Pyber's base size conjecture

Tim Burness
University of Bristol

Joint work with Ákos Seress

Group Theory Seminar
Ecole Polytechnique Fédérale de Lausanne
March 26th 2015

University of BRISTOL

Introduction

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a permutation group.

A subset B of Ω is a **base** if the pointwise stabiliser of B in G is trivial.

The **base size** of G, denoted $b(G)$, is the minimal size of a base.

Examples:

- G has a regular orbit on $\Omega \iff b(G) = 1$
- $G = S_n,\ \Omega = \{1, \ldots, n\} = [1, n] \implies b(G) = n - 1$
- $G = D_{2n}$, $\Omega = \{$vertices of a regular n-gon$\} \implies b(G) = 2$
- $G = \mathrm{GL}(V),\ \Omega = V \implies b(G) = \dim V$

Preliminary bounds

Let G be a permutation group on a finite set Ω with a base B.

If $x, y \in G$ and $\alpha^x = \alpha^y$ for all $\alpha \in B$, then $x = y$. So $|G| \leqslant |\Omega|^{|B|}$ and

$$b(G) \geqslant \frac{\log|G|}{\log|\Omega|}$$

For an upper bound, let $B = \{\alpha_1, \ldots, \alpha_{b(G)}\} \subseteq \Omega$ be a base. Then

$$G > G_{\alpha_1} > G_{\alpha_1,\alpha_2} > \cdots > G_{\alpha_1,\ldots,\alpha_{b(G)-1}} > G_{\alpha_1,\ldots,\alpha_{b(G)}} = 1$$

so $|G| \geqslant 2^{b(G)}$ and thus

$$\frac{\log|G|}{\log|\Omega|} \leqslant b(G) \leqslant \log|G|$$

$$\frac{\log |G|}{\log |\Omega|} \leqslant b(G) \leqslant \log |G|$$

Example 1. Let $G = S_n$ and $\Omega = \{1, \dots, n\}$. Then

$$b(G) = n - 1 < 2\frac{\log |G|}{\log |\Omega|}$$

Example 2. Let $G = Z_2 \wr Z_k = Z_2^k \rtimes Z_k$ and $\Omega = \{1, \dots, 2k\}$, so G preserves the partition

$$\Omega = \{1, 2\} \cup \{3, 4\} \cup \cdots \cup \{2k - 1, 2k\}$$

Then

$$b(G) = k = \log |G| - \log k > \frac{1}{2} \log |G|$$

Primitivity

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite transitive permutation group.

The **degree** of G is $|\Omega|$, and the **socle** of G, denoted $\mathrm{Soc}(G)$, is the product of the minimal normal subgroups of G.

Definition. G is **primitive** if the only G-invariant partitions of Ω are $\{\Omega\}$ and $\{\{\alpha\} : \alpha \in \Omega\}$, otherwise G is **imprimitive**.

Equivalently, G is primitive iff G_α is a maximal subgroup of G.

Examples:

- **Primitive:** $G = S_n$, $\Omega = \{1, \dots, n\}$
- **Imprimitive (but transitive)**: $G = Z_2 \wr Z_k$, $\Omega = \{1, \dots, 2k\}$

Pyber's conjecture

"All **primitive** permutation groups have small bases"

Conjecture (Pyber, 1993)

There is a constant c such that

$$b(G) \leqslant c \frac{\log |G|}{\log n}$$

for any primitive group G of degree n.

Primitivity is essential, e.g. $G = Z_2 \wr Z_k$, $k = 2^\ell$, $n = 2k = 2^{\ell+1}$:

$$b(G) = 2^\ell, \quad \frac{\log |G|}{\log n} = \frac{2^\ell + \ell}{\ell + 1}$$

Primitive groups

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite primitive permutation group.

The possibilities for G are described by the **O'Nan-Scott Theorem**

(Here T is a nonabelian simple group, $k \geqslant 2$, $V = (\mathbb{F}_p)^d$)

Type	Description
I	**Almost simple:** $T \leqslant G \leqslant \mathrm{Aut}(T)$
II	**Diagonal:** $T^k \leqslant G \leqslant T^k.(\mathrm{Out}(T) \times P)$, $P \leqslant S_k$
III	**Affine:** $G = V \rtimes G_0 \leqslant \mathrm{AGL}(V)$, $G_0 \leqslant \mathrm{GL}(V)$ irreducible
IV	**Product-type:** $G \leqslant H \wr P$, H primitive type I or II, $P \leqslant S_k$
V	**Twisted wreath product**

Almost simple groups

Let $G \leqslant \mathrm{Sym}(\Omega)$ be an **almost simple** primitive group with socle T, so

$$T \leqslant G \leqslant \mathrm{Aut}(T)$$

Say G is **standard** if one of the following holds:

- $T = A_n$ and Ω is an orbit of subsets or partitions of $\{1, \ldots, n\}$
- $T = \mathrm{Cl}(V)$ is a classical group and Ω is an orbit of subspaces of V

Theorem

Pyber's conjecture holds for almost simple groups.

- **Benbenishty, 2005:** Standard groups (with explicit constant)
- **Liebeck & Shalev, 1999:** There is a constant c such that $b(G) \leqslant c$ for all non-standard G ($c = 7$ is optimal)

Diagonal groups

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a primitive **diagonal-type** group of degree n, so

$$T^k \leqslant G \leqslant T^k.(\mathrm{Out}(T) \times P)$$

where $k \geqslant 2$, T is a nonabelian simple group, and $P \leqslant S_k$ is the group induced by the conjugation action of G on the k factors of T^k.

Theorem (Fawcett, 2013)

Pyber's conjecture holds for diagonal groups:

$$b(G) \leqslant \left\lceil \frac{\log |G|}{\log n} \right\rceil + 2$$

In fact, $b(G) = 2$ *if* $P \neq A_k, S_k$.

Affine groups

Let $G = V \rtimes G_0 \leqslant \mathrm{AGL}(V)$ be a primitive **affine** group, so $V = (\mathbb{F}_p)^d$ and $G_0 \leqslant \mathrm{GL}(V)$ is irreducible.

Seress, 1996:	G_0 soluble	$\Longrightarrow$	$b(G) \leqslant 4$
Gluck & Magaard, 1998:	$p \nmid \lvert G_0 \rvert$	$\Longrightarrow$	$b(G) \leqslant 95$
Halasi & Podoski, 2014:	$p \nmid \lvert G_0 \rvert$	$\Longrightarrow$	$b(G) \leqslant 3$

Theorem (Liebeck & Shalev, 2002)

Pyber's conjecture holds (with an explicit constant) for all affine groups $G = V \rtimes G_0$ such that $G_0 \leqslant \mathrm{GL}(V)$ is primitive (as a linear group).

Pyber's conjecture: The remaining cases

To complete the proof, we need to handle the following cases:

(A) $G = V \rtimes G_0 \leqslant \mathrm{AGL}(V)$ insoluble, where $V = (\mathbb{F}_p)^d$, $G_0 \leqslant \mathrm{GL}(V)$ is imprimitive and p divides $|G_0|$

(B) G is a product-type group

(C) G is a twisted wreath product

Our main theorem deals with cases (B) and (C):

Theorem (B & Seress, 2013)

Pyber's conjecture holds for all non-affine groups.

Product-type groups

Let $H \leqslant \mathrm{Sym}(\Gamma)$ be a primitive group.

Fix $k \geqslant 2$ and set

$$W = H \wr S_k = H^k \rtimes S_k = \{(h_1, \ldots, h_k)\pi \,:\, h_i \in H, \pi \in S_k\}$$

Product action: Combine the natural actions of H^k and S_k on $\Omega = \Gamma^k$:

$$(\gamma_1, \ldots, \gamma_k)^{(h_1,\ldots,h_k)\pi} = \left(\left(\gamma_{1^{\pi^{-1}}}\right)^{h_{1^{\pi^{-1}}}}, \ldots, \left(\gamma_{k^{\pi^{-1}}}\right)^{h_{k^{\pi^{-1}}}} \right)$$

This defines a faithful, primitive action of W on Ω.

Product-type groups

Let $H \leqslant \mathrm{Sym}(\Gamma)$ be a primitive **almost simple** or **diagonal** group.

Set $W = H \wr S_k$, $\Omega = \Gamma^k$ and view $W \leqslant \mathrm{Sym}(\Omega)$ via the product action.

Let $A = \mathrm{Soc}(H)$ and $B = \mathrm{Soc}(W)$, so $B = A^k$.

A subgroup $G \leqslant W \leqslant \mathrm{Sym}(\Omega)$ is a primitive **product-type** group if

- $A^k \leqslant G$; and
- G induces a transitive group $P \leqslant S_k$ on the k factors of A^k.

In particular,

$$\mathrm{Soc}(G) = A^k \leqslant G \leqslant H \wr P$$

The Key Lemma

Let $G \leqslant \mathrm{Sym}(X)$ be a finite permutation group and let $\rho = (X_1, \ldots, X_m)$ be an ***m*-partition** of X.

Define $G_\rho = \bigcap_{i=1}^{m} G_{\{X_i\}}$: the intersection of the setwise stabilisers of the X_i.

Note: If $m = 2$ then $G_\rho = G_{\{X_1\}}$ is just the setwise stabiliser of X_1

Key Lemma

There is a constant c with the following property:

If $G \leqslant \mathrm{Sym}(X)$ is a transitive permutation group of degree k then there exist 2-partitions $\{\rho_1, \ldots, \rho_\ell\}$ of X such that

$$\bigcap_{i=1}^{\ell} G_{\rho_i} = 1 \text{ and } \ell \leqslant c\left(1 + \frac{\log|G|}{k}\right)$$

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This is essentially **best possible**:

Example. If $G = S_k$ then at least $\lceil \log k \rceil > \dfrac{\log|G|}{k}$ distinct 2-partitions are required, e.g. if $k = 8$ take

$$\begin{aligned}
\rho_1 &= (\{1,2,3,4\},\{5,6,7,8\}) \\
\rho_2 &= (\{1,2,5,6\},\{3,4,7,8\}) \\
\rho_3 &= (\{1,3,5,7\},\{2,4,6,8\})
\end{aligned}$$

Applying the Key Lemma

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a primitive **product-type** group, where $\Omega = \Gamma^k$,

$$T^k \leqslant G \leqslant H \wr P = H^k \rtimes P$$

and $H \leqslant \mathrm{Sym}(\Gamma)$ is an **almost simple** primitive group with socle T.

Recall that $P \leqslant S_k$ is transitive and G acts on Ω via

$$(\gamma_1, \ldots, \gamma_k)^{(h_1,\ldots,h_k)\pi} = \left(\left(\gamma_{1^{\pi^{-1}}}\right)^{h_{1^{\pi^{-1}}}}, \ldots, \left(\gamma_{k^{\pi^{-1}}}\right)^{h_{k^{\pi^{-1}}}} \right)$$

Applying the Key Lemma

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a primitive **product-type** group, where $\Omega = \Gamma^k$,

$$T^k \leqslant G \leqslant H \wr P = H^k \rtimes P$$

and $H \leqslant \mathrm{Sym}(\Gamma)$ is an **almost simple** primitive group with socle T.

Let $\{\gamma_1, \ldots, \gamma_b\} \subseteq \Gamma$ be a base for H with $b = b(H)$.

For $1 \leqslant i \leqslant b$ set $\alpha_i = (\gamma_i, \ldots, \gamma_i) \in \Omega$.

If $g = (h_1, \ldots, h_k)\pi \in G$ fixes each α_i then

$$(\gamma_i, \ldots, \gamma_i) = (\gamma_i, \ldots, \gamma_i)^g = \left((\gamma_i)^{h_{1\pi^{-1}}}, \ldots, (\gamma_i)^{h_{k\pi^{-1}}}\right)$$

so h_j fixes γ_i for all i and all j. Therefore $g = (1, \ldots, 1)\pi$.

Since $P \leqslant S_k$ is transitive, let $\{\rho_1, \dots, \rho_\ell\}$ be 2-partitions of $X = [1, k]$ provided by the **Key Lemma**, where

$$\ell \leqslant c_1 \left(1 + \frac{\log |P|}{k}\right)$$

for some constant c_1. Set $r = \lfloor \log |\Gamma| \rfloor$ and assume $\ell \geqslant r$.

- Set $Y = \{\rho_1, \dots, \rho_r\}$ and let $\sigma = (\sigma_1, \dots, \sigma_s)$ be the **common refinement** of the partitions in Y.

  i.e., if $\rho_i = (A_i, B_i)$, $1 \leqslant i \leqslant r$, then the σ_j are the nonempty subsets in the collection

  $$\{C_1 \cap C_2 \cap \cdots \cap C_r \,:\, C_i \in \{A_i, B_i\}, 1 \leqslant i \leqslant r\}$$

- By construction, σ is an s-partition of X, $s \leqslant 2^r \leqslant |\Gamma|$ and each σ_j is contained in one of the two parts of each 2-partition in Y.

- **Note:** $\pi \in S_k$ fixes $\sigma \implies \pi$ fixes each $\rho_i \in Y$

An example. $k = 8$, $r = 2$

$$\rho_1 = (\{1,3,6\},\{2,4,5,7,8\}) = (A_1, B_1)$$

$$\rho_2 = (\{4,6,7,8\},\{1,2,3,5\}) = (A_2, B_2)$$

$$\begin{aligned} \sigma &= (A_1 \cap A_2, A_1 \cap B_2, B_1 \cap A_2, B_1 \cap B_2) \\ &= (\{6\},\{1,3\},\{4,7,8\},\{2,5\}) \end{aligned}$$

$\pi \in S_8$ fixes $\sigma \implies \pi$ fixes ρ_1 and ρ_2

Since $P \leqslant S_k$ is transitive, let $\{\rho_1, \ldots, \rho_\ell\}$ be 2-partitions of $X = [1, k]$ provided by the **Key Lemma**, where

$$\ell \leqslant c_1 \left(1 + \frac{\log |P|}{k}\right)$$

for some constant c_1. Set $r = \lfloor \log |\Gamma| \rfloor$ and assume $\ell \geqslant r$.

- $Y = \{\rho_1, \ldots, \rho_r\}$, $\sigma = (\sigma_1, \ldots, \sigma_s)$, $s \leqslant 2^r \leqslant |\Gamma|$ and each σ_j is contained in one of the two parts of each 2-partition in Y.
- Choose distinct $\gamma_1, \ldots, \gamma_s \in \Gamma$ and define $\beta \in \Omega$ so that all the coordinates in β corresponding to points in σ_i are equal to γ_i.

e.g. if $k = 8$ and $\sigma = (\{1, 3, 4\}, \{2, 7\}, \{5, 6, 8\})$, then $s = 3$ and

$$\beta = (\gamma_1, \gamma_2, \gamma_1, \gamma_1, \gamma_3, \gamma_3, \gamma_2, \gamma_3) \in \Gamma^8 = \Omega$$

Note. $\pi \in S_8$ fixes $\beta \implies \pi$ fixes $\sigma \implies \pi$ fixes each $\rho_i \in Y$

Since $P \leqslant S_k$ is transitive, let $\{\rho_1, \ldots, \rho_\ell\}$ be 2-partitions of $X = [1, k]$ provided by the **Key Lemma**, where

$$\ell \leqslant c_1 \left(1 + \frac{\log |P|}{k}\right)$$

for some constant c_1. Set $r = \lfloor \log |\Gamma| \rfloor$ and assume $\ell \geqslant r$.

- $Y = \{\rho_1, \ldots, \rho_r\}$, $\sigma = (\sigma_1, \ldots, \sigma_s)$, $s \leqslant 2^r \leqslant |\Gamma|$ and each σ_j is contained in one of the two parts of each 2-partition in Y.
- Choose distinct $\gamma_1, \ldots, \gamma_s \in \Gamma$ and define $\beta \in \Omega$ so that all the coordinates in β corresponding to points in σ_i are equal to γ_i.

$$\begin{aligned} g = (1, \ldots, 1)\pi \in G \text{ fixes } \beta &\implies \pi \text{ fixes } \sigma \\ &\implies \pi \text{ fixes each } \rho_i \in Y \end{aligned}$$

In this way, we obtain $\{\beta_1, \ldots, \beta_{\lceil \ell/r \rceil}\} \subseteq \Omega$ so that if $g = (1, \ldots, 1)\pi \in G$ fixes each β_i then π fixes $\rho_1, \ldots, \rho_\ell$, so $\pi = 1$ and thus $g = 1$.

We now have a base $\{\alpha_1, \dots, \alpha_b, \beta_1, \dots, \beta_{\lceil \ell/r \rceil}\}$ for G, where

$$\ell \leqslant c_1 \left(1 + \frac{\log|P|}{k}\right), \quad r = \lfloor \log|\Gamma| \rfloor, \quad b = b(H) \leqslant c_2 \frac{\log|H|}{\log|\Gamma|}$$

for constants c_1, c_2 (via Pyber for almost simple groups).

Note that $|H| \leqslant |\mathrm{Aut}(T)| \leqslant |T|^2$ (H is almost simple with socle T), so

$$|G| \geqslant |T|^k |P| \geqslant \left(|H|^k |P|\right)^{\frac{1}{2}}$$

$$\begin{aligned} b(G) \leqslant \lceil \ell/r \rceil + b &\leqslant \left\lceil c_1 \frac{1}{\lfloor \log|\Gamma| \rfloor} + c_1 \frac{\log|P|}{k \lfloor \log|\Gamma| \rfloor} \right\rceil + c_2 \frac{\log|H|}{\log|\Gamma|} \\ &\leqslant c_3 \frac{\log|P|}{\log|\Omega|} + c_4 \frac{\log|H|^k}{\log|\Omega|} \\ &\leqslant c_5 \frac{\log|G|}{\log|\Omega|} \end{aligned}$$

Proof of the Key Lemma

Key Lemma

There is a constant c with the following property:

If $G \leqslant \mathrm{Sym}(\Delta)$ is a transitive permutation group of degree k then there exist 2-partitions $\{\rho_1, \ldots, \rho_\ell\}$ of Δ such that

$$\bigcap_{i=1}^{\ell} G_{\rho_i} = 1 \quad \text{and} \quad \ell \leqslant c \left(1 + \frac{\log |G|}{k}\right)$$

Distinguishing number

The **distinguishing number** $D(G)$ of a permutation group $G \leqslant \mathrm{Sym}(\Delta)$ is the minimal m such that Δ admits an m-partition with trivial stabiliser.

Examples:

- $D(G) = 1 \iff G = 1$
- $D(G) = 2 \iff G$ has a regular orbit on the power set of Δ
- $D(S_k) = k$ and $D(A_k) = k - 1$

Let G be a primitive group of degree k, with $G \neq A_k, S_k$.

Cameron, Neumann & Saxl, 1984: $D(G) = 2$ if $k \gg 0$

Seress, 1997: $D(G) = 2$ if $k > 32$

Dolfi, 2000: $D(G) \leqslant 4$ for all k

Key Lemma: The primitive case

Let $G \leqslant \mathrm{Sym}(\Delta)$ be a **primitive** group of degree k.

Case 1. If $G \neq A_k, S_k$ then $D(G) \leqslant 4$ by **Dolfi**, so let (X_1, X_2, X_3, X_4) be a distinguishing partition of Δ. Then the stabiliser of the 2-partitions

$$(X_1 \cup X_2, X_3 \cup X_4)\,, \;\; (X_1 \cup X_3, X_2 \cup X_4)$$

is trivial.

Case 2. If $G = A_k$ or S_k then $\lceil \log k \rceil < 2\left(1 + \frac{\log |G|}{k}\right)$ 2-partitions are sufficient, e.g. if $k = 8$ then take

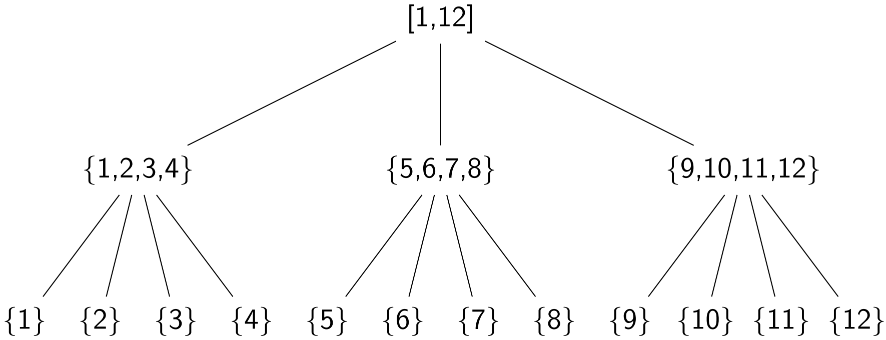
$$(\{1,2,3,4\},\{5,6,7,8\})\,,\; (\{1,2,5,6\},\{3,4,7,8\})\,,\; (\{1,3,5,7\},\{2,4,6,8\})$$

The imprimitive case

Let $G \leqslant \mathrm{Sym}(\Delta)$ be **imprimitive**, where $\Delta = [1, k]$.

Fix a **structure tree** for G. This is a rooted tree T, with levels $T_0 = \{\Delta\}$ (the **root**), $T_1, \ldots, T_{s-1}$ and $T_s = \{\{x\} : x \in \Delta\}$ (the **leaves**).

Example: $G = S_4 \wr S_3$, $\Delta = [1, 12]$:



Structure trees

Fix a **structure tree** for G. This is a rooted tree T, with levels $T_0 = \{\Delta\}$ (the **root**), $T_1, \ldots, T_{s-1}$ and $T_s = \{\{x\} : x \in \Delta\}$ (the **leaves**).

- The vertices on each fixed level of T correspond to subsets in a G-invariant partition of Δ
- The action of G on Δ extends naturally to an action on T
- If $x \in T_{i-1}$ is a non-leaf vertex with children $\Delta(x) \subseteq T_i$ then
  - $\Delta(x)$ is a partition of x
  - G_x, the setwise stabiliser of x in G, acts primitively on $\Delta(x)$; the induced group is denoted by $G(x) \leqslant \mathrm{Sym}(\Delta(x))$
- Level T_i is **large** if $|\Delta(x)| \geqslant 7$ and $G(x) = \mathrm{Alt}(\Delta(x))$ or $\mathrm{Sym}(\Delta(x))$ for some (hence all) $x \in T_{i-1}$

The Key Lemma: G imprimitive, no large levels

Key Lemma

There is a constant c with the following property:

If $G \leqslant \mathrm{Sym}(\Delta)$ is a transitive permutation group of degree k then there exist 2-partitions $\{\rho_1, \ldots, \rho_\ell\}$ of Δ such that

$$\bigcap_{i=1}^{\ell} G_{\rho_i} = 1 \quad \text{and} \quad \ell \leqslant c\left(1 + \frac{\log|G|}{k}\right)$$

Theorem. *If $G \leqslant \mathrm{Sym}(\Delta)$ is imprimitive with a structure tree T with* **no large levels***, then the Key Lemma holds with $\ell = 6$.*

G imprimitive, no large levels

$G \leqslant \mathrm{Sym}(\Delta)$ is imprimitive with a structure tree T with **no large levels**.

Let $x \in T$ be a non-leaf vertex with children $\Delta(x)$.

Then $G(x) \leqslant \mathrm{Sym}(\Delta(x))$ is primitive and $D(G(x)) \leqslant 6$ by **Dolfi**, so there exist three 2-partitions

$$\Delta(x) = \Delta_j(x) \cup \Delta_j(x)', \quad 1 \leqslant j \leqslant 3$$

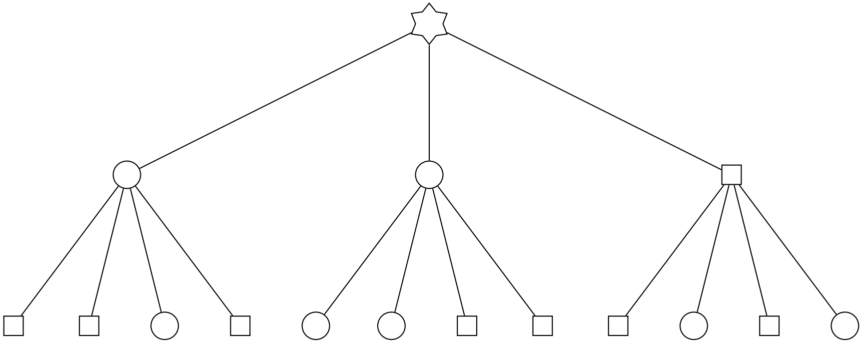
such that the intersection of their stabilisers in $G(x)$ is trivial.

We inductively define three 3-colourings of the vertices of T, denoted $F_j : T \to \mathbb{F}_3$, $1 \leqslant j \leqslant 3$.

Set $F_j(T_0) = 0$. For $x \in T_i$ and $y \in \Delta(x)$ we define

$$F_j(y) = \begin{cases} F_j(x) & \text{if } y \in \Delta_j(x) \\ F_j(x) + 1 & \text{if } y \in \Delta_j(x)' \end{cases}$$

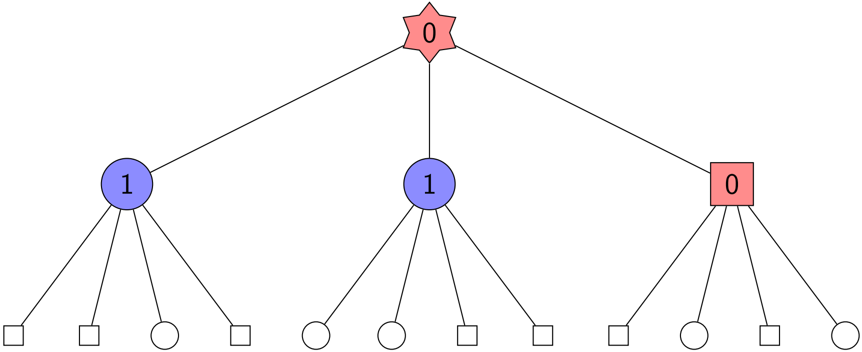
An example. Fix $j \in \{1, 2, 3\}$



Every non-root vertex in T is either a square or circle:

Squares: children of x in $\Delta_j(x)$
Circles: children of x in $\Delta_j(x)'$

An example. Fix $j \in \{1, 2, 3\}$



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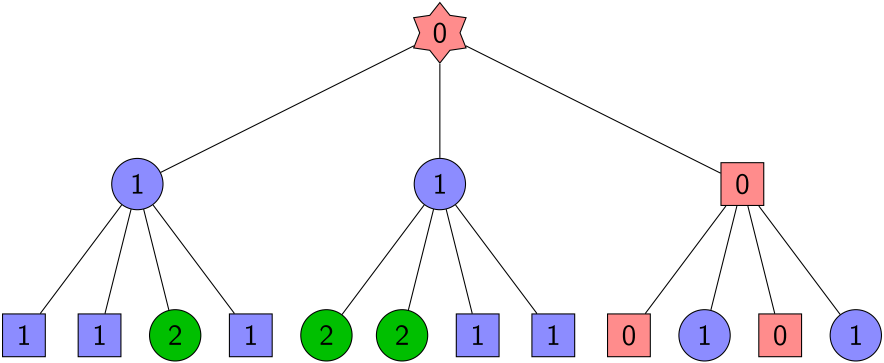


Every non-root vertex in T is either a square or circle:

Squares: children of x in $\Delta_j(x)$
Circles: children of x in $\Delta_j(x)'$

Claim. If $g \in G$ fixes each F_j-colouring of T, then $g = 1$.

Let $x = T_0$ be the root vertex. Then g stabilises the 2-partitions

$$\Delta(x) = \Delta_j(x) \cup \Delta_j(x)', \quad j = 1, 2, 3$$

so g fixes T_1 pointwise.

Let $y \in T_1$. Then $g \in G_y$ stabilises the 2-partitions

$$\Delta(y) = \Delta_j(y) \cup \Delta_j(y)', \quad j = 1, 2, 3$$

so g fixes $\Delta(y)$ pointwise. Therefore g fixes T_2 pointwise.

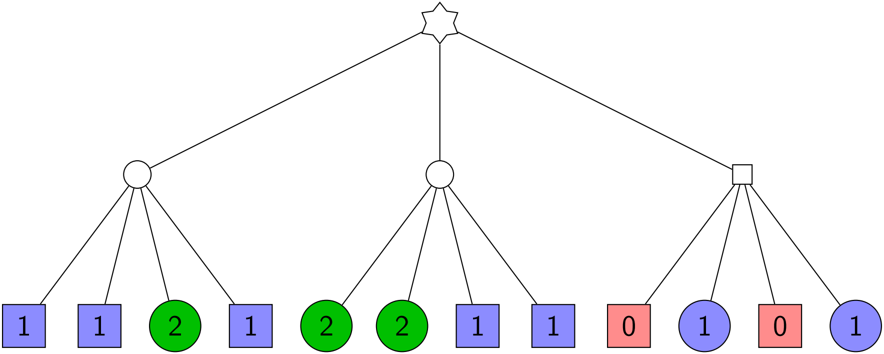
By induction on $i = 0, 1, \ldots, s$, g fixes $T_0 \cup T_1 \cup \cdots \cup T_i$ pointwise, so g fixes $T_s = \Delta$ pointwise and thus $g = 1$.

Claim. If $g \in G$ fixes each F_j-colouring of $T_s = \Delta$, then $g = 1$.

Each F_j-colouring of T can be reconstructed from the corresponding colouring of the leaves.

By induction on $i = s, s-1, \ldots, 0$, $g \in G$ fixes the F_j-colouring of $T_i \cup T_{i+1} \cup \cdots \cup T_s$.

An example. Fix $j \in \{1, 2, 3\}$



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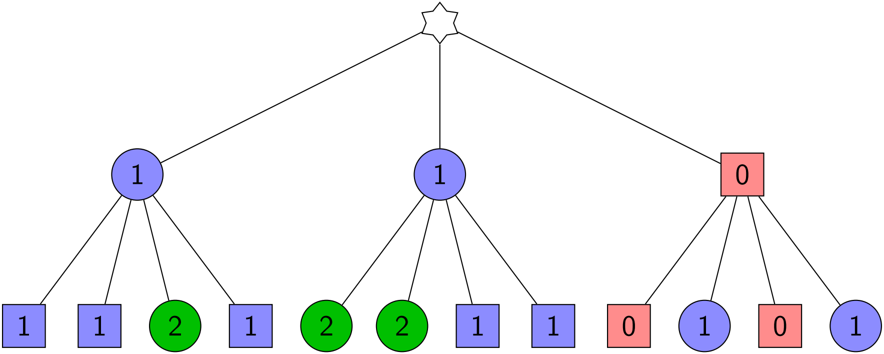

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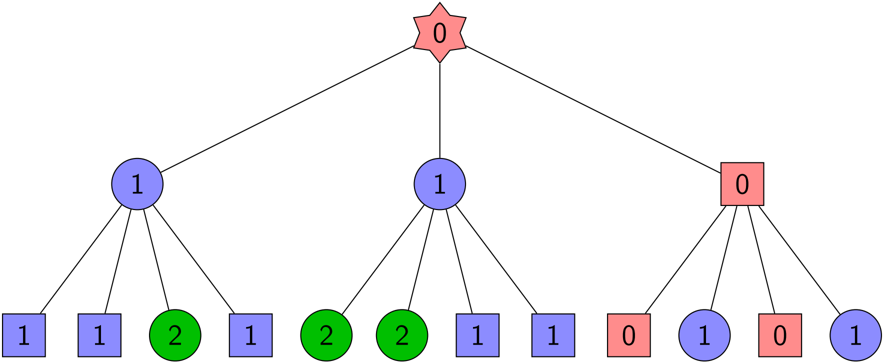

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Squares: children of x in $\Delta_j(x)$
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Claim. If $g \in G$ fixes each F_j-colouring of $T_s = \Delta$ then $g = 1$.

Each F_j-colouring of T can be reconstructed from the corresponding colouring of the leaves.

By induction on $i = s, s-1, \ldots, 0$, g fixes each F_j-colouring of $T_i \cup T_{i+1} \cup \cdots \cup T_s$.

Therefore, g fixes all three F_j-colourings of T, so $g = 1$.

The final step!

$$\begin{aligned} G\text{-stabiliser of } F_j\text{-colouring of } \Delta &= G\text{-stabiliser of a 3-partition of } \Delta \\ &= G_{\rho_1} \cap G_{\rho_2}, \text{ where } \rho_i \text{ is a 2-partition of } \Delta \end{aligned}$$

Theorem. *There are 2-partitions $\{\rho_1, \ldots, \rho_6\}$ of Δ such that*

$$\bigcap_{i=1}^{6} G_{\rho_i} = 1$$

Concluding remarks

- A similar, but more complicated, argument via tree colourings applies if T has large levels.
- We can argue by induction on the number of large levels – the base case (a unique large level) is the most difficult.
- For product-type groups G, the constant c such that

$$b(G) \leqslant c\frac{\log|G|}{\log n}$$

  is undetermined.
- **A proof of Pyber's conjecture?**