ is of type $A_{10}$ then the proof of [5, 2.5] yields

$$\max_{x \in G^{\#}} \mathrm{fpr}(x) \leqslant \frac{45}{255} = \frac{3}{17} < 2.2^{-\frac{12}{5}}.$$

Now assume $G_0 = \mathrm{PSL}_{l+1}^{\epsilon}(q)$, with $l \geqslant 6$. If $\iota = \iota(G, H) = 0$ then Theorem 1 and Lemma 1.1 imply that

$$\mathrm{fpr}(x) < |x^G|^{-\frac{1}{2}+\frac{1}{l+1}} < (4q^{-2l})^{\frac{1}{2}-\frac{1}{l+1}} < 2q^{-\frac{l(l-1)}{l+1}}$$

as claimed. If $\iota > 0$ then we may assume $H$ is of type $\mathrm{Sp}_{l+1}(q)$ (see Table 1) in which case $\iota = 1/(l+1)$ and $l \geqslant 7$ is odd. Now if $x \in H \cap \mathrm{PGL}(V)$ has prime order $r$ and $\nu(x) = 1$ (see (4)) then $r = p$ and $x$ is a long root element. Then [3, 3.22] (and its proof) implies that $|x^G \cap H| < q^{l+1}$, $|x^G| > \frac{1}{2}(q+1)^{-1}q^{2l+1}$ and the result follows. Similarly, if $\nu(x) > 1$ then [3, 3.20, 3.24] and [3, Tables 3.7, 3.8] imply that $|x^G| > \frac{1}{2}(q+1)^{-1}q^{4l-3}$ (minimal if $r = p$ and $x$ has Jordan form $[J_2^2, J_1^{l-3}]$ on $V$, where $J_i$ is a standard Jordan block of size $i$) and thus Theorem 1 yields

$$\mathrm{fpr}(x) < (2(q+1)q^{-(4l-3)})^{\frac{1}{2}-\frac{2}{l+1}} < 2q^{-\frac{l(l-1)}{l+1}}$$

for all $l \geqslant 7$ and $q \geqslant 2$. Finally, if $x \in H - \mathrm{PGL}(V)$ then $|x^G| > \frac{1}{2}(q+1)^{-1}q^{(l^2+l-2)/2}$ (see [3, 3.49]) and the result follows via Theorem 1.

Now assume $G_0 = \mathrm{PSp}_{2l}(q)$. If $\iota = 0$ then the result follows from Theorem 1 and Lemma 1.1 so let us assume $\iota > 0$. In view of our earlier work, we may assume $H$ is of type $\mathrm{Sp}_l(q) \wr S_2$ or $\mathrm{Sp}_l(q^2)$ (see Table 1). First consider the $\mathscr{C}_2$-subgroup of type $\mathrm{Sp}_l(q) \wr S_2$. Here $\iota = 1/2l$ and we find that the desired result follows via Theorem 1 if

$$|x^G| > \frac{1}{2}(q+1)^{-1}q^{4l-3}. \tag{5}$$

In view of [3, 3.49] and [3, Tables 3.4-5, 3.7-8], it is clear that (5) holds unless $r = p$ and $x \in H \cap \mathrm{PGL}(V)$ is a long root element. In this case, the bounds $|x^G| > \frac{1}{2d}q^{2l}$, where $d = 2 - \delta_{2,p}$ (see [3, 3.18, 3.20] and the proof of [3, 3.22]), and $|x^G \cap H| < 2q^l$ (see Case 2 in the proof of [4, 2.9]) are sufficient. If $H$ is the $\mathscr{C}_3$-subgroup of type $\mathrm{Sp}_l(q^2)$ then $\iota = 1/(2l+2)$ and the proof of [20, 4.2] implies that $\nu(x) \geqslant 2$ for all non-trivial $x \in H \cap \mathrm{PGL}(V)$. In particular, (5) holds for all $x \in H$ of prime order and the result follows as before.

To complete the proof let us assume $G_0$ is an orthogonal group. Here, arguing as before, we quickly reduce to the case $\iota > 0$, so we may assume $G_0 = \mathrm{P}\Omega_{2l}^{\epsilon}(q)$ and $H$ is a subgroup of type $\mathrm{GL}_l^{\epsilon'}(q)$ (see Table 1). Then $\iota = 1/(2l-2)$ and the result follows via Theorem 1 if (5) holds. As in the proof of [20, 4.2], we deduce that $\nu(x) \geqslant 2$ for all $x \in H \cap \mathrm{PGL}(V)$ and thus we can assume $r = p$ and $x$ is a long root element (see [3, 3.49] and [3, Tables 3.4-5, 3.7-8]). Then [3, 3.20, 3.22] imply that $|x^G| > \frac{1}{2d}q^{4l-6}$, where $d = 2 - \delta_{2,p}$, and the desired result follows since $|x^G \cap H| < 2q^{2l-2}$ (see [3, 3.18] and the proof of [4, 3.3]). $\square$

## 1.2 Minimal base sizes

Let $G$ be a permutation group on a finite set $\Omega$ and recall that a *base* for $G$ is a subset $B \subseteq \Omega$ whose pointwise stabilizer $G_{(B)}$ is trivial. Let $b(G)$ be the minimal size of a base for $G$. The Cameron-Kantor conjecture [8, 9] concerns almost simple primitive groups $G$; it asserts that for such groups, either $b(G)$ is bounded by some absolute constant or $G$ lies in a prescribed list of exceptions. The conjecture was finally confirmed by Liebeck and Shalev, with their result on fixed point ratios [20, Theorem ($\star$)] playing a central role in the proof.

**Theorem 1.2 ([20, 1.3]).** *There is a constant $c$ such that if $G$ is any almost simple primitive permutation group on a set $\Omega$ then one of the following holds:*

*(i) $G$ is $A_n$ or $S_n$ acting on $k$-subsets of $\{1,\ldots,n\}$ or an orbit of partitions of $\{1,\ldots,n\}$;*

*(ii) $G$ is a classical group in a subspace action;*

*(iii) $b(G) \leqslant c$.*

*Furthermore, in (iii), the probability that a random $c$-tuple of elements from $\Omega$ forms a base for $G$ tends to 1 as $|G| \to \infty$.*

**Remark 1.3.** In general, (iii) does not hold for the examples in (i) and (ii). Indeed, the definition of a base implies that $|G| \leqslant |\Omega|^{b(G)}$ and in most cases the orders of the groups in (i) and (ii) are not bounded by some fixed polynomial function of their degree $|\Omega|$.

In [9] it is shown that if the socle $G_0$ of $G$ is an alternating group then the probabilistic statement in Theorem 1.2 holds with a best possible constant $c = 2$. In fact, Guralnick and Saxl have recently shown that if $G_0 = A_n$ and $n > 12$ then $b(G) = 2$ for all primitive actions of $G$, with the obvious exclusion of the examples in (i). According to the proof of [20, 1.3], if $G$ is a finite almost simple classical group then the Cameron-Kantor conjecture holds with a constant $c = 11\epsilon^{-1}$, where $\epsilon$ is the undetermined constant given in [20, Theorem ($\star$)] (see (3)). Of course, Theorem 1 states that $\epsilon \approx 1/2$ and Liebeck and Shalev have used this result to establish a much stronger version of Theorem 1.2 for classical groups.

**Theorem 1.4 ([21, 1.11]).** *Let $G$ be a finite almost simple classical group, with natural module of dimension greater than 15. If $G$ acts primitively on a set $\Omega$ in a non-subspace action then the probability that 3 randomly chosen points in $\Omega$ form a base for $G$ tends to 1 as $|G| \to \infty$. In particular, for $G$ sufficiently large we have $b(G) \leqslant 3$.*

**Remark 1.5.** Observe that $|G| \geqslant |G_\omega|^2$ if $b(G) = 2$. Therefore Theorem 1.4 is best possible since there are infinitely many primitive non-subspace actions with $|G| < |G_\omega|^2$. For example, the action of $\mathrm{Sp}_n(q)$ on the set of cosets of $\mathrm{Sp}_{n/2}(q) \wr S_2$ has this property.

As observed in the proof of [20, 1.3], the connection between fixed point ratios and base sizes arises as follows: if $Q(G,c)$ denotes the probability that a randomly chosen $c$-tuple in $\Omega$ does not form a base for $G$ then

$$Q(G,c) \leqslant \sum_{i=1}^{k} |x_i^G| \cdot \mathrm{fpr}(x_i)^c, \tag{6}$$

where $x_1,\ldots,x_k$ represent the distinct $G$-classes of elements of prime order in $G$. Of course, $G$ admits a base of size $c$ if and only if $Q(G,c) < 1$. If $G$ is a classical group and $\Omega$ is a non-subspace $G$-set then we can use Theorem 1 to bound $Q(G,c)$ via (6). In this way, the "zeta function"

$$\zeta_G(t) = \sum_{C \in \mathscr{C}} |C|^{-t}$$

arises naturally, where $\mathscr{C}$ denotes the set of conjugacy classes in $G$ of elements of prime order. Evidently there exists a real number $0 < t_G < 1$ such that $\zeta_G(t_G) = 1$ and Theorem 1 implies that $G$ admits a base of size $c$ if $c(1/2 - 1/n - \iota) - 1 > t_G$. In particular, we can bound $b(G)$ by bounding the function $t_G$. Proceeding in this manner, we have proved the following result.

**Theorem 1.6 ([7]).** *Let $G$ be a finite almost simple classical group, $\Omega$ a primitive non-subspace $G$-set and $H = G_\omega$ for some $\omega \in \Omega$. Then either $b(G) \leqslant 4$ or $G = \mathrm{PSU}_6(2).2$, $H = \mathrm{PSU}_4(3).2^2$ and $b(G) = 5$.*

Of course, Theorem 1.6 only concerns classical groups and in [7] we also consider base sizes in actions of finite groups of exceptional Lie type. Here a similar approach is possible, utilising the detailed results on fixed point ratios in [17].

### 1.3 Monodromy groups

Following [15], we say that a finite permutation group $G$ has *genus* $g$ if it is isomorphic to the monodromy group of a branched covering $\varphi : X \to \mathbb{P}^1\mathbb{C}$, where $X$ is a compact connected Riemann surface of genus $g$. According to the Riemann Existence Theorem, if $G$ has genus $g$ then $G$ is generated by elements $x_1, \ldots, x_k$ such that $\prod_1^k x_i = 1$ and

$$2(d + g - 1) = \sum_{i=1}^{k} \mathrm{ind}(x_i),$$

where $d$ is the degree of $G$ and $\mathrm{ind}(x) = \sum_1^l (r_i - 1)$ if $x \in G$ acts on $\Omega$ with cycles of length $r_1, \ldots, r_l$. The Guralnick-Thompson genus conjecture [15] asserts that if $\mathscr{E}(g)$ is the set of non-abelian, non-alternating composition factors of groups of genus $g$, then $\mathscr{E}(g)$ is finite for each $g$. This was finally established in [12] and [20], with bounds on fixed point ratios playing a key role in the proof (see [11] and [20, 1.2] for example).

It is known that for any $g \geqslant 0$ there are only finitely many primitive permutation groups of fixed genus $g$ whose socle is not a product of alternating groups. In recent work, Frohardt, Magaard and Guralnick have completely classified the primitive groups of genus at most two (with the above condition on the socle) and furthermore, they have shown that the genus of such primitive groups grows as a linear function of the degree.

A basic result suggests that Theorem 1 may be useful in classifying the primitive permutation groups of higher genus (see [15, Theorem E] for the case $g = 0$). This result states that if $G$ is a finite almost simple primitive permutation group of genus $g$ and degree $d$ then there exists a non-identity element $x \in G$ such that

$$\mathrm{fpr}(x) > \frac{1}{85}\left(1 - \frac{84g}{d}\right). \tag{7}$$

Therefore for each $g$, using Corollary 2, we can compile a finite list of almost simple classical groups which could possibly admit a primitive non-subspace action of genus $g$. Moreover, since $G = \langle x_1, \ldots, x_k \rangle$, with the properties described above, we can derive more precise lower bounds on the size of conjugacy classes in $G$ and thus strong upper bounds on fixed point ratios via Theorem 1. In this way, the list of possible candidates can be further refined. This may provide an effective means to study primitive classical groups of arbitrary genus.

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