THE CLASSIFICATION OF EXTREMELY PRIMITIVE GROUPS

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Dedicated to the memory of Jan Saxl

Abstract. Let G be a finite primitive permutation group on a set Ω with nontrivial point stabilizer G_α. We say that G is extremely primitive if G_α acts primitively on each of its orbits in $\Omega \setminus \{\alpha\}$. These groups arise naturally in several different contexts and their study can be traced back to work of Manning in the 1920s. In this paper, we determine the almost simple extremely primitive groups with socle an exceptional group of Lie type. By combining this result with earlier work of Burness, Praeger and Seress, this completes the classification of the almost simple extremely primitive groups. Moreover, in view of results by Mann, Praeger and Seress, our main theorem gives a complete classification of all finite extremely primitive groups, up to finitely many affine exceptions (and it is conjectured that there are no exceptions). Along the way, we also establish several new results on base sizes for primitive actions of exceptional groups, which may be of independent interest.

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1. Introduction

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite primitive permutation group with point stabilizer $H = G_\alpha \neq 1$. We say that G is *extremely primitive* if H acts primitively on each of its orbits in $\Omega \setminus \{\alpha\}$ (this term was coined by Mann, Praeger and Seress in [65]). Equivalently, G is extremely primitive if and only if $H \cap H^x$ is a maximal subgroup of H for all $x \in G \setminus H$. For example, the natural action of $G = \mathrm{PGL}_2(q)$ on the projective line over $\mathbb{F}_q$ is extremely primitive (here G is 2-transitive and H is a Borel subgroup). These groups arise naturally in several different contexts, including the construction of some of the sporadic simple groups (in particular J_2 and HS) and the study of permutation groups with restricted movement (see [68]).

By a theorem of Manning [66, Corollary I, p.821] from 1927, if G is extremely primitive then H acts faithfully on each of its orbits in $\Omega \setminus \{\alpha\}$. In particular, $H \cap H^x$ is a core-free maximal subgroup of H for all $x \in G \setminus H$. In turn, this implies that H is itself a primitive permutation group on each of its nontrivial orbits and thus the O'Nan-Scott Theorem imposes strong restrictions on the structure of H. In particular, H has at most two minimal normal

Date: October 20, 2020.

subgroups, and the socle of H (denoted by $\mathrm{soc}(H)$) is a direct product of isomorphic simple groups.

Recall that G is *almost simple* if $G_0 \leqslant G \leqslant \mathrm{Aut}(G_0)$ for some nonabelian simple group G_0, which is the unique minimal normal subgroup of G. Also recall that G is *affine* if Ω has the structure of a vector space over a prime field $\mathbb{F}_p$ and G acts by affine transformations. A major step towards the classification of the extremely primitive groups is a theorem of Mann, Praeger and Seress [65], which states that each extremely primitive group is either almost simple or of affine type. Furthermore, they classify the affine examples up to the possibility of finitely many extremely primitive groups of the form $G = V{:}H$, where $V = \mathbb{F}_2^d$ and $H \leqslant \mathrm{GL}(V)$ is irreducible and almost simple (as discussed below, it is conjectured that there are no additional examples). In later work [19, 20], Burness, Praeger and Seress determined all the extremely primitive almost simple groups with socle an alternating, classical or sporadic group.

In this paper, we complete the picture by determining the extremely primitive almost simple groups with socle an exceptional group of Lie type. Note that in the statement of Theorem 1, we exclude the Ree groups with socle ${}^2G_2(3)' \cong \mathrm{L}_2(8)$.

Theorem 1. *Let $G \leqslant \mathrm{Sym}(\Omega)$ be an almost simple group with stabilizer H and socle an exceptional group of Lie type. Then G is extremely primitive if and only if $(G,H) = (G_2(4), \mathrm{J}_2)$ or $(G_2(4).2, \mathrm{J}_2.2)$.*

By combining Theorem 1 with the results in [19, 20], we can now complete the classification of the almost simple extremely primitive groups.

Theorem 2. *Let $G \leqslant \mathrm{Sym}(\Omega)$ be an almost simple group with stabilizer H and socle G_0. Then G is extremely primitive if and only if (G,H) is one of the cases in Table 1.*

Remark 1. Let us make some comments on the statement of Theorem 2.

(i) In view of the isomorphisms
$$\mathrm{L}_2(4) \cong \mathrm{L}_2(5) \cong \mathrm{Alt}_5, \;\; \mathrm{L}_2(9) \cong \mathrm{Alt}_6$$
we assume $q \geqslant 7$ and $q \neq 9$ in the fifth and sixth rows of Table 1 with $G_0 = \mathrm{L}_2(q)$. Similarly, we assume $G_0 \neq \mathrm{L}_4(2), \mathrm{L}_3(2), {}^2G_2(3)'$ since $\mathrm{L}_4(2) \cong \mathrm{Alt}_8$, $\mathrm{L}_3(2) \cong \mathrm{L}_2(7)$ and ${}^2G_2(3)' \cong \mathrm{L}_2(8)$.

(ii) In the fifth row of the table, P_1 denotes a Borel subgroup of G.

(iii) In the final column we describe the extremely primitive groups with socle G_0 and $H \cap G_0$ as given in the second column (if no conditions are recorded, then every almost simple group with socle G_0 is extremely primitive).

(iv) In the third column we record the rank of G, which is the number of orbits of H on Ω (so the almost simple 2-primitive groups have rank 2). In the special case $G = G_2(4).\alpha$ arising in Theorem 1, we have
$$|G : H| = 1 + 100 + 315$$
and the stabilizers for the nontrivial orbits of H are the maximal subgroups $\mathrm{U}_3(3).\alpha$ and $2^{1+4}{:}\mathrm{Alt}_5.\alpha$ of $H = \mathrm{J}_2.\alpha$.

(v) In the eighth row, $G_0 = \mathrm{U}_4(3)$ and $(G,H) = (G_0.2^2, \mathrm{L}_3(4).2^2)$ or $(G_0.2, \mathrm{L}_3(4).2)$. More precisely, $G = G_0.\langle x,y\rangle$ or $G_0.\langle y\rangle$, where x is an involutory diagonal automorphism (class `2B` in the notation of [26]) and y is an involutory graph automorphism with centralizer of type $\mathrm{O}_4^-(3)$ (class `2F`).

(vi) In the ninth row, $G_0 = \mathrm{L}_3(4)$ and $(G,H) = (G_0.2^2, \mathrm{Alt}_6.2^2)$ or $(G_0.2, \mathrm{Alt}_6.2)$, where in the latter case, $G = G_0.\langle x\rangle$ and x is an involutory graph or graph-field automorphism (classes `2B` or `2D`).

G_0	$H \cap G_0$	Rank	Conditions
Alt_n	$(\mathrm{Sym}_{n/2} \wr \mathrm{Sym}_2) \cap G_0$	$\frac{1}{4}(n+2)$	$n \equiv 2 \pmod 4$
Alt_n	Alt_{n-1}	2	$G = \mathrm{Sym}_n$ or Alt_n
Alt_6	$\mathrm{L}_2(5)$	2	$G = \mathrm{Sym}_6$ or Alt_6
Alt_5	D_{10}	2	
$\mathrm{L}_2(q)$	P_1	2	
$\mathrm{L}_2(q)$	$D_{2(q+1)}$	$\frac{1}{2}q$	$G = G_0$, $q+1$ is a Fermat prime
$\mathrm{Sp}_n(2)$	$\mathrm{O}_n^{\pm}(2)$	2	$n \geqslant 6$
$\mathrm{U}_4(3)$	$\mathrm{L}_3(4)$	3	See Remark 1(v)
$\mathrm{L}_3(4)$	Alt_6	3	See Remark 1(vi)
$\mathrm{L}_2(11)$	Alt_5	2	$G = G_0$
$G_2(4)$	J_2	3	
M_{11}	Sym_6	2	
M_{11}	$\mathrm{L}_2(11)$	2	
M_{12}	M_{11}	2	$G = G_0$
M_{22}	$\mathrm{L}_3(4)$	2	
M_{23}	M_{22}	2	
M_{24}	M_{23}	2	
J_2	$\mathrm{U}_3(3)$	3	
HS	M_{22}	3	
HS	$\mathrm{U}_3(5)$	2	$G = G_0$
Suz	$G_2(4)$	3	
McL	$\mathrm{U}_4(3)$	3	
Ru	${}^2F_4(2)$	3	
Co_2	$\mathrm{U}_6(2)$	3	
Co_2	McL	6	
Co_3	McL	2	

Table 1. The extremely primitive almost simple groups

(vii) It is worth noting the following unrefinable chain of subgroups of Conway's sporadic group Co_3:

$$\mathrm{Co}_3 > \mathrm{McL}.2 > \mathrm{U}_4(3).2 > \mathrm{L}_3(4).2 > \mathrm{PGL}_2(9) > 3^2{:}8$$

For each inclusion $K > H$ in this chain, K is extremely primitive on K/H.

(viii) Let us also highlight the remarkable rank 6 example that arises when $G = \mathrm{Co}_2$ and $H = \mathrm{McL}$. Here

$$|G : H| = 47104 = 1 + 275 + 2025 + 7128 + 15400 + 22275$$

and the respective stabilizers for the nontrivial orbits of H are $\mathrm{U}_4(3)$, M_{22}, $\mathrm{U}_3(5)$, $3^4{:}\mathrm{M}_{10}$ and $2^4{:}\mathrm{Alt}_7$, each of which is a maximal subgroup of H.

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite primitive permutation group and let $e(G) \geqslant 0$ be the largest integer k with the property that for every ℓ-set $\Delta \subseteq \Omega$ with $1 \leqslant \ell \leqslant k$, the pointwise stabilizer G_Δ is nontrivial and acts primitively on its orbits in $\Omega \setminus \Delta$. Note that G is extremely primitive if and only if $e(G) \geqslant 1$. Suppose $G = V{:}H$ is affine and $e(G) \geqslant 3$, where $V = \mathbb{F}_p^d$. Then H_{v_1,v_2,v_3} is a maximal subgroup of H_{v_1,v_2} for all triples of distinct nonzero vectors in V. By setting $v_3 = \lambda v_1$ if $p > 2$ and $v_3 = v_1 + v_2$ if $p = 2$, where $1 \neq \lambda \in \mathbb{F}_p^\times$, we get $H_{v_1,v_2,v_3} = H_{v_1,v_2}$ and so there are no affine groups with $e(G) \geqslant 3$. By inspecting the almost simple groups in Table 1, we obtain the following corollary.

Corollary 3. *Let G be a finite primitive group of degree n with $e(G) \geqslant 3$. Then G is 4-transitive and one of the following holds:*

G_0	$H \cap G_0$	Conditions
Alt_6	$(\mathrm{Sym}_3 \wr \mathrm{Sym}_2) \cap G_0$	
Alt_5	Alt_4	
Alt_5	D_{10}	
$\mathrm{L}_2(q)$	P_1	
$\mathrm{L}_2(q)$	$D_{2(q+1)}$	$G = G_0$, $q+1$ is a Fermat prime

TABLE 2. The extremely primitive almost simple groups with solvable point stabilizer

(i) $G \cong \mathrm{Sym}_n$ *and* $e(G) = n-2$.
(ii) $G \cong \mathrm{Alt}_n$, $n \geqslant 6$ *and* $e(G) = n-3$.
(iii) $(G,n) = (\mathrm{M}_{12}, 12)$ *or* $(\mathrm{M}_{24}, 24)$ *and* $e(G) = 3$.

By combining Theorem 2 with the main results from [65] on affine groups, we obtain the following theorem. Note that in part (ii)(a), a prime divisor r of p^d-1 is a *primitive prime divisor* if r does not divide p^i-1 for all $i = 1,\ldots,d-1$ (in other words, the order of p mod r is d). Also recall that a primitive group is *simply primitive* if it is not 2-transitive.

Theorem 4. *Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite extremely primitive group with point stabilizer H. Then either*

(i) *G is almost simple and (G,H) is one of the cases in Table 1; or*
(ii) *$G = V{:}H \leqslant \mathrm{AGL}_d(p)$ is affine, p is a prime and one of the following holds:*
   (a) *$H = Z_r.Z_e$, where e divides d and r is a primitive prime divisor of p^d-1.*
   (b) *$p=2$ and $H = \mathrm{SL}_d(2)$ with $d \geqslant 3$, or $H = \mathrm{Sp}_d(2)$ with $d \geqslant 4$.*
   (c) *$p=2$ and $(d,H) = (4,\mathrm{Alt}_6)$, $(4,\mathrm{Alt}_7)$, $(6,\mathrm{U}_3(3))$ or $(6,\mathrm{U}_3(3).2)$.*
   (d) *$p=2$ and (d,H) is one of the following:*

$(10,\mathrm{M}_{12})$	$(10,\mathrm{M}_{22})$	$(10,\mathrm{M}_{22}.2)$	$(11,\mathrm{M}_{23})$
$(11,\mathrm{M}_{24})$	$(22,\mathrm{Co}_3)$	$(24,\mathrm{Co}_1)$	$(2k,\mathrm{Alt}_{2k+1})$
$(2k,\mathrm{Sym}_{2k+1})$	$(2\ell,\mathrm{Alt}_{2\ell+1})$	$(2\ell,\mathrm{Sym}_{2\ell+1})$	$(2\ell,\Omega^{\pm}_{2\ell}(2))$
$(2\ell,\mathrm{O}^{\pm}_{2\ell}(2))$	$(8,\mathrm{L}_2(17))$	$(8,\mathrm{Sp}_6(2))$	

   where $k \geqslant 2$ and $\ell \geqslant 3$.
   (e) *$p=2$, H is almost simple and G is simply primitive.*

Moreover, every group in parts (i) and (ii)(a,b,c,d) is extremely primitive.

As an immediate corollary we get the following result, which shows that if $G \leqslant \mathrm{Sym}(\Omega)$ is extremely primitive, then in almost every case G_α acts as a primitive group of almost simple or affine type on each of its orbits in $\Omega \setminus \{\alpha\}$ (the exceptions arise in part (iii), where G_α acts as a product-type primitive group).

Corollary 5. *Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite extremely primitive group with point stabilizer H. Then one of the following holds:*

(i) *H is almost simple.*
(ii) *H is solvable and either*
   (a) *G is almost simple with socle G_0 and (G,H) is recorded in Table 2; or*
   (b) *$G \leqslant \mathrm{AGL}_d(p)$ is affine and $H = Z_r.Z_e$, where e divides d and r is a primitive prime divisor of p^d-1.*
(iii) *$G = \mathrm{Sym}_n$ or Alt_n, $n \equiv 2 \pmod 4$, $n \geqslant 10$ and $H = (\mathrm{Sym}_{n/2} \wr \mathrm{Sym}_2) \cap G$.*

By Theorem 4, in order to complete the classification of extremely primitive groups, it remains to handle the affine groups $G = V{:}H$ arising in part (ii)(e), where $H \leqslant \mathrm{GL}(V)$ is almost simple and $V = \mathbb{F}_2^d$. Let $\mathcal{M}$ be the set of maximal subgroups of H, and for $M \in \mathcal{M}$, let $\mathrm{fix}(M)$ be the space of vectors in V fixed by M. By [65, Lemma 4.1], we have

$$\sum_{M \in \mathcal{M}} (|\mathrm{fix}(M)| - 1) \leqslant 2^d - 1,$$

with equality if and only if G is extremely primitive. Since $\dim \mathrm{fix}(M) \leqslant d/2$ for each $M \in \mathcal{M}$, it follows that G is not extremely primitive if $|\mathcal{M}| < 2^{d/2}$. In this way, upper bounds on $|\mathcal{M}|$, combined with lower bounds on the dimensions of irreducible modules for H, play an important role in the analysis. In particular, a theorem of Liebeck and Shalev [58] implies that $|\mathcal{M}| < |H|^{8/5}$ for all sufficiently large almost simple groups H and this is a key ingredient in the proof that there are at most finitely many extremely primitive affine groups arising in part (ii)(e) of Theorem 4.

A well known conjecture of G.E. Wall from 1961 asserts that $|\mathcal{M}| < |H|$. Wall's conjecture was originally formulated for all finite groups, but counterexamples have recently been constructed, see [62]. However, the conjecture has been established for all sufficiently large alternating and symmetric groups (see [58]) and a theorem of Liebeck, Martin and Shalev [49] implies that $|\mathcal{M}| < |H|^{1+o(1)}$ for all almost simple groups H of Lie type. If one assumes Wall's conjecture for almost simple groups, then [65, Theorem 4.8] identifies a very short and explicit list of affine groups that can arise in part (ii)(e) of Theorem 4 (see [65, Table 2]). In each case, H is an almost simple group of Lie type (and defined over the field $\mathbb{F}_2$, with just one exception) and it is conjectured in [65] that none of these groups are extremely primitive. In other words, the list of extremely primitive groups in parts (i) and (ii)(a,b,c,d) of Theorem 4 is conjectured to be complete. See Remark 2 below for some additional comments on the cases in [65, Table 2].

Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite primitive permutation group with stabilizer $H = G_\alpha$. There are several different methods for showing that G is not extremely primitive. As mentioned previously, Manning's result [66] implies that if G is extremely primitive, then H acts faithfully and primitively on each of its nontrivial orbits and this imposes strong restrictions on the socle of H, via the O'Nan-Scott Theorem (see Lemma 2.1, for example). If the socle is compatible with extreme primitivity, then it may be the case that the rank of G and the indices of the maximal subgroups of H are incompatible (for instance, see Lemma 2.3). In other situations, it may be possible to identify an explicit element $x \in G$ such that $H \cap H^x < H$ is non-maximal (see Lemma 2.4, for example).

Recall that a subset B of Ω is a *base* for G if the pointwise stabilizer of B in G is trivial. The *base size* of G, denoted $b(G,H)$, is then the minimal size of a base for G. If $b(G,H) = 2$, then this implies that there exists $x \in G$ such that $H \cap H^x = 1$, which is maximal in H if and only if H has prime order. Since no maximal subgroup of an almost simple group has prime order, the base-two property rules out extreme primitivity in this situation. This criterion, combined with a probabilistic approach for bounding the base size (see Lemma 2.7), provides a powerful technique for showing that a given group is not extremely primitive.

There is a substantial literature on bases for almost simple primitive groups, see [9, 10, 15, 17, 18], for example. In particular, there has been significant interest in determining the primitive permutation groups with a base of size 2, which is a far-reaching project initiated by Jan Saxl in the 1990s. This remains an open problem, although there has been a lot of progress in recent years. In order to prove Theorem 1, we will establish several new base results for primitive groups with socle an exceptional group of Lie type. These results make an important contribution to ongoing efforts to determine all the base-two almost simple groups and they significantly strengthen some of the results presented in [17], where the general bound $b(G,H) \leqslant 6$ is established. We anticipate that Propositions 4.2 and 5.5, as well as Theorem 7.1, will be of independent interest and applicable to other problems (see [11, 13] for some immediate applications). A systematic study of bases for almost simple exceptional groups will be the subject of a future paper.

Remark 2. The base-two problem has also been studied for primitive groups of affine type. Here $G = V{:}H$, where $H \leqslant \mathrm{GL}(V)$ is irreducible, and in this setting we have $b(G,H) = 2$ if and only if H has a regular orbit on the module V. In particular, determining if G admits a base of size 2 is a very natural problem in the representation theory of finite groups. For example, it plays an important role in the solution to the famous $k(GV)$ problem [69], which in turn proves part of a conjecture of Brauer on defect groups of blocks [7].

In recent work, Lee [46, 47, 48] has conducted an in-depth study of base sizes for affine groups of the form $G = V{:}H$, where H is an almost simple group of Lie type. In particular, as a corollary of her much more detailed results, she is able to eliminate some of the extremely primitive candidates in [65, Table 2]. More precisely, if $G = V{:}H$ with $V = \mathbb{F}_2^d$ then Lee proves that $b(G,H) = 2$ in each of the following cases $(d, \mathrm{soc}(H))$ listed in [65, Table 2]:

$$(40, \mathrm{PSp}_4(9)),\ (40, \mathrm{L}_5(2)),\ (48, \Omega_8^{\pm}(2)),\ (100, \mathrm{Sp}_{10}(2)),\ (126, \mathrm{L}_9(2)),$$

together with the cases $d = \binom{k}{3}$ and $H = \mathrm{L}_k(2)$ with $10 \leqslant k \leqslant 14$. For the remaining groups, either the precise base size is undetermined, or it is known to be at least 3.

By adopting a different approach, we prove in [22] that none of the groups recorded in [65, Table 2] are extremely primitive. In particular, this reduces the classification of the affine extremely primitive groups to Wall's conjecture for almost simple groups.

Notation. Let G be a finite group and let n be a positive integer. Our group theoretic notation is standard. In particular, we will write Z_n, or just n, for a cyclic group of order n and G^n will denote the direct product of n copies of G. An unspecified extension of G by a group H will be denoted by $G.H$. If X is a subset of G, then $i_n(X)$ is the number of elements of order n in X. We adopt the standard notation for simple groups from [36]. The Fitting subgroup of G will be denoted $F(G)$ and the socle of G is $\mathrm{soc}(G)$. For positive integers a and b, we write (a, b) for the greatest common divisor of a and b. Further notation will be introduced as and when needed in the main text.

Organisation. In Section 2 we record some preliminary results, which will be needed in the proof of Theorem 1. This includes a discussion of some general techniques for proving that a given primitive group is not extremely primitive. We also present several results on conjugacy classes in almost simple exceptional groups of Lie type, which will be applied repeatedly later in the paper. In Sections 3 and 4 we prove Theorem 1 in the cases where a point stabilizer is a maximal parabolic or maximal rank subgroup of G, respectively. It is worth noting that the latter subgroups require considerably more effort and the proof of Theorem 4.1 spans almost 30 pages. In Sections 5, 6 and 7, we complete the proof of Theorem 1 for the groups with socle $E_8(q)$, $E_7(q)$, $E_6^{\epsilon}(q)$, $F_4(q)$ or $G_2(q)$. Here we organise our analysis in accordance with a key theorem of Liebeck and Seitz on the subgroup structure of exceptional groups (Theorem 5.1), which partitions the remaining possibilities for G_α into several families. Finally, we complete the proof in Section 8, where the remaining twisted groups are handled.

Acknowledgements. The authors thank David Craven, Martin Liebeck, Alastair Litterick and Jay Taylor for helpful conversations regarding the content of this paper. They also thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme *Groups, Representations and Applications: New perspectives*, when some of the work on this paper was undertaken. This work was supported by: EPSRC grant number EP/R014604/1.

2. Preliminaries

2.1. **Extremely primitive groups.** Let $G \leqslant \mathrm{Sym}(\Omega)$ be a finite primitive permutation group with point stabilizer $H = G_\alpha$. In this section, we record three results which can be used to show that G is not extremely primitive.

By a theorem of Guralnick [33, Theorem 3], H acts faithfully on at least one of its orbits in $\Omega \setminus \{\alpha\}$. Moreover, if we assume G is extremely primitive then Manning's theorem [66, Corollary I, p.821] implies that H acts faithfully and primitively on *all* of its orbits in $\Omega \setminus \{\alpha\}$. In particular, if G is extremely primitive then we can view H itself as a primitive permutation group and this allows us to apply the O'Nan-Scott Theorem to impose strong restrictions on the structure of H.

Lemma 2.1. *Suppose one of the following holds:*

(i) $Z(H) \neq 1$;
(ii) $F(H)$ *is not elementary abelian;*
(iii) $F(H) = Z_p^d$ *is elementary abelian and* p^d *does not divide* $|\Omega| - 1$;
(iv) $F(H) = Z_p^d$ *is elementary abelian and* $H/F(H)$ *is not isomorphic to an irreducible subgroup of* $\mathrm{GL}_d(p)$;
(v) $\mathrm{soc}(H)$ *is not a direct product of isomorphic simple groups.*

Then G *is not extremely primitive.*

Proof. As noted above, if G is extremely primitive then H is a primitive permutation group on all of its orbits in $\Omega \setminus \{a\}$ and by applying the O'Nan-Scott Theorem we deduce that either

(a) $F(H) = 1$ and $\mathrm{soc}(H)$ is a direct product of isomorphic nonabelian simple groups; or
(b) $H = F(H)K$ is an affine group, where $K \leqslant \mathrm{GL}_d(p)$ is irreducible and $\mathrm{soc}(H) = F(H) = Z_p^d$ acts regularly on each H-orbit in $\Omega \setminus \{\alpha\}$.

The result follows (also see [19, Lemma 2.2]). □

Remark 2.2. Suppose G is extremely primitive with $F(H) \neq 1$. Then as in case (b) in the proof of Lemma 2.1, we have $H = F(H)K$ with $\mathrm{soc}(H) = F(H) = Z_p^d$ and $K \leqslant \mathrm{GL}_d(p)$. Since $F(H)$ acts regularly on the H-orbits in $\Omega \setminus \{\alpha\}$, every nontrivial element in $F(H)$ has a unique fixed point on Ω and we deduce that $F(H) \cap K^g = 1$ for all $g \in G$. We thank an anonymous referee for making this observation.

The next result records an elementary observation which will be useful when we can compute the rank r of G (that is, the number of orbits of H on Ω) and we know the indices of all the core-free maximal subgroups of H. For example, if the character tables of G and H are available and we can compute the fusion of H-classes in G (using the GAP Character Table Library [8], for example), then we can use the Orbit Counting Lemma to compute r. See the proofs of Lemmas 4.13, 4.20, 5.4, 7.5 and 7.6, for example.

Lemma 2.3. *Suppose* G *has rank* r *and let* $\{M_1, \ldots, M_k\}$ *be representatives of the* H*-classes of core-free maximal subgroups of* H*. Set* $n_i = |H : M_i|$ *for* $i = 1, \ldots, k$*. Suppose there is no* k*-tuple of non-negative integers* $[a_1, \ldots, a_k]$ *with* $\sum_i a_i = r - 1$ *and* $\sum_i a_i n_i = |\Omega| - 1$*. Then* G *is not extremely primitive.*

Proof. Since G has rank r, there exist $\beta_i \in \Omega$ such that

$$\Omega = \{\alpha\} \cup \beta_1^H \cup \cdots \cup \beta_{r-1}^H$$

is a disjoint union of H-orbits. If G is extremely primitive, then each stabilizer H_{β_j} is a maximal core-free subgroup of H, whence

$$|\Omega| = 1 + \sum_{j=1}^{r-1} |\beta_j^H| = 1 + \sum_{j=1}^{r-1} |H : H_{\beta_j}| = 1 + \sum_{i=1}^{k} a_i n_i$$

for some k-tuple $[a_1, \ldots, a_k]$ of non-negative integers with $\sum_i a_i = r-1$. The result follows. □

The next lemma is a key tool in the proof of Theorem 1.

Lemma 2.4. *Let $G = G_0.A$ be an almost simple group with socle G_0 and let $H = H_0.A$ be a maximal subgroup of G with $H_0 = H \cap G_0$. Let K be a proper A-stable subgroup of H_0 and let $\mathcal{M}$ be the set of maximal overgroups of K in H_0. Assume that each of the following conditions are satisfied:*

(i) *H_0 is a maximal subgroup of G_0;*
(ii) *Each $M \in \mathcal{M}$ is A-stable;*
(iii) *There exists $g \in N_{G_0}(K)$ such that $M^g \not\leqslant H_0$ for all $M \in \mathcal{M}$.*

Then $H \cap H^{g^{-1}}$ is a non-maximal subgroup of H and thus the action of G on G/H is not extremely primitive.

Proof. Let L be a maximal subgroup of H containing K. If L contains H_0 then L/H_0 is maximal in $H/H_0 = A$ and thus $L = H_0.B$ with $B < A$ maximal. On the other hand, if L does not contain H_0 then $H = H_0L$ and we deduce that $L = (H_0 \cap L).A$. Then since $K \leqslant H_0 \cap L$, the maximality of L in H implies that $L = M.A$ for some $M \in \mathcal{M}$ (here we are using (ii)).

Let $g \in N_{G_0}(K)$ be an element satisfying the condition in (iii). Seeking a contradiction, suppose $H \cap H^{g^{-1}}$ is a maximal subgroup of H. From the above description of the maximal overgroups of H containing K, it follows that either $(M.A)^g$ or $(H_0.B)^g$ must be contained in H for some $M \in \mathcal{M}$ or maximal subgroup $B < A$. If $(M.A)^g < H$ then $M^g < H \cap G_0 = H_0$ and this contradicts (iii). Similarly, if $(H_0.B)^g < H$ then $\langle H_0, H_0^g \rangle \leqslant H$, but $\langle H_0, H_0^g \rangle = G_0$ since H_0 is a maximal subgroup of G_0 and $g \notin H_0$ (by (i) and (iii)). In both cases we reach a contradiction and this completes the proof of the lemma. □

The following example demonstrates how we will apply Lemma 2.4.

Example 2.5. Suppose $G = E_8(q)$ and $H = \Omega_{16}^+(q)$ with q even, so $G = G_0$ and $H = H_0$. Set $K = \Omega_8^+(q)^2 < H$ and observe that $M = N_H(K) = K.2^2$ is the unique maximal overgroup of K in H. Since $L = K.(\mathrm{Sym}_3 \times 2)$ is a maximal subgroup of G (see [50, Table 5.1]), it follows that $N_G(K) = L$. Now M is not normal in L, so there exists $g \in N_G(K)$ such that $M^g \neq M$. Since M is the unique maximal overgroup of K in H, it follows that $M^g \not\leqslant H$ and thus Lemma 2.4 implies that $H \cap H^{g^{-1}}$ is non-maximal in H. We refer the reader to Lemma 4.6, where the general case with $G_0 = E_8(q)$ and H of type $D_8(q)$ is handled.

2.2. **Base-two groups.** Recall that the *base size* of G, denoted by $b(G, H)$, is the smallest size of a subset $B \subseteq \Omega$ such that $\bigcap_{\alpha \in B} G_\alpha = 1$. In particular, $b(G, H) = 2$ if and only if there exists an element $x \in G$ such that $H \cap H^x = 1$.

Since no maximal subgroup of an almost simple group has prime order, we obtain the following result.

Lemma 2.6. *If G is almost simple and $b(G, H) = 2$, then G is not extremely primitive.*

As discussed in Section 1, there is an extensive literature on bases for almost simple primitive groups and there has been a special interest in determining the groups with $b(G, H) = 2$. For the exceptional groups of Lie type, the main references are [10, 17]. In particular, the main theorem of [17] states that $b(G,H) \leqslant 6$ if G is any almost simple primitive group with socle of exceptional type. This is a key step in the proof of an influential conjecture of Cameron on bases for so-called *non-standard* almost simple primitive groups (see [17] and the references therein).

Probabilistic methods play a key role in the proof of Cameron's base size conjecture. This approach arises from an elementary observation due to Liebeck and Shalev (see the proof of Theorem 1.3 in [59]). Fix a positive integer c and notice that a c-tuple of points in Ω is *not* a base for G if and only if there exists an element $x \in G$ of prime order fixing each element in the tuple. Now the probability that a given element $x \in G$ fixes a uniformly random element in Ω is given by the expression

$$\mathrm{fpr}(x, G/H) = \frac{|C_\Omega(x)|}{|\Omega|} = \frac{|x^G \cap H|}{|x^G|},$$

which is the *fixed point ratio* of x (here $C_\Omega(x)$ is the set of fixed points of x on Ω). It follows easily that if $\mathcal{Q}(G,c)$ is the probability that a randomly chosen c-tuple in Ω is not a base then

$$\mathcal{Q}(G,c) \leqslant \sum_{x \in \mathcal{P}} \mathrm{fpr}(x, G/H)^c,$$

where $\mathcal{P}$ is the set of elements of prime order in G. In particular, if this upper bound is less than 1, then $b(G,H) \leqslant c$ and thus upper bounds on fixed point ratios can be used to bound the base size. For exceptional groups of Lie type, which are the main focus of this paper, we refer the reader to [44] for a systematic study of fixed point ratios in this setting.

As a special case, we record the following lemma.

Lemma 2.7. *Let $x_1, \ldots, x_k$ be representatives of the G-classes of elements of prime order in H and set*

$$\mathcal{Q}(G,H) = \sum_{i=1}^{k} |x_i^G| \cdot \left(\frac{|x_i^G \cap H|}{|x_i^G|} \right)^2. \tag{1}$$

If $\mathcal{Q}(G,H) < 1$ then $b(G,H) = 2$.

The next result is [9, Lemma 2.1], which is a useful tool for bounding $\mathcal{Q}(G,H)$.

Lemma 2.8. *Suppose $x_1, \ldots, x_m$ represent distinct G-classes such that $\sum_i |x_i^G \cap H| \leqslant A$ and $|x_i^G| \geqslant B$ for all i. Then*

$$\sum_{i=1}^{m} |x_i^G| \cdot \left(\frac{|x_i^G \cap H|}{|x_i^G|} \right)^2 \leqslant A^2/B.$$

Remark 2.9. We have now introduced several methods for showing that a given permutation group G with point stabilizer $H = G_\alpha$ is not extremely primitive. Let us briefly summarise how we will apply these techniques in the proof of Theorem 1.

(i) First we will seek to apply Lemma 2.1, noting that if the structure of H is incompatible, then it can be quickly eliminated. The remaining groups will then be partitioned into two collections, according to the order of H. If $|H|$ is small, which will typically mean $|H| \ll |G|^{1/2}$, then it is often possible to force $b(G,H) = 2$ via Lemma 2.7 and a careful analysis of the conjugacy classes of elements of prime order in H (in particular, we are interested in the corresponding G-classes of these elements).

(ii) For the remaining groups, it may be possible to apply Lemma 2.3 in some special cases; typically, this will depend on whether or not we can access the relevant character tables in [8]. But in general, our main aim will be to identify a subgroup K of H_0 with the desired properties in Lemma 2.4. To do this, it will often be convenient to work first with the ambient algebraic groups and then descend to the relevant finite groups by taking the fixed points of a suitable Steinberg endomorphism.

x_1	Long root element
x_2	Unipotent element of order p, not a long root element (nor a short root element if $(\bar{G},p) = (F_4,2)$ or $(G_2,3)$)
x_3	Semisimple involution
x_4	Semisimple element of odd prime order
x_5	Prime order element in $G \setminus \mathrm{Inndiag}(G_0)$

TABLE 3. The elements $x_i \in G$ in Proposition 2.11

(iii) In order to handle some special cases where the underlying field $\mathbb{F}_q$ is small, we will sometimes use computational methods. See Section 2.4 for further details.

2.3. **Conjugacy classes.** Let G be an almost simple group with socle G_0, an exceptional group of Lie type over $\mathbb{F}_q$, where $q = p^f$ with p prime. Write $G_0 = (\bar{G}_\sigma)'$, where $\bar{G}$ is a simple algebraic group of adjoint type over the algebraic closure $\bar{\mathbb{F}}_p$ and σ is an appropriate Steinberg endomorphism of $\bar{G}$. Recall that $\bar{G}_\sigma = \mathrm{Inndiag}(G_0)$ is the group of inner-diagonal automorphisms of G_0.

In order to effectively apply the base-two criterion discussed in Section 2.2, we will need detailed information on the centralizers and conjugacy classes of elements of prime order in G. Here there is an extensive literature to draw upon and our primary sources will be [57] for an in-depth treatment of unipotent classes and [61] for information on semisimple classes. The centralizers of prime order graph, field and graph-field automorphisms of exceptional groups are described in [43, Proposition 1.1]. We refer the reader to [14, Chapter 3] for a convenient source of information on conjugacy classes in the finite classical groups.

Remark 2.10. The terminology we adopt for automorphisms in this paper is fairly standard, although there are differences in the literature. In particular, we will refer to graph automorphisms of $F_4(q)$ and $G_2(q)$ when $p = 2$ and 3, respectively. This is consistent with [24], for example, but not [32], where the term graph-field automorphism is preferred.

The next result gives lower bounds on the sizes of conjugacy classes in G, according to the type of elements in the class. Note that in Table 4, we set $\alpha = (q-1)/q$, $\beta = (2, q-1)$, $\gamma = (3, q-1)$ and $\delta = (3, q+1)$.

Proposition 2.11. *Let G be an almost simple group with socle G_0, an exceptional group of Lie type over $\mathbb{F}_q$, where $q = p^f$ with p prime. Let $x_i \in G$ be an element of prime order, as described in Table 3. Then $|x_i^G| > \ell_i$, where the ℓ_i are given in Table 4.*

Proof. This is an entirely straightforward computation, using the available information on conjugacy classes and centralizers in [57, 61] and [43, Proposition 1.1].

For example, suppose $\bar{G}_\sigma = G_0 = F_4(q)$ and q is odd, so

$$|G_0| = q^{24}(q^2-1)(q^6-1)(q^8-1)(q^{12}-1).$$

If $x \in G_0$ is a long root element, then x is contained in the unipotent class labelled A_1 in [57, Table 22.2.4] and we read off

$$|x^G| = \frac{|G_0|}{q^{15}|\mathrm{Sp}_6(q)|} = (q^4+1)(q^{12}-1) > q^{16} = \ell_1.$$

The next smallest unipotent class is labelled $\tilde{A}_1$ in $\bar{G}$ (these are the short root elements); since p is odd, this $\bar{G}$-class splits into two $\bar{G}_\sigma$-classes and we have

$$|x^G| = \frac{|G_0|}{2q^{15}|\mathrm{SL}_4^\epsilon(q)|} = \frac{1}{2}q^3(q^3+\epsilon)(q^4+1)(q^{12}-1) > \frac{1}{2}(q-1)q^{21} = \ell_2.$$

G_0	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5
$E_8(q)$	q^{58}	q^{92}	q^{112}	αq^{114}	q^{124}
$E_7(q)$	q^{34}	q^{52}	$\frac{\alpha}{2}q^{54}$	αq^{54}	$\frac{1}{\beta}q^{133/2}$
$E_6(q)$	q^{22}	q^{32}	q^{32}	q^{32}	$\frac{\alpha}{\gamma}q^{26}$
${}^2E_6(q)$	αq^{22}	αq^{32}	αq^{32}	αq^{32}	$\frac{1}{\delta}q^{26}$
$F_4(q)$	q^{16}	$\frac{\alpha}{2}q^{22}$	q^{16}	αq^{30}	q^{26}
$G_2(q)$	αq^6	αq^8	q^8	αq^6	q^7
${}^3D_4(q)$	αq^{10}	q^{16}	q^{16}	αq^{18}	q^{14}
${}^2F_4(q)$, $q \geqslant 2^3$	αq^{11}	αq^{14}	–	αq^{18}	$q^{52/3}$
${}^2G_2(q)$, $q \geqslant 3^3$	αq^4	$\frac{\alpha}{2}q^5$	αq^4	$\frac{1}{2}q^6$	$q^{14/3}$
${}^2B_2(q)$	αq^3	–	–	$\frac{1}{2}q^4$	$q^{10/3}$

TABLE 4. The lower bounds $|x_i^G| > \ell_i$ in Proposition 2.11

Now assume $x \in G_0$ is a semisimple involution. There are two classes of involutions in G_0, with $C_{\bar{G}}(x) = A_1C_3$ or B_4, and we see that $|x^G|$ is minimal when $C_{\bar{G}}(x) = B_4$ ([32, Table 4.5.1] is an excellent source of information on semisimple involutions). Therefore,

$$|x^G| \geqslant \frac{|G_0|}{|\mathrm{SO}_9(q)|} = q^8(q^8+q^4+1) > q^{16} = \ell_3.$$

If $x \in G_0$ is a semisimple element of odd order then by inspecting [61] we deduce that $|x^G|$ is minimal when $C_{\bar{G}}(x) = B_3T_1$ or C_3T_1 (here T_1 denotes a 1-dimensional torus). This yields

$$|x^G| \geqslant \frac{|G_0|}{|\mathrm{SO}_7(q)|(q+1)} > (q-1)q^{29} = \ell_4.$$

Finally, if $x \in G \setminus G_0$ has prime order r, then x is a field automorphism, $q = q_0^r$ and $C_{G_0}(x) = F_4(q_0)$, so $|x^G|$ is minimal when $r = 2$ and we get

$$|x^G| \geqslant \frac{|G_0|}{|F_4(q^{1/2})|} = q^{12}(q+1)(q^3+1)(q^4+1)(q^6+1) > q^{26} = \ell_5.$$

The other groups are handled in a similar fashion and we omit the details. □

In almost every case, we see that $|x^G|$ is minimal when x is a long root element (or a short root element when $(\bar{G}, p) = (F_4, 2)$ or $(G_2, 3)$). Therefore, it will be important to know when certain maximal subgroups of G contain such elements. With this in mind, we present the following result for algebraic groups, which is a simplified version of [43, Proposition 1.13].

Proposition 2.12. *Let $\bar{G}$ be a simple algebraic group, let $\bar{M}$ be a connected reductive subgroup of $\bar{G}$ and assume $u \in N_{\bar{G}}(\bar{M})$ is a long root element.*

- (i) *If $u \in \bar{M}$ and $\bar{M}$ is semisimple, then u is a root element in one of the simple factors of $\bar{M}$.*
- (ii) *If $u \notin \bar{M}$, then $p = 2$ and $\bar{M} = \bar{X}\bar{Y}$ is a commuting product such that u centralizes $\bar{X}$, and $\bar{Y}$ is either a simple factor of type D_n or a 1-dimensional torus.*

In the special case where $\bar{M}$ is a maximal torus of $\bar{G}$, we get the following corollary (this is based on an observation in the proof of [44, Lemma 4.3]).

Corollary 2.13. *Let $\bar{G}$ be a simple algebraic group and let σ be a Steinberg endomorphism of $\bar{G}$. Let $G = \bar{G}_\sigma$ and let $H = N_G(\bar{T}_\sigma)$, where $\bar{T}$ is a σ-stable maximal torus of $\bar{G}$. If $x \in H$ is a root element in G, then $p = 2$ and*

$$|x^G \cap H| \leqslant |\Sigma^+(\bar{G})|\,|\bar{T}_\sigma|,$$

where $\Sigma^+(\bar{G})$ is the set of positive roots in the root system of $\bar{G}$.

Proof. By Proposition 2.12(ii), if $w \in N_{\bar{G}}(\bar{T})$ is a root element, then $p = 2$ and w centralizes a subtorus in $\bar{T}$ of codimension 1. In particular, w corresponds to a reflection in the Weyl group $W(\bar{G}) = N_{\bar{G}}(\bar{T})/\bar{T}$ and the result follows since there are precisely $|\Sigma^+(\bar{G})|$ reflections in $W(\bar{G})$. □

The following result is [44, Proposition 1.3].

Proposition 2.14. *Let $G_0 = (\bar{G}_\sigma)'$ be a finite simple group of Lie type and let $N = |\Sigma^+(\bar{G})|$ be the number of positive roots in the root system of $\bar{G}$. Set*

$$N_2 = \frac{1}{\alpha}(\dim \bar{G} - N), \;\; N_3 = \frac{1}{\alpha}(\dim \bar{G} - \frac{2}{3}N),$$

where $\alpha = 2$ if $G_0 = {}^2F_4(q)$, ${}^2G_2(q)$ or ${}^2B_2(q)$, otherwise $\alpha = 1$.

(i) *For $r \in \{2,3\}$, we have $i_r(\mathrm{Aut}(G_0)) < 2(q+1)q^{N_r-1}$.*
(ii) *The number of unipotent elements in $\bar{G}_\sigma$ is equal to $q^{2N/\alpha}$.*

2.4. **Computational methods.** As previously remarked, it is feasible to use computational methods to handle certain groups defined over small fields and these computations can be implemented in Magma [4] or GAP [31]. Here we briefly outline the main techniques we will use, referring the reader to [21] for a more detailed discussion, which includes the relevant code we used to obtain the results.

2.4.1. *Permutation representations.* In some cases, we can work with a suitable permutation representation of G in Magma and we can construct H as a subgroup of G. Then by random search, we can seek an element $x \in G$ such that $H \cap H^x$ is non-maximal in H (or $H \cap H^x = 1$ if we wish to show that $b(G,H) = 2$). Typically, we will use the Magma command `AutomorphismGroupSimpleGroup` to construct $\mathrm{Aut}(G_0)$ as a permutation group and we then identify G as a subgroup of $\mathrm{Aut}(G_0)$. We can then construct H via the `MaximalSubgroups` command, or by a direct construction if needed. For example, we may have $H = N_G(C_G(x))$ for some element $x \in G$, which provides a way to construct H directly.

2.4.2. *Character tables.* In order to effectively apply Lemma 2.3, we need to know the rank of G (or a suitable bound on the rank). If the character tables of G and H are available in the GAP Character Table Library [8] then we may be able to use GAP [31] to compute the fusion map from H-classes to G-classes via the command `FusionConjugacyClasses`. If this is possible, then we can calculate $|x^G \cap H|$ for each $x \in H$, which yields

$$|C_\Omega(x)| = \frac{|x^G \cap H|}{|x^G|} \cdot |G : H|.$$

(If the fusion map is not stored, then it may still be possible to proceed in the same way by using the command `PossibleClassFusions`.) We can then calculate the rank r of G since

$$r = \frac{1}{|H|} \sum_{x \in H} |C_\Omega(x)|$$

by the Orbit Counting Lemma. Finally, it may be feasible to determine the indices of the maximal subgroups of H, for example by working with a suitable permutation representation and the Magma command `MaximalSubgroups`. Typically, H is almost simple and the relevant information on maximal subgroups may also be available in the Web Atlas [78], for example.

2.4.3. *Lie type computations.* In Magma, there are sophisticated in-built functions for working with groups of Lie type in terms of their associated Lie structures, such as their root subgroups and Weyl group, etc. Therefore, if H can be defined in terms of this data (for example, if $H = \bar{H}_\sigma$ and $\bar{H}$ is a σ-stable subsystem subgroup of $\bar{G}$), then it may be possible to construct H as a subgroup of an appropriate group of Lie type via the `GroupOfLieType` command (in practice, it may be easier to construct H as a subgroup of a larger group of Lie type which contains G). We can then obtain detailed information on the conjugacy classes of H and the action of class representatives on certain modules for G_0, such as the adjoint module. In turn, this will allow us to estimate $|x^G \cap H|$ and $|x^G|$ for all elements $x \in H$ of prime order and we can use these estimates to derive an upper bound on the function $\mathcal{Q}(G,H)$ in (1). For example, if $x \in H$ is unipotent, then we can often use the Jordan form of x on the adjoint module for G_0 to determine the $\bar{G}$-class of x via [40]. If the bound we obtain gives $\mathcal{Q}(G,H) < 1$, then $b(G,H) = 2$ by Lemma 2.7 and G is not extremely primitive.

2.4.4. *Small groups.* To close this preliminary section, it is convenient to use computational methods to establish Theorem 1 for some small groups.

Theorem 2.15. *Let G be an almost simple primitive group with socle G_0 and point stabilizer H, where G_0 is one of*

$$ {}^2B_2(8),\ {}^2B_2(32),\ {}^2F_4(2)',\ {}^3D_4(2),\ G_2(2)',\ G_2(3),\ G_2(4),\ G_2(5). \tag{2}$$

Then G is extremely primitive if and only if $(G,H) = (G_2(4), \mathrm{J}_2)$ or $(G_2(4).2, \mathrm{J}_2.2)$.

Proof. This is an entirely straightforward Magma [4] computation, working with a suitable permutation representation of G (see [21, Theorem 2.1] for further details). □

Remark 2.16. Suppose G is one of the groups in Theorem 2.15 and let $b = b(G,H)$ denote the base size of G. If $G_0 \neq G_2(2)'$ then b is recorded in [17, Tables 11 and 12]. For completeness, let us record that if $G_0 = G_2(2)' \cong \mathrm{U}_3(3)$ then $b \leqslant 3$, with equality if and only if H is one of

$$3^{1+2}{:}8.\alpha,\ \mathrm{GU}_2(3).\alpha, 4^2{:}\mathrm{Sym}_3.\alpha,\ \mathrm{L}_2(7).\alpha$$

where $\alpha = |G : G_0|$. In each case, this is an easy Magma computation.

3. Parabolic subgroups

Let G be an almost simple primitive permutation group with socle G_0 and point stabilizer H, where G_0 is a simple exceptional group of Lie type over $\mathbb{F}_q$ and $q = p^f$ with p a prime. In this section, we begin the proof of Theorem 1 by handling the groups where H is a maximal parabolic subgroup.

We begin with a preliminary lemma. The following result is presumably well known, but we include a proof because we were unable to find one in the literature. Note that we are not assuming P is a maximal parabolic subgroup of G_0 (although the subgroups that arise in parts (i) and (ii) are maximal and they are labelled in the usual manner).

Lemma 3.1. *Let P be a parabolic subgroup of G_0. Then the unipotent radical of P is abelian if and only if (G_0, P) is one of the following:*

(i) *$G_0 = E_6(q)$ and $P = P_1$ or P_6.*
(ii) *$G_0 = E_7(q)$ and $P = P_7$.*

Proof. Let Q be the unipotent radical of P. To begin with, let us assume G_0 is not one of the following groups:

$$F_4(q)\,(p=2),\ G_2(q)\,(p=2,3),\ {}^2F_4(q),\ {}^2G_2(q),\ {}^2B_2(q). \tag{3}$$

Let r be the untwisted Lie rank of G_0 and fix a set of simple roots $\alpha_1, \ldots, \alpha_r$ for the corresponding root system, labelled as in [5]. By [2, Theorems 2 and 3], the nilpotency class of Q, which we denote by $c(Q)$, is independent of the field and it can be calculated as follows. First recall that the conjugacy classes of parabolic subgroups of G_0 are in bijective correspondence with the subsets of $[r] = \{1, \ldots, r\}$; under this correspondence, the maximal parabolic subgroups line up with subsets of size $r-1$, while the Borel subgroups correspond to the empty set. Now, if P corresponds to the subset $I = \{i_1, \ldots, i_k\}$, then [2] gives $c(Q) = \sum_{i \in [r] \setminus I} c_i$, where $\alpha_0 = \sum_{i=1}^r c_i\alpha_i$ is the highest root for G_0. It is now a routine calculation with the root system of G_0 to show that the only cases with $c(Q) = 1$ are the maximal parabolic subgroups in the statement of the lemma.

To complete the proof, let us consider the groups in (3). First assume $G_0 = F_4(q)$ (with $p = 2$) or $G_2(q)$ (with $p = 2, 3$). These groups are called *special* in [2] due to the existence of degeneracies in Chevalley's commutator relations. However, a straightforward calculation with these relations shows that if P is maximal then Q is nonabelian. For example, if $G_0 = F_4(q)$ and $P = P_1$ (with $p = 2$) then $u = x_{\alpha_1}(1)$ and $v = x_{\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4}(1)$ are contained in Q and we have $[u, v] = x_{\alpha_0}(1)$, where $X_\alpha = \{x_\alpha(c) : c \in \mathbb{F}_q\}$ is the root subgroup of G_0 corresponding to the root α. The result now follows because the unipotent radical of any non-maximal parabolic subgroup contains the unipotent radical of a maximal parabolic subgroup.

Similarly, for $G_0 = {}^2F_4(q)$ it suffices to check that Q is nonabelian when P is maximal. Generators and relations for these subgroups are given in [72, (2.2) and (2.3)] and the desired conclusion follows immediately. Finally, we note that the parabolic subgroups of ${}^2G_2(q)$ and ${}^2B_2(q)$ are Borel subgroups and so in these cases Q is a Sylow p-subgroup of G_0. These Sylow subgroups are nonabelian by [76, Main Theorem (2)] and [74, Section 13], respectively. □

Theorem 3.2. *If H is a parabolic subgroup of G, then G is not extremely primitive.*

Proof. Let $H_0 = QL$ be a Levi decomposition of $H_0 = H \cap G_0$, so $F(H) = Q$ is the Fitting subgroup of H. In view of Lemma 2.1(ii), we may as well assume Q is elementary abelian, in which case we can apply Lemma 3.1.

If $G_0 = E_6(q)$ and H is a P_1 or P_6 parabolic then

$$|F(H)| = q^{16}, \ \ |\Omega| = |G_0 : H_0| = (q^8+q^4+1)(q^6+q^3+1)(q^2+q+1),$$

and similarly if $G_0 = E_7(q)$ and H is a P_7-parabolic then

$$|F(H)| = q^{27}, \ \ |\Omega| = |G_0 : H_0| = \frac{(q^{14}-1)(q^9+1)(q^5+1)}{q-1}.$$

In both cases, we see that $|F(H)|$ does not divide $|\Omega| - 1$, so Lemma 2.1(iii) implies that G is not extremely primitive.

Alternatively, we can appeal to Remark 2.2, noting that both L and $F(H) = Q$ contain long root elements, which are conjugate in G. □

4. Maximal rank subgroups

Let $G_0 = (\bar{G}_\sigma)'$ be a simple exceptional group of Lie type over $\mathbb{F}_q$, where $\bar{G}$ is a simple algebraic group of adjoint type and σ is an appropriate Steinberg endomorphism of $\bar{G}$. In this section, we prove Theorem 1 in the cases where $H = N_G(\bar{H}_\sigma)$ and $\bar{H}$ is a σ-stable non-parabolic maximal rank subgroup of $\bar{G}$ (in particular, the connected component $\bar{H}^0$ contains a σ-stable maximal torus of $\bar{G}$). The possibilities for H are determined in [50] (see [50, Tables 5.1 and 5.2]) and there are two subcases to consider, according to whether or not H is the normalizer of a maximal torus. Throughout this section, we will continue to exclude the groups in (2).

Our main result is the following.

Theorem 4.1. *If H is a maximal rank subgroup of G, then G is not extremely primitive.*

G_0	Type of H
$E_8(q)$	$A_4^{\epsilon}(q)^2, A_4^-(q^2), D_4(q)^2, D_4(q^2), {}^3D_4(q)^2, {}^3D_4(q^2)$
	$A_2^{\epsilon}(q)^4, A_2^-(q^2)^2, A_2^-(q^4), A_1(q)^8$
$E_7(q)$	$A_1(q^7), A_1(q)^7$
$E_6^{\epsilon}(q)$	$A_2^{\epsilon}(q)^3, A_2(q^2)A_2^{-\epsilon}(q), A_2^{\epsilon}(q^3)$
$F_4(q)$	$A_2^{\epsilon}(q)^2, C_2(q)^2\ (p=2), C_2(q^2)\ (p=2)$
${}^2F_4(q)$	$A_2^-(q), {}^2B_2(q)^2, C_2(q)$

TABLE 5. Some maximal rank subgroups H with $b(G,H)=2$

Recall that $b(G,H)$ denotes the *base size* of G and the condition $b(G,H)=2$ is equivalent to the existence of an element $x \in G$ with $H \cap H^x = 1$. In particular, if $b(G,H)=2$ then G is not extremely primitive (see Lemma 2.6). In proving Theorem 4.1, we will establish the following result on base-two groups, which may be of independent interest. In particular, this result significantly strengthens various bounds on $b(G,H)$ presented in [17]. (Note that in the second column of Table 5 we record the *type* of H, which gives an approximate description of the structure of H.)

Proposition 4.2. *Let $G \leqslant \mathrm{Sym}(\Omega)$ be an almost simple primitive group with point stabilizer H and socle G_0, a simple exceptional group of Lie type over $\mathbb{F}_q$. Suppose*

(i) *H is the normalizer of a maximal torus of G; or*
(ii) *(G,H) is one of the maximal rank cases recorded in Table 5.*

Then $b(G,H)=2$ and the probability that two randomly chosen points in Ω form a base for G tends to 1 as q tends to infinity.

In order to prove Proposition 4.2, we will apply the probabilistic approach explained in Section 2.2. More precisely, in view of Lemma 2.7, we will aim to show that $\mathcal{Q}(G,H) < 1$, where

$$\mathcal{Q}(G,H) = \sum_{i=1}^{k} |x_i^G| \cdot \left(\frac{|x_i^G \cap H|}{|x_i^G|}\right)^2$$

and $x_1,\ldots,x_k$ are representatives for the G-classes of elements of prime order. Of course, if x_i^G does not meet H, then the contribution to $\mathcal{Q}(G,H)$ from x_i^G is zero, so we are interested in the elements of prime order in H. To estimate $\mathcal{Q}(G,H)$ effectively, we will apply Lemma 2.8, using information on the conjugacy classes of elements of prime order in both H and G. In particular, the lower bounds in Proposition 2.11 will be applied repeatedly. In some cases, we will need more detailed information from [57] (for unipotent elements) and [61] (for semisimple elements). Finally, in order to establish the asymptotic statement in Proposition 4.2, it suffices to show that $\mathcal{Q}(G,H)$ tends to 0 as q tends to infinity. In every case, we will derive an explicit upper bound on $\mathcal{Q}(G,H)$ as a function of q and the desired asymptotic property will follow immediately.

4.1. $G_0 = E_8(q)$.

Lemma 4.3. *If $G_0 = E_8(q)$ and H is the normalizer of a maximal torus, then $b(G,H)=2$.*

Proof. Let $W(\bar{G}) = 2.\mathrm{O}_8^+(2)$ be the Weyl group of $\bar{G} = E_8$ and note that the possibilities for H are recorded in [50, Table 5.2]. If $x \in G$ is a long root element, then Proposition 2.11 gives $|x^G| > q^{58} = b_1$ and Corollary 2.13 implies that $|x^G \cap H| \leqslant 120(q+1)^8 = a_1$. For all other nontrivial elements we have $|x^G| > q^{92} = b_2$ and we note that

$$|H| \leqslant (q+1)^8 |W(\bar{G})|.\log_2 q = a_2.$$

By applying Lemma 2.8, we deduce that

$$\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2 < q^{-1} \qquad (4)$$

and the result follows via Lemma 2.7. □

Lemma 4.4. *Suppose $G_0 = E_8(q)$ and H is of type*

$$A_4^{\epsilon}(q)^2,\ A_4^{-}(q^2),\ A_2^{\epsilon}(q)^4,\ A_2^{-}(q^2)^2,\ A_2^{-}(q^4),\ A_1(q)^8.$$

Then $b(G,H) = 2$.

Proof. In each case, the precise structure of $H_0 = H \cap G_0$ is presented in [50, Table 5.1].

First assume H is of type $A_1(q)^8$. If $x \in G$ is a long root element, then $|x^G| > q^{58} = b_1$ and Proposition 2.12 implies that $|x^G \cap H| = 8(q^2-1) = a_1$. Otherwise, $|x^G| > q^{92} = b_2$ and

$$|H| \leqslant (q(q^2-1))^8|\mathrm{AGL}_3(2)|.\log_2 q = a_2.$$

These bounds imply that (4) holds and the result follows. An entirely similar argument applies when H is of type $A_2^{\epsilon}(q)^4$ and we omit the details.

Next assume H is of type $A_2^{-}(q^2)^2$ or $A_2^{-}(q^4)$. By considering the structure of H_0 and its embedding in $\bar{H} = A_2^4.\mathrm{GL}_2(3)$, it follows via Proposition 2.12 that H contains no long root elements of G_0. Therefore, $|x^G| > q^{92} = b_1$ for all nontrivial $x \in H$ and we observe that $|H| < 8\log_2 q.q^{32} = a_1$. This gives $\mathcal{Q}(G,H) < a_1^2/b_1 < q^{-1}$.

Finally, let us assume H is of type $A_4^{-}(q^2)$ or $A_4^{\epsilon}(q)^2$, so $\bar{H}^0 = A_4^2$ and $\bar{H} = \bar{H}^0.4$. The total contribution to $\mathcal{Q}(G,H)$ from elements $x \in G$ with $|x^G| > q^{112} = b_1$ is less than a_1^2/b_1, where $a_1 = 4\log_2 q.q^{48}$ is an upper bound on $|H|$. So it remains to consider the contribution from the elements with $|x^G| \leqslant q^{112}$, which implies that x is a unipotent element in one of the classes labelled A_1 and A_1^2 (see [57, 61]).

Let V be the adjoint module for $\bar{G}$. By considering the composition factors of the restriction of V to $\bar{H}^0$ (see [75, Table 5], for example), we calculate that if $p = 2$ then each involution in $\bar{H} \setminus \bar{H}^0$ has Jordan form (J_2^{120}, J_1^8) on V and by inspecting [40, Table 9] we deduce that they are contained in the class labelled A_1^4 in [57, Table 22.2.1]. In particular, the condition $|x^G| \leqslant q^{112}$ implies that $x^G \cap H \subseteq \bar{H}^0$.

Suppose H is of type $A_4^{-}(q^2)$. By considering the embedding of H_0 in G_0, we deduce that H does not contain any long root elements of G_0, so we may assume x is in the A_1^2 class and thus $|x^G| > q^{92} = b_2$. Here $|x^G \cap H|$ coincides with the number of long root elements in $\mathrm{U}_5(q^2)$ (we noted above that $\bar{H} \setminus \bar{H}^0$ contains no elements in the A_1^2 class), so

$$|x^G \cap H| = (q^2-1)(q^4+1)(q^{10}+1) < 4q^{16} = a_2$$

and we conclude that (4) holds.

Finally, suppose H is of type $A_4^{\epsilon}(q)^2$. By applying Proposition 2.12, we deduce that if $x \in G$ is a long root element, then

$$|x^G \cap H| \leqslant 2(q+1)(q^2+1)(q^5-1) = a_3$$

and $|x^G| > q^{58} = b_3$. Now assume $x \in G$ is a unipotent element in the A_1^2 class. Suppose $y \in \mathrm{L}_5^{\epsilon}(q)$ has Jordan form (J_2^2, J_1) on the natural module and let $z \in \mathrm{L}_5^{\epsilon}(q)$ be a long root element (so z has Jordan form (J_2, J_1^3)). Then by appealing to [41, Section 4.17], we calculate that

$$|x^G \cap H| = 2|y^{\mathrm{L}_5^{\epsilon}(q)}| + |z^{\mathrm{L}_5^{\epsilon}(q)}|^2 = 2\left(\frac{|\mathrm{SL}_5^{\epsilon}(q)|}{q^8|\mathrm{GL}_2^{\epsilon}(q)|}\right) + \left(\frac{|\mathrm{SL}_5^{\epsilon}(q)|}{q^7|\mathrm{GL}_3^{\epsilon}(q)|}\right)^2 < a_2$$

and thus

$$\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2 + a_3^2/b_3 < q^{-1}$$

as required. □

Lemma 4.5. *Suppose $G_0 = E_8(q)$ and H is of type*
$$D_4(q)^2,\ D_4(q^2),\ {}^3D_4(q)^2,\ {}^3D_4(q^2).$$
Then $b(G,H) = 2$.

Proof. Here $\bar{H} = \bar{H}^0.(2 \times \mathrm{Sym}_3)$ and $\bar{H}^0 = D_4^2$. The precise structure of H_0 is presented in [50, Table 5.1] and in each case, one checks that
$$|H| < 12q^{56}\log_2 q = a_1.$$
Therefore, the contribution to $\mathcal{Q}(G,H)$ from elements $x \in G$ with $|x^G| > \frac{1}{2}q^{124} = b_1$ is less than a_1^2/b_1. For the remainder, let $x \in H$ be an element of prime order r with $|x^G| \leqslant b_1$. Note that the bound on $|x^G|$ implies that $x \in G_0$ (see Proposition 2.11).

Suppose $x \in H_0$ is a long root element of G_0, so $|x^G| > q^{58} = b_2$. By considering Proposition 2.12 and the embedding of H_0, we see that H is of type $D_4(q)^2$ or ${}^3D_4(q)^2$. Since there are fewer than q^{10} long root elements in both $\mathrm{P}\Omega_8^+(q)$ and ${}^3D_4(q)$, it follows that $|x^G \cap H| < 2q^{10} = a_2$. For the remainder, we may assume $q^{92} < |x^G| \leqslant b_1$.

Next suppose $r \in \{2,3\}$. By inspecting each possibility for H_0 in turn and applying [44, Proposition 1.3], we deduce that
$$i_r(H_0) \leqslant 4(q+1)^2q^{38} = a_3.$$
For example, suppose H is of type $D_4(q)^2$, so [50, Table 5.1] gives
$$H_0 = d^2.\mathrm{P}\Omega_8^+(q)^2.d^2.(\mathrm{Sym}_3 \times 2)$$
with $d = (2, q-1)$. Let Z be the normal subgroup of order d^2. Then
$$i_3(H_0) = i_3(H_0/Z) \leqslant i_3(\mathrm{Aut}(\mathrm{P}\Omega_8^+(q)) \wr \mathrm{Sym}_2)$$
and by applying Proposition 2.14(i) we deduce that
$$i_3(H_0) \leqslant (1 + i_3(\mathrm{Aut}(\mathrm{P}\Omega_8^+(q))))^2 \leqslant (2(q+1)q^{19})^2 = 4(q+1)^2q^{38}.$$
Similarly,
$$\begin{aligned} i_2(H_0) &\leqslant |Z| \cdot (1 + i_2(\mathrm{Aut}(\mathrm{P}\Omega_8^+(q)) \wr \mathrm{Sym}_2)) \\ &\leqslant |Z| \cdot (1 + (1 + i_2(\mathrm{Aut}(\mathrm{P}\Omega_8^+(q))))^2 + |\mathrm{Aut}(\mathrm{P}\Omega_8^+(q))|) \\ &< 4\left(4(q+1)^2q^{30} + 6\log_2 q.q^{28}\right) \end{aligned}$$
and the desired bound follows. The other cases are very similar. We conclude that the combined contribution to $\mathcal{Q}(G,H)$ from elements $x \in G$ of order 2 and 3 with $|x^G| > q^{92} = b_3$ is less than $2a_3^2/b_3$.

Now assume $r \geqslant 5$, so $i_r(H_0) = i_r(L)$, where $L = \mathrm{P}\Omega_8^+(q)^2$, $\mathrm{P}\Omega_8^+(q^2)$, ${}^3D_4(q)^2$ and ${}^3D_4(q^2)$ in each of the respective cases. If $r \neq p$ then $|x^G| > \frac{1}{2}q^{114} = b_4$ and $|L| < q^{56} = a_4$. Now assume $r = p$. If x is not in the class A_1^2, then $|x^G| > q^{112} = b_5$ and we note that L contains fewer than $q^{48} = a_5$ elements of order p (see Proposition 2.14(ii)).

Finally, suppose x is a unipotent element in the A_1^2 class, so $|x^G| > q^{92} = b_6$. Let V be the adjoint module for $\bar{G}$ and note that x has Jordan form $(J_3^{14}, J_2^{64}, J_1^{78})$ on V (see [40, Table 9]). By considering the restriction of V to $\bar{H}^0 = D_4^2$ (see [75, Table 5]), we deduce that $x \in \bar{H}^0$ is of the form (u,u'), $(v,1)$ or $(1,v)$, where $u,u' \in D_4$ are long root elements and $v \in D_4$ is in the class labelled A_1^2 (that is, v has Jordan form (J_3, J_1^5) on the natural module for D_4). See [16, p.2327] for further details.

If $L = \mathrm{P}\Omega_8^+(q^2)$ or ${}^3D_4(q^2)$, it follows that $|x^G \cap H| = |z^L|$, where $z \in L$ is a long root element, and we deduce that $|x^G \cap H| < 2q^{20} = a_6$. Similarly, if $L = {}^3D_4(q)^2$ then $|x^G \cap H| = |z^{{}^3D_4(q)}|^2$, where $z \in {}^3D_4(q)$ is a long root element, and the same bound holds. Finally, suppose $L = \mathrm{P}\Omega_8^+(q)^2$. Here
$$|x^G \cap H| = 2|y^{\mathrm{P}\Omega_8^+(q)}| + |z^{\mathrm{P}\Omega_8^+(q)}|^2 < a_6$$

where $y \in \mathrm{P}\Omega_8^+(q)$ is a unipotent element in the A_1^2 class of D_4 and $z \in \mathrm{P}\Omega_8^+(q)$ is a long root element.

By bringing the above estimates together, we conclude that

$$\mathcal{Q}(G,H) < \sum_{i=1}^{6} a_i^2/b_i + a_3^2/b_3 < q^{-1}$$

for all $q \geqslant 3$. In addition, the given bound is less than 1 when $q = 2$. $\square$

Lemma 4.6. *Suppose $G_0 = E_8(q)$ and H is of type*

$$D_8(q),\ A_1(q)E_7(q),\ A_8^{\epsilon}(q),\ A_2^{\epsilon}(q)E_6^{\epsilon}(q).$$

Then G is not extremely primitive.

Proof. Write $G = G_0.A$, where A is a group of field automorphisms of G_0. Set $d = (2, q-1)$.

First assume H is of type $A_1(q)E_7(q)$, so $H_0 = d.(\mathrm{L}_2(q) \times E_7(q)).d$ (see [50, Table 5.1]). If q is odd then H is the centralizer of an involution, so $Z(H) \neq 1$ and G is not extremely primitive by Lemma 2.1(i). On the other hand, if q is even then $\mathrm{soc}(H)$ is not a direct product of isomorphic simple groups (indeed, either $q = 2$ and $\mathrm{soc}(H) = 3 \times E_7(2)$, or $q \geqslant 4$ and $\mathrm{soc}(H) = \mathrm{L}_2(q) \times E_7(q)$), so extreme primitivity is ruled out by Lemma 2.1(v). A very similar argument shows that G is not extremely primitive if H is of type $A_2^{\epsilon}(q)E_6^{\epsilon}(q)$.

Next assume H is of type $D_8(q)$, so $H_0 = d.\mathrm{P}\Omega_{16}^+(q).d$. In view of Lemma 2.1(i), we may assume q is even, so $H_0 = \Omega_{16}^+(q)$ and $H = H_0.A$. Let $K = \Omega_8^+(q)^2 < H_0$ and observe that $M = N_{H_0}(K) = K.2^2$ is the unique maximal overgroup of K in H_0. In addition, we note that $N_{G_0}(K) = K.(\mathrm{Sym}_3 \times 2)$, which is a maximal subgroup of G_0 (see [50, Table 5.1]). Clearly, M is not a normal subgroup of $N_{G_0}(K)$ and so there exists $g \in N_{G_0}(K)$ which does not normalize M. Finally, since M is the unique maximal overgroup of K in H_0, it follows that $M^g \not\leqslant H_0$ and we conclude by applying Lemma 2.4, noting that K and M are both A-stable.

Finally, let us assume H is of type $A_8^{\epsilon}(q)$, so $H_0 = h.\mathrm{L}_9^{\epsilon}(q).e.2$, where $e = (3, q-\epsilon)$ and $h = (9, q-\epsilon)/e$. In view of Lemma 2.1, we may assume that $h = 1$.

Fix a set of simple roots $\alpha_1, \ldots, \alpha_8$ for $\bar{G}$, labelled in the usual way (see [5]), and let X_α be the root subgroup of $\bar{G}$ corresponding to the root α. We may assume that $\bar{H} = \bar{H}^0.2$, where

$$\bar{H}^0 = \langle X_{\pm\alpha_1}, X_{\pm\alpha_3}, X_{\pm\alpha_4}, X_{\pm\alpha_5}, X_{\pm\alpha_6}, X_{\pm\alpha_7}, X_{\pm\alpha_8}, X_{\pm\alpha_0} \rangle$$

is of type A_8 and α_0 is the highest root. Let $\bar{P} = \bar{Q}\bar{L}$ be the parabolic subgroup of $\bar{H}^0$ corresponding to the simple roots α_4 and α_7 with Levi factor $\bar{L} = A_2^3T_2$. Note that $\bar{L}$ is contained in a maximal rank subgroup $\bar{J}$ of $\bar{H}^0$ of type A_2^4 (indeed, as noted in [52, Table 2], we have $C_{\bar{G}}(A_2)^0 = E_6$ and $C_{E_6}(A_2)^0 = A_2^2$). The subgroup $Z(\bar{L})^0 = T_2$ centralizes $\bar{K}$ and is therefore a maximal torus in the fourth A_2 factor of $\bar{J}$, which we denote by $\bar{M}$. So $C_{\bar{G}}(\bar{K})^0 = \bar{M}$ and $N_{\bar{M}}(Z(\bar{L})^0) = T_2.\mathrm{Sym}_3$. A straightforward calculation in the Weyl group of $\bar{H}^0$ shows that $N_{\bar{H}^0}(\bar{L}) = \bar{L}.\mathrm{Sym}_3$, where Sym_3 acts naturally on the factors of $\bar{K} = \bar{L}' = A_2^3$ and it acts on $Z(\bar{L})^0$ in the same way as the Weyl group of $\bar{M}$. By working in the Weyl group of $\bar{G}$ we see that $N_{\bar{G}}(\bar{L}) = \bar{L}.(\mathrm{Sym}_3 \times \mathrm{Sym}_3 \times 2)$, where the first Sym_3 factor naturally permutes the A_2 factors, the second acts on $Z(\bar{L})^0$ as before and the central involution acts as a graph automorphism on each A_2 factor and inverts $Z(\bar{L})^0$. In particular, $N_{\bar{H}^0}(\bar{L})/\bar{L}$ is isomorphic to a diagonal subgroup of $\mathrm{Sym}_3 \times \mathrm{Sym}_3$.

We now have a maximal rank subgroup $\bar{L}$ with $N_{\bar{H}^0}(\bar{L})/\bar{L} \cong \mathrm{Sym}_3$ and we note that $\bar{H}$ and $\bar{L}$ are σ-stable. As explained in [50, Section 1], we may compose the Steinberg endormorphism σ of $\bar{G}$ with the inner automorphism of $\bar{G}$ corresponding to the lift of an element in $N_{\bar{H}^0}(\bar{L})/\bar{L}$. By a slight abuse of notation, we will write σ to denote this composition. Then by choosing the element in $N_{\bar{H}^0}(\bar{L})/\bar{L}$ appropriately, we will obtain the two different possibilities for H_0 by taking the fixed points in $\bar{H}$ of the modified map σ. We consider the cases $\epsilon = +$ and $\epsilon = -$ separately.

First assume $\epsilon = +$. Here we take σ to be the product of the standard Frobenius morphism of $\bar{G}$ with an inner automorphism corresponding to the lift of an element of order 3 in $N_{\bar{H}^0}(\bar{L})/\bar{L}$. Set $K = (\bar{K}_\sigma)' = \mathrm{L}_3(q^3) < H_0$. By inspecting [6, Tables 8.54 and 8.55], we see that K is contained in a unique maximal subgroup of H_0, namely

$$S = N_{H_0}(K) = (Z_{q^2+q+1} \times \mathrm{L}_3(q^3)).3.e.2$$

(recall that we may assume $h = 1$). It follows that $C_{H_0}(K) = C_S(K) = Z_{q^2+q+1}$, which we know from above is a maximal torus of $\bar{M}_\sigma = \mathrm{SL}_3(q) \leqslant C_{G_0}(K)$. Therefore, we may choose $g \in \bar{M}_\sigma$ so that it does not normalize $C_{H_0}(K)$. It follows that g does not normalize S because otherwise $C_S(K)^g = C_{S^g}(K^g) = C_S(K)$, which contradicts our choice of g. Therefore, $S^g \not\leqslant H_0$. Since K and S are A-stable, we can now conclude by applying Lemma 2.4.

Finally, let us assume $\epsilon = -$. Let σ be the product of the standard Frobenius morphism with the lift of an involution in $N_{\bar{H}^0}(\bar{L})/\bar{L}$. Set $K = \bar{L}_\sigma$. Then by considering [6, Tables 8.56 and 8.57], we deduce that $S = N_{H_0}(K) = K.(\mathrm{Sym}_3 \times 2)$ is the unique maximal overgroup of K in H_0. As noted above, we have $N_{\bar{G}}(\bar{L}) = \bar{L}.(\mathrm{Sym}_3 \times \mathrm{Sym}_3 \times 2)$ and thus $S < R \leqslant N_{G_0}(K)$ with

$$R = K.(\mathrm{Sym}_3 \times \mathrm{Sym}_3 \times 2).$$

Since S/K is not normal in R/K, there exists $g \in R$ which does not normalize S and we now conclude as above, via Lemma 2.4. □

4.2. $G_0 = E_7(q)$.

Lemma 4.7. *If $G_0 = E_7(q)$ and H is the normalizer of a maximal torus, then $b(G, H) = 2$.*

Proof. Let $W(\bar{G}) = \mathrm{Sp}_6(2) \times 2$ be the Weyl group of $\bar{G}$. If $x \in G$ is a long root element, then $|x^G| > q^{34} = b_1$ and Corollary 2.13 implies that $|x^G \cap H| \leqslant 63(q+1)^7 = a_1$. For all other nontrivial elements we have $|x^G| > q^{52} = b_2$ (see Proposition 2.11) and we note that

$$|H| \leqslant (q+1)^7.|W(\bar{G})|.\log_2 q = a_2.$$

For $q \geqslant 3$ we deduce that $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2 < q^{-1}$.

Finally, let us assume $q = 2$, so $G = E_7(2)$ and $H = 3^7.W(\bar{G})$. Here Lemma 2.1(iv) implies that G is not extremely primitive, but we need to work harder to show that $b(G, H) = 2$. Using Magma [4], we can construct H as a subgroup of $E_7(4)$ and we can then determine the Jordan form of each element $x \in H$ of prime order r on the adjoint module V for $\bar{G}$ (see [21, Proposition 2.2] for further details on this computation). For $r = 2$, we inspect [40, Table 8] to determine the G_0-class of x. Similarly, if r is odd then $\dim C_{\bar{G}}(x) = \dim C_V(x)$ and by inspecting [3, Table 2] we can read off the G_0-class of x (note that if $r = 3$ and x is in one of the classes labelled `3C` or `3D` in [3, Table 2], then $\dim C_{\bar{G}}(x) = 49$ in both cases, so the eigenvalues on V do not allow us to distinguish between these classes). The results are as follows (here we label the involution classes as in [57, Table 22.2.2]):

A_1	189	`3A`	56	`5A`	3919104
A_1^2	8505	`3B`	6174	`7C`	151165440
$(A_1^3)^{(1)}$	8505	`3C` or `3D`	1392372		
$(A_1^3)^{(2)}$	127575	`3E`	3992352		
A_1^4	583929				

So for example, if $x \in G$ is an involution in the class labelled A_1^4, then $|x^G \cap H| = 583929$. It is now entirely straightforward to check that $\mathcal{Q}(G,H) < 1$ and thus $b(G,H) = 2$. □

Lemma 4.8. *If $G_0 = E_7(q)$ and H is of type $A_1(q)^7$ or $A_1(q^7)$, then $b(G, H) = 2$.*

Proof. First assume H is of type $A_1(q)^7$. If $x \in G$ is a long root element, then by applying Proposition 2.12 we deduce that $|x^G \cap H| = 7(q^2-1) = a_1$ and we have $|x^G| > q^{34} = b_1$. Otherwise, $|x^G| > q^{52} = b_2$ and we note that

$$|H| \leqslant (q(q^2-1))^7.|\mathrm{L}_3(2)|.\log_2 q < q^{30} = a_2.$$

Therefore, $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2 < q^{-1}$ and thus $b(G,H) = 2$.

Now assume H is of type $A_1(q^7)$, so $|H| \leqslant 7q^7(q^{14}-1).\log_2 q = a_1$. Here H does not contain any long root elements of G_0, so $|x^G| > q^{52} = b_1$ for all nontrivial $x \in H$ and we get $\mathcal{Q}(G,H) < a_1^2/b_1 < q^{-1}$. □

Lemma 4.9. *Suppose $G_0 = E_7(q)$ and H is of type*

$$A_1(q)D_6(q),\ A_2^{\epsilon}(q)A_5^{\epsilon}(q),\ D_4(q)A_1(q)^3,\ {}^3D_4(q)A_1(q^3),\ E_6^{\epsilon}(q).(q-\epsilon).$$

Then G is not extremely primitive.

Proof. Set $H_0 = H \cap G_0$ and note that the structure of $N_L(H_0)$ is recorded in [50, Table 5.1], where $L = \mathrm{Inndiag}(G_0)$. Set $d = (2, q-1)$.

First assume H is of type $A_1(q)D_6(q)$, so $H_0 = d.(\mathrm{L}_2(q) \times \mathrm{P\Omega}_{12}^+(q))$. If q is odd then H is the centralizer of an involution, so $Z(H) \neq 1$ and thus G is not extremely primitive by Lemma 2.1(i). On the other hand, if q is even then $\mathrm{soc}(H)$ is not a direct product of isomorphic simple groups, so Lemma 2.1(v) implies that G is not extremely primitive. Very similar reasoning rules out extreme primitivity when H is of type $A_2^{\epsilon}(q)A_5^{\epsilon}(q)$, $D_4(q)A_1(q)^3$ and ${}^3D_4(q)A_1(q^3)$.

Finally, let us assume H is of type $E_6^{\epsilon}(q).(q-\epsilon)$, so

$$H_0 = e.(E_6^{\epsilon}(q) \times (q-\epsilon)/de).e.2$$

with $e = (3, q-\epsilon)$. If $e = 3$ then $F(H) = Z_3$ and we apply Lemma 2.1(iv). For the remainder, we may assume $e = 1$. If $q \geqslant 4$ is even then $H_0 = (E_6^{\epsilon}(q) \times (q-\epsilon)).2$ and $\mathrm{soc}(H)$ is not a direct product of isomorphic simple groups, hence G is not extremely primitive by Lemma 2.1(v). The same argument applies if $(\epsilon, q) = (-, 2)$. Similarly, if q is odd then we may assume $(\epsilon, q) = (+, 3)$. Note that if $G = E_7(3).2$ then $H = (E_6(3) \times 2).2$ and the structure of $\mathrm{soc}(H)$ is incompatible with extreme primitivity.

To complete the proof, we may assume $\epsilon = +$, $q \in \{2, 3\}$ and $G = G_0$, in which case $H = \langle E_6(q), \tau \rangle = \mathrm{Aut}(E_6(q))$, where τ is an involutory graph automorphism of $S = \mathrm{soc}(H)$ with $K = C_S(\tau) = F_4(q)$. Now K is a maximal subgroup of S and $\mathcal{M} = \{M, S\}$ is the set of maximal overgroups of K in H, where $M = K \times \langle \tau \rangle$. By considering the embedding of S in G, we observe that K centralizes a subgroup $L \cong \mathrm{L}_2(q)$ of G. In particular, $K \times L < G$ and we see that $\tau \in L$.

Since $H = N_{G_0}(S)$ it follows that τ is the only nontrivial element of L which normalizes S and hence the only nontrivial element normalizing H. Since $\langle \tau \rangle$ is non-normal in L, it follows that there exists $g \in L$ which centralizes K but does not normalize M nor S. We can now apply Lemma 2.4 to conclude that G is not extremely primitive. □

In order to complete the proof of Theorem 4.1 for $G_0 = E_7(q)$, we may assume H is of type $A_7^{\epsilon}(q)$.

Lemma 4.10. *If $G_0 = E_7(q)$ and H is of type $A_7^{\epsilon}(q)$, then G is not extremely primitive.*

Proof. First let us record that the structure of $N_{\bar{G}_\sigma}(H_0)$ given in [50, Table 5.1] is incorrect. The correct structure is

$$N_{\bar{G}_\sigma}(H_0) = h.\mathrm{L}_8^{\epsilon}(q).g.2,$$

where $h = (4, q-\epsilon)/d$ and $g = (8, q-\epsilon)/h$. Therefore, by appealing to Lemma 2.1(iv), we may assume that $q \not\equiv \epsilon \pmod 4$ for the remainder of the proof. Write $G = G_0.A$ and $H = H_0.A$, where A is a group of automorphisms of G_0.

Fix a set of simple roots $\alpha_1, \ldots, \alpha_7$ for $\bar{G}$, labelled in the usual way (see [5]). For each root α, let X_α be the corresponding root subgroup of $\bar{G}$ and note that

$$\bar{H}^0 = \langle X_{\pm\alpha_0}, X_{\pm\alpha_1}, X_{\pm\alpha_3}, X_{\pm\alpha_4}, X_{\pm\alpha_5}, X_{\pm\alpha_6}, X_{\pm\alpha_7} \rangle$$

is of type A_7, where α_0 is the highest root. Let $\bar{P} = \bar{Q}\bar{L}$ be the standard P_4 maximal parabolic subgroup of $\bar{H}^0$ with Levi factor $\bar{L} = A_3^2T_1$. Then $N_{\bar{H}^0}(\bar{L}) = \bar{L}.2$, where the outer involution swaps the two A_3 factors and inverts the T_1 torus (as can be calculated in the Weyl group of $\bar{H}^0$). Let $\bar{K} = \bar{L}' = A_3^2$ and note that $\bar{K}.2 < N_{\bar{H}^0}(\bar{L})$. It is straightforward to check that $C_{\bar{H}^0}(\bar{K})^0 = Z(\bar{L})^0 = T_1$.

By inspecting [52, Table 2], we see that A_3A_1 is the connected centralizer in $\bar{G}$ of the first A_3 factor of $\bar{K}$ and it follows that $\bar{M} = C_{\bar{G}}(\bar{K})^0$ is of type A_1 and contains $Z(\bar{L})^0$ as a maximal torus. In fact, since $\langle X_{\pm\alpha_2}\rangle$ clearly centralizes $\bar{K}$, we have $\bar{M} = \langle X_{\pm\alpha_2}\rangle$. Now

$$\bar{L} = A_3^2T_1 < A_3^2A_1 < D_6A_1 < \bar{G},$$

where D_6A_1 is a maximal subsystem subgroup of $\bar{G}$, and the Weyl group of D_6 contains an involution swapping the two A_3 factors of $\bar{K}$. So $N_{\bar{G}}(\bar{L})/\bar{L}$ has a subgroup $Z_2 \times Z_2 = \langle a\rangle \times \langle b\rangle$, where a swaps the two A_3 factors and b inverts the torus $Z(\bar{L})^0$. The diagonal subgroup $Z_2 = \langle ab\rangle$ is contained in the Weyl group of $\bar{H}^0$.

The case $q = 2$ requires special attention and it will be handled at the end of the proof. So for now, let us assume $q \geqslant 3$.

First assume $\epsilon = +$, so $q \not\equiv 1 \pmod 4$. Here we replace the standard Frobenius morphism σ of $\bar{G}$ defining G_0 by the product of σ and the inner automorphism of $\bar{G}$ induced by the lift of an outer involution in $\bar{K}.2$ which swaps the two A_3 factors. We will abuse notation by writing σ for this modified map. Then $(\bar{G}_\sigma)' = G_0$, $N_{\bar{G}_\sigma}(H_0) = (\bar{H}.2)_\sigma = \mathrm{L}_8(q).d.2$ and $\bar{K}_\sigma = d.\mathrm{L}_4(q^2).d$. Set $K = (\bar{K}_\sigma)' = d.\mathrm{L}_4(q^2) < H_0 = \mathrm{L}_8(q).2$.

By inspecting [6, Tables 8.44 and 8.45], we see that K is contained in a unique maximal subgroup of H_0, namely its normalizer

$$L = (\bar{L}.2)_\sigma = ((q+1)\circ K).d.2^2 = d.((q+1)/d \times \mathrm{L}_4(q^2)).d.2^2.$$

Since $q \geqslant 3$, we can choose an element $g \in \bar{M}_\sigma = \mathrm{SL}_2(q) \leqslant C_{G_0}(K)$ that does not normalize the non-normal subgroup $(Z(\bar{L})^0)_\sigma = C_{H_0}(K) = Z_{q+1}$ of $\bar{M}_\sigma$. Suppose L^g is contained in H_0. Then $K = K^g$ is contained in L^g, which is a maximal subgroup of H_0, so $L = L^g$ and thus g normalizes L. But $C_{H_0}(K) = C_L(K)$ and so g also normalizes $C_{H_0}(K)$, which is a contradiction. Therefore, $L^g \not\leqslant H_0$ and the desired result now follows by applying Lemma 2.4, noting that K and L are A-stable.

Next suppose $\epsilon = -$ and let us continue to assume $q \geqslant 3$. Recall that $q \not\equiv 3 \pmod 4$. In this case, we replace the standard Frobenius morphism σ by the product of σ with a lift of the longest element of the Weyl group of $\bar{G}$ (we will continue to write σ for the modified map). Define $K = (\bar{K}_\sigma)' = d.\mathrm{U}_4(q)^2 < H_0$. By inspecting [6, Tables 8.46 and 8.47], we see that

$$L = N_{H_0}(K) = ((q+1)\circ K).d.2^2 = d.((q+1)/d \times \mathrm{U}_4(q)^2.d).2^2$$

is the unique maximal overgroup of K in H_0. Since $q \geqslant 3$, it follows that $(Z(\bar{L})^0)_\sigma = C_{H_0}(K) = Z_{q+1}$ is a non-normal subgroup of $\bar{M}_\sigma = \mathrm{SL}_2(q) \leqslant C_{G_0}(K)$, so there exists $g \in \bar{M}_\sigma$ which does not normalize $C_{H_0}(X)$. We now complete the argument as we did in the $\epsilon = +$ case above.

To complete the proof of the lemma, we may assume $q = 2$ and thus $G = G_0$. For $\epsilon = +$ we take σ to be the standard Frobenius morphism, and we take the product of this with a lift of the following Weyl group element

$$w = (1,6)(2)(3,5)(7,126)\ldots \in W(\bar{G}) \qquad (5)$$

when $\epsilon = -$. Here we are expressing w as a permutation of the set of roots of $\bar{G}$, where our labelling is consistent with Magma (we only give part of the permutation, but this is enough to uniquely determine it). Set

$$K = \bar{K}_\sigma = \begin{cases} \mathrm{SL}_4(2) \times \mathrm{SL}_4(2) & \text{if } \epsilon = + \\ \mathrm{SL}_4(4) & \text{if } \epsilon = -. \end{cases}$$

Clearly, $M = N_{H_0}(K) = K.2$ is a maximal overgroup of K in H_0. By inspecting [6], we see that every other maximal overgroup of K is a parabolic subgroup of the form $2^{16}{:}K$.

We claim that K is contained in precisely two maximal parabolic subgroups of H_0. Clearly, we can take the fixed points under σ of the standard maximal parabolic subgroup of $\bar{H}^0$ containing $\bar{K}$, as well as the opposite parabolic subgroup, so there are at least two such subgroups. Let P be a maximal parabolic subgroup of H_0 and suppose $K \leqslant P \cap P^h$ for some $h \in H_0$. Then $K, K^{h^{-1}} < P$ and [64, Proposition 26.1(b)] implies that K and $K^{h^{-1}}$ are P-conjugate. So $K^x = K^{h^{-1}}$ for some $x \in P$ and thus $xh \in N_{H_0}(K) = K.2$. If $xh \in K$ then $P^h = P$. On the other hand, if $xh \notin K$ then $P^{xh} \neq P$ since $N_{H_0}(P) = P$. But for each y in the coset Kxh we have $P^y = P^{xh}$ and thus K is contained in precisely two maximal parabolic subgroups of H_0 as claimed.

In view of the claim, let us write $\mathcal{M} = \{M, M_1, M_2\}$ where M_1 and M_2 are the maximal parabolic subgroups of H_0 containing K. To complete the proof, we will exhibit an element $g \in N_{G_0}(K)$ such that none of the subgroups M^g, M_1^g, M_2^g are contained in H_0. The result will then follow from Lemma 2.4.

By considering the above set up at the algebraic group level, we see that $C_{G_0}(K)$ contains $\bar{M}_\sigma = \mathrm{SL}_2(2) \cong \mathrm{Sym}_3$. In addition, we note that σ induces a standard Frobenius morphism on $\bar{M}$ since the Weyl group element w in (5) fixes the roots α_2 and $-\alpha_2$. Therefore, we may choose $g = x_{\alpha_2}(1)x_{-\alpha_2}(1) \in \bar{M}_\sigma$. Since g has order 3, it does not commute with the involution $n_{\alpha_2} \in \bar{M}_\sigma$ (here n_{α_2} is the standard lift of the fundamental reflection in $W(\bar{G})$ corresponding to α_2). The lift of the involution in $N_{\bar{H}^0}(\bar{K})/\bar{K} \cong \mathrm{Sym}_2$ is the product of n_{α_2}, which inverts a maximal torus of $\bar{M}$, and an involution that swaps the two A_3 factors of $\bar{K}$. Therefore, g does not normalize $\bar{K}.2$ and so it does not normalize $M = K.2$. In particular, $M^g \not\leqslant H_0$.

Finally, we need to consider M_1 and M_2. They are the fixed points under σ of the standard maximal parabolic subgroups of $\bar{H}^0$ containing $\bar{K}$. In particular, the unipotent radicals of M_1 and M_2 contain $x_1 = x_{\alpha_4}(1)$ and $x_2 = x_{-\alpha_4}(1)$, respectively (since both elements are σ-stable). A straightforward calculation shows that $x_1^g = x_{\alpha_2+\alpha_4}(1)$ and x_2^g is the product of x_2 with $x_{-(\alpha_2+\alpha_4)}(1)$. Therefore, if $M_1^g \leqslant H_0$ then $x_{\alpha_2+\alpha_4}(1) \in H_0$. Similarly, if $M_2^g \leqslant H_0$ then $x_{-(\alpha_2+\alpha_4)}(1) \in H_0$. However, $x_{\pm(\alpha_2+\alpha_4)}(1)^{n_{\alpha_4}} = x_{\pm\alpha_2}(1)$ and thus $x_{\pm(\alpha_2+\alpha_4)}(1)$ is not even contained in $\bar{H}^0$. This implies that $M_i^g \not\leqslant H_0$ for $i = 1, 2$ and the proof of the lemma is complete. □

4.3. $G_0 = E_6^\epsilon(q)$.

Lemma 4.11. *If $G_0 = E_6^\epsilon(q)$ and H is the normalizer of a maximal torus, then $b(G, H) = 2$.*

Proof. Set $H_0 = H \cap G_0$ and note that the structure of $N_L(H_0)$ is recorded in [50, Table 5.2], where $L = \mathrm{Inndiag}(G_0)$. Let $W(\bar{G}) = \mathrm{O}_6^-(2)$ be the Weyl group of $\bar{G}$.

Let $x \in G$ be an element of prime order. If x is a long root element, then $|x^G| > (q-1)q^{21} = b_1$ and Corollary 2.13 implies that $|x^G \cap H| \leqslant 36(q+1)^6 = a_1$. If x is not a long root element, nor an involutory graph automorphism with $C_{\bar{G}}(x) = F_4$, then $|x^G| > (q-1)q^{31} = b_2$ and we have

$$|H| \leqslant (q+1)^6|W(\bar{G})|.2\log_2 q = a_2.$$

Finally, suppose x is an F_4-type graph automorphism, so $|x^G| > \frac{1}{3}(q-1)q^{25} = b_3$. Since $W(\bar{G})$ is centralized by a graph automorphism, it follows that

$$|x^G \cap H| \leqslant (q+1)^6 i_2(W(\bar{G}) \times 2) = 3567(q+1)^6 = a_3$$

and we conclude that $\mathcal{Q}(G, H) < \sum_{i=1}^{3} a_i^2/b_i < q^{-1}$ for all $q \geqslant 5$.

To complete the proof, we may assume $q \leqslant 4$. First let us consider the case where

$$N_L(H_0) = (q^2+\epsilon q+1)^3.3^{1+2}.\mathrm{SL}_2(3),$$

so $(q,\epsilon) \neq (2,-)$ (see [50, Table 5.2]). By arguing as in the previous paragraph, we see that the contribution to $\mathcal{Q}(G,H)$ from all elements other than F_4-type graph automorphisms is less than $a_1^2/b_1 + a_2^2/b_2$, where b_1 and b_2 are defined as above, $a_1 = 36(q^2-\epsilon q+1)^3$ and

$$a_2 = (q^2+\epsilon q+1)^3 3^3 |\mathrm{SL}_2(3)|.2\log_2 q$$

is an upper bound on $|H|$. Now assume x is a graph automorphism with $C_{\bar{G}}(x) = F_4$. Here $|x^G| > b_3$ as above and we have

$$|x^G \cap H| \leqslant (q^2+\epsilon q+1)^3 3^3.i_2(\mathrm{SL}_2(3)\times 2) = 81(q^2+\epsilon q+1)^3 = a_3.$$

This gives $\mathcal{Q}(G,H) < \sum_{i=1}^{3} a_i^2/b_i < 1$ for $q \geqslant 3$.

Suppose $(q,\epsilon) = (2,+)$, so $G_0 = E_6(2)$ and $H_0 = 7^3{:}3^{1+2}.\mathrm{SL}_2(3)$. To handle this case, we view $\bar{G}.2 < E_7$ and we use Magma to construct $H_0.2$ as a subgroup of $E_7(8)$ (see [21, Example 1.11]). In this way, we calculate that $i_2(H) \leqslant 847 = a_1$. Since $|x^G| > 2^{41} = b_2$ for all $x \in G$ of odd prime order (see [17, Table 9]), it follows that

$$\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2 < 1,$$

where $b_1 = 2^{21}$ and $a_2 = 2|H_0|$.

For the remainder of the proof, we may assume $\epsilon = -$ and $N_L(H_0) = (q+1)^6.W(\bar{G})$ with $q \leqslant 4$ (according to [50, Table 5.2], the condition $q \leqslant 4$ implies that $\epsilon = -$). First assume $q = 2$, so either $G = G_0.3$ and $H = 3^6.W(\bar{G})$, or $G = G_0.\mathrm{Sym}_3$ and $H = 3^6.(W(\bar{G})\times 2)$ (see [50, Table 5.2]). To get started, let us assume $G = G_0.3$. As explained in [21, Proposition 2.2], we can use Magma to construct H as a subgroup of $E_6(4)$ and we then compute the action of each element $x \in H$ of prime order r on the adjoint module V for $\bar{G}$. More precisely, if $r = 2$ we compute the Jordan form of x on V and we inspect [40, Table 6] to determine the G_0-class of x (in the table below, we use the labelling of unipotent classes in [57, Table 22.2.3]). For $r \in \{3,5\}$ we compute $\dim C_V(x) = \dim C_{\bar{G}}(x)$ and this allows us to identify the structure of $C_{\bar{G}}(x)^0$. The results we obtain are summarised in the following table:

$r=2$		$r=3$		$r=5$	
A_1	108	D_5T_1	54	A_3T_3	419904
A_1^2	2430	A_5T_1	2232		
A_1^3	18225	D_4T_2	47610		
		$A_4A_1T_1$	39312		
		A_2^3	144800		

In each case, it is easy to determine a lower bound on $|x^G|$ and the desired bound $\mathcal{Q}(G,H) < 1$ quickly follows. For example, one checks that the contribution from elements of order 3 is less than $\sum_{i=1}^{5} a_i^2/b_i$, where

$$a_1 = 54,\ a_2 = 2232,\ a_3 = 47610,\ a_4 = 39312,\ a_5 = 144800,$$
$$b_1 = 2^{31},\ b_2 = 2^{41},\ b_3 = 2^{45},\ b_4 = 2^{44},\ b_5 = 2^{52}.$$

To complete the analysis of this case, let us now assume $G = G_0.\mathrm{Sym}_3$ and $x \in G$ is an involutory graph automorphism. Since the algebraic group E_7 contains a subgroup $E_6.2$, we can use Magma to construct H as a subgroup of $E_7(4)$ (once again, see [21]) and we find that there are 5 conjugacy classes of involutions in $H \setminus H_0$; the size of each class and the Jordan form of a representative on the adjoint module $\mathcal{L}(E_7)$ for E_7 are as follows:

$$(J_2^{53}, J_1^{27}){:}\ 405$$
$$(J_2^{63}, J_1^{7}){:}\ 729 + 8748 + 14580 + 21870 = 45927$$

If $C_{\bar{G}}(x) = F_4$, then [45, Table 7] indicates that x is contained in the E_7-class labelled $(A_1^3)''$ in [40], whence x has Jordan form (J_2^{53}, J_1^{27}) on $\mathcal{L}(E_7)$ (see [40, Table 8]). Similarly, if

$C_{\bar{G}}(x) \neq F_4$ then x has Jordan form (J_2^{63}, J_1^7). It follows that the contribution to $\mathcal{Q}(G,H)$ from involutory graph automorphisms is less than $a_1^2/b_1 + a_2^2/b_2 < 0.02$, where

$$a_1 = 405,\ b_1 = \frac{1}{3}2^{25},\ a_2 = 45927,\ b_2 = \frac{1}{3}2^{41}.$$

By combining this with the above estimates, we conclude that $\mathcal{Q}(G,H) < 1$ and the result follows.

Next assume $q = 3$. Here $G = G_0$ and $H = 4^6.W(\bar{G})$, or $G = G_0.2$ and $H = 4^6.(W(\bar{G}) \times 2)$. First assume $G = G_0$. Here we construct H as a subgroup of $E_6(9)$ (see [21]) and by considering the action of H on the adjoint module for $\bar{G}$ we obtain the following results:

$r = 2$		$r = 3$		$r = 5$	
D_5T_1	5211	A_2	3840	A_3T_3	1327104
A_1A_5	60516	A_2^2	122880		
		$A_2^2A_1$	327680		

The desired bound $\mathcal{Q}(G,H) < 1$ quickly follows.

Now assume $G = G_0.2$. Here we construct H as a subgroup of $E_7(9)$ (see [21] again for the details) and we deduce that $i_2(H \setminus H_0) = 147520$. Moreover, by calculating the eigenvalues of the involutions in $H \setminus H_0$ on the adjoint module for E_7, we see that 720 have a 79-dimensional 1-eigenspace and the remainder have a 63-dimensional 1-eigenspace. If $x \in H$ is such an involution, then we may view x as a semisimple involution in E_7 via the embedding $E_6.2 < E_7$ and we recall that the connected component of the centralizer of an involution in E_7 is one of A_1D_6, A_7 or E_6T_1 (see [43, Proposition 1.2]). Since F_4 is not contained in A_1D_6 or A_7, it follows that if $C_{\bar{G}}(x) = F_4$ then $C_{E_7}(x)^0 = E_6T_1$ and thus $|x^G \cap H| = 720 = a_1$. On the other hand, if $C_{\bar{G}}(x) \neq F_4$ then $|x^G \cap H| = 146800 = a_2$. Therefore, the combined contribution to $\mathcal{Q}(G,H)$ from graph automorphisms is less than $a_1^2/b_1 + a_2^2/b_2 < 10^{-6}$, where $b_1 = \frac{1}{2}3^{26}$ and $b_2 = \frac{1}{2}3^{42}$. It is now easy to check that $\mathcal{Q}(G,H) < 1$ and the result follows.

Finally, let us assume $q = 4$, so either $G = G_0$ and $H = 5^6.W(\bar{G})$, or $G = G_0.2$ and $H = 5^6.(W(\bar{G}) \times 2)$, or $G = G_0.4$ and $H = 5^6.W(\bar{G}).4$. First assume $G = G_0$. Here we construct H as a subgroup of $E_6(16)$ and as before we study the action of H on the adjoint module for $\bar{G}$ (as usual, see [21] for the details). If $x \in G$ is a long root element, then $|x^G| > \frac{1}{2}4^{22} = b_1$ and we find that $|x^G \cap H| = 180 = a_1$. On the other hand, if x is not a long root element then Proposition 2.11 gives $|x^G| > 3.4^{31} = b_2$ and we set $a_2 = |H|$. Now assume $G \neq G_0$. Since every element of prime order in $G_0.4$ is contained in $G_0.2$, we may assume that $G = G_0.2$. We now construct H as a subgroup of $E_7(16)$ and we find that $i_2(H \setminus H_0) = 365500 = a_3$. Since $|x^G| > \frac{1}{2}4^{26} = b_3$ for all involutory graph automorphisms $x \in G$, we conclude that $\mathcal{Q}(G,H) < \sum_{i=1}^{3} a_i^2/b_i < 1$ and thus $b(G,H) = 2$. □

Lemma 4.12. *Suppose $G_0 = E_6^{\epsilon}(q)$ and H is of type $A_2^{\epsilon}(q)^3$, $A_2(q^2)A_2^{-\epsilon}(q)$ or $A_2^{\epsilon}(q^3)$. Then $b(G,H) = 2$.*

Proof. Here $\bar{H} = A_2^3.\mathrm{Sym}_3$, where $\mathrm{Sym}_3 = \langle a, b\rangle$ and the action of a and b on $\bar{H}^0 = A_2^3$ is given by

$$a:\ (x_1,x_2,x_3) \mapsto (x_3,x_1,x_2),\ \ b:\ (x_1,x_2,x_3) \mapsto (x_2^{\tau},x_1^{\tau},x_3^{\tau}) \quad (6)$$

where x_i denotes an arbitrary element in the ith A_2 factor of $\bar{H}^0$ and τ is an involutory graph automorphism of A_2 (in terms of matrices, we may view τ as the inverse-transpose map $y \mapsto y^{-T}$). Let $V = \mathcal{L}(\bar{G})$ be the adjoint module for $\bar{G}$ and let V_{27} be one of the 27-dimensional minimal modules. As noted in [75, Table 3], we have

$$V{\downarrow}\bar{H}^0 = \mathcal{L}(\bar{H}^0) \oplus (W \otimes W \otimes W) \oplus (W^* \otimes W^* \otimes W^*) \quad (7)$$

and

$$V_{27}{\downarrow}\bar{H}^0 = (W \otimes W^* \otimes 0) \oplus (W^* \otimes 0 \otimes W) \oplus (0 \otimes W \otimes W^*)$$

where W is the natural module for A_2, W^* its dual and 0 the trivial module. Before we begin the main analysis, it will be useful to record some preliminary observations concerning the embedding of $\bar{H}$ in $\bar{G}$.

Suppose $x \in \bar{H}$ has order p. If $x \in \bar{H}^0$ then the $\bar{G}$-class of x is determined in [41, Section 4.9]. The nontrivial unipotent classes in A_2 are labelled A_1 and A_2, which gives a corresponding labelling of the classes in $\bar{H}^0$. It turns out that the $\bar{G}$-class containing a given $\bar{H}^0$-class inherits the same label, with the exception of the regular $\bar{H}^0$-class labelled A_2^3 (the latter is contained in the $\bar{G}$-class $D_4(a_1)$ if $p \neq 3$ and $A_2^2A_1$ if $p = 3$).

Now assume $x \in \bar{H} \setminus \bar{H}^0$ has order p, so $p = 2$ or 3. Suppose $p = 3$ and note that x is $\bar{H}$-conjugate to a. Now (6) indicates that x cyclically permutes the three A_2 factors of $\bar{H}$, whence x has Jordan form (J_3^9) on V_{27}. By inspecting [40, Table 5], it follows that x is in one of the $\bar{G}$-classes A_2^2 or $A_2^2A_1$. Visibly, x centralizes a diagonally embedded A_2 subgroup of $\bar{H}^0$ and so by considering the possibilities for $C_{\bar{G}}(x)^0$ (see [57, Table 22.1.3]) we conclude that x is in the A_2^2 class.

Now assume $p = 2$ and $x = (x_1, x_2, x_3)b \in \bar{H}$ has order 2, so $x_2 = x_1^T$ and $x_3 = x_3^T$. Since every invertible symmetric matrix is congruent to the identity matrix, it is easy to see that x is $\bar{H}^0$-conjugate to b, so there is a unique $\bar{H}$-class of involutions in $\bar{H} \setminus \bar{H}^0$. Note that

$$C_{\bar{H}^0}(b) = \{(x_1, x_2, x_3) \in \bar{H}^0 \,:\, x_2 = x_1^\tau,\, x_3 = x_3^\tau\}$$

and thus $C_{\bar{H}^0}(b)^0$ is of type A_2A_1. Now b has Jordan form $(J_2^8) \oplus (J_2^3, J_1^2)$ on $\mathcal{L}(\bar{H}^0)$ and it interchanges the two 27-dimensional summands in (7), so b has Jordan form (J_2^{38}, J_1^2) on V. By inspecting [40, Table 6], we conclude that every involution in $\bar{H} \setminus \bar{H}^0$ is contained in the class labelled A_1^3.

Next we turn to the semisimple elements in $\bar{H}$. Suppose $x \in \bar{H}^0$ has prime order $r \neq p$. By working with the decomposition in (7), it is easy to compute the eigenvalues of x on V, which allows us to read off $\dim C_V(x) = \dim C_{\bar{G}}(x)$. For example, if $x = (x_1, x_2, x_3) \in \bar{H}^0$ has order 2, where $x_1 = x_2 \neq 1$ and $x_3 = 1$ then the dimension of the 1-eigenspace of x on $\mathcal{L}(\bar{H}^0)$ is equal to $\dim C_{\bar{H}^0}(x) = 4 + 4 + 8 = 16$ and we calculate that x acts as $(-I_{12}, I_{15})$ on the two 27-dimensional summands in (7). Therefore, $\dim C_{\bar{G}}(x) = 16 + 30 = 46$ and thus $C_{\bar{G}}(x)^0 = D_5T_1$. Now assume $x \in \bar{H} \setminus \bar{H}^0$, so $r \in \{2, 3\}$. If $r = 2$ then x is $\bar{H}$-conjugate to b and we calculate that x acts as $(-I_{40}, I_{38})$ on V, so $C_{\bar{G}}(x)^0 = A_5A_1$. Similarly, if $r = 3$ then x is conjugate to a and we have $\dim C_{\bar{G}}(x) = 30$, so $C_{\bar{G}}(x)^0 = D_4T_2$.

Finally, let γ be an involutory graph automorphism of $\bar{G}$. Then the normalizer of $\bar{H}$ in $\bar{G}.2 = \bar{G}.\langle\gamma\rangle$ is $\bar{H}.2 = \bar{H}^0.(\mathrm{Sym}_3 \times 2)$, where Sym_3 acts naturally on the three factors of $\bar{H}^0$ and a generator c for the cyclic group of order 2 acts as a simultaneous graph automorphism on all three factors. There are three classes of involutions in $\bar{H}.2 \setminus \bar{H}$, represented by c, bc and $(1, 1, t)bc$, where $t \in A_2$ is an involution. Here bc acts as a transposition on the three factors of $\bar{H}^0$, swapping the first two and centralizing the third. Working with (7), we calculate that bc has Jordan form (J_2^{26}, J_1^{26}) or $(-I_{26}, I_{52})$ on V, according to the parity of p, and we deduce that $C_{\bar{G}}(bc) = F_4$. On the other hand, if $x = c$ or $(1, 1, t)bc$ then x has Jordan form (J_2^{36}, J_1^6) or $(-I_{42}, I_{36})$ and we see that $C_{\bar{G}}(x) \neq F_4$.

Case 1. H is of type $A_2^\epsilon(q)^3$.

We are now ready to begin the proof of the lemma. First assume H is of type $A_2^\epsilon(q)^3$, so

$$H_0 = e.\mathrm{L}_3^\epsilon(q)^3.e.\mathrm{Sym}_3 \;\text{ and }\; K = N_L(H_0) = e.\mathrm{L}_3^\epsilon(q)^3.e^2.\mathrm{Sym}_3,$$

where $e = (3, q - \epsilon)$ and $L = \mathrm{Inndiag}(G_0) = G_0.e$. We proceed by estimating the contribution to $\mathcal{Q}(G, H)$ from the different types of elements of prime order in G.

First we consider the contribution from unipotent elements. Suppose $p = 2$ and let $x \in H$ be a unipotent involution. Both $\mathrm{SL}_3^\epsilon(q)$ and $\mathrm{L}_3^\epsilon(q)$ contain $(q^2 - 1)(q^2 + \epsilon q + 1) < 2q^4$ involutions, which form a single conjugacy class, and we recall from above that every involution in $\bar{H} \setminus \bar{H}^0$ is contained in the class labelled A_1^3. If x is in the G_0-class labelled A_1 then $|x^G \cap H| < 3.2q^4 = a_1$ and $|x^G| > (q-1)q^{21} = b_1$. Similarly, if x is in the A_1^2 class, then $|x^G \cap H| < 3.(2q^4)^2 = 12q^8 = a_2$ and $|x^G| > (q-1)q^{31} = b_2$. Finally, if x is in the class labelled A_1^3, then

$$|x^G \cap H| < (2q^4)^3 + 3 \cdot \frac{|\mathrm{SL}_3^\epsilon(q)|^3}{|\mathrm{SL}_3^\epsilon(q)||\mathrm{SL}_2(q)|} < 8q^{12} + 6q^{13} = a_3, \;\; |x^G| > (q-1)q^{39} = b_3$$

and thus the unipotent contribution when $p = 2$ is less than $\sum_{i=1}^3 a_i^2/b_i$.

Now assume $p \geqslant 3$. As above, the contribution to $\mathcal{Q}(G,H)$ from elements in the classes labelled A_1, A_1^2 and A_1^3 is less than $\sum_{i=1}^3 a_i^2/b_i$. Now $|x^G| > \frac{1}{4}q^{42} = b_4$ for all other nontrivial elements of order p in G_0 (see [57, Table 22.2.3]) and using Proposition 2.14(ii) we note that

$$i_p(e.\mathrm{L}_3^\epsilon(q)^3.e) = i_p(\mathrm{L}_3^\epsilon(q)^3) < q^{18} = a_4.$$

In addition, if $p = 3$ then there are elements of order p in $\bar{H} \setminus \bar{H}^0$ which transitively permute the factors of $\bar{H}^0$. As explained above, such an element x is in the $\bar{G}$-class labelled A_2^2, so $|x^G| > \frac{1}{6}q^{48} = b_5$ and we note that there are at most $2|\mathrm{SL}_3^\epsilon(q)|^2 < 2q^{16} = a_5$ of these elements in H_0.

We conclude that the unipotent contribution is less than $\sum_{i=1}^5 a_i^2/b_i < q^{-4} + q^{-5}$ for all p.

Next let us turn to the contribution to $\mathcal{Q}(G,H)$ from semisimple elements. Let $x \in H$ be a semisimple element of prime order r. First assume $r = 2$, so $C_{\bar{G}}(x)^0 = A_5A_1$ or D_5T_1. Recall that the involutions in $\bar{H} \setminus \bar{H}^0$ are of type A_5A_1 and note that both $\mathrm{SL}_3^\epsilon(q)$ and $\mathrm{L}_3^\epsilon(q)$ contain $q^2(q^2 + \epsilon q + 1) < 2q^4$ involutions, which form a single class. By considering the decomposition in (7), we can calculate $\dim C_V(x)$ for each involution $x \in H$ and this allows us to determine the G_0-class of x. In this way, we deduce that if x is of type D_5T_1 then $|x^G \cap H| < 3.(2q^4)^2 = 12q^8 = a_6$ and $|x^G| > (q-1)q^{31} = b_6$, whereas

$$|x^G \cap H| < 3.2q^4 + (2q^4)^3 + 3 \cdot \frac{|\mathrm{SL}_3^\epsilon(q)|^3}{|\mathrm{SL}_3^\epsilon(q)||\mathrm{SL}_2(q)|} < 6q^4 + 8q^{12} + 6q^{13} = a_7$$

and $|x^G| > (q-1)q^{39} = b_7$ if x is of type A_5A_1. It follows that the contribution to $\mathcal{Q}(G,H)$ from semisimple involutions is less than $a_6^2/b_6 + a_7^2/b_7 < q^{-9}$.

Next assume $r = 3$, so $p \neq 3$. First assume $e = 3$, so we have $H_0 = 3.\mathrm{L}_3^\epsilon(q)^3.3.\mathrm{Sym}_3$ and $K = 3.\mathrm{L}_3^\epsilon(q)^3.3^2.\mathrm{Sym}_3$. Let $x \in K$ be a semisimple element of order 3, whence

$$C_{\bar{G}}(x)^0 = A_5T_1,\ D_4T_2,\ A_2^3,\ D_5T_1 \text{ or } A_1A_4T_1$$

(see [32, Table 4.7.1]) and we recall that $C_{\bar{G}}(x)^0 = D_4T_2$ if $x \in \bar{H} \setminus \bar{H}^0$. Let Z be the normal subgroup of K of order 3, so

$$K/Z = \mathrm{L}_3^\epsilon(q)^3.3^2.\mathrm{Sym}_3 < \mathrm{PGL}_3^\epsilon(q)^3.\mathrm{Sym}_3$$

and

$$i_3(K) < |Z| \cdot (1 + i_3(\mathrm{PGL}_3^\epsilon(q) \wr Z_3)).$$

By applying Proposition 2.14(i), we deduce that

$$1 + i_3(\mathrm{PGL}_3^\epsilon(q) \wr Z_3) \leqslant (i_3(\mathrm{Aut}(\mathrm{L}_3^\epsilon(q))) + 1)^3 + 2|\mathrm{PGL}_3^\epsilon(q)|^2 < 8(q+1)^3q^{15} + 2q^{16}$$

and thus $i_3(K) < 24(q+1)^3q^{15} + 6q^{16}$. If $q = 2$ then $\epsilon = -$ and we can use MAGMA to construct K as a subgroup of $E_6(4)$ (see [21, Lemma 2.3]); in this way, we calculate that $i_3(K) = 492074$. Set

$$a_8 = \begin{cases} 24(q+1)^3q^{15} + 6q^{16} & \text{if } q > 2, \\ 492074 & \text{if } q = 2. \end{cases}$$

If $\dim x^{\bar{G}} > 42$ then $|x^G| > \frac{1}{6}q^{48} = b_8$ and the contribution to $\mathcal{Q}(G,H)$ from these elements is less than a_8^2/b_8.

Now assume $\dim x^{\bar{G}} \leqslant 42$, so $x \in \bar{H}^0$ and we have $C_{\bar{G}}(x)^0 = D_5T_1$ or A_5T_1. By considering (7), one can check that $C_{\bar{G}}(x)^0 = D_5T_1$ if and only if $Zx \in \mathrm{PGL}_3^\epsilon(q)^3$ is of the form $(x_1, x_2, 1)$

(up to permutations of the coordinates), where $x_1 \in \mathrm{PGL}_3^{\epsilon}(q)$ is conjugate to the image modulo scalars of a diagonal matrix $\mathrm{diag}(1,1,\omega) \in \mathrm{GL}_3^{\epsilon}(q)$ with ω a primitive cube root of unity (so $x_1 \in \mathrm{PGL}_3^{\epsilon}(q) \setminus \mathrm{L}_3^{\epsilon}(q)$ is a diagonal automorphism) and x_2 is conjugate to x_1^{-1}. Now $|x^G| > (q-1)q^{31} = b_9$ and there are at most

$$3! \cdot \left(\frac{|\mathrm{SL}_3^{\epsilon}(q)|}{|\mathrm{GL}_2^{\epsilon}(q)|}\right)^2 < 24q^8 = a_9$$

such elements in K. Now assume $C_{\bar{G}}(x)^0 = A_5T_1$, so $|x^G| > (q-1)q^{41} = b_{10}$. Here we find that $C_{\bar{G}}(x)^0 = A_5T_1$ if and only if $Zx \in \mathrm{PGL}_3^{\epsilon}(q)^3$ is one of the following, up to permutations:

(a) $(x_1, 1, 1)$, where x_1 is conjugate to the image of $\mathrm{diag}(1,\omega,\omega^2) \in \mathrm{GL}_3^{\epsilon}(q)$; or

(b) (x_1, x_2, x_3), where x_1 and x_2 are conjugate to the image of $\mathrm{diag}(1,1,\omega)$ and x_3 is conjugate to x_1^{-1}.

Therefore, there are at most

$$3 \cdot \frac{|\mathrm{SL}_3^{\epsilon}(q)|}{|\mathrm{GL}_1^{\epsilon}(q)|^2} + 3 \cdot 2 \cdot \left(\frac{|\mathrm{SL}_3^{\epsilon}(q)|}{|\mathrm{GL}_2^{\epsilon}(q)|}\right)^3 < 12q^6 + 48q^{12} = a_{10}$$

such elements in K.

Bringing these estimates together, we conclude that if $e = 3$, then the contribution to $\mathcal{Q}(G,H)$ from semisimple elements of order 3 is less than $\sum_{i=8}^{10} a_i^2/b_i < q^{-4}$.

Now assume $e = 1$, so $H_0 = \mathrm{L}_3^{\epsilon}(q)^3.\mathrm{Sym}_3$ and we note that $\mathrm{L}_3^{\epsilon}(q)$ contains $q^3(q^3+\epsilon) < 2q^6$ elements of order 3, which form a single class (note that these elements are regular). The possibilities for $C_{\bar{G}}(x)^0$ are A_5T_1, A_2^3 and D_4T_2. By working with (7) we calculate that if $C_{\bar{G}}(x)^0 = A_5T_1$ then $|x^G \cap H| < 3.2q^6 = 6q^6$ and $|x^G| > (q-1)q^{41}$. Similarly, if $C_{\bar{G}}(x)^0 = A_2^3$ then $|x^G \cap H| < (2q^6)^3 = 8q^{18}$ and $|x^G| > \frac{1}{6}q^{54}$. And for $C_{\bar{G}}(x)^0 = D_4T_2$ we get

$$|x^G \cap H| < 3.(2q^6)^2 + 2|\mathrm{SL}_3^{\epsilon}(q)|^2 < 12q^{12} + 2q^{16}, \quad |x^G| > \frac{1}{6}q^{48}.$$

From the above estimates, it is straightforward to check that the combined contribution to $\mathcal{Q}(G,H)$ from semisimple elements of order 3 is less than q^{-4} for all q.

To complete the analysis of semisimple elements, let us assume $r \geqslant 5$. If $\dim x^{\bar{G}} \geqslant 50$ then $|x^G| > (q-1)q^{49} = b_{11}$ and there are clearly fewer than $q^{24} = a_{11}$ such elements in H. Now assume $\dim x^{\bar{G}} < 50$, so $C_{\bar{G}}(x)^0 = D_5T_1$, A_5T_1 or D_4T_2. Since $r \geqslant 5$ we have $x \in \bar{H}^0$, say $x = (x_1, x_2, x_3)$, and by working with the decomposition in (7) we deduce that $\dim C_{\bar{G}}(x) \leqslant 36$. Therefore, $C_{\bar{G}}(x)^0 = A_5T_1$ or D_4T_2 and thus $|x^G| > (q-1)q^{41} = b_{12}$. If each x_i is nontrivial then we calculate that $\dim C_{\bar{G}}(x) \leqslant 24$, which is a contradiction. Therefore, there are fewer than $3|\mathrm{SL}_3^{\epsilon}(q)| + 3|\mathrm{SL}_3^{\epsilon}(q)|^2 < 3q^8(q^8+1) = a_{12}$ such elements in H and we conclude that the entire contribution to $\mathcal{Q}(G,H)$ from semisimple elements of order at least 5 is less than $a_{11}^2/b_{11} + a_{12}^2/b_{12} < q^{-1} + q^{-4}$.

Finally, let us assume $x \in G$ is a field, graph or graph-field automorphism of G_0. First assume x is a field or graph-field automorphism of order r, so $q = q_0^r$. If $r \geqslant 3$ then $|x^G| > \frac{1}{6}q^{52} = b_{13}$ and we note that there are fewer than $\log_2 q.6q^{24} = a_{13}$ such elements in H. Now assume $r = 2$, so $\epsilon = +$ and it is convenient to use the bound

$$\mathrm{fpr}(x, G/H) < \frac{|\bar{H} : \bar{H}^0| \cdot q^{24}}{(q^{1/2}-1)^6 q^9 |x^{G_0}|} < 36q^{-24}(q^{1/2}-1)^{-6},$$

which is explained in the proof of [44, Lemma 6.1]. Since $|x^G| < 2q^{39}$, it follows that the contribution to $\mathcal{Q}(G,H)$ from field and graph-field automorphisms is less than

$$\eta\left(a_{13}^2/b_{13}\right) + 2 \cdot 2q^{39} \cdot \left(36q^{-24}(q^{1/2}-1)^{-6}\right)^2 < 4\eta q^{-1} + \mu q^{-2},$$

where $\eta = 1$ if $q = p^f$ and f is divisible by an odd prime, otherwise $\eta = 0$, and similarly $\mu = 1$ if q is a square, otherwise $\mu = 0$.

Now assume $x \in G$ is an involutory graph automorphism. As explained earlier, if $C_{\bar{G}}(x) \neq F_4$ then either x induces a graph automorphism on each A_2 factor of $\bar{H}^0$, or it swaps two of the factors and acts nontrivially on the third. Therefore, $|x^G| > \frac{1}{6}q^{42} = b_{14}$ and there are at most

$$\left(\frac{|\mathrm{SL}_3^{\epsilon}(q)|}{|\mathrm{SL}_2(q)|}\right)^3 + 3|\mathrm{SL}_3^{\epsilon}(q)| \cdot (q^2-1)(q^2+\epsilon q+1) < 2q^{15} = a_{14}$$

such elements in H. On the other hand, if $C_{\bar{G}}(x) = F_4$ then $|x^G| > \frac{1}{6}q^{26} = b_{15}$ and x acts as a transposition on the factors of $\bar{H}^0$, centralizing the fixed factor, whence H contains at most $3|\mathrm{SL}_3^{\epsilon}(q)| < 3q^8 = a_{15}$ such elements. Therefore, the contribution to $\mathcal{Q}(G,H)$ from graph automorphisms is less than $a_{14}^2/b_{14} + a_{15}^2/b_{15} < q^{-4}$.

Bringing together all of the above bounds, we conclude that

$$\mathcal{Q}(G,H) < (1+4\eta)q^{-1} + \mu q^{-2} + 4q^{-4} + q^{-5} + q^{-9} < 1$$

and the result follows.

Case 2. H is of type $A_2(q^2)A_2^{-\epsilon}(q)$.

As before, set $L = \mathrm{Inndiag}(G_0)$ and observe that

$$K = N_L(H_0) = g.(\mathrm{L}_3(q^2) \times \mathrm{L}_3^{-\epsilon}(q)).h.2,$$

where $g = (3, q+\epsilon)$ and $h = (3, q^2-1)$. In particular, note that if $e = (3, q-\epsilon) = 3$ then $H_0 = (\mathrm{L}_3(q^2) \times \mathrm{L}_3^{-\epsilon}(q)).2$. Let us also observe that there is a unique class of involutions in $K \setminus (g.(\mathrm{L}_3(q^2) \times \mathrm{L}_3^{-\epsilon}(q)).h)$, which acts on $\mathrm{L}_3(q^2) \times \mathrm{L}_3^{-\epsilon}(q)$ by inducing a graph-field automorphism on the first factor and a graph automorphism on the second. This corresponds to the involution $b \in \bar{H}$ discussed earlier (see (6)).

First assume $p = 2$ and $x \in G_0$ is an involution. If x is in the class labelled A_1, then $|x^G| > (q-1)q^{21} = b_1$ and $|x^G \cap H| = i_2(\mathrm{L}_3^{-\epsilon}(q)) < 2q^4 = a_1$. Similarly, if x is in the A_1^2 class then $|x^G \cap H| = i_2(\mathrm{L}_3(q^2)) < 2q^8 = a_2$ and $|x^G| > (q-1)q^{31} = b_2$. Now assume x is in the class labelled A_1^3, so $|x^G| > (q-1)q^{39} = b_3$. Here we get

$$|x^G \cap H| \leqslant i_2(\mathrm{L}_3(q^2)) \cdot i_2(\mathrm{L}_3^{-\epsilon}(q)) + \frac{|\mathrm{SL}_3(q^2)|}{|\mathrm{SU}_3(q)|} \cdot \frac{|\mathrm{SL}_3^{-\epsilon}(q)|}{|\mathrm{SL}_2(q)|} < 4q^{13} = a_3.$$

By arguing as in Case 1, we see that the unipotent contribution to $\mathcal{Q}(G,H)$ for any p is less than $\sum_{i=1}^{4} a_i^2/b_i < q^{-4} + q^{-5}$, where $a_4 = q^{18}$ and $b_4 = \frac{1}{4}q^{42}$.

Next assume $x \in G$ is a semisimple element of prime order r. Suppose $r = 2$, so $C_{\bar{G}}(x)^0 = D_5T_1$ or A_5A_1. If x is a D_5T_1 involution, then $|x^G \cap H| = i_2(\mathrm{L}_3(q^2)) < 2q^8 = a_5$ and $|x^G| > (q-1)q^{31} = b_5$. Similarly, if x is an A_5A_1 involution then $|x^G| > (q-1)q^{39} = b_6$ and we see that

$$|x^G \cap H| \leqslant i_2(\mathrm{L}_3^{-\epsilon}(q)) + i_2(\mathrm{L}_3(q^2)) \cdot i_2(\mathrm{L}_3^{-\epsilon}(q)) + \frac{|\mathrm{SL}_3(q^2)|}{|\mathrm{SU}_3(q)|} \cdot \frac{|\mathrm{SL}_3^{-\epsilon}(q)|}{|\mathrm{SL}_2(q)|} < 4q^{13} = a_6.$$

Now suppose $r = 3$. Let Z be the normal subgroup of K of order $g = (3, q+\epsilon)$, so K/Z is a subgroup of $(\mathrm{PGL}_3(q^2) \times \mathrm{PGL}_3^{-\epsilon}(q)).2$ and thus

$$i_3(K) < |Z| \cdot (1 + i_3(\mathrm{PGL}_3(q^2) \times \mathrm{PGL}_3^{-\epsilon}(q))).$$

For $q \in \{2,3\}$ we compute

$$i_3(\mathrm{PGL}_3(q^2) \times \mathrm{PGL}_3^{-\epsilon}(q)) \leqslant \begin{cases} 387420488 & \text{if } q = 3 \\ 391472 & \text{if } q = 2 \end{cases}$$

and by applying Proposition 2.14(i) we deduce that

$$a_7 = \begin{cases} 12(q+1)(q^2+1)q^{15} & \text{if } q > 3 \\ 387420489 & \text{if } q = 3 \\ 1174419 & \text{if } q = 2 \end{cases}$$

is an upper bound on $i_3(K)$. If $\dim x^{\bar{G}} \geqslant 42$ then $|x^G| > (q-1)q^{41} = b_7$ and so the contribution to $\mathcal{Q}(G,H)$ from these elements is less than a_7^2/b_7. Now assume $\dim x^{\bar{G}} < 42$, so $C_{\bar{G}}(x)^0 = D_5T_1$ is the only option and thus $|x^G| > (q-1)q^{31} = b_8$. By arguing as in Case 1, we see that there are at most $|Z| \cdot i_3(\mathrm{PGL}_3(q^2)) < 6(q^2+1)q^{10}$ such elements in H. Set $a_8 = 6(q^2+1)q^{10}$ if $q > 2$ and $a_8 = 3 \cdot i_3(\mathrm{PGL}_3(4)) = 14496$ if $q = 2$.

Now assume $r \geqslant 5$ and note that $i_r(K) = i_r(\mathrm{L}_3(q^2) \times \mathrm{L}_3^{-\epsilon}(q))$. As in Case 1, if $\dim x^{\bar{G}} \geqslant 50$ then $|x^G| > (q-1)q^{49} = b_9$ and H contains fewer than q^{24} such elements. In fact, for $q = 2$ we calculate that $\mathrm{L}_3(q^2) \times \mathrm{L}_3^{-\epsilon}(q)$ contains at most 290352 elements of prime order at least 5, so we set $a_9 = q^{24}$ if $q > 2$ and $a_9 = 290352$ if $q = 2$. By arguing as before, if $\dim x^{\bar{G}} < 50$ then $|x^G| > (q-1)q^{41} = b_{10}$ and there are fewer than $|\mathrm{SL}_3^{\epsilon}(q^2)| + |\mathrm{SL}_3^{-\epsilon}(q)| < q^{16} = a_{10}$ such elements in H.

Putting all of the above estimates together, we conclude that the contribution to $\mathcal{Q}(G,H)$ from semisimple elements is less than $\sum_{i=5}^{10} a_i^2/b_i < q^{-1} + q^{-2}$.

Finally, let us assume x is a field, graph or graph-field automorphism. By repeating the argument in Case 1, we see that the contribution from field and graph-field automorphisms is less than $4\eta q^{-1} + \mu q^{-2}$, where η and μ are defined as in Case 1. Now assume x is an involutory graph automorphism. Recall that if $C_{\bar{G}}(x) = F_4$ then $|x^G| > \frac{1}{6}q^{26} = b_{11}$ and x acts on the factors of $\bar{H}^0$ by swapping the first two and centralizing the third. Given the structure of H_0, this implies that x induces a field automorphism on the $\mathrm{L}_3(q^2)$ factor and thus

$$|x^G \cap H| \leqslant \frac{|\mathrm{SL}_3(q^2)|}{|\mathrm{SL}_3(q)|} < q^8 = a_{11}.$$

Similarly, if $C_{\bar{G}}(x) \neq F_4$ then $|x^G| > \frac{1}{6}q^{42} = b_{12}$ and x either induces graph automorphisms on both $\mathrm{L}_3(q^2)$ and $\mathrm{L}_3^{-\epsilon}(q)$, or it acts as a field automorphism on $\mathrm{L}_3(q^2)$ and as an involutory inner automorphism on $\mathrm{L}_3^{-\epsilon}(q)$. It follows that

$$|x^G \cap H| \leqslant \frac{|\mathrm{SL}_3(q^2)|}{|\mathrm{SL}_2(q^2)|} \cdot \frac{|\mathrm{SL}_3^{-\epsilon}(q)|}{|\mathrm{SL}_2(q)|} + \frac{|\mathrm{SL}_3(q^2)|}{|\mathrm{SL}_3(q)|} \cdot (q^2-1)(q^2-\epsilon q+1) < 2q^{15} = a_{12}$$

and thus the total contribution to $\mathcal{Q}(G,H)$ from graph automorphisms is less than $a_{11}^2/b_{11} + a_{12}^2/b_{12} < q^{-6}$.

In conclusion, the above estimates show that

$$\mathcal{Q}(G,H) < (1+4\eta)q^{-1} + (1+\mu)q^{-2} + q^{-4} + q^{-5} + q^{-6}$$

and one checks that this upper bound is always less than 1.

Case 3. H is of type $A_2^{\epsilon}(q^3)$.

To complete the proof, we may assume H is of type $A_2^{\epsilon}(q^3)$, so $H_0 = \mathrm{L}_3^{\epsilon}(q^3).3$ and

$$K = N_L(H_0) = \mathrm{L}_3^{\epsilon}(q^3).(e \times 3) \leqslant \mathrm{PGL}_3^{\epsilon}(q^3).3$$

where $L = \mathrm{Inndiag}(G_0)$ and $e = (3, q-\epsilon)$ as before. Note that H_0 contains an element of order 3, which acts as a field automorphism on $\mathrm{soc}(H_0) = \mathrm{L}_3^{\epsilon}(q^3)$; this corresponds to the element $a \in \bar{H}$, which transitively permutes the three factors of $\bar{H}^0$ (see (6)).

First assume $x \in H$ is a unipotent element of G. If x is a long root element in $\mathrm{L}_3^{\epsilon}(q^3)$ then x is contained in the A_1^3 class of G_0, so $|x^G| > (q-1)q^{39} = b_1$ and

$$|x^G \cap H| = \frac{|\mathrm{SL}_3^{\epsilon}(q^3)|}{q^9(q^3-\epsilon)} < 2q^{12} = a_1.$$

If $p \geqslant 3$ and $x \in \mathrm{L}_3^{\epsilon}(q^3)$ is regular, then [41, Section 4.9] implies that x is contained in the $\bar{G}$-class labelled $A_2^2A_1$ (if $p = 3$) or $D_4(a_1)$ (if $p > 3$). Similarly, if $p = 3$ and $x \in H_0$ is a field automorphism of order 3, then x is in the class A_2^2. In both cases, $|x^G| > \frac{1}{6}q^{48} = b_2$ and there are fewer than

$$i_p(\mathrm{L}_3^\epsilon(q^3)) + 2 \cdot \frac{|\mathrm{SL}_3^\epsilon(q^3)|}{|\mathrm{SL}_3^\epsilon(q)|} < 2q^{18} = a_2$$

such elements in H. Therefore, the unipotent contribution to $\mathcal{Q}(G,H)$ is less than $a_1^2/b_1 + a_2^2/b_2 < q^{-7}$.

Next assume $x \in H$ is a semisimple element of prime order r. If $r = 2$ then $|x^G \cap H| = i_2(\mathrm{L}_3^\epsilon(q^3)) < 2q^{12} = a_3$ and $C_{\bar{G}}(x)^0 = A_5A_1$, so $|x^G| > (q-1)q^{39} = b_3$. Now suppose $r = 3$. By considering the structure of K and the action of $\bar{H}^0$ on the adjoint module for $\bar{G}$ (see (7)), it is easy to see that $C_{\bar{G}}(x)^0 \neq D_5T_1$, so $|x^G| > (q-1)q^{41} = b_4$ and by applying Proposition 2.14(i) we deduce that

$$i_3(K) \leqslant i_3(\mathrm{Aut}(\mathrm{L}_3^\epsilon(q^3))) < 2(q^3+1)q^{15} = a_4.$$

Similarly, if $r \geqslant 5$ then $\dim x^{\bar{G}} \geqslant 50$, so $|x^G| > (q-1)q^{49} = b_5$ and we note that $|\mathrm{L}_3^\epsilon(q^3)| < q^{24} = a_5$. Therefore, the contribution to $\mathcal{Q}(G,H)$ from semisimple elements is less than $\sum_{i=3}^5 a_i^2/b_i < q^{-1} + q^{-2}$.

Finally, let us assume $x \in G$ is a field, graph or graph-field automorphism. Suppose x is a field automorphism of prime order r, so $q = q_0^r$ and we may assume $r \neq 3$ since every element in H of order 3 is contained in K. If $r = 2$ then $\epsilon = +$ and x acts as an involutory field automorphism on $\mathrm{L}_3(q^3)$, so $|x^G| > \frac{1}{6}q^{39} = b_6$ and

$$|x^G \cap H| \leqslant \frac{|\mathrm{SL}_3(q^3)|}{|\mathrm{SL}_3(q^{3/2})|} < 2q^{12} = a_6.$$

If $r \geqslant 5$ then $|x^G| > \frac{1}{6}q^{312/5} = b_7$ and we note that $|H| < 6\log_2 q.q^{24} = a_7$. Next assume $x \in G$ is an involutory graph-field automorphism, so $\epsilon = +$ and $q = q_0^2$. Here $|x^G| > \frac{1}{6}q^{39} = b_8$ and x induces a graph-field automorphism on $\mathrm{L}_3(q^3)$, so $|x^G \cap H| < 2q^{12} = a_8$. Finally, if x is an involutory graph automorphism then x acts as a graph automorphism on $\mathrm{L}_3(q^3)$, so

$$|x^G \cap H| \leqslant \frac{|\mathrm{SL}_3^\epsilon(q^3)|}{|\mathrm{SL}_2(q^3)|} < 2q^{15} = a_9.$$

In terms of the ambient algebraic groups, x must act as a simultaneous graph automorphism on the three factors of $\bar{H}^0$ (since $\mathrm{L}_3^\epsilon(q^3)$ is not normalized by a graph automorphism that also induces a nontrivial permutation of the factors of $\bar{H}^0$). Therefore, $C_{\bar{G}}(x) \neq F_4$ and thus $|x^G| > \frac{1}{6}q^{42} = b_9$. It follows that the contribution to $\mathcal{Q}(G,H)$ from field, graph and graph-field automorphisms is less than $\sum_{i=6}^9 a_i^2/b_i < q^{-5}$.

To conclude, it follows that

$$\mathcal{Q}(G,H) < \sum_{i=1}^{9} a_i^2/b_i < q^{-1} + q^{-2} + q^{-5} + q^{-7}$$

and the proof of the lemma is complete. □

Lemma 4.13. *Suppose $G_0 = E_6^\epsilon(q)$ and H is of type*

$$A_1(q)A_5^\epsilon(q),\ D_4(q).(q-\epsilon)^2,\ {}^3D_4(q).(q^2+\epsilon q+1),\ D_5^\epsilon(q).(q-\epsilon).$$

Then G is not extremely primitive.

Proof. Write $H_0 = H \cap G_0$ and $L = \mathrm{Inndiag}(G_0)$. The structure of $N_L(H_0)$ is presented in [50, Table 5.1].

First assume H is of type $A_1(q)A_5^\epsilon(q)$, so $H_0 = d.(\mathrm{L}_2(q) \times \mathrm{L}_6^\epsilon(q)).d$, where $d = (2, q-1)$. If q is odd then $Z(H) \neq 1$, whereas $\mathrm{soc}(H)$ is not a direct product of isomorphic simple groups if q is even. Therefore, G is not extremely primitive by Lemma 2.1. Similar reasoning handles the cases where H is of type $D_4(q).(q-\epsilon)^2$ or ${}^3D_4(q).(q^2+\epsilon q+1)$.

Finally, let us assume H is of type $D_5^\epsilon(q).(q-\epsilon)$, so
$$H_0 = h.(\mathrm{P}\Omega_{10}^\epsilon(q) \times (q-\epsilon)/eh).h$$
with $e = (3, q-\epsilon)$ and $h = (4, q-\epsilon)$. If q is odd then $F(H) = Z_h \neq 1$ and G is not extremely primitive by Lemma 2.1(iv). So for the remainder of the proof we may assume q is even and thus $H_0 = \Omega_{10}^\epsilon(q) \times (q-\epsilon)/e$. If $(q,\epsilon) \neq (2,-), (2,+), (4,+)$ then the structure of $\mathrm{soc}(H)$ is incompatible with extreme primitivity, so it remains to consider the three special cases.

First assume $(q,\epsilon) = (2,-)$. If G contains $\mathrm{Inndiag}(G_0)$ then $\mathrm{soc}(H)$ is incompatible, so assume otherwise, in which case $(G,H) = ({}^2E_6(2), \Omega_{10}^-(2))$ or $({}^2E_6(2).2, \mathrm{O}_{10}^-(2))$. Let r be the rank of G on $\Omega = G/H$. In both cases, the character tables of G and H are available in the GAP Character Table Library [8]. For $G = G_0$, the fusion map from H-classes to G-classes is also stored and this allows us to compute the number of fixed points of each $x \in H$ on $\Omega = G/H$. In turn, we deduce that $r = 13$ via the Orbit Counting Lemma. On the other hand, if $G = G_0.2$ then there are two possible fusion maps and they both give $r = 12$. With the aid of Magma, we can determine the indices $n_1, \ldots, n_k$ of a set of representatives of the H-classes of core-free maximal subgroups of H and then it is routine to rule out extreme primitivity via Lemma 2.3. For example, if $G = {}^2E_6(2).2$ and $H = \mathrm{O}_{10}^-(2)$ then $n_i \leqslant |H : \mathrm{M}_{12}.2| = 263208960$ and one checks that $1 + 11.263208960 < |G : H|$. We refer the reader to [21, Lemma 2.4] for further details on the GAP and Magma computations used in this case.

Next assume $(q,\epsilon) = (2,+)$, so $G = \mathrm{Aut}(E_6(2)) = E_6(2).2$ and $H = \mathrm{O}_{10}^+(2)$ (as noted in [50, Table 5.1], if $\epsilon = +$ then H is maximal only if G contains a graph automorphism). We can handle this case in a similar fashion. Working with the character tables of G_0 and $H_0 = \Omega_{10}^+(2)$ in [8], we calculate that G_0 has rank 35 and thus G has rank at most 35 (the character table of G is not available in [8] and we have not computed the exact rank of G). Let M be a core-free maximal subgroup of G. Then using Magma, we find that $|H : M|$ is maximal when $M = \mathrm{O}_6^+(2) \times \mathrm{O}_4^+(2)$, giving $|H : M| \leqslant 16189440$. But $1 + 34.16189440 < |G : H|$, so G is not extremely primitive by Lemma 2.3.

Finally, let us turn to the case $(q,\epsilon) = (4,+)$. Here we may assume G contains a graph automorphism (so that H is maximal) but does not contain $\mathrm{Inndiag}(G_0)$ (so that the structure of $\mathrm{soc}(H)$ is compatible with extreme primitivity). Therefore, $(G,H) = (E_6(4).2, \mathrm{O}_{10}^+(4))$ or $(E_6(4).2^2, \mathrm{O}_{10}^+(4).2)$.

Fix a set of simple roots $\alpha_1, \ldots, \alpha_6$ for $\bar{G}$ and let $X_\alpha = \{x_\alpha(c) : c \in \mathbb{F}_4\}$ be the root subgroup of G_0 corresponding to the root α. By replacing H by a suitable conjugate, we may assume that
$$H_0 = \langle X_{\pm\alpha_1}, X_{\pm\alpha_2}, X_{\pm\alpha_3}, X_{\pm\alpha_4}, X_{\pm\alpha_5} \rangle = \Omega_{10}^+(4).$$
Let $g \in G_0 \setminus H_0$ be the involution $x_{\alpha_6}(1)$. By applying Chevalley's commutator relations (see [24, Theorem 5.2.2]), we see that g centralizes X_α for all $\alpha \in \{\pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_4, -\alpha_5\}$ and thus g centralizes the subgroup $K = 4^{10}{:}\mathrm{SL}_5(4) < H_0$. By inspecting [6, Table 8.66], we see that the only maximal subgroup of $H_0.2 = \mathrm{O}_{10}^+(4)$ containing K is H_0 itself (note that the maximal parabolic subgroup $4^{10}{:}\mathrm{GL}_5(4)$ of H_0 does not extend to a maximal subgroup of $H_0.2$). Similarly, the only maximal subgroups of $\mathrm{O}_{10}^+(4).2 = H_0.2^2$ containing K are the three index-two subgroups of the form $H_0.2$.

Suppose $H \cap H^g$ is a maximal subgroup of H. Then since g normalizes K, we have $K \leqslant H \cap H^g < H$ and thus $H \cap H^g = H_0$ (if $G = G_0.2$) or $H_0.2$ (if $G = G_0.2^2$). Since g is an involution, it normalizes $H \cap H^g$ and so it must also normalize the characteristic subgroup H_0. But from the explicit description of H_0 above in terms of root subgroups, it is easy to see that g does not normalize H_0 and so we have reached a contradiction. We conclude that $H \cap H^g$ is non-maximal in H and the proof is complete. □

Remark 4.14. In the proof of the previous lemma, we applied Lemma 2.3 to handle the case $(G,H) = (E_6(2).2, \mathrm{O}_{10}^+(2))$. It is worth noting that this case can also be treated by making minor modifications to the argument we used for $(q,\epsilon) = (4,+)$.

4.4. $G_0 = F_4(q)$.

Lemma 4.15. *If $G_0 = F_4(q)$ and H is the normalizer of a maximal torus, then $b(G,H) = 2$.*

Proof. Let $W(\bar{G}) = \mathrm{O}_4^+(3)$ be the Weyl group of $\bar{G}$ and note that q is even and G contains graph automorphisms (see [50, Table 5.2]). It will be useful to note that if $x \in G$ has prime order, and x is not a long or short root element, then $|x^G| > q^{22}$ (minimal if x is an involution in the class labelled $(\tilde{A}_1)_2$).

If $x \in G$ is a long (or short) root element, then $|x^G| > q^{16} = b_1$ and Corollary 2.13 gives $|x^G \cap H| \leqslant 24(q+1)^4 = a_1$. As noted above, for all other nontrivial elements we have $|x^G| > q^{22}$ and we observe that

$$|H| \leqslant (q+1)^4.|W(\bar{G})|.2\log_2 q = a_2.$$

This yields $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2$, which is less than 1 for $q \geqslant 8$ (it is also less than q^{-1} for $q \geqslant 8$). Therefore, to complete the proof we may assume $q \in \{2,4\}$.

Suppose $q = 2$, so $G = F_4(2).2$ and $H = 7^2{:}(3{\times}2.\mathrm{Sym}_4)$. Here $H_0 = H \cap G_0 = 7^2{:}(3{\times}\mathrm{SL}_2(3))$ has a unique involution, so by [23, Corollary 4.4] we see that there are precisely 49 involutions in H_0 and they are all contained in the largest G_0-class of involutions (this is the class labelled $A_1\tilde{A}_1$). Therefore, $|x^G| > 2^{26} = b_1$ for all $x \in H$ of prime order and thus $\mathcal{Q}(G,H) < a_1^2/b_1 < 1$, where $a_1 = |H|$. In particular, $b(G,H) = 2$.

Now assume $q = 4$, so $G = F_4(4).2$ or $F_4(4).4$. Up to conjugacy, there are 5 possibilities for H_0 and we will inspect each case in turn. If $H_0 = (4^2 \pm 4 + 1)^2{:}(3 \times \mathrm{SL}_2(3))$ then by arguing as above we see that $|x^G| > 4^{26} = b_1$ and the result follows since $\mathcal{Q}(G,H) < a_1^2/b_1 < 1$ with $a_1 = 4|H_0|$. The case $H_0 = 241{:}12$ is entirely similar. Next assume $H_0 = 17^2{:}(4 \circ \mathrm{GL}_2(3))$. If $x \in G$ is not a root element, then $|x^G| > 4^{22} = b_1$ and the contribution from these elements is less than a_1^2/b_1, where $a_1 = 4|H_0|$. On the other hand, if x is a root element, then $|x^G| > 4^{16} = b_2$ and we have the trivial bound $|x^G \cap H| \leqslant |H_0| = a_2$. This gives $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2 < 1$ as required.

Finally, let us assume $H_0 = 5^4{:}W(\bar{G})$. As in the previous case, the contribution from non-long root elements is less than a_1^2/b_1, where $a_1 = 4|H_0|$ and $b_1 = 4^{22}$. Now assume x is a root element, so $|x^G| > 4^{16} = b_2$. Here we can use Magma to construct H_0 as a subgroup of $F_4(16)$ (see [21, Proposition 2.2]) and this allows us to compute the Jordan form of each involution in H_0 on the adjoint module for $\bar{G}$. By inspecting [40, Table 4], we deduce that $|x^G \cap H| = 120 = a_2$. Therefore, $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2 < 1$ and thus $b(G,H) = 2$. □

Lemma 4.16. *If $G_0 = F_4(q)$ and H is of type $A_2^{\epsilon}(q)^2$, then $b(G,H) = 2$.*

Proof. Here $\bar{H}^0 = A_2\tilde{A}_2$, where the first factor is generated by long root subgroups and the second by short root subgroups. Set $e = (3, q-\epsilon)$ and observe that

$$H_0 = (\mathrm{SL}_3^{\epsilon}(q) \circ \mathrm{SL}_3^{\epsilon}(q)).e.2 = e.\mathrm{L}_3^{\epsilon}(q)^2.e.2,$$

where the outer involution acts as a graph automorphism on both copies of $\mathrm{SL}_3^{\epsilon}(q)$. Let $V = \mathcal{L}(\bar{G})$ be the adjoint module for $\bar{G}$ and note that

$$V{\downarrow}A_2\tilde{A}_2 = \mathcal{L}(A_2\tilde{A}_2) \oplus (W \otimes S^2(W^*)) \oplus (W^* \otimes S^2(W)), \tag{8}$$

where W is the natural module for A_2, W its dual and $S^2(W)$ is the symmetric-square of W (see [75, Table 2]).

Let $x \in H$ be an element of prime order r. To begin with, let us assume $r = p = 2$. Note that if $x \in \bar{H}^0$ then the $\bar{G}$-class and $\bar{H}^0$-class of x have the same label (see [41, Section 4.7]). If x acts as a graph automorphism on the two $\mathrm{SL}_3^{\epsilon}(q)$ factors, then x has Jordan form (J_2^6, J_1^4) on $\mathcal{L}(A_2\tilde{A}_2)$ and it interchanges the two 18-dimensional summands in (8). Therefore, x has Jordan form (J_2^{24}, J_1^4) on V and by inspecting [40, Table 4] we deduce that x is in the $\bar{G}$-class labelled $A_1\tilde{A}_1$. Putting this together, it follows that if x is a long or short root element, then

$|x^G| > q^{16} = b_1$ and there are $2 \cdot i_2(\mathrm{L}_3^{\epsilon}(q)) < 3q^4 = a_1$ such elements in H. On the other hand, if x is in the class labelled $A_1\tilde{A}_1$ then $|x^G| > q^{28} = b_2$ and H contains

$$i_2(\mathrm{L}_3^{\epsilon}(q))^2 + \left(\frac{|\mathrm{SL}_3^{\epsilon}(q)|}{|\mathrm{SL}_2(q)|}\right)^2 = (q+\epsilon)^2(q^3-\epsilon)^2 + q^4(q^3-\epsilon)^2 < 2q^{10} = a_2$$

such elements (no involution in H is contained in the class labelled $(\tilde{A}_1)_2$).

Next assume $r = p > 2$. If $\dim x^{\bar{G}} \geqslant 30$ then $|x^G| > \frac{1}{4}q^{30} = b_3$ and we note that H_0 contains $q^{12} = a_3$ unipotent elements (see Proposition 2.14(ii)). Now assume $\dim x^{\bar{G}} < 30$, so x is contained in one of the G_0-classes labelled A_1, $\tilde{A}_1$ or $A_1\tilde{A}_1$. As in the previous paragraph, the contribution to $\mathcal{Q}(G,H)$ from these elements is less than $a_1^2/b_1 + a_2^2/b_2$.

Now let us turn to semisimple elements. Suppose $x \in H$ is a semisimple element of prime order r. First assume $r = 2$, so $C_{\bar{G}}(x) = A_1C_3$ or B_4. If x acts as a graph automorphism on both $\mathrm{SL}_3^{\epsilon}(q)$ factors, then x has Jordan form $(-I_{10}, I_6)$ on $\mathcal{L}(A_2\tilde{A}_2)$ and it swaps the two 18-dimensional summands in (8), so $\dim C_V(x) = 24$ and thus $C_{\bar{G}}(x) = A_1C_3$. Now assume $x = (x_1, x_2) \in \bar{H}^0$. If either $x_2 = 1$, or x_1 and x_2 are both nontrivial, then $C_{\bar{G}}(x) = A_1C_3$. On the other hand, if $x_1 = 1$ then $C_{\bar{G}}(x) = B_4$. We conclude that if x is a B_4-type involution, then $|x^G| > q^{16} = b_4$ and $|x^G \cap H| = i_2(\mathrm{L}_3^{\epsilon}(q)) < 2q^4 = a_4$. Similarly, if $C_{\bar{G}}(x) = A_1C_3$ then $|x^G| > q^{28} = b_5$ and

$$|x^G \cap H| \leqslant i_2(\mathrm{L}_3^{\epsilon}(q)) + i_2(\mathrm{L}_3^{\epsilon}(q))^2 + \left(\frac{|\mathrm{SL}_3^{\epsilon}(q)|}{|\mathrm{SL}_2(q)|}\right)^2 < 2q^{10} = a_5.$$

Now assume $r \geqslant 3$. If $\dim x^{\bar{G}} \geqslant 36$ then $|x^G| > \frac{1}{2}q^{36} = b_6$ and we note that there are fewer than $q^{16} = a_6$ such elements in H. So we may assume $\dim x^{\bar{G}} < 36$, in which case $C_{\bar{G}}(x) = B_3T_1$ or C_3T_1 and $|x^G| > (q-1)q^{29} = b_7 = b_8$. Let us also observe that r divides $|Z(C_{G_0}(x))| = q \pm 1$.

Suppose $r = 3$ and let Z be the normal subgroup of H_0 of order e. Then by applying Proposition 2.14(i) we deduce that

$$i_3(H_0) < |Z| \cdot (1 + i_3(\mathrm{PGL}_3^{\epsilon}(q)^2)) \leqslant 12(q+1)^2q^{10}.$$

In fact, one checks that $i_3(\mathrm{PGL}_3^{\epsilon}(2)^2) \leqslant 6560$ and thus $i_3(H_0) \leqslant 19683$ when $q = 2$. It follows that the contribution to $\mathcal{Q}(G,H)$ from elements $x \in G$ of order 3 with $\dim x^{\bar{G}} < 36$ is less than a_7^2/b_7, where $a_7 = 12(q+1)^2q^{10}$ if $q \geqslant 4$ and $a_7 = 19683$ if $q = 2$. Now assume $r \geqslant 5$ (and $C_{\bar{G}}(x) = B_3T_1$ or C_3T_1). An easy calculation using (8) shows that an element $x = (x_1, x_2) \in \bar{H}^0$ of order r has the appropriate centralizer in $\bar{G}$ if and only if x_1 or x_2 is trivial. Therefore, there are fewer than $2|\mathrm{L}_3^{\epsilon}(q)| < 2q^8 = a_8$ such elements in H.

To complete the proof, let us assume $p = 2$ and x is an involutory graph automorphism, so $|x^G| > q^{26} = b_9$. In terms of the ambient algebraic groups, x interchanges the two A_2 factors of $\bar{H}^0$, so there are two classes of involutions in $\bar{H}.2 \setminus \bar{H}$, represented by x and yx, where $y \in \bar{H}$ acts as a simultaneous graph automorphism on both factors. Now $C_{\bar{H}^0}(x)$ and $C_{\bar{H}^0}(yx)$ are both isomorphic to A_2 and we deduce that $|x^G \cap H| \leqslant 2|\mathrm{SL}_3^{\epsilon}(q)| < 2q^8 = a_9$.

We conclude that $\mathcal{Q}(G,H) < \sum_{i=1}^{9} a_i^2/b_i$ and it is routine to verify that this upper bound is less than 1 for all $q \geqslant 2$ (and it tends to zero as q tends to infinity). □

Remark 4.17. It is worth noting that the cases with $q = 2$ in Lemma 4.16 can also be handled using Magma. We refer the reader to [21, Lemma 2.5] for the details.

Lemma 4.18. *If $G_0 = F_4(q)$ and H is of type $C_2(q)^2$, then $b(G,H) = 2$.*

Proof. Here $p = 2$, G contains a graph automorphism and

$$H_0 = \mathrm{Sp}_4(q) \wr \mathrm{Sym}_2 < \mathrm{Sp}_8(q) < G_0.$$

	1	b_1	a_2	c_2
1	1	$\tilde{A}_1$	$\tilde{A}_1$	$(\tilde{A}_1)_2$
b_1	A_1	$(\tilde{A}_1)_2$	$A_1\tilde{A}_1$	$A_1\tilde{A}_1$
a_2	A_1	$A_1\tilde{A}_1$	$(\tilde{A}_1)_2$	$A_1\tilde{A}_1$
c_2	$(\tilde{A}_1)_2$	$A_1\tilde{A}_1$	$A_1\tilde{A}_1$	$A_1\tilde{A}_1$

TABLE 6. The involutions in $\mathrm{Sp}_4(q)^2 < F_4(q)$, $p = 2$

In particular, H_0 is a non-maximal subgroup of G_0. Set $\bar{H}^0 = B_2^2 < B_4 < \bar{G}$ and let W be the 4-dimensional natural module for B_2, with $\rho : B_2 \to \mathrm{GL}(W)$ the corresponding representation. Let τ be the standard graph automorphism of B_2 and let W^τ be the B_2-module W afforded by the representation $\rho\tau$. Then by inspecting [75, Table 2] we see that the restriction of the adjoint module $V = \mathcal{L}(\bar{G})$ to $\bar{H}^0$ has the form

$$V{\downarrow}B_2^2 = \mathcal{L}(B_2^2) \oplus (W \otimes W) \oplus (W^\tau \otimes W^\tau) \tag{9}$$

The case $q = 2$ can be checked using MAGMA (see [21, Lemma 2.5]) and so we may assume $q \geqslant 4$. Our goal is to show that $\mathcal{Q}(G,H) < 1$.

Let $x \in H$ be an element of prime order r. First assume $x \in H_0$ and $r = 2$. Recall that $\mathrm{Sp}_4(q)$ has three classes of involutions, labelled b_1, a_2 and c_2 in [1]. It is straightforward to determine the $\mathrm{Sp}_8(q)$-class of each involution in $\mathrm{Sp}_4(q)^2$ and then the G_0-class can be read off from [43, p.373] (also see [41, Section 4.4]). For example, if $x = (x_1, x_2) \in \mathrm{Sp}_4(q)^2$, where x_1 is of type a_2 and x_2 is of type b_1, then x is a b_3 involution in $\mathrm{Sp}_8(q)$ and is therefore contained in the G_0-class labelled $A_1\tilde{A}_1$. For the reader's convenience, the G_0-class of each involution in $\mathrm{Sp}_4(q)^2$ is recorded in Table 6. Similarly, if $x \in H_0$ interchanges the two copies of $\mathrm{Sp}_4(q)$ then x embeds in $\mathrm{Sp}_8(q)$ as an a_4-type involution and thus x is in the G_0-class labelled $(\tilde{A}_1)_2$.

To summarise, if $x \in G_0$ is a long or short root element, then $|x^G| > q^{16} = b_1$ and there are precisely $4(q^4-1) = a_1$ such elements in H. Similarly, if x is in the G_0-class $(\tilde{A}_1)_2$ then $|x^G| > q^{22} = b_2$ and we have

$$|x^G \cap H| = 2|c_2^{\mathrm{Sp}_4(q)}| + |b_1^{\mathrm{Sp}_4(q)}|^2 + |\mathrm{Sp}_4(q)| < q^6(q^4+2q^2+2) = a_2.$$

Finally, if x is in the $A_1\tilde{A}_1$ class then $|x^G| > q^{28} = b_3$ and $|x^G \cap H| < q^8(q^4+4q^2+2) = a_3$.

Next assume $x \in H_0$ has order $r \geqslant 3$, so $|x^G| > (q-1)q^{29} = b_4$. Since

$$i_3(\mathrm{Sp}_4(q)) \leqslant 2 \cdot \frac{|\mathrm{Sp}_4(q)|}{|\mathrm{GL}_2(q)|} = 2q^3(q+1)(q^2+1)$$

it follows that

$$i_3(H_0) = i_3(\mathrm{Sp}_4(q)^2) \leqslant (2q^3(q+1)(q^2+1)+1)^2 - 1 < 5q^{12} = a_4$$

and thus the contribution to $\mathcal{Q}(G,H)$ from semisimple elements of order 3 is less than a_4^2/b_4.

Now assume $x \in \mathrm{Sp}_4(q)^2$ has prime order $r \geqslant 5$. If $\dim x^{\bar{G}} \geqslant 42$ then $|x^G| > \frac{1}{2}q^{42} = b_5$ and we note that there are fewer than $|\mathrm{Sp}_4(q)|^2 < q^{20} = a_5$ semisimple elements in H. Therefore, to complete the analysis for semisimple elements, we may assume $r \geqslant 5$ and $\dim x^{\bar{G}} < 42$, in which case

$$C_{\bar{G}}(x) = B_2T_2,\ A_2A_1T_1,\ B_3T_1 \text{ or } C_3T_1. \tag{10}$$

In order to work with the decomposition in (9), it will be useful to observe that the action of τ on semisimple elements of B_2 is as follows (up to conjugacy, and with respect to the natural module):

$$\tau : \mathrm{diag}(\lambda, \mu, \mu^{-1}, \lambda^{-1}) \mapsto \mathrm{diag}(\lambda\mu, \lambda^{-1}\mu, \lambda\mu^{-1}, \lambda^{-1}\mu^{-1}). \tag{11}$$

Let $i \geqslant 1$ be minimal such that r divides $q^i - 1$, so $i \in \{1,2,4\}$. First let us consider the contribution to $\mathcal{Q}(G,H)$ from the elements with $i = 4$ and $\dim x^{\bar{G}} < 42$. Note that $C_{\bar{G}}(x) = B_2T_2$ is the only option since r divides q^2+1 and thus $\dim Z(C_{\bar{G}}(x)) \geqslant 2$. In particular, $|x^G| > \frac{1}{2}q^{40} = b_6$. Write $x = (x_1,x_2) \in \mathrm{Sp}_4(q)^2$. Using (11) and the decomposition in (9), we deduce that $\dim C_V(x) = 12$ if and only if x_1 and x_2 are $\mathrm{Sp}_4(q)$-conjugate, or if $x_j = 1$ for some j. If π denotes the set of primes $s \geqslant 5$ dividing q^2+1, then $|\pi| < 2\log_2 q$ and we deduce that there are at most

$$2|\mathrm{Sp}_4(q)| + \sum_{r\in\pi} \frac{1}{4}(r-1)\cdot\left(\frac{|\mathrm{Sp}_4(q)|}{q^2+1}\right)^2 < 2q^{10} + \frac{1}{2}\log_2 q.q^{10}(q^2-1)^4 = a_6$$

such elements in H. Therefore, the contribution to $\mathcal{Q}(G,H)$ from these elements is less than a_6^2/b_6.

Now assume $i \in \{1,2\}$ and $\dim x^{\bar{G}} < 42$. As above, write $x = (x_1,x_2) \in \mathrm{Sp}_4(q)^2$ and let us assume $C_{\bar{G}}(x) = B_3T_1$ or C_3T_1. By considering the various possibilities for x_1 and x_2 we deduce that $\dim C_V(x) = 22$ if and only if x_1 is non-regular and x_2 is either trivial or conjugate to x_1 (or vice versa). Therefore, if α denotes the number of such elements in H, then

$$\alpha \leqslant \sum_{r\in\pi}\frac{1}{2}(r-1)\cdot\left(\left(2\frac{|\mathrm{Sp}_4(q)|}{|\mathrm{GL}_2(q)|}\right)^2 + 4\frac{|\mathrm{Sp}_4(q)|}{|\mathrm{GL}_2(q)|}\right),$$

where π is the set of primes $s \geqslant 5$ dividing q^2-1. If $q = 4$, then $\pi = \{5\}$ and the above bound yields $\alpha \leqslant 236792320$. Now assume $q \geqslant 8$. Since $r \leqslant q+1$, it follows that

$$\alpha < 2\log_2 q.q(4q^6(q+1)^2(q^2+1)^2 + 4q^3(q+1)(q^2+1)). \tag{12}$$

Now $|x^G| > (q-1)q^{29} = b_7$ and so the contribution to $\mathcal{Q}(G,H)$ from the semisimple elements $x \in H$ with $r \geqslant 5$ and $C_{\bar{G}}(x) = B_3T_1$ or C_3T_1 is less than a_7^2/b_7, where $a_7 = 236792320$ if $q = 4$ and a_7 is the upper bound on α in (12) if $q \geqslant 8$.

Finally, let us assume $i \in \{1,2\}$ and $C_{\bar{G}}(x) = B_2T_2$ or $A_2A_1T_1$, so $|x^G| > \frac{1}{2}q^{40} = b_8$. First observe that $\mathrm{Sp}_4(q)$ contains at most

$$\frac{1}{2}(r-1)\cdot 2\frac{|\mathrm{Sp}_4(q)|}{|\mathrm{GL}_2(q)|} < 2(r-1)q^6$$

non-regular semisimple elements of order r. Therefore, there are fewer than $4(r-1)^2q^{12}$ elements in $\mathrm{Sp}_4(q)^2$ of order r of the form (x_1,x_2), where neither x_1 nor x_2 is regular. Since r divides q^2-1, there are less than $8\log_2 q.q^{14}$ semisimple elements of this form in H.

Now assume $x = (x_1,x_2) \in \mathrm{Sp}_4(q)^2$ has order r and x_1 is regular. By considering (9), we deduce that $\dim C_{\bar{G}}(x) = 12$ if and only if one of the following holds, where $\sim$ denotes B_2-conjugacy:

(a) x_2 is either trivial or conjugate to x_1;
(b) $x_1 \sim \mathrm{diag}(\lambda,\lambda^2,\lambda^{-2},\lambda^{-1})$ and $x_2 \sim \mathrm{diag}(\lambda,1,1,\lambda^{-1})$ or $\mathrm{diag}(\lambda,\lambda,\lambda^{-1},\lambda^{-1})$ for some primitive rth root of unity λ (with $r = 5$ in the latter case).

Let α and β the total number of elements in H satisfying the conditions in (a) and (b), respectively (allowing for the conditions on x_1 and x_2 to be interchanged) and let π be the set of primes $s \geqslant 5$ dividing q^2-1. Then

$$\alpha \leqslant 2|\mathrm{Sp}_4(q)| + \sum_{r\in\pi}\binom{\frac{1}{2}(r-1)}{2}\left(\frac{|\mathrm{Sp}_4(q)|}{(q-1)^2}\right)^2 < 2q^{10} + 2\log_2 q.\frac{1}{8}q^2.(2q^8)^2 = 2q^{10} + \log_2 q.q^{18}$$

and

$$\beta \leqslant \sum_{r\in\pi}\frac{1}{2}(r-1)\cdot 2\cdot\frac{|\mathrm{Sp}_4(q)|}{(q-1)^2}\cdot 2\frac{|\mathrm{Sp}_4(q)|}{|\mathrm{GL}_2(q)|} < 16\log_2 q.q^{15}.$$

By combining the above estimates, we conclude that H contains fewer than

$$a_8 = 2q^{10} + \log_2 q.q^{14}(q^4 + 16q + 8)$$

semisimple elements of prime order $r \geqslant 5$ with $i \in \{1,2\}$ and $C_{\bar{G}}(x) = B_2T_2$ or $A_2A_1T_1$.

To complete the proof of the lemma, we need to estimate the contribution to $\mathcal{Q}(G,H)$ from field and graph automorphisms. First assume $x \in H$ is a field automorphism of order r, so $q = q_0^r$ and $|x^G| > \frac{1}{2}q^{52(1-r^{-1})}$. If r is odd then x acts on $\mathrm{Sp}_4(q)^2$ as a field automorphism on both factors, so H contains

$$(r-1)\cdot\left(\frac{|\mathrm{Sp}_4(q)|}{|\mathrm{Sp}_4(q^{1/r})|}\right)^2 < 4(r-1)q^{20\left(1-r^{-1}\right)}$$

such elements. Since $|H| < 4\log_2 q.q^{20}$, we see that the contribution from odd order field automorphisms is less than $\sum_{i=9}^{11} a_i^2/b_i$, where

$$a_9 = 8q^{40/3},\ b_9 = \frac{1}{2}q^{104/3},\ a_{10} = 16q^{16},\ b_{10} = \frac{1}{2}q^{208/5},\ a_{11} = 4\log_2 q.q^{20},\ b_{11} = \frac{1}{2}q^{312/7}.$$

Now assume $r = 2$, so $|x^G| > q^{26} = b_{12}$ and there are two H_0-classes of involutions in the coset H_0x. It follows that

$$|x^G \cap H| = \left(\frac{|\mathrm{Sp}_4(q)|}{|\mathrm{Sp}_4(q^{1/2})|}\right)^2 + |\mathrm{Sp}_4(q)| < 5q^{10} = a_{12}.$$

Finally, suppose x is an involutory graph automorphism of G_0, so $|x^G| > q^{26} = b_{13}$ and $q = 2^{2m+1}$ with $m \geqslant 1$. Now x induces a graph automorphism on both $\mathrm{Sp}_4(q)$ factors and we note that there are two classes of involutions in H_0x. In particular,

$$|x^G \cap H| = \left(\frac{|\mathrm{Sp}_4(q)|}{|{}^2B_2(q)|}\right)^2 + |\mathrm{Sp}_4(q)| < 5q^{10} = a_{13}.$$

Bringing the above estimates together, we conclude that

$$\mathcal{Q}(G,H) < \sum_{i=1}^{13} a_i^2/b_i < 1$$

for all $q \geqslant 4$ (and the upper bound tends to 0 as q tends to infinity). The result follows. □

Lemma 4.19. *If $G_0 = F_4(q)$ and H is of type $C_2(q^2)$, then $b(G,H) = 2$.*

Proof. Here $p = 2$, $H_0 = \mathrm{Sp}_4(q^2).2 < \mathrm{Sp}_8(q) < G_0$ and G contains graph automorphisms. Note that

$$(H_0)' = \mathrm{Sp}_4(q^2) = \{(x, x^{\varphi}) \,:\, x \in \mathrm{Sp}_4(q^2)\} < \bar{H}^0 = B_2^2, \qquad (13)$$

where φ is an involutory field automorphism of $\mathrm{Sp}_4(q^2)$. In particular, the outer involution in H_0, which acts as a field automorphism on $\mathrm{Sp}_4(q^2)$, corresponds to an involution in $\bar{H}$ that interchanges the two B_2 factors of $\bar{H}^0$. The case $q = 2$ can be checked using Magma (see [21, Lemma 2.5] for the details of this computation) and so we may assume $q \geqslant 4$.

Let $x \in H$ be an element of prime order r. First assume $r = 2$ and $x \in H_0$. There are three classes of involutions in $\mathrm{Sp}_4(q^2)$ and it is easy to determine the corresponding class in $\mathrm{Sp}_8(q)$, which in turn allows us to identify the G_0-class of x (see [43, p.373], for example). Using the notation from [1], we find that the b_1-involutions in $\mathrm{Sp}_4(q^2)$ are contained in the $\mathrm{Sp}_8(q)$-class labelled c_2, which is contained in the G_0-class $(\tilde{A}_1)_2$. Similarly, the a_2-type involutions in $\mathrm{Sp}_4(q^2)$ are also in the $(\tilde{A}_1)_2$ class, while the c_2 involutions are in the class $A_1\tilde{A}_1$. In addition, the involutory field automorphisms in H_0 embed in $\mathrm{Sp}_8(q)$ as a_4-type involutions, so they are contained in the G_0-class $(\tilde{A}_1)_2$.

In conclusion, if x is an involution in the class $(\tilde{A}_1)_2$, then $|x^G| > q^{22} = b_1$ and

$$|x^G \cap H| = 2(q^8-1) + \frac{|\mathrm{Sp}_4(q^2)|}{|\mathrm{Sp}_4(q)|} < 2q^{10} = a_1.$$

And for x in the class $A_1\tilde{A}_1$ we have $|x^G \cap H| < q^{12} = a_2$ and $|x^G| > q^{28} = b_2$.

Next assume $x \in G_0$ has order $r \geqslant 3$, so $|x^G| > (q-1)q^{29} = b_3$. Since

$$i_3(H_0) \leqslant 2\frac{|\mathrm{Sp}_4(q^2)|}{|\mathrm{GL}_2(q^2)|} = 2q^6(q^2+1)(q^4+1) = a_3,$$

it follows that the contribution to $\mathcal{Q}(G,H)$ from semisimple elements of order 3 is less than a_3^2/b_3. Now assume $r \geqslant 5$. If $\dim x^{\bar{G}} \geqslant 42$ then $|x^G| > \frac{1}{2}q^{42} = b_4$ and there are fewer than $q^{20} = a_4$ semisimple elements in H, so the contribution from these elements is less than $a_4^2/b_4 = 2q^{-2}$.

To complete the analysis of semisimple elements, we may assume $r \geqslant 5$ and $\dim x^{\bar{G}} < 42$, so the possibilities for $C_{\bar{G}}(x)$ are listed in (10). In terms of the embedding of $\mathrm{Sp}_4(q^2)$ in $\bar{H}^0$ (see (13)), we may write $x = (x_1, x_2) \in \bar{H}^0$, where up to conjugacy we have

$$x_1 = \mathrm{diag}(\lambda, \mu, \mu^{-1}, \lambda^{-1}), \;\; x_2 = x_1^{\varphi} = \mathrm{diag}(\lambda^q, \mu^q, \mu^{-q}, \lambda^{-q}).$$

By expressing x in this form, we can use (9) to determine the dimension of the 1-eigenspace of x on the adjoint module $V = \mathcal{L}(\bar{G})$. Note that r divides q^8-1.

Suppose $C_{\bar{G}}(x) = B_3T_1$ or C_3T_1. By considering (9) and the various possibilities for x_1 and x_2, we deduce that x_1 and x_2 must be $\mathrm{Sp}_4(q^2)$-conjugate and non-regular, so r divides q^2-1. Let π be the set of primes $s \geqslant 5$ dividing q^2-1 and note that $|x^G| > (q-1)q^{29} = b_5$. Then H contains at most

$$\sum_{r \in \pi} \frac{1}{2}(r-1) \cdot 2\frac{|\mathrm{Sp}_4(q^2)|}{|\mathrm{GL}_2(q^2)|} < f(q).q^6(q^2+1)(q^4+1) = a_5$$

such elements, where $f(q) = 2\log_2 q.q^2$ if $q \geqslant 8$ and $f(4) = 4$ (note that $\pi = \{5\}$ if $q = 4$).

Now assume $C_{\bar{G}}(x) = B_2T_2$ or $A_2A_1T_1$, so $|x^G| > \frac{1}{2}q^{40} = b_6$. Suppose $x \in \mathrm{Sp}_4(q^2)$ is regular. If r divides q^4+1 then by considering the possibilities for $|Z(C_{G_0}(x))|$ we deduce that $C_{\bar{G}}(x) = T_4$, which is a contradiction. Now assume r divides q^4-1. If x_1 and x_2 are not $\mathrm{Sp}_4(q^2)$-conjugate, then by considering the 1-eigenspace of x on V we deduce that $\dim x^{\bar{G}} \geqslant 42$. On the other hand, if x_1 and x_2 are conjugate (which is always the case if r divides q^2-1) then $\dim C_{\bar{G}}(x) = 12$. Therefore, if π is the set of primes $s \geqslant 5$ dividing q^4-1, then there are at most

$$\sum_{r \in \pi} \binom{\frac{1}{2}(r-1)}{2} \frac{|\mathrm{Sp}_4(q^2)|}{(q^2-1)^2} < 4\log_2 q.\frac{1}{8}q^2.q^8(q^2+1)^2(q^4+1) = \frac{1}{2}\log_2 q.q^{10}(q^2+1)^2(q^4+1)$$

such elements in H. Finally, by arguing as in the previous paragraph, we calculate that there are fewer than $2a_5$ non-regular semisimple elements in $\mathrm{Sp}_4(q^2)$ of order r, where $r \geqslant 5$ is a prime divisor of q^4-1. We conclude that there are less than

$$\frac{1}{2}\log_2 q.q^{10}(q^2+1)^2(q^4+1) + 2a_5 = a_6$$

semisimple elements $x \in H$ of prime order $r \geqslant 5$ with $C_{\bar{G}}(x) = B_2T_2$ or $A_2A_1T_1$.

Now suppose $x \in G$ is a field automorphism of prime order r, so $q = q_0^r$. If $r = 2$ and $L = \langle G_0, x\rangle$, then every involution in $N_L(H_0) = \mathrm{Sp}_4(q^2).4$ is contained in H_0, so we may assume r is odd. Since x induces a field automorphism of order r on $\mathrm{Sp}_4(q^2)$, we see that the contribution to $\mathcal{Q}(G,H)$ from field automorphisms is less than $\sum_{i=7}^{9} a_i^2/b_i$, where

$$a_7 = 4q^{40/3}, \; b_7 = \frac{1}{2}q^{104/3}, \; a_8 = 8q^{16}, \; b_8 = \frac{1}{2}q^{208/5}, \; a_9 = 4\log_2 q.q^{20}, \; b_9 = \frac{1}{2}q^{312/7}.$$

Finally, suppose $x \in G$ is an involutory graph automorphism, so $q = 2^{2m+1}$ with $m \geqslant 1$. As noted above for involutory field automorphisms, if $G = G_0.2$ then every involution in $H = H_0.2 = \mathrm{Sp}_4(q^2).4$ is contained in H_0 (note that $\mathrm{Out}(\mathrm{Sp}_4(q^2))$ is cyclic). In particular, there is no contribution to $\mathcal{Q}(G,H)$ from graph automorphisms.

In view of the above estimates, we conclude that

$$\mathcal{Q}(G,H) < \sum_{i=1}^{9} a_i^2/b_i$$

and one checks that this bound is sufficient. □

Lemma 4.20. *Suppose $G_0 = F_4(q)$ and H is of type*

$$A_1(q)C_3(q)\,(p \neq 2),\ B_4(q),\ D_4(q),\ {}^3D_4(q).$$

Then G is not extremely primitive.

Proof. If H is of type $A_1(q)C_3(q)$ (with q odd) then H is the centralizer of an involution, so $Z(H) \neq 1$ and G is not extremely primitive by Lemma 2.1(i).

In the three remaining cases, the maximality of H implies that G does not contain any graph automorphisms (see [50, Table 5.1]). In other words, we may assume that $G = G_0.A$ and $H = H_0.A$, where A is a group of field automorphisms of G_0.

Suppose H is of type $B_4(q)$, so $H_0 = d.\Omega_9(q)$ with $d = (2, q-1)$ and we may assume q is even (otherwise $Z(H) \neq 1$). To handle this case, we will appeal to Lemma 2.4. Let $K = \Omega_8^+(q) < H_0$. By inspecting [6, Tables 8.58 and 8.59], for example, it is easy to see that $M = N_{H_0}(K) = K.2$ is the only maximal overgroup of K in H_0. Since M is a non-normal subgroup of $N_{G_0}(K) = K.\mathrm{Sym}_3$ (see [50, Table 5.1]), we may choose $g \in N_{G_0}(K)$ such that $M^g \neq M$. If $M^g \leqslant H_0$ then $K = K^g < M^g$ and thus $M = M^g$ since M is the unique maximal overgroup of K in H_0, which is a contradiction. Therefore, $M^g \not\leqslant H_0$ and the result follows by applying Lemma 2.4 (noting that K and M are both A-stable).

Now assume H is of type $D_4(q)$, so $H_0 = d^2.\mathrm{P}\Omega_8^+(q).\mathrm{Sym}_3$ and once again we may assume q is even. Consider the subgroup $\bar{K} = B_3 < \bar{H}^0 = D_4$ and set

$$K = \bar{K}_\sigma = \mathrm{Sp}_6(q) < S = (\bar{H}^0)_\sigma = \Omega_8^+(q) < H_0.$$

Let $\mathcal{M}$ be the set of maximal overgroups of K in H_0. By inspecting [6, Table 8.50] we see that $N_{H_0}(K) = K \times Z_2$ and $C_{H_0}(K) = Z_2$. In particular, each $M \in \mathcal{M}$ is of the form $S.2$ (three such groups) or $S.3$. Note that K and each subgroup in $\mathcal{M}$ is A-stable. By inspecting [52, Table 2] we see that $C_{\bar{G}}(\bar{K})^0 = A_1$ and thus $C_{G_0}(K) \geqslant \mathrm{SL}_2(q)$. Therefore, we can choose $g \in C_{G_0}(K) \setminus C_{H_0}(K)$. Then g normalizes K, but it does not normalize S (since $H_0 = N_{G_0}(S)$), so $M^g \not\leqslant H_0$ for all $M \in \mathcal{M}$. Now apply Lemma 2.4.

Finally, let us assume H is of type ${}^3D_4(q)$, so $H_0 = {}^3D_4(q).3$. Write $H_0 = S.\langle\tau\rangle$, where $S = {}^3D_4(q)$ and τ is a triality graph automorphism of S with $C_S(\tau) = G_2(q) = K$. Note that $M = N_{H_0}(K) = K \times \langle\tau\rangle$ and S are the only maximal overgroups of K in H_0. Let us also observe that K and M are A-stable.

We claim that

$$C_{H_0}(K) = \langle\tau\rangle < \mathrm{PGL}_2(q) \leqslant C_{G_0}(K).$$

To see this, first observe that $K = \bar{K}_\sigma$ for a σ-stable subgroup $\bar{K} = C_{\bar{G}}(\tau)$ of type G_2, which contains long root subgroups of $\bar{G}$. So by [52, Table 3], $C_{\bar{G}}(\bar{K})^0 = \bar{J}$ is of type A_1. In addition, note that if $p \neq 2$ then $\bar{J} \times \bar{K}$ is a maximal subgroup of $\bar{G}$ by [56, Theorem 1] and thus $\bar{J}$ must be of adjoint type. It remains to prove that $\tau \in \bar{J}$, since then τ will be contained in $\bar{J}_\sigma \cong \mathrm{PGL}_2(q)$. Suppose this is not the case. Then τ must centralize $\bar{J}$ and so it centralizes $\bar{J} \times \bar{K}$. As noted above, if $p \neq 2$ then $\bar{J} \times \bar{K}$ is maximal, so this is not possible. On the other hand, if $p = 2$ then [32, Table 4.7.1] implies that $C_{\bar{G}}(\tau) = B_3T_1$, C_3T_1 or $A_2\tilde{A}_2$. But once again we reach a contradiction since none of these groups contain a subgroup of type A_1G_2. This justifies the claim.

Suppose $q \geqslant 3$. Then $\langle \tau \rangle$ is non-normal in $\mathrm{PGL}_2(q)$, so we can choose $g \in C_{G_0}(K) \setminus C_{H_0}(K)$ that does not normalize $Z(M) = \langle \tau \rangle$. In particular, g does not normalize M. If $M^g \leqslant H_0$ then $K = K^g < M^g \leqslant H_0$ and thus $M = M^g$, which is a contradiction. Similarly, if $S^g \leqslant H_0$ then $S = S^g$ and thus $g \in N_{G_0}(S) = H_0$. But we have $g \in C_{G_0}(K) \setminus C_{H_0}(K)$, so once again we have reached a contradiction. Therefore, $M^g \not\leqslant H_0$ and $S^g \not\leqslant H_0$, so Lemma 2.4 implies that G is not extremely primitive.

To complete the proof, let us assume $q = 2$. The character tables of G and H are available in the GAP Character Table Library [8] and we calculate that G has rank 7 (there are four possible fusion maps from H-classes to G-classes, but this does not affect the computation of the rank; see [21, Lemma 2.6] for the details). In addition, the indices of the core-free maximal subgroups of H are as follows:

$$\{n_1, \ldots, n_9\} = \{4064256, 978432, 179712, 163072, 89856, 69888, 17472, 2457, 819\}.$$

Now $|G : H| = 5222400$ and it is routine to check that there is no tuple $[a_1, \ldots, a_9]$ of non-negative integers such that $\sum_i a_i = 6$ and $1 + \sum_i a_i n_i = |G : H|$. Therefore, G is not extremely primitive by Lemma 2.3. □

4.5. $G_0 = G_2(q)$.

Lemma 4.21. *If $G_0 = G_2(q)$ and H is the normalizer of a maximal torus, then $b(G, H) = 2$.*

Proof. Here $p = 3$, $q \geqslant 9$ and G contains graph automorphisms (see [50, Table 5.2]). By Corollary 2.13, there are no root elements in H, so $|x^G| \geqslant q^3(q + 1)(q^3 - 1) = b_1$ for all $x \in H$ of prime order (see [57, 61]) and we note that $|H| \leqslant 12(q+1)^2.2 \log_3 q = a_1$. Therefore, $\mathcal{Q}(G, H) < a_1^2/b_1$ and this upper bound is less than 1 if $q > 9$ (and it is less than q^{-1} for $q > 81$).

To complete the proof, we may assume $q = 9$. Here $|x^G| \geqslant q^3(q + 1)(q^3 + 1)$ for all $x \in H$ of prime order (minimal if x is an involutory field automorphism) and by arguing as above we reduce to the cases where $H_0 = (q \pm 1)^2.D_{12}$. If $x \in H$ is not an involutory field automorphism, then $|x^G| \geqslant (q^6 - 1)(q^2 - 1) = b_1$ and we note that $|H| \leqslant (q + 1)^2.12.4 = a_1$. Now assume x is an involutory field automorphism, so $|x^G| = q^3(q + 1)(q^3 + 1) = b_2$ and there are at most $|H_0x| \leqslant (q + 1)^2.12 = a_2$ such elements in H. Therefore, $\mathcal{Q}(G, H) < a_1^2/b_1 + a_2^2/b_2 < 1$ and the result follows. □

Lemma 4.22. *If $G_0 = G_2(q)$ and H is of type $A_1(q)^2$, then G is not extremely primitive.*

Proof. If q is odd then H is the centralizer of an involution, so $Z(H) \neq 1$ and G is not extremely primitive by Lemma 2.1(i). For the remainder, let us assume q is even. Write $G = G_0.A$ and $H = H_0.A$, where $A = \langle \varphi \rangle$ and φ is either trivial or a field automorphism.

Let α_1, α_2 be simple roots for G_0 with α_1 short, α_2 long and let $X_\alpha = \{x_\alpha(c) : c \in \mathbb{F}_q\}$ be the root subgroup corresponding to the root α. Then up to conjugacy, we may assume that

$$H_0 = \langle X_{\pm(3\alpha_1+2\alpha_2)}, X_{\pm\alpha_1} \rangle = \mathrm{L}_2(q) \times \mathrm{L}_2(q).$$

Set $g = x_{\alpha_2}(1)x_{\alpha_1+\alpha_2}(1)x_{-\alpha_2}(1) \in G_2(2) \leqslant G_0$. By [12, Theorem A.1], we have $H_0 \cap H_0^g = 1$ and thus $b(G_0, H_0) = 2$. Moreover, since $g \in G_2(2) \leqslant C_G(\varphi)$ it follows that $H \cap H^g = \langle \varphi \rangle$. Clearly, this is not a maximal subgroup of H and thus G is not extremely primitive. □

Lemma 4.23. *If $G_0 = G_2(q)$ and H is of type $A_2^\epsilon(q)$, then G is not extremely primitive.*

Proof. Here $H_0 = \mathrm{SL}_3^\epsilon(q).2$ and the maximality of H implies that $G = G_0.A$ and $H = H_0.A$, where A is a group of field automorphisms (see [50, Table 5.1]). In view of Theorem 2.15, we may assume that $q \geqslant 7$. If $q \equiv \epsilon \pmod 3$ then $F(H) = Z_3$ and extreme primitivity is ruled out by Lemma 2.1(iv), so we may assume that $(3, q - \epsilon) = 1$. Let $S = \mathrm{soc}(H_0) = \mathrm{SL}_3^\epsilon(q)$.

By inspecting [6, Tables 8.3 and 8.5], we see that H_0 has a maximal subgroup $M = \mathrm{GL}_2^{\epsilon}(q).2$. Set
$$K = \mathrm{SL}_2(q) < L = \mathrm{GL}_2^{\epsilon}(q) < M$$
and let $\mathcal{M}$ be the set of maximal overgroups of K in H_0. Note that $N_{H_0}(K) = M$. We claim that $\mathcal{M} = \{M, S\}$. Plainly $S \in \mathcal{M}$ and using [6] it is clear that every other subgroup in $\mathcal{M}$ is a conjugate of M. Suppose K is contained in M^h for some $h \in H_0$. Then $K, K^{h^{-1}} \leqslant M$, but K is the only subgroup of M isomorphic to $\mathrm{SL}_2(q)$, so $K = K^{h^{-1}}$ and thus $h \in N_{H_0}(K) = M$. Therefore, $M = M^h$ and this justifies the claim. Note that K, M and S are all A-stable. In addition, let us observe that $C_S(K) = C_L(K) = Z(L) = Z_{q-\epsilon}$ is a characteristic subgroup of $C_{H_0}(K) = C_M(K) = Z_{q-\epsilon}.2$ and M.

We claim that $C_S(K) < \mathrm{SL}_2(q) < G_0$. Firstly, we note that $L = \bar{L}_\sigma$ for a σ-stable subgroup $\bar{L} < \bar{G}$ of type A_1T_1 and so $K = (\bar{L}')_\sigma$. By inspecting [52, Table 2], we see that $C_{\bar{G}}(\bar{L}')^0 = \bar{J}$, where $\bar{J}$ is a subgroup of type A_1, whence $T_1 = Z(\bar{L}) < \bar{J}$ and $C_S(K) = Z(\bar{L})_\sigma < \bar{J}_\sigma = \mathrm{SL}_2(q)$, as required.

By the claim, it follows that $C_L(K)$ is a non-normal subgroup of $C_{G_0}(K)$. Therefore, we may choose an element $g \in C_{G_0}(K)$ which does not normalize $C_L(K)$. Suppose $M^g \leqslant H_0$. Then $K = K^g < M^g \leqslant H_0$ and thus $M = M^g$ (since M and S are the only maximal overgroups of K in H_0). But $C_L(K) = Z(L)$ is a characteristic subgroup of M, so this would imply that g normalizes $C_L(K)$, which is a contradiction. Similarly, if $S^g \leqslant H_0$ then $S^g = S$ and thus $C_S(K)^g = C_S(K)$. But $C_S(K) = C_L(K)$ and g does not normalize $C_L(K)$, so once again we have reached a contradiction. We conclude that $M^g \not\leqslant H_0$ and $S^g \not\leqslant H_0$, so the desired result follows from Lemma 2.4. □

4.6. $G_0 = {}^3D_4(q)$.

Lemma 4.24. *If $G_0 = {}^3D_4(q)$ and H is the normalizer of a maximal torus, then $b(G,H) = 2$.*

Proof. The possibilities for H are described in [50, Table 5.2] and in each case we observe that $H_0 = S.N$, where S is a torus of odd order and N has a unique involution. If $q = 2$ then we refer the reader to [17, Table 12]. For the remainder, we may assume $q \geqslant 3$.

Suppose H_0 contains a long root element x. Then $p = 2$ by Corollary 2.13, so x is an involution. But [23, Corollary 4.4] implies that every involution in H_0 is contained in the largest class of involutions in G_0 and it follows that there are no long root elements in H_0. Therefore, Proposition 2.11 gives $|x^G| > q^{14} = b_1$ for all $x \in H$ of prime order and we note that
$$|H| \leqslant (q^2+q+1)^2.|\mathrm{SL}_2(3)|.3\log_2 q = a_1.$$
This gives $\mathcal{Q}(G,H) < a_1^2/b_1$, which is less than 1 if $q \geqslant 7$ (and it is less than q^{-1} for $q \geqslant 11$).

Now assume $q \in \{3,4,5\}$. If $H_0 = (q^4-q^2+1){:}4$, then we can replace a_1 in the previous bound by $3\log_2 q.|H_0|$ and this is sufficient. Next suppose $H_0 = (q^2-q+1)^2{:}\mathrm{SL}_2(3)$. If $q = 5$ then $|H| \leqslant 21^2.|\mathrm{SL}_2(3)|.3 = a_1$ and we get $\mathcal{Q}(G,H) < a_1^2/b_1 < 1$ with $b_1 = 5^{14}$. If $q \in \{3,4\}$ then $H = N_G(L)$, where L is a Sylow (q^2-q+1)-subgroup of G_0, and we can use Magma to show that $b(G,H) = 2$. Finally, the case $H_0 = (q^2+q+1)^2{:}\mathrm{SL}_2(3)$ can be handled in a similar fashion, noting that $H = N_G(L)$ for a Sylow r-subgroup of G_0 with $r = 13$, 7 and 31 when $q = 3$, 4 and 5, respectively. We refer the reader to [21, Lemmas 2.7 an 2.8] for the details of these Magma computations. □

Lemma 4.25. *Suppose $G_0 = {}^3D_4(q)$ and H is of type $A_1(q)A_1(q^3)$ or $A_2^{\epsilon}(q).(q^2+\epsilon q+1)$. Then G is not extremely primitive.*

Proof. In view of Theorem 2.15, we may assume $q \geqslant 3$. First assume H is of type $A_1(q)A_1(q^3)$, so $H_0 = d.(\mathrm{L}_2(q) \times \mathrm{L}_2(q^3)).d$ with $d = (2, q-1)$. If q is odd then H is the centralizer of an involution, so $Z(H) \neq 1$ and G is not extremely primitive. On the other hand, if q is even then the structure of $\mathrm{soc}(H)$ is incompatible with extreme primitivity. Similarly, if H is of type $A_2^\epsilon(q).(q^2+\epsilon q+1)$, then $H_0 = (\mathrm{SL}_3^\epsilon(q) \circ (q^2+\epsilon q+1)).h.2$ with $h = (q^2+\epsilon q+1, 3)$ and the result follows since $\mathrm{soc}(H)$ is not a direct product of isomorphic simple groups. □

4.7. $G_0 = {}^2F_4(q)'$.

Lemma 4.26. *If $G_0 = {}^2F_4(q)'$ and H is the normalizer of a maximal torus, then $b(G,H) = 2$.*

Proof. If $q = 2$ then the result follows from [17, Table 12], so we may assume $q \geqslant 8$. By inspecting [50, Table 5.2], we see that

$$|H| \leqslant (q+\sqrt{2q}+1)^2.|4 \circ \mathrm{GL}_2(3)|.2\log_2 q = a_1.$$

Now $|x^G| > (q-1)q^{10} = b_1$ for all $x \in G$ of prime order, so $\mathcal{Q}(G,H) < a_1^2/b_1$ and one checks that this upper bound is less than 1 for all $q \geqslant 8$ (in addition, it is less than q^{-1} for all $q \geqslant 32$). □

Lemma 4.27. *If $G_0 = {}^2F_4(q)'$ and H is of type $A_2^-(q)$, then $b(G,H) = 2$.*

Proof. Here $H_0 = \mathrm{SU}_3(q).2$ or $\mathrm{PGU}_3(q).2$. In view of Remark 2.16, we may assume $q \geqslant 8$. Let $x \in H$ be an element of prime order r.

First assume $r = 2$, so $x \in H_0$ and $|C_{H_0}(x)|$ is divisible by 3. In the notation of [71, Table II], it follows that each involution in H_0 is G_0-conjugate to u_2, whence $|x^G| > (q-1)q^{13} = b_1$ and

$$|x^G \cap H| = i_2(H) = \frac{|\mathrm{SU}_3(q)|}{q^3(q+1)^2} + \frac{|\mathrm{SU}_3(q)|}{|\mathrm{SL}_2(q)|} < 2q^5 = a_1.$$

Similarly, if $x \in H_0$ has order 3 then $|x^G| > (q-1)q^{17} = b_2$ and since G_0 has a unique class of elements of order 3, it follows that

$$|x^G \cap H| = i_3(H_0) \leqslant i_3(\mathrm{PGU}_3(q)) < 2(q+1)q^5 = a_2.$$

If $x \in H_0$ has order $r \geqslant 5$ then $|x^G| > \frac{1}{2}q^{20} = b_3$ (minimal if x is conjugate to the element denoted t_9 in [71, Table IV]) and we record the bound $|\mathrm{SU}_3(q)| < q^8 = a_3$.

Finally, suppose $x \in H$ is a field automorphism of order r. If $r = 3$ then $|x^G| > \frac{1}{2}q^{52/3} = b_4$ and H contains fewer than

$$2 \cdot \frac{|\mathrm{SU}_3(q)|}{|\mathrm{SU}_3(q^{1/3})|} < 4q^{16/3} = a_4$$

such elements. For $r \geqslant 5$, $|x^G| > \frac{1}{2}q^{104/5} = b_5$ and we note that $|H| < 2\log_2 q.q^8 = a_5$.

We conclude that $\mathcal{Q}(G,H) < \sum_{i=1}^5 a_i^2/b_i < q^{-1}$ and the result follows. □

Lemma 4.28. *If $G_0 = {}^2F_4(q)'$ and H is of type $C_2(q)$, then $b(G,H) = 2$.*

Proof. Here $H_0 = \mathrm{Sp}_4(q).2$, where the outer involution acts as a graph automorphism on $\mathrm{Sp}_4(q)$. For $q = 2$, we refer the reader to [17, Table 12]. For the remainder, let us assume $q \geqslant 8$.

It will be convenient to view H_0 as a subgroup of $F_4(q)$. Then in terms of the ambient algebraic groups, we have

$$B_2 = \{(x, x^\tau) \,:\, x \in B_2\} < B_2^2 < B_4 < F_4, \tag{14}$$

where τ is an involutory graph automorphism of B_2. Let V be the adjoint module for $\bar{G}$ and note that the restriction of V to B_2^2 is given in (9).

Let $x \in H$ be an element of prime order r. First assume $r = 2$, so $x \in H_0$. If x acts as a graph automorphism on $\mathrm{Sp}_4(q)$, then $C_{\mathrm{Sp}_4(q)}(x) = {}^2B_2(q)$ and we deduce that x is G_0-conjugate to u_1 (see [71, Table II]). Similarly, if x is a long or short root element in $\mathrm{Sp}_4(q)$, then x is conjugate to u_2. Now assume $x \in \mathrm{Sp}_4(q)$ is a c_2-type involution, in the notation of [1]. From the embedding $B_2 < B_4$ in (14), we see that x is in the B_4-class labelled c_4, and by considering the fusion of B_4-classes of involutions in F_4 (see [43, p.373], for example), we deduce that x is in the F_4-class labelled $A_1\tilde{A}_1$. It follows that x is G_0-conjugate to u_1 (the involutions in ${}^2F_4(q)$ conjugate to u_2 are contained in the F_4-class $(\tilde{A}_1)_2$). To summarise: if x is G_0-conjugate to u_1, then

$$|x^G \cap H| = \frac{|\mathrm{Sp}_4(q)|}{|{}^2B_2(q)|} < q^5 = a_1, \;\; |x^G| > (q-1)q^{10} = b_1,$$

whereas

$$|x^G \cap H| = 2(q^4-1) + (q^2-1)(q^4-1) < 2q^6 = a_2, \;\; |x^G| > (q-1)q^{13} = b_2$$

if x is conjugate to u_2.

Next assume $x \in H_0$ has order 3. Here $|x^G| > (q-1)q^{17} = b_3$ and

$$|x^G \cap H| = i_3(H_0) = 2q^3(q^2+1)(q-1) < 2q^6 = a_3.$$

Now suppose $x \in H_0$ has order $r \geqslant 5$, so [71, Table IV] indicates that $C_{\bar{G}}(x) = B_2T_2$, $A_1\tilde{A}_1T_2$ or T_4. It will be useful to recall the action of τ on semisimple elements of B_2 in (11).

If x is not regular in $\mathrm{Sp}_4(q)$ then by working with the decomposition in (9), we calculate that $\dim C_V(x) = 12$ and thus $\dim C_{\bar{G}}(x) = 12$. For example, if r divides $q-1$ and $x = \mathrm{diag}(1,1,\omega,\omega^{-1}) \in \mathrm{Sp}_4(q)$, then $x^\tau = \mathrm{diag}(\omega,\omega,\omega^{-1},\omega^{-1})$ and we find that x has an 8-dimensional 1-eigenspace on $\mathcal{L}(B_2B_2)$ and a 4-dimensional 1-eigenspace on $W \otimes W$, where W is the natural 4-dimensional module for B_2. Furthermore, x^τ acts on W^τ as $\mathrm{diag}(1,1,\omega^2,\omega^{-2})$ and thus $\dim C_{W^\tau \otimes W^\tau}(x) = 0$, giving $\dim C_V(x) = 12$ as claimed. It follows that $C_{\bar{G}}(x) = B_2T_2$ and by inspecting the relevant tables in [70, 71] we deduce that $|x^G| > \frac{1}{2}q^{20} = b_4$ (note that x is of type t_1, t_7 or t_9 in [71, Table IV]). Similarly, we find that $C_{\bar{G}}(x) = B_2T_2$ if $r = 5$ (note that 5 divides q^2+1, so every element of order 5 is regular). Since r must divide q^2-1 if x is non-regular, it follows that there are at most

$$\frac{|\mathrm{Sp}_4(q)|}{q^2+1} + \sum_{r \in \pi} \frac{1}{2}(r-1) \cdot 2 \cdot \frac{|\mathrm{Sp}_4(q)|}{|\mathrm{GL}_2(q)|} < q^8 + 8\log_2 q . q^7 = a_4$$

semisimple elements in H_0 with $C_{\bar{G}}(x) = B_2T_2$, where π is the set of primes $s \geqslant 5$ dividing q^2-1.

Now assume $x \in H_0$ is a regular semisimple element with $r \geqslant 7$. Then by considering (9), we deduce that $\dim C_{\bar{G}}(x) \leqslant 8$ and thus $C_{\bar{G}}(x) = A_1\tilde{A}_1T_2$ or T_4. This implies that $|x^G| > \frac{1}{2}q^{22} = b_5$ and we note that $|\mathrm{Sp}_4(q)| < q^{10} = a_5$.

Finally, let us assume $x \in G$ is a field automorphism of prime order r, so $q = q_0^r$, $r \geqslant 3$ and x acts as a field automorphism on $\mathrm{Sp}_4(q)$. If $r = 3$ then $|x^G| > \frac{1}{2}q^{52/3} = b_6$ and there are fewer than $2|\mathrm{Sp}_4(q) : \mathrm{Sp}_4(q^{1/3})| < 4q^{20/3} = a_6$ such elements in H. Similarly, if $r = 5$ then $|x^G| > \frac{1}{2}q^{104/5} = b_7$ and H contains fewer than $8q^8 = a_7$ such elements. For $r \geqslant 7$ we have $|x^G| > \frac{1}{2}q^{156/7} = b_8$ and we note that $|H| < 2\log_2 q . q^{10} = a_8$.

Bringing together the above bounds, we conclude that $\mathcal{Q}(G,H) < \sum_{i=1}^{8} a_i^2/b_i < 1$ and the result follows. □

Lemma 4.29. *If $G_0 = {}^2F_4(q)'$ and H is of type ${}^2B_2(q)^2$, then $b(G,H) = 2$.*

Proof. Here $H_0 = {}^2B_2(q) \wr \mathrm{Sym}_2$ and $q \geqslant 8$ (see [50, Table 5.1]). As in the previous case, it will be useful to view H_0 as a subgroup of $F_4(q)$ via $B_2^2 < B_4 < F_4$. Let $x \in H$ be an element of prime order r and let $V = \mathcal{L}(\bar{G})$ be the adjoint module for $\bar{G} = F_4$.

First assume $r = 2$, so $x \in H_0$ and we note that ${}^2B_2(q)$ contains $(q^2+1)(q-1)$ involutions, which form a single conjugacy class. If x interchanges the two ${}^2B_2(q)$ factors, then $C_{H_0}(x)$ contains ${}^2B_2(q)$ and thus x is G_0-conjugate to u_1 in the notation of [71, Table II]. Similarly, each involution in ${}^2B_2(q)^2$ of the form $(x_1,1)$ or $(1,x_1)$ is of type u_1. However, if $x = (x_1,x_2) \in {}^2B_2(q)^2$ and each x_i is an involution, then x is a c_4-type involution in B_4 (since each x_i is a

c_2-type involution in B_2) and thus x is in the $\bar{G}$-class labelled $A_1\tilde{A}_1$. In particular, we deduce that x is G_0-conjugate to u_2. It follows that if x is of type u_1, then

$$|x^G \cap H| = 2(q^2+1)(q-1) + |{}^2B_2(q)| < q^5 = a_1,\ \ |x^G| > (q-1)q^{10} = b_1$$

and for x of type of u_2 we get $|x^G \cap H| < q^6 = a_2$ and $|x^G| > (q-1)q^{13} = b_2$.

Next assume $x \in H_0$ and $r \geqslant 5$ (observe that H_0 does not contain any elements of order 3). Write $x = (x_1, x_2) \in {}^2B_2(q)^2$ and note that if $x_i \neq 1$ then x_i is a regular element of B_2 (since $x_i \in C_{B_2}(\tau)$, this follows from (11)). If $x_2 = 1$ then using (9) we deduce that $\dim C_V(x) = 12$, so $C_{\bar{G}}(x) = B_2T_2$. Similarly, if x_1 and x_2 are conjugate, then $C_{\bar{G}}(x) = B_2T_2$. On the other hand, if the x_i are nontrivial and non-conjugate, then $C_{\bar{G}}(x) = A_1\tilde{A}_1T_2$ or T_4, so $|x^G| > \frac{1}{2}q^{22} = b_3$ and we note that $|{}^2B_2(q)^2| < q^{10} = a_3$.

Now assume $C_{\bar{G}}(x) = B_2T_2$, so x is of type t_1, t_7 or t_9 with respect to the notation in [71, Table IV], where $|C_{G_0}(t_i)| = q^2(q^2+1)(q-1)f_i(q)$ and

$$f_1(q) = q-1,\ \ f_7(q) = q - \sqrt{2q}+1,\ \ f_9(q) = q+\sqrt{2q}+1.$$

In particular, $|x^G| > q^{20}$ if x is of type t_1 or t_7, whereas $|x^G| > \frac{1}{2}q^{20}$ if x is of type t_9. Let us also observe that if $y \in {}^2B_2(q)$ has order r, then $|C_{{}^2B_2(q)}(y)| = q-1$ or $q \pm \sqrt{2q}+1$ (see [74]). Moreover, there are $\frac{1}{2}(q-2)$ distinct ${}^2B_2(q)$-classes of semisimple elements with centralizer of order $q-1$, and $\frac{1}{4}(q \pm \sqrt{2q})$ classes with a centralizer of order $q \pm \sqrt{2q}+1$. By Lagrange's theorem, it follows that if $x = (x_1, x_2) \in {}^2B_2(q)^2$ and the x_i are conjugate elements of order r, then x is of type t_1 if $|C_{{}^2B_2(q)}(x_i)| = q-1$. Similarly, x is of type t_7 if $|C_{{}^2B_2(q)}(x_i)| = q - \sqrt{2q}+1$, and type t_9 if $|C_{{}^2B_2(q)}(x_i)| = q+\sqrt{2q}+1$. Therefore, if x is of type t_1 then $|x^G| > q^{20} = b_4$ and there are at most

$$2|{}^2B_2(q)| + \frac{1}{2}(q-2)\cdot(q^2(q^2+1))^2 = a_4$$

such elements in H_0. Similarly, if x is of type t_7 then $|x^G| > q^{20} = b_5$ and H_0 contains no more than

$$2|{}^2B_2(q)| + \frac{1}{4}(q-\sqrt{2q})\cdot(q^2(q-1)(q+\sqrt{2q}+1))^2 = a_5$$

such elements. Finally, if x is of type t_9 then $|x^G| > \frac{1}{2}q^{20} = b_6$ and there are at most

$$2|{}^2B_2(q)| + \frac{1}{4}(q+\sqrt{2q})\cdot(q^2(q-1)(q-\sqrt{2q}+1))^2 = a_6$$

such elements in H_0.

Finally, suppose $x \in G$ is a field automorphism of prime order r, so $q = q_0^r$, $r \geqslant 3$ and x acts as a field automorphism on both ${}^2B_2(q)$ factors. If $r = 3$ then $|x^G| > \frac{1}{2}q^{52/3} = b_7$ and there are at most $2|{}^2B_2(q) : {}^2B_2(q^{1/3})|^2 < 8q^{20/3} = a_7$ such elements in H. Similarly, if $r = 5$ then $|x^G| > \frac{1}{2}q^{104/5} = b_8$ and H contains fewer than $16q^8 = a_8$ such elements. Finally, $|x^G| > \frac{1}{2}q^{156/7} = b_9$ if $r \geqslant 7$ and we note that $|H| < 2\log_2 q.q^{10} = a_9$.

We conclude that

$$\mathcal{Q}(G,H) < \sum_{i=1}^{9} a_i^2/b_i < 1$$

and the result follows. □

4.8. $G_0 = {}^2G_2(q)'$ or ${}^2B_2(q)$.

Lemma 4.30. *Suppose $G_0 = {}^2G_2(q)'$ or ${}^2B_2(q)$ and H is the normalizer of a maximal torus. Then $b(G,H) = 2$.*

Proof. See [17, Lemmas 4.37 and 4.39]. In both cases, it is easy to check that the upper bound on $\mathcal{Q}(G,H)$ given in [17] tends to 0 as q tends to infinity. □

Lemma 4.31. *If $G_0 = {}^2G_2(q)'$ and H is of type $2 \times \mathrm{L}_2(q)$, then G is not extremely primitive.*

Proof. Here $q > 3$ and H is the centralizer of an involution, so $Z(H) \neq 1$ and G is not extremely primitive by Lemma 2.1(i). □

We have now completed the proof of Theorem 4.1 and Proposition 4.2.

5. Lower rank subgroups

At this point, we have now established Theorem 1 in the cases where H contains a maximal torus of G_0 and we will now consider the remaining possibilities for H. It will be convenient to postpone the analysis of the twisted groups ${}^2B_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$ and ${}^3D_4(q)$ to Section 8, so in the next three sections we will assume

$$G_0 \in \{E_8(q), E_7(q), E_6^{\epsilon}(q), F_4(q), G_2(q)'\}, \tag{15}$$

where $q = p^f$ with p a prime.

For $G_0 = G_2(q)'$, the maximal subgroups of G have been determined up to conjugacy by Cooperstein [27] (for $p = 2$) and Kleidman [34] (for $p \neq 2$). In the remaining cases, there is a complete description of the maximal subgroups when $G_0 = F_4(2)$, $E_6(2)$, ${}^2E_6(2)$ or $E_7(2)$; see [67], [37], [26, 77] and [3], respectively.

To proceed in the general case, we will apply the following fundamental result (see [51, Theorem 2]), which partitions the remaining maximal subgroups into various types.

Theorem 5.1. *Let G be an almost simple group with socle $G_0 = (\bar{G}_\sigma)'$ as in (15). Let H be a maximal subgroup of G with $G = HG_0$ and assume H does not contain a maximal torus of G_0. Set $H_0 = H \cap G_0$. Then one of the following holds:*

- (I) *$H = N_G(\bar{H}_\sigma)$, where $\bar{H}$ is a maximal closed σ-stable positive dimensional subgroup of $\bar{G}$ (not parabolic nor maximal rank);*
- (II) *H is of the same type as G (possibly twisted) over a subfield of $\mathbb{F}_q$;*
- (III) *H is an exotic local subgroup (determined in [25]);*
- (IV) *$G_0 = E_8(q)$, $p \geqslant 7$ and $H_0 = (\mathrm{Alt}_5 \times \mathrm{Alt}_6).2^2$;*
- (V) *H is almost simple, and not of type (I) or (II).*

In view of this result, for the remainder of the paper we will refer to Type I and Type II subgroups of G, etc. We direct the reader to Theorem 7.2 for more detailed information on Type V subgroups.

We begin by handling the subgroups of Type III and IV, which are easily eliminated.

Theorem 5.2. *If H is a Type III or IV subgroup of G, then G is not extremely primitive.*

Proof. In view of Lemma 2.1 (specifically, parts (ii) and (v)), we may assume that one of the following holds, where $H_0 = H \cap G_0$:

- (a) $G_0 = G_2(q)$, $H_0 = 2^3.\mathrm{SL}_3(2)$, $q = p \geqslant 3$;
- (b) $G_0 = F_4(q)$, $H_0 = 3^3.\mathrm{SL}_3(3)$, $p \geqslant 5$;
- (c) $G_0 = E_8(q)$, $H_0 = 5^3.\mathrm{SL}_3(5)$, $p \neq 2, 5$.

First consider case (a). For $q \in \{3, 5\}$, we appeal to Theorem 2.15 (note that $b(G, H) = 3$ when $q = 3$). Now assume $q \geqslant 7$, so $G = G_2(q)$ and $H = 2^3.\mathrm{SL}_3(2)$. We claim that $b(G, H) = 2$. For $q = 7$ we use Magma to construct G as a permutation group of degree 19608 and we construct $H = N_G(L)$ by identifying an appropriate elementary abelian subgroup L of order 8 in a Sylow 2-subgroup of G. It is then routine to find an element $g \in G$ by random search such that $H \cap H^g = 1$ (see [21, Lemma 2.9] for more details). Now assume $q \geqslant 11$ and let $x \in H$ be an element of prime order r, so $r \in \{2, 3, 7\}$ and we note that $i_2(H) = 91 = a_1$,

G_0	$\mathrm{soc}(H_0)$
$E_8(q)$	$\mathrm{L}_2(q)$ (3 classes, $p \geqslant 23, 29, 31$), $\Omega_5(q)$ ($p \geqslant 5$)
$E_7(q)$	$\mathrm{L}_2(q)$ (2 classes, $p \geqslant 17, 19$), $\mathrm{L}_3^{\epsilon}(q)$ ($p \geqslant 5$), $\mathrm{L}_2(q)^2$ ($p \geqslant 5$), ${}^3D_4(q)$ ($p \geqslant 3$)
$E_6^{\epsilon}(q)$	$\mathrm{L}_3^{\pm}(q)$ ($p \geqslant 5$), $G_2(q)$ ($p \neq 7$), $\mathrm{PSp}_8(q)$ ($p \geqslant 3$), $F_4(q)$,
$F_4(q)$	$\mathrm{L}_2(q)$ ($p \geqslant 13$), $G_2(q)$ ($p = 7$)
$G_2(q)$	$\mathrm{L}_2(q)$ ($p \geqslant 7$)

Table 7. Type I subgroups with a compatible socle

$i_3(H) = 224 = a_2$ and $i_7(H) = 384 = a_3$. If $r = 2$ then $|x^G| = q^4(q^4+q^2+1) = b_1$ and similarly $|x^G| \geqslant q^3(q^3-1) = b_2$ if $r = 3$ and $|x^G| > \frac{1}{2}q^{10} = b_3$ if $r = 7$. Therefore,

$$\mathcal{Q}(G,H) < \sum_{i=1}^{3} a_i^2/b_i < q^{-1}$$

and the result follows.

In cases (b) and (c) we also claim that $b(G,H) = 2$. For example, in (c) we note that $|H| \leqslant \log_2 q.5^3|\mathrm{SL}_3(5)| = a_1$ and $|x^G| > q^{58} = b_1$ for all nontrivial $x \in G$, which yields $\mathcal{Q}(G,H) < a_1^2/b_1 < q^{-1}$. Case (b) is entirely similar, noting that $|x^G| > q^{16}$ for all nontrivial $x \in G$ (see Proposition 2.11). □

For the remainder of this section, we will focus on the subgroups arising in part (I) of Theorem 5.1. Following [55, Theorem 8], we partition these subgroups into three cases:

(a) $G_0 = E_7(q)$, $p \geqslant 3$ and $\bar{H}_\sigma = (2^2 \times \mathrm{P}\Omega_8^+(q).2^2).\mathrm{Sym}_3$ or ${}^3D_4(q).3$;
(b) $G_0 = E_8(q)$, $p \geqslant 7$ and $\bar{H}_\sigma = \mathrm{PGL}_2(q) \times \mathrm{Sym}_5$; (16)
(c) $(G_0, \mathrm{soc}(H_0))$ is one of the cases listed in [55, Table 3].

Our main theorem is the following.

Theorem 5.3. *If H is a Type I subgroup of G, then G is not extremely primitive.*

Suppose G is extremely primitive and H is a Type I subgroup of G. By Lemma 2.1(v), the socle of H is a direct product of isomorphic simple groups. Therefore, by considering the groups in (a) and (b) above, together with the cases in [55, Table 3], we deduce that the only subgroups of this type are the ones listed in Table 7.

First we handle two special cases with $G_0 = E_6^{\epsilon}(q)$.

Lemma 5.4. *Suppose $G_0 = E_6^{\epsilon}(q)$ and $\mathrm{soc}(H) = F_4(q)$ or $\mathrm{PSp}_8(q)$. Then G is not extremely primitive.*

Proof. Set $H_0 = H \cap G_0$ and first assume $\mathrm{soc}(H) = F_4(q)$, so $H_0 = F_4(q)$ is the centralizer in G_0 of a graph automorphism. Note that $\mathrm{Inndiag}(G_0) \cap G = G_0$ by the maximality of H in G. In addition, if G contains a graph automorphism, then $\mathrm{soc}(H)$ is not a direct product of isomorphic simple groups and thus G is not extremely primitive by Lemma 2.1(v). Write $G = G_0.A$ and $H = H_0.A$. Set $e = (3, q-\epsilon)$.

The action of the quasisimple group $e.G_0$ on the set of cosets of $F_4(q)$ is studied in some detail by Lawther in [39] and he computes all the subdegrees (see [39, Table 1] for $\epsilon = +$ and [39, Table 3] for $\epsilon = -$). In both cases, there is a subdegree $s(q) = q^4(q^8-1)(q^{12}-1)$. This is also a subdegree for the action of G_0, so there exists $g \in G_0$ such that

$$|H_0 \cap H_0^g| = |H_0|/s(q) = q^{20}(q^2-1)(q^6-1).$$

Suppose G is extremely primitive, so $M = H \cap H^g$ is a maximal subgroup of H such that $|M \cap H_0| = |H_0 \cap H_0^g|$. First we observe that M is non-parabolic since $|H_0 \cap H_0^g|$ is indivisible by q^{24}. Then since $|M| > q^{22}$, it follows that M is one of the subgroups listed in [17, Lemma 4.23]. But none of these subgroups have order divisible by q^{20}, so we have reached a contradiction and we conclude that G is not extremely primitive.

To complete the proof, we may assume p is odd and $\mathrm{soc}(H) = \mathrm{PSp}_8(q)$. As in the previous case, $H_0 = \mathrm{PGSp}_8(q) = \mathrm{PSp}_8(q).2$ is the centralizer in G_0 of an involutory graph automorphism and we may assume that $\mathrm{Inndiag}(G_0) \cap G_0 = G_0$ and G does not contain any graph automorphisms. Once again, write $G = G_0.A$ and $H = H_0.A$.

Set $\bar{H} = C_4 < \bar{G} = E_6$ and let $\bar{L}$ be a Levi subgroup of $\bar{H}$ of type A_3T_1. Then $N_{\bar{H}}(\bar{L}) = \bar{L}.2$ and the outer involution induces a graph automorphism on $\bar{K} = \bar{L}' = A_3$ and inverts the torus $Z(\bar{L})^0 = T_1$. It is straightforward to check that $C_{\bar{H}}(\bar{K}) = Z(\bar{L})^0$. We claim that $\bar{K}.2$ is a subgroup of a maximal subsystem subgroup A_1A_5 of $\bar{G}$. By inspecting [53, Table 8.3], we see that $\bar{G}$ has two conjugacy classes of A_3 subgroups, both of which are contained in a maximal subsystem subgroup A_1A_5. The subgroups in the two classes differ in the structure of their connected centralizers in $\bar{G}$, which is either A_1 or $A_1^2T_1$. By considering the composition factors of the A_3 subgroups on the adjoint module for $\bar{G}$ (see [53, Table 8.3]), we deduce that $C_{\bar{G}}(\bar{K})^0 = \bar{J}$ is of type A_1. It follows that $\bar{K}$ is contained in the A_5 factor of a subsystem subgroup A_1A_5, with $\bar{K}$ acting irreducibly on the natural module for A_5. In particular, $\bar{K}.2 < A_5 < A_1A_5$ and $C_{\bar{G}}(\bar{K}.2)^0 = \bar{J}$, which contains $Z(\bar{L})^0$ as a maximal torus.

As usual, in order to descend from the algebraic groups discussed above to the corresponding finite groups we will take the fixed points of an appropriate Steinberg endomorphism σ. Specifically, we define σ to be the product of the standard Frobenius map and a lift of the central involution in the Weyl group of $\bar{H}$. Referring to the subgroups introduced above, we may assume that $\bar{H}$, $\bar{L}$, $\bar{K}$ and $\bar{J}$ are all σ-stable. Then $(\bar{G}_\sigma)' = G_0$, $\bar{H}_\sigma = H_0$ and $\bar{K}_\sigma = h.\mathrm{U}_4(q).h$, where $h = (4, q+1)/2$. Set $K = (\bar{K}_\sigma)' = h.\mathrm{U}_4(q)$. By considering [6, Tables 8.48 and 8.49], we deduce that

$$M = N_{H_0}(K) = h.((q+1)/2h \times K).2^2.2$$

is the unique maximal overgroup of K in H_0.

Now $C_{H_0}(K) = (Z(\bar{L})^0)_\sigma = Z_{q+1}$ is contained in the subgroup $J = \bar{J}_\sigma = \mathrm{SL}_2(q)$ of $C_{G_0}(K)$. Since $C_{H_0}(K)$ is non-normal in J, we can choose $g \in J$ which does not normalize $C_{H_0}(K)$. If $M^g \leqslant H_0$ then M^g is a maximal subgroup of H_0 containing K, so $M^g = M$ and thus g normalizes $C_M(K) = C_{H_0}(K)$, which is a contradiction. Therefore, $M^g \not\leqslant H_0$ and we complete the proof by applying Lemma 2.4, noting that both K and M are A-stable. □

In each of the remaining cases, we claim that $b(G, H) = 2$.

Proposition 5.5. *If (G, H) is one of the cases in Table 7 and $(G_0, \mathrm{soc}(H)) \neq (E_6^\epsilon(q), F_4(q))$, $(E_6^\epsilon(q), \mathrm{PSp}_8(q))$, then $b(G, H) = 2$.*

We will prove Proposition 5.5 in a sequence of lemmas.

Lemma 5.6. *If $G_0 = E_7(q)$ and $\mathrm{soc}(H) = {}^3D_4(q)$, then $b(G, H) = 2$.*

Proof. Here $p \geqslant 3$ and $H_0 = H \cap G_0 = {}^3D_4(q).3$. We proceed by estimating the various contributions to $\mathcal{Q}(G, H)$ from elements of prime order.

Let $x \in H_0$ be an element of prime order r. If $r = 2$ then $|x^G \cap H| = q^8(q^8 + q^4 + 1) = a_1$ and $|x^G| > \frac{1}{2}(q-1)q^{53} = b_1$ (note that H_0 has a unique conjugacy class of involutions). Similarly, if x is unipotent then x is not a long root element of G_0 (see the proof of [16, Proposition 5.12]), so $|x^G| > q^{52} = b_2$ and we note that H_0 contains fewer than $q^{24} = a_2$ elements of order p (see Proposition 2.14(ii)).

Next assume $x \in H_0$ and $r \neq 2, p$. If $r = 3$ then $|x^G| > (q-1)q^{53} = b_3$ and Proposition 2.14(i) gives $i_3(H_0) < 2(q+1)q^{19} = a_3$. Similarly, if $r \geqslant 5$ and $C_{\bar{G}}(x)^0 \neq E_6T_1$, then $|x^G| > \frac{1}{2}q^{66} = b_3$ and we note that $|{}^3D_4(q)| < q^{28} = a_3$. Now assume $C_{\bar{G}}(x)^0 = E_6T_1$, so

$|x^G| > \frac{1}{2}q^{54} = b_4$. We claim that x is not a regular element of ${}^3D_4(q)$, so there are fewer than $q^{25} = a_4$ such elements in H_0. Suppose x is regular and let $V = \mathcal{L}(\bar{G})$ be the adjoint module for $\bar{G}$. Let $\lambda_1, \ldots, \lambda_4$ be fundamental dominant weights for $\bar{H}^0 = D_4$, labelled in the usual manner. As noted in [75, Table 4], we have

$$V{\downarrow}\bar{H}^0 = \mathcal{L}(\bar{H}^0) \oplus V(2\lambda_1) \oplus V(2\lambda_3) \oplus V(2\lambda_4),$$

where $V(2\lambda_i)$ is an irreducible module for $\bar{H}^0$ with highest weight $2\lambda_i$. Now $V(2\lambda_i)$ is the unique nontrivial composition factor of $S^2(V(\lambda_i))$ and by considering the action of a regular semisimple element on this latter module we deduce that $\dim C_{V(2\lambda_i)}(x) \leqslant 5$. Since $\dim C_{\mathcal{L}(\bar{H}^0)}(x) = 4$, it follows that $\dim C_{\bar{G}}(x) = \dim C_V(x) \leqslant 19$ and the claim follows.

Finally, suppose $x \in G$ is a field automorphism. Then $|x^G| > \frac{1}{2}q^{133/2} = b_5$ and we note that $|H| < 3\log_3 q.q^{28} = a_5$. We conclude that

$$\mathcal{Q}(G,H) < \sum_{i=1}^{5} a_i^2/b_i < q^{-1} \tag{17}$$

as required. □

Lemma 5.7. *If $G_0 = F_4(q)$ and $\mathrm{soc}(H) = G_2(q)$, then $b(G,H) = 2$.*

Proof. Here $p = 7$, $H_0 = G_2(q)$ and we proceed as in the proof of the previous lemma. To get started, let $x \in H_0$ be an element of prime order r. First assume $r = 7$, so x is unipotent. The G-class of x is determined in [41, Section 5.2] and we deduce that $|x^G| > q^{28} = b_1$ (minimal if x is contained in the class labelled $A_1\tilde{A}_1$), so the contribution to $\mathcal{Q}(G,H)$ is less than a_1^2/b_1, where $a_1 = q^{12}$ is the total number of unipotent elements in H_0 (see Proposition 2.14(ii)). If $r \neq 2, 7$ then Proposition 2.11 gives $|x^G| > (q-1)q^{29} = b_2$ and we note that $|H_0| < q^{14} = a_2$.

Now assume $r = 2$, so $|x^G \cap H| = q^4(q^4+q^2+1) = a_3$ since H_0 has a unique class of involutions. There are two classes of involutions in G_0 and we claim that $C_{\bar{G}}(x) = A_1C_3$ rather than B_4. To see this, let V_{26} be the minimal module for $\bar{G}$. By considering the restriction of V_{26} to $C_{G_2}(x) = A_1\tilde{A}_1$, one checks that x has trace 2 on V (see [75, Section 6.6]) and thus $C_{\bar{G}}(x) = A_1C_3$ as claimed (as noted in [54, Proposition 1.2], the involutions $y \in \bar{G}$ with $C_{\bar{G}}(y) = B_4$ have trace -6 on V_{26}). Therefore, $|x^G| > q^{28} = b_3$.

Finally, let us assume $x \in G$ is a field automorphism of order r. If $r = 2$ then $|x^G| > \frac{1}{2}q^{26} = b_4$ and $|x^G \cap H| = |G_2(q) : G_2(q^{1/2})| < 2q^7 = a_4$. On the other hand, if $r \geqslant 3$ then $|x^G| > \frac{1}{2}q^{34} = b_5$ and we note that $|H| < \log_7 q.q^{14} = a_5$.

By bringing these estimates together, we conclude that (17) holds and the result follows. □

Lemma 5.8. *If $(G_0, \mathrm{soc}(H)) = (E_6^{\epsilon}(q), G_2(q))$ or $(G_2(q), \mathrm{L}_2(q))$, then $b(G,H) = 2$.*

Proof. First assume $G_0 = G_2(q)$ and $\mathrm{soc}(H) = \mathrm{L}_2(q)$, in which case $p \geqslant 7$. If $x \in H_0$ has order 3 then $|x^G| \geqslant q^3(q^3-1) = b_1$ and we record that $i_3(H_0) \leqslant q(q+1) = a_1$. As explained in [41, Section 5.1], the elements of order p in H_0 are regular in G. Therefore, if $x \in H$ has prime order and x is not a semisimple element of order 3, then $|x^G| \geqslant q^3(q+1)(q^3+1) = b_2$ (minimal if x is an involutory field automorphism) and we note that $|H| \leqslant q(q^2-1).\log_7 q = a_2$. This gives $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2 < q^{-1/2}$ and we are done.

Finally, let us assume $G_0 = E_6^{\epsilon}(q)$ and $\mathrm{soc}(H) = G_2(q)$. From [41, Section 5.5], we see that H does not contain any long root elements of G_0. Let $x \in H$ be an element of prime order r. If $r \geqslant 3$ then $|x^G| > (q-1)q^{31} = b_1$ and we note that $|H| < 2\log_2 q.q^{14} = a_1$. On the other hand, if $r = 2$ then $|x^G| > \frac{1}{3}(q-1)q^{25} = b_2$ and Proposition 2.14(i) implies that $i_2(H) \leqslant i_2(\mathrm{Aut}(G_2(q))) < 2(q+1)q^7 = a_2$. Therefore, $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2 < q^{-1/2}$ and the result follows. □

We are now in a position to complete the proof of Proposition 5.5.

Proof of Proposition 5.5. In each of the remaining cases we find that the trivial bound $\mathcal{Q}(G,H) < a_1^2/b_1$ is sufficient, where $a_1 = |\mathrm{Aut}(\mathrm{soc}(H))|$ is an upper bound on $|H|$ and b_1 is the size of the smallest nontrivial conjugacy class in G, which can be read off from Proposition 2.11. For example, if $G_0 = E_6^\epsilon(q)$ and $\mathrm{soc}(H) = \mathrm{L}_3^{\pm}(q)$, then $p \geqslant 5$ and the bounds

$$|H| \leqslant |\mathrm{Aut}(\mathrm{L}_3^{\pm}(q))| < 2\log_5 q.q^8 = a_1, \;\; |x^G| > (q-1)q^{21} = b_1$$

are sufficient. □

6. Subfield subgroups

In this section, we prove Theorem 1 when H is one of the subgroups in part (II) of Theorem 5.1. In particular, G_0 is one of the groups in (15) and H is either a subfield subgroup corresponding to a maximal subfield of $\mathbb{F}_q$, or H is a twisted subgroup of type ${}^2E_6(q^{1/2})$ or ${}^2F_4(q)$ with $G_0 = E_6(q)$ or $F_4(q)$ (for $q = 2^{2m+1}$), respectively. The remaining subfield subgroups will be handled in Section 8.

Our main result is the following.

Theorem 6.1. *If H is a Type II subgroup of G, then G is not extremely primitive.*

We partition the proof of Theorem 6.1 into a sequence of lemmas.

Lemma 6.2. *The conclusion to Theorem 6.1 holds when $G_0 = G_2(q)$ and H is of type $G_2(q_0)$.*

Proof. Here $H_0 = G_2(q_0)$ and $q = q_0^k$, where k is a prime. For k odd, we claim that $b(G,H) = 2$, so G is not extremely primitive by Lemma 2.6.

First assume $k \geqslant 5$. If $x \in G_0$ is either a long root element (or a short root element when $p = 3$) or a semisimple element of order 3 with $C_{\bar{G}}(x) = A_2$, then $|x^G| > (q-1)q^5 = b_1$ and we note that H contains fewer than $2q_0^6 \leqslant 2q^{6/5} = a_1$ such elements. For all other nontrivial elements $x \in G$ we have $|x^G| > q^7 = b_2$ and thus $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2$, where $a_2 = 2\log_2 q.q^{14/5}$ is an upper bound on $|H|$. One checks that the given upper bound on $\mathcal{Q}(G,H)$ is always less than 1 (and it tends to zero as q tends to infinity).

A more detailed analysis is required when $k = 3$. Let $x \in G_0$ be an element of prime order r and first assume $r = 2$. If $p \neq 2$ then G_0 and H_0 both have a unique conjugacy class of involutions and we get $|x^G \cap H| < 2q^{8/3} = a_1$ and $|x^G| > q^8 = b_1$. Similarly, if $p = 2$ and x is a long root element, then $|x^G \cap H| < q^2 = a_2$ and $|x^G| > (q-1)q^5 = b_2$, whereas $|x^G \cap H| < q^{8/3} = a_3$ and $|x^G| > (q-1)q^7 = b_3$ if x is a short root element.

Next assume $r = p \geqslant 3$. As above, the contribution from long root elements is less than a_2^2/b_2. Similarly, short root elements contribute at most a_2^2/b_2 if $p = 3$ and at most a_3^2/b_3 if $p \geqslant 5$. Since $a_3^2/b_3 < a_2^2/b_2$, it follows that the combined contribution from long and short root elements is less than $2a_2^2/b_2$ for all p. If $p = 3$ and x is in the class labelled $(\tilde{A}_1)_3$ in [57, Table 22.2.6] then $|x^G \cap H| < a_3$ and $|x^G| > b_3$. For all other unipotent elements, we have $|x^G| > \frac{1}{7}q^{10} = b_4$ and we note that H_0 contains precisely $q^4 = a_4$ unipotent elements (see Proposition 2.14(ii)).

Now assume $x \in G_0$ is semisimple with $r \geqslant 3$. If $r = 3$ and $C_{\bar{G}}(x) = A_2$ then $|x^G \cap H| < 2q^2 = a_5$ and $|x^G| > (q-1)q^5 = b_5$. In every other case, $|x^G| > (q-1)q^9 = b_6$ and we note that $|H_0| < q^{14/3} = a_6$.

To complete the analysis for $k = 3$, we may assume x is a field or graph automorphism of G_0. Suppose x is a field automorphism of order r, so $q = q_1^r$. If $r = 2$ then $|x^G| > q^7 = b_7$ and there are fewer than $2q^{7/3} = a_7$ such elements in H. Similarly, if $r \geqslant 5$ then $|x^G| > \frac{1}{2}q^{56/5} = b_8$ and we note that $|H| < 2\log_2 q.q^{14/3} = a_8$. If $r = 3$ then $|x^G| > \frac{1}{2}q^{28/3} = b_9$ and we may assume x centralizes H_0, so there are precisely $2(i_3(H_0)+1)$ such elements in H. By applying Proposition 2.14(i), we deduce that $2(i_3(H_0)+1) \leqslant 4(q^{1/3}+1)q^3 = a_9$. Finally, if

x is an involutory graph-field automorphism of G_0 then x acts as an involutory graph-field automorphism on H_0, whence $|x^G \cap H| < 2q^{7/3} = a_{10}$ and $|x^G| > q^7 = b_{10}$.

Bringing the above bounds together, we conclude that

$$\mathcal{Q}(G,H) < a_2^2/b_2 + \sum_{i=1}^{10} a_i^2/b_i$$

and one checks that this bound is sufficient.

Finally, let us assume $k = 2$. In [38], Lawther calculates the subdegrees for G_0 and by inspecting [38, Tables 2-4] we find that H_0 has an orbit of length $\frac{1}{2}q^2(q^6-1)(q^2-1)$. In particular, there exists $g \in G_0$ such that $|H_0 \cap H_0^g| = 2q^4$. By inspecting [6, Tables 8.30, 8.41 and 8.42], we see that H does not have a maximal subgroup M with $|M \cap H_0| = 2q^4$. Therefore, $H \cap H^g$ is a non-maximal subgroup of H and we conclude that G is not extremely primitive. □

Lemma 6.3. *The conclusion to Theorem 6.1 holds when $G_0 = G_2(q)$ and H is of type ${}^2G_2(q)$.*

Proof. Here $p = 3$ and $H_0 = {}^2G_2(q)$. The subdegrees for the action of G_0 are recorded in [38, Table 1] and we see that H_0 has an orbit of length $(q^3+1)(q-1)$. Therefore, there exists an element $g \in G_0$ such that $|H_0 \cap H_0^g| = q^3$. By inspecting [6, Table 8.43], we deduce that there is no maximal subgroup M of H with $|M \cap H_0| = q^3$. The result follows. □

Now let us turn to the remaining subfield subgroups that we are handling in this section, so one of the following holds:

(a) $G_0 = L(q)$ and H_0 is of type $L(q_0)$, where $L \in \{F_4, E_6^\epsilon, E_7, E_8\}$ and $q = q_0^k$ with k a prime.
(b) $G_0 = E_6(q)$, $q = q_0^2$ and H_0 is of type ${}^2E_6(q_0)$.
(c) $G_0 = F_4(q)$, $q = 2^{2m+1}$ and $H_0 = {}^2F_4(q)$.

First let us consider the cases in (a). As before, let $\bar{G}$ be the ambient simple algebraic group and fix a Steinberg endomorphism ψ of $\bar{G}$ such that $\mathrm{soc}(H_0) = (\bar{G}_\psi)'$ and $G_0 = (\bar{G}_{\psi^k})'$. Here $\psi = \sigma\tau$, where σ is a standard Frobenius morphism corresponding to the map $\lambda \mapsto \lambda^{q_0}$ on $\mathbb{F}_q$ and either $\tau = 1$, or $G_0 = {}^2E_6(q)$, k is odd and τ is an involutory graph automorphism of $\bar{G}$.

Remark 6.4. Let us record that H_0 is simple, unless one of the following holds:

(i) $G_0 = E_6^\epsilon(q)$, $k = 3$ and $q \equiv \epsilon \pmod 3$, in which case $H_0 = \mathrm{Inndiag}(E_6^\epsilon(q_0))$.
(ii) $G_0 = E_7(q)$, $k = 2$ and q is odd, in which case $H_0 = \mathrm{Inndiag}(E_7(q_0))$.

Set $\bar{X} = \langle X_{\pm\alpha_0}\rangle$ and $\bar{Y} = C_{\bar{G}}(\bar{X})^0$, where α_0 is the highest root of $\bar{G}$. Then $\bar{M} = \bar{X}\bar{Y}$ is a maximal rank subgroup of $\bar{G}$ of type A_1C_3, A_1A_5, A_1D_6 or A_1E_7 for $\bar{G} = F_4, E_6, E_7$ or E_8, respectively. Since $\bar{X}$ is ψ-stable, it follows that $\bar{M}$ is ψ-stable and by taking fixed points we get

$$\bar{M}_\psi = \begin{cases} d.(\mathrm{L}_2(q_0) \times \mathrm{PSp}_6(q_0)).d & \text{if } \bar{G} = F_4 \\ d.(\mathrm{L}_2(q_0) \times \mathrm{L}_6^\epsilon(q_0)).de & \text{if } \bar{G} = E_6 \\ d.(\mathrm{L}_2(q_0) \times \mathrm{P}\Omega_{12}^+(q_0)).d & \text{if } \bar{G} = E_7 \\ d.(\mathrm{L}_2(q_0) \times E_7(q_0)).d & \text{if } \bar{G} = E_8, \end{cases}$$

where $d = (2, q-1)$ and $e = (3, q-\epsilon)$. By inspecting [50, Table 5.1], it follows that $\bar{M}_\psi$ is a maximal subgroup of $\bar{G}_\psi$, unless $(\bar{G}, p) = (F_4, 2)$, in which case $\bar{M}_\psi < \mathrm{Sp}_8(q_0) < \bar{G}_\psi$. Set

$$M = \begin{cases} \mathrm{Sp}_8(q_0) & \text{if } (\bar{G},p) = (F_4,2) \\ \bar{M}_\psi \cap H_0 & \text{otherwise} \end{cases}$$

and $Y = (\bar{Y}_\psi)'$, so that $\mathrm{SL}_2(q_0) \circ Y$ is a subgroup of M. We also set $K = T \circ Y$, where $T = Z_{q_0+1}$ is a maximal torus of $\mathrm{SL}_2(q_0)$.

Lemma 6.5. *The only maximal overgroup of K in H_0 is M. In addition, if $(\bar{G}, p) = (F_4, 2)$ then $\bar{M}_\psi$ is the unique maximal overgroup of K in M.*

Proof. By construction, K is contained in M, which in turn is a maximal subgroup of H_0 by [50]. By inspection, we observe that K is not contained in a parabolic subgroup of H_0.

Suppose K is contained in a non-parabolic maximal subgroup J of H_0. If $H_0 = E_8(q_0)$ then $|J| \geqslant |K| > q_0^{88}$ and thus [17, Lemma 4.2] implies that J is of type $E_8(q_0^{1/2})$, $A_1(q_0)E_7(q_0)$ or $D_8(q_0)$ and we immediately deduce that J must be H_0-conjugate to M. A very similar argument applies if $H_0 = E_7(q_0)$ or $E_6^\epsilon(q_0)$, using [17, Lemmas 4.6 and 4.13]. For $H_0 = F_4(q_0)$ with $q_0 \geqslant 3$ we appeal to [10, Lemma 3.15], noting that $|J| > q_0^{17}$, and we use [67] for $q_0 = 2$. Note that if $H_0 = F_4(q_0)$ and q_0 is even, then H_0 has two classes of subgroups isomorphic to $\mathrm{Sp}_8(q_0)$, but K is only contained in conjugates of M. To see this, first note that the subgroups in the other class are of the form $\bar{L}_\psi$, where $\bar{L}$ is of type B_4. Let $V = V_{26}$ be the irreducible 26-dimensional module for $\bar{G} = F_4$ with highest weight λ_4 (in terms of the usual labelling). By [75, Chapter 12, Table 2], K acts on V with composition factors of dimension $14, 6, 6$ and this is incompatible with the action of $\bar{L}$, which has composition factors of dimension $16, 8, 1, 1$.

Next we claim that K is contained in a unique conjugate of M. If $(\bar{G}, p) \neq (F_4, 2)$ then Y is normal in M, which in turn is a maximal subgroup of H_0, so $M = N_{H_0}(Y)$. Since $N_{H_0}(K) \leqslant N_{H_0}(Y)$, we quickly deduce that K is contained in a unique conjugate of M. Indeed, suppose $K < M^g$, so $K, K^{g^{-1}} < M$. Then since M contains a unique class of subgroups H_0-conjugate to K we have $K = K^{mg}$ for some $m \in M$. But $mg \in N_{H_0}(K) \leqslant M$ and thus $g \in M$.

Finally, let us assume that $(\bar{G}, p) = (F_4, 2)$, so

$$K = Z_{q_0+1} \times \mathrm{Sp}_6(q_0) < \bar{M}_\psi = \mathrm{SL}_2(q_0) \times \mathrm{Sp}_6(q_0) < M = \mathrm{Sp}_8(q_0) < H_0 = F_4(q_0).$$

We claim that $N_{H_0}(K) \leqslant \bar{M}_\psi$ and the H_0-class of K does not split in either $\bar{M}_\psi$ or M. Then by repeating the argument in the previous paragraph, we deduce that M is the unique maximal overgroup of K in H_0, and $\bar{M}_\psi$ is the unique maximal overgroup of K in M, which completes the proof of the lemma.

To prove the claim, we first show that M contains a unique class of subgroups isomorphic to K and that $N_M(K) = K.2$. To do this, we use [6, Tables 8.48 and 8.49] to check that any maximal subgroup of M containing a subgroup isomorphic to K is necessarily conjugate to $\bar{M}_\psi$. It follows that $N_M(K)$ must be contained in a conjugate of $\bar{M}_\psi$ and since $N_{\bar{M}_\psi}(K) = K.2$ we conclude that $N_M(K) = K.2 \leqslant \bar{M}_\psi$. Moreover, it is clear that $\bar{M}_\psi$ contains a unique class of subgroups isomorphic to K, so it follows that the H_0-class of K does not split in M nor in $\bar{M}_\psi$. Finally, let us consider $N_{H_0}(K)$. Since this is a proper subgroup of H_0 containing K, it is therefore contained in a conjugate of M. So $N_{H_0}(K) = N_{M^g}(K)$ for some $g \in H_0$ and we deduce that

$$N_{H_0}(K) = N_M(K^{g^{-1}})^g = (K^{g^{-1}}.2)^g = K.2 = N_{\bar{M}_\psi}(K) \leqslant \bar{M}_\psi.$$

This justifies the claim and the proof of the lemma is complete. □

Proposition 6.6. *The conclusion to Theorem 6.1 holds if $G_0 = F_4(q)$, $E_6^\epsilon(q)$, $E_7(q)$ or $E_8(q)$ and H is of type $F_4(q_0)$, $E_6^\epsilon(q_0)$, $E_7(q_0)$ or $E_8(q_0)$, respectively.*

Proof. Write $G = G_0.A$, $H = H_0.A$ and let us adopt the notation introduced above.

Suppose there exists $g \in N_{G_0}(K)$ that does not normalize M. Then $M^g \not\leqslant H_0$ and the result now follows via Lemma 2.4, unless $(\bar{G}, p) = (F_4, 2)$ and G contains graph automorphisms (indeed, we may assume K and M are A-stable in all the other cases). Let us assume we are in the latter situation, so $K = Z_{q_0+1} \times Y$ with $Y = \mathrm{Sp}_6(q_0)$. Suppose $H \cap H^g$ is a maximal subgroup of H. We claim that $H \cap H^g = H_0.B$ for some maximal subgroup B of A. Now A contains a graph automorphism τ such that Y and Y^τ are non-conjugate subgroups of H_0, so $\langle Y, Y^\tau \rangle = H_0$ and thus H_0 is the only A-stable subgroup of H_0 containing K. This justifies the claim. Therefore, H^g contains H_0 and H_0^g. But H_0 and H_0^g generate G_0 since they are both maximal subgroups of G_0 and g does not normalize H_0 (this follows from Lemma 6.5; if g normalizes H_0 then it must also normalize the unique maximal overgroup of K in H_0, which is M). This is absurd since H^g does not contain G_0. So we have reached a contradiction and we conclude that $H \cap H^g$ is not maximal in H, whence G is not extremely primitive.

In view of the above remarks, in order to prove the proposition it suffices to show that there exists $g \in N_{G_0}(K)$ with $M \neq M^g$. To do this, take an element $g \in \bar{X}_{\psi^k} = \mathrm{SL}_2(q)$ that centralizes $T = Z_{q_0+1}$ but does not normalize $\bar{X}_\psi = \mathrm{SL}_2(q_0)$. Such an element g exists because the centralizer of T in $\bar{X}_{\psi^k}$ is a cyclic torus of order $q_0^k - (-1)^k$ which is clearly not a subgroup of $N_{\bar{X}_{\psi^k}}(\bar{X}_\psi) = \bar{X}_\psi.a$ (where $a \in \{1, 2\}$).

Since $g \in \bar{X}$, it follows that g centralizes Y and $K = T \circ Y$. In addition, since $\bar{X}_\psi \circ Y$ is characteristic in $\bar{M}_\psi \cap H_0$, it follows that g does not normalize $\bar{M}_\psi \cap H_0$. If $(\bar{G}, p) \neq (F_4, 2)$ then $\bar{M}_\psi \cap H_0 = M$ and we have proved the claim. So suppose that $(\bar{G}, p) = (F_4, 2)$. Then Lemma 6.5 shows that $\bar{M}_\psi$ is the unique maximal overgroup of K in M. Therefore, if g normalizes M then it must also normalize the unique maximal overgroup of K in M. But we have just observed that g does not normalize $\bar{M}_\psi \cap H_0 = \bar{M}_\psi$ and this completes the proof of the proposition. □

Finally, we handle the twisted maximal subgroups that arise when $G_0 = E_6(q)$ or $F_4(q)$ (with $p = 2$ in the latter case).

Proposition 6.7. *The conclusion to Theorem 6.1 holds if $G_0 = E_6(q)$ or $F_4(q)$ and H is of type ${}^2E_6(q^{1/2})$ or ${}^2F_4(q)$, respectively.*

Proof. First assume $G_0 = E_6(q)$, where $q = q_0^2$. Set $\psi = \sigma\tau$, where σ is a standard Frobenius morphism of $\bar{G} = E_6$ and τ is a graph automorphism such that $(\bar{G}_\psi)' = \mathrm{soc}(H_0) = {}^2E_6(q_0)$ and $(\bar{G}_{\psi^2})' = G_0$.

Set $\bar{M} = \bar{X}\bar{Y} = A_1A_5$, where $\bar{X} = \langle X_{\pm\alpha_0} \rangle$ and $\bar{Y} = C_{\bar{G}}(\bar{X})^0$. Then $\bar{M}$ is ψ-stable and [50] implies that

$$M = \bar{M}_\psi \cap H_0 = (\mathrm{SL}_2(q_0) \circ d.\mathrm{U}_6(q_0)).d = d.(\mathrm{L}_2(q_0) \times \mathrm{U}_6(q_0)).d$$

is a maximal subgroup of H_0, where $d = (2, q-1)$. Set $K = Z_{q_0+1} \circ \mathrm{U}_6(q_0) < M$. By arguing as in the proof of Lemma 6.5, we deduce that M is the unique maximal overgroup of K in H_0. Similarly, by repeating the argument in the proof of Proposition 6.6, we see that there exists $g \in N_{G_0}(K)$ such that $M \neq M^g$ and we conclude that G is not extremely primitive by applying Lemma 2.4.

Finally, let us assume $G_0 = F_4(q)$ and $H_0 = {}^2F_4(q)$, where $q = 2^{2m+1}$ and $m \geqslant 0$. Let $\bar{G} = F_4$ and set $\psi = \sigma\tau$, where σ is the standard Frobenius morphism of $\bar{G}$ corresponding to the map $\lambda \mapsto \lambda^{2^m}$ on $\mathbb{F}_q$ and τ is the standard graph automorphism of $\bar{G}$, which interchanges long root and short root subgroups. Then $G_0 = \bar{G}_{\psi^2}$ and $H_0 = \bar{G}_\psi$.

The case $q = 2$ can be handled in Magma (see [21, Lemma 2.10]). More precisely, we construct $H_0 < G_0$ as permutation groups and we use random search to find an element $g \in G_0$ such that $H_0 \cap H_0^g = 1$. Therefore, $b(G_0, H_0) = 2$ and more generally we have $|H \cap H^g| \leqslant 2$. In particular, $H \cap H^g$ is not a maximal subgroup of H and thus G is not extremely primitive. For the remainder, we will assume $q \geqslant 8$.

Fix a set of simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ for $\bar{G} = F_4$ and consider the following roots

$$\beta_1 = \alpha_1 + \alpha_2 + \alpha_3 \qquad \beta_2 = \alpha_2 + 2\alpha_3 + 2\alpha_4$$
$$\beta_3 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \quad \beta_4 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 = \alpha_0$$

where β_1, β_3 are short and β_2, β_4 are long. Note that ψ acts on root elements as follows

$$x_{\beta_1}(c) \mapsto x_{\beta_2}(c^{2^{m+1}}),\ x_{\beta_2}(c) \mapsto x_{\beta_1}(c^{2^m}),\ x_{\beta_3}(c) \mapsto x_{\beta_4}(c^{2^{m+1}}),\ x_{\beta_4}(c) \mapsto x_{\beta_3}(c^{2^m})$$

for all $c \in \bar{\mathbb{F}}_2$. Let $\bar{P} = P_{1,4}$ be the standard parabolic subgroup of $\bar{G}$ with $\bar{L}' = \langle X_{\pm\alpha_2}, X_{\pm\alpha_3}\rangle$ of type B_2, where $\bar{L}$ is a Levi factor of $\bar{P}$. Then $C_{\bar{G}}(\bar{L}')^0 = \langle X_{\pm\beta_1}, X_{\pm\beta_2}, X_{\pm\beta_3}, X_{\pm\beta_4}\rangle = B_2$.

Consider the maximal subgroup $M = {}^2B_2(q) \wr \mathrm{Sym}_2$ of H_0 (see [63]) and set

$$K = {}^2B_2(q) \times Z_{q-1} < M.$$

By choosing M appropriately, we may assume that K is a Levi subgroup of the maximal parabolic $P = \bar{P}_\psi$ of H_0. Let $\mathcal{M}$ be the set of maximal overgroups of K in H_0. We claim that $\mathcal{M} = \{M, P, P^{\rm op}\}$, where $P^{\rm op} = (\bar{P}^{\rm op})_\psi$ is the opposite parabolic subgroup to P.

Firstly, by inspecting [63] we deduce that each subgroup in $\mathcal{M}$ is conjugate to M or P. Let X be the ${}^2B_2(q)$ factor in K and set $N = {}^2B_2(q) \times {}^2B_2(q) < M$. Now $N \leqslant N_{H_0}(X)$ since X is normal in N, but M is clearly the unique maximal overgroup of N in H_0, and X is not normal in M, so $N_{H_0}(X) = N$ and we deduce that $N_{H_0}(K) \leqslant N$. By [74, Proposition 3], the normalizer of the torus $Z_{q-1} < {}^2B_2(q)$ is a dihedral group $D_{2(q-1)}$ and so

$$N_{H_0}(K) = N_N(K) = K.2 = {}^2B_2(q) \times D_{2(q-1)}.$$

Since $N_{H_0}(K) = K.2 < M$, it follows that M is the only H_0-conjugate of M containing K. Now let us turn to the conjugates of P in $\mathcal{M}$. Here we proceed as in the proof of Lemma 4.10 (see the argument for the case $q = 2$), noting that if $K \leqslant P \cap P^h$ for some $h \in H_0$, then K and $K^{h^{-1}}$ are P-conjugate (this follows from [64, Proposition 26.1(b)]).

By the claim, each subgroup in $\mathcal{M}$ contains $x_1 = x_{\beta_3}(1)x_{\beta_4}(1)$ or $x_2 = x_{-\beta_3}(1)x_{-\beta_4}(1)$.

The subgroup K is contained in a Levi subgroup $\mathrm{Sp}_4(q) \times Z_{q-1}^2$ of the parabolic subgroup $\bar{P}_{\psi^2}$ of G_0, where $Z_{q-1}^2 = \langle h_{\beta_1}(c), h_{\beta_2}(c') \,:\, c, c' \in \mathbb{F}_q^\times\rangle$ in terms of the standard Lie notation. Since $X < \mathrm{Sp}_4(q)$ it follows that $Z_{q-1}^2 \leqslant N_{G_0}(K)$. Fix $1 \neq c \in \mathbb{F}_q^\times$ and set $g = h_{\beta_3}(c) \in N_{G_0}(K)$ so $x_1^g = x_{\beta_3}(c^2)x_{\beta_4}(c^2)$ (here we are using the fact that $q \geqslant 8$). Now

$$\psi(x_{\beta_3}(c^2)x_{\beta_4}(c^2)) = x_{\beta_4}((c^2)^{2^{m+1}})x_{\beta_3}((c^2)^{2^m}) = x_{\beta_3}(c^{2^{m+1}})x_{\beta_4}(c^{2^{m+2}})$$

and thus x_1^g is not fixed by ψ. In particular, $x_1^g \notin H_0$. An entirely similar calculation shows that $\psi(x_2^g) \neq x_2^g$. Therefore, for each $J \in \mathcal{M}$ we have $J^g \not\leqslant H_0$. Finally, since K and the three subgroups in $\mathcal{M}$ are stable under all automorphisms of G_0, the desired conclusion follows from Lemma 2.4. □

7. Almost simple subgroups

In this section, we complete the proof of Theorem 1 for the groups with socle G_0 as in (15). To do this, it remains to handle the subgroups of Type V in Theorem 5.1. Our main result is the following (we have been unable to determine the exact base size in (ii)).

Theorem 7.1. *If H is a Type V subgroup of G, then G is not extremely primitive. Moreover, $b(G,H) = 2$ unless one of the following holds:*

(i) $G_0 = {}^2E_6(2)$, $S = \mathrm{Fi}_{22}$ *and* $b(G,H) = 3$;
(ii) $G_0 = F_4(2)$, $S = \mathrm{L}_4(3)$ *and* $b(G,H) \leqslant 3$.

The following theorem, which is part of [55, Theorem 8], describes the possibilities for the socle of a Type V subgroup (note that the value of $u(E_8(q))$ in part (ii)(c) is taken from [42]). In the statement, if X is a simple group of Lie type then $\mathrm{rk}(X)$ is the (untwisted) Lie rank of X. In addition, $\mathrm{Lie}(p)$ denotes the set of finite simple groups of Lie type defined over fields of characteristic p.

Theorem 7.2. *Let G be an almost simple group with socle G_0, a simple exceptional group of Lie type over $\mathbb{F}_q$, where $q = p^f$ with p a prime. Assume G_0 is one of the groups in* (15) *and let H be a maximal almost simple subgroup of G as in part (V) of Theorem 5.1, with socle S. Then one of the following holds:*

(i) $S \notin \mathrm{Lie}(p)$ *and the possibilities for S are described in* [54]*;*

(ii) $S = H(q_0) \in \mathrm{Lie}(p)$*,* $\mathrm{rk}(S) \leqslant \frac{1}{2}\mathrm{rk}(G_0)$ *and one of the following holds:*
  (a) $q_0 \leqslant 9$*;*
  (b) $S = \mathrm{L}_3^{\epsilon}(16)$*;*
  (c) $S = \mathrm{L}_2(q_0)$*,* ${}^2B_2(q_0)$ *or* ${}^2G_2(q_0)$*, where $q_0 \leqslant (2, q-1)u(G_0)$ and $u(G_0)$ is defined in the following table.*

G_0	$G_2(q)$	$F_4(q)$	$E_6^{\epsilon}(q)$	$E_7(q)$	$E_8(q)$
$u(G_0)$	12	68	124	388	1312

More recently, the list of possibilities for S in parts (i) and (ii) of Theorem 7.2 has been significantly refined. For the so-called *non-generic* subgroups arising in (i), we refer the reader to Litterick [60] and Craven [28]. For instance, the main theorem of [28] states that $S = \mathrm{Alt}_n$ only if $n = 6$ or 7. Craven has also made substantial progress in eliminating many *generic* subgroups in (ii). Indeed, by combining the main results of [29, 30], we get the following theorem.

Theorem 7.3 (Craven)**.** *Let G be an almost simple group with socle G_0, a simple exceptional group of Lie type over $\mathbb{F}_q$, where $q = p^f$ with p a prime. Assume G_0 is one of the groups in* (15) *and let H be a maximal almost simple subgroup of G as in part (V) of Theorem 5.1, with socle $S \in \mathrm{Lie}(p)$. Then one of the following holds:*

(i) $G_0 = E_8(q)$ *and either* $S = \mathrm{L}_2(q_0)$ *with* $q_0 \leqslant (2, q-1)u(G_0)$*, or*
$$S \in \{\mathrm{L}_3^{\epsilon}(3), \mathrm{L}_3^{\epsilon}(4), \mathrm{U}_3(8), \mathrm{PSp}_4(2)', \mathrm{U}_4(2), {}^2B_2(8)\};$$

(ii) $G_0 = E_7(q)$ *and* $S = \mathrm{L}_2(q_0)$ *with* $q_0 \in \{7, 8, 25\}$*.*

We begin the proof of Theorem 7.1 by handling the groups with socle $G_0 = G_2(q)$.

Proposition 7.4. *The conclusion to Theorem 7.1 holds when $G_0 = G_2(q)$.*

Proof. In view of Theorem 2.15 (and Remark 2.16), we may assume $q \geqslant 7$. By inspecting [27, 34] (also see [6, Tables 8.30, 8.41 and 8.42]), we observe that there are four cases to consider (in each case $S \notin \mathrm{Lie}(p)$):

(a) $S = \mathrm{L}_2(13)$ and either $q = p \equiv 1, 3, 4, 9, 10, 12 \pmod{13}$, or $q = p^2$ and $p \equiv 2, 5, 6, 7, 8, 11 \pmod{13}$;

(b) $S = \mathrm{L}_2(8)$ and either $q = p \equiv 1, 8 \pmod 9$, or $q = p^3$ and $p \equiv 2, 4, 5, 7 \pmod 9$;

(c) $G = G_2(q)$, $H = \mathrm{U}_3(3){:}2 = G_2(2)$ and $q = p \geqslant 7$;

(d) $G = G_2(11)$ and $H = \mathrm{J}_1$.

In all four cases, we claim that $b(G, H) = 2$.

In (a) and (b), we have $|H| \leqslant |\mathrm{PGL}_2(13)| = 2184 = a_1$ and $|x^G| \geqslant q^3(q^3-1) = b_1$ for all nontrivial $x \in G$. Moreover, the given conditions imply that $q \geqslant 17$ and thus $\mathcal{Q}(G, H) < a_1^2/b_1 < 1$ (in addition, this upper bound is less than q^{-1} if $q \geqslant 23$).

Next consider (c), so $|H| = 12096$. Let $x \in H$ be an element of prime order r and note that $r \in \{2, 3, 7\}$. Let us also observe that
$$i_2(H) = 315 = a_1, \; i_3(H) = 728 = a_2, \; i_7(H) = 1728 = a_3.$$
If $r = 2$ then G has a unique conjugacy class of elements of order r and we have $|x^G| = q^4(q^4+q^2+1) = b_1$. Similarly, if $r = 3$ then $|x^G| \geqslant q^3(q^3-1) = b_2$ and for $r = 7$ and $q \geqslant 11$

we get $|x^G| \geqslant q^5(q^3-1)(q^2-q+1) = b_3$. Putting these estimates together, we deduce that $\mathcal{Q}(G,H) < \sum_{i=1}^{3} a_i^2/b_i$, which is less than 1 if $q \geqslant 11$ (and it is less than q^{-1} if $q \geqslant 17$).

To complete the proof in case (c), we may assume $q = 7$. Here we need to be more careful when estimating the contributions to $\mathcal{Q}(G,H)$ from elements of order 3 and 7. To do this, let V be the minimal 7-dimensional module for G over $\mathbb{F}_7$ and observe that H acts irreducibly on V (see [34, Theorem A], for example). With the aid of MAGMA [4], we can compute the action of each $x \in H$ on V (we refer the reader to [21, Lemma 2.11] for the details of these computations). If x has order 7, then we find that x has Jordan form (J_7) on V and thus x is a regular unipotent element in G (see [40, Table 1]). Therefore, the contribution to $\mathcal{Q}(G,H)$ from elements of order 7 is precisely a_2^2/b_2, where $a_2 = 1728$ and $b_2 = 7^4(7^2-1)(7^6-1)$.

Finally, suppose $x \in H$ has order 3 and note that both H and G have two conjugacy classes of elements of order 3. We will write 3A and 3B to denote the two H-classes (they have sizes $56 = a_3$ and $672 = a_4$, respectively). We find that 3A-elements and 3B-elements have Jordan form $(I_1, \omega I_3, \omega^2 I_3)$ and $(I_3, \omega I_2, \omega^2 I_2)$ on V, respectively, where $\omega \in \mathbb{F}_7$ is a primitive cube root of unity. In particular, we see that the two classes are not fused in G. Moreover, we deduce that if $x \in H$ is a 3A-element then $C_G(x) = \mathrm{SL}_3(7)$, so $|x^G| = 7^3(7^3+1) = b_3$, whereas $|x^G| = 7^5(7^6-1)/6 = b_4$ for the elements in 3B. Setting $a_1 = 315$ and $b_1 = 7^4(7^4+7^2+1)$ as before, we conclude that

$$\mathcal{Q}(G,H) = \sum_{i=1}^{4} a_i^2/b_i = \frac{4649}{103243} < 1.$$

Finally, let us turn to case (d). Suppose $x \in H$ has prime order r, so $r \in \{2,3,5,7,11,19\}$ and we note that

$$i_2(H) = 1463 = a_1,\ i_3(H) = 5852 = a_2,\ i_5(H) = 9704 = a_3,$$
$$i_7(H) = 25080 = a_4,\ i_{11}(H) = 27720 = a_5,\ i_{19}(H) = 27720 = a_6.$$

If $r = 2$ then $|x^G| = 11^4(11^4+11^2+1) = b_1$. Similarly, if $r \in \{5,7,19\}$ then $C_{\bar{G}}(x) = A_1T_1$ or T_2 and thus $|x^G| \geqslant 11^5(11^3-1)(11^2-11+1) = b_3 = b_4 = b_6$.

Now assume $r \in \{3,11\}$. Let V be the minimal module for G over $\mathbb{F}_{11}$ and note that H acts irreducibly on V. Using MAGMA, we can compute the action of x on V (see [21, Lemma 2.11]). If $r = 3$ then x has Jordan form $(I_3, \omega I_2, \omega^2 I_2)$ on $V \otimes \bar{\mathbb{F}}_{11}$, so $C_{\bar{G}}(x) \neq A_2$ and thus $|x^G| = 11^5(11^3-1)(11^2-11+1) = b_2$. Finally, if $r = 11$ then x has Jordan form (J_7) on V, so x is a regular unipotent element in G and $|x^G| = 11^4(11^2-1)(11^6-1) = b_5$.

We conclude that

$$\mathcal{Q}(G,H) < \sum_{i=1}^{6} a_i^2/b_i < 1$$

and the proof of the proposition is complete. □

Next we consider the two special cases highlighted in the statement of Theorem 7.1.

Lemma 7.5. *The conclusion to Theorem 7.1 holds if* $G_0 = {}^2E_6(2)$ *and* $S = \mathrm{Fi}_{22}$.

Proof. Here $(G,H) = ({}^2E_6(2), \mathrm{Fi}_{22})$ or $({}^2E_6(2).2, \mathrm{Fi}_{22}.2)$. In both cases, we observe that

$$\frac{\log|G|}{\log|G:H|} > 2$$

which implies that $b(G,H) \geqslant 3$ (in fact, we have $b(G,H) = 3$, as noted in the proof of [17, Proposition 4.21]).

First assume $G = {}^2E_6(2)$ and $H = \mathrm{Fi}_{22}$. Set $\Omega = G/H$. The character tables of G and H are available in the GAP Character Table Library [8], together with the corresponding fusion map from H-classes to G-classes. This allows us to compute $|C_\Omega(x)|$ for each $x \in H$, where $C_\Omega(x)$ is the set of fixed points of x on Ω. In this way, via the Orbit Counting Lemma, we deduce that H has 8 orbits on Ω (see [21, Lemma 2.12] for the details of this computation). Let $d_1, \ldots, d_7$ be the lengths of the nontrivial H-orbits, ordered so that $d_i \leqslant d_{i+1}$ for each i.

Seeking a contradiction, suppose G is extremely primitive. Then each d_i must be the index of a maximal subgroup of $H = \mathrm{Fi}_{22}$ and by inspecting the Web-Atlas [78] we deduce that each d_i is one of the following:

$$3510, 14080, 61776, 142155, 694980, 1216215, 1647360,$$

$$3592512, 3648645, 12812800, 17791488, 679311360.$$

Since

$$1 + 6{\cdot}17791488 + 679311360 < |\Omega|$$

it follows that $d_6 = d_7 = 679311360$. But $1 + 2{\cdot}679311360 > |\Omega|$ and we have reached a contradiction.

The case $G = {}^2E_6(2).2$ with $H = \mathrm{Fi}_{22}.2$ is entirely similar. Once again, by computing fixed points, we find that H has 8 orbits on $\Omega = G/H$ and by inspecting [78] we deduce that if G is extremely primitive then the length of each nontrivial H-orbit is one of

$$3510, 61776, 142155, 694980, 1216215, 1647360, 3612614,$$

$$3648645, 5125120, 12812800, 15206400, 17791488.$$

But $1 + 7{\cdot}17791488 < |\Omega|$, so G is not extremely primitive. □

Lemma 7.6. *The conclusion to Theorem 7.1 holds if* $G_0 = F_4(2)$ *and* $S = \mathrm{L}_4(3)$.

Proof. First assume $G = F_4(2)$, so $H = \mathrm{L}_4(3).2$ (see [67]). The character tables of G and H are available in the GAP Character Table Library [8]. Although the precise fusion of H-classes in G is not available in [8], we can use `PossibleClassFusions` to compute

$$\mathrm{fpr}(x, G/H) = \frac{|x^G \cap H|}{|x^G|}$$

for all $x \in G$ of prime order (there are two possible fusion maps and they both give the same fixed point ratios). This allows us to compute $\mathcal{Q}(G,H)$ precisely and we find that $\mathcal{Q}(G,H) > 1$ (indeed, just the contribution from involutions is greater than 1). If $x_1, \ldots, x_k$ are representatives of the G-classes of elements of prime order, then

$$\sum_{i=1}^{k} |x_i^G| \cdot \mathrm{fpr}(x_i, G/H)^3 < 1$$

and thus $b(G,H) \leqslant 3$ by [18, Corollary 2.4]. Since $\log|G| < 2\log|G/H|$, we cannot rule out $b(G,H) = 2$ and we have been unable to determine the exact base size in this case.

To show that G is not extremely primitive, we can argue as in the proof of the previous lemma. By applying the Orbit Counting Lemma, we deduce that H has 94 orbits on G/H. However, if M is a core-free maximal subgroup of H then $|H : M| \leqslant 10530$ and we have $1 + 93{\cdot}10530 < |G : H|$. We conclude that G is not extremely primitive. (See [21, Lemma 2.13] for further details on the computation.)

The case $G = F_4(2).2$, $H = \mathrm{L}_4(3).2^2$ is entirely similar and we omit the details (here H has 66 orbits on G/H). □

Lemma 7.7. *The conclusion to Theorem 7.1 holds when* $G_0 = E_7(2)$, $E_6^{\epsilon}(2)$ *or* $F_4(2)$.

Proof. By inspecting [3, 37, 67, 77], we see that $(G_0, S) = ({}^2E_6(2), \Omega_7(3))$, $({}^2E_6(2), \mathrm{Fi}_{22})$ or $(F_4(2), \mathrm{L}_4(3))$. The latter two cases were handled in Lemma 7.5 and 7.6, so we may assume that $G_0 = {}^2E_6(2)$ and $S = \Omega_7(3)$, so by [77] we have $(G,H) = ({}^2E_6(2), \Omega_7(3))$ or $({}^2E_6(2).2, \mathrm{SO}_7(3))$. In both cases, by proceeding as in the proof of Lemma 7.6 we can compute $\mathcal{Q}(G,H)$ precisely and we deduce that $\mathcal{Q}(G,H) < 1$ (see [21, Lemma 2.14]). Therefore, $b(G,H) = 2$. □

S	$\bar{G}$
M_{11}	$E_6\,(p=3,5),\ E_8\,(p=3,11)$
M_{12}	$E_6\,(p=5)$
M_{22}	$E_7\,(p=5)$
J_1	$E_6\,(p=11)$
J_2	$E_6\,(p=2),\ E_7\,(p=2)$
J_3	$E_6(p=2),\ E_8\,(p=2)$
Ru	$E_7\,(p=5)$
Fi_{22}	$E_6\,(p=2)$
HS	$E_7\,(p=5)$
Th	$E_8\,(p=3)$

TABLE 8. The possibilities for $\bar{G}$ and S, where S is sporadic

For the remainder of this section, we will assume

$$G_0 \in \{E_8(q), E_7(q), E_6^{\epsilon}(q), F_4(q)\} \setminus \{E_7(2), E_6^{\epsilon}(2), F_4(2)\}. \tag{18}$$

We partition the remainder of the proof of Theorem 7.1 into two parts, according to the cases $S \notin \mathrm{Lie}(p)$ and $S \in \mathrm{Lie}(p)$. Before launching into the details, let us record the following result on long root elements, which will be useful in the subsequent analysis.

Proposition 7.8. *Suppose $p > 2$ and let H be a Type V subgroup of G. Then H does not contain a long root element of G.*

Proof. This follows immediately from [52, Corollary 6.2]. Indeed, if $x \in H$ is a long root element of G, then $x^2 \in H$ is also a long root element and thus [52, Corollary 6.2] implies that $H \leqslant N_G(\bar{H}_\sigma) < G$, where $\bar{H}$ is a σ-stable positive dimensional maximal closed subgroup of $\bar{G}$. But this is incompatible with the definition of a Type V subgroup. □

Remark 7.9. For $p = 2$, it is worth noting that the conclusion to Proposition 7.8 is false in general. For example, $G = {}^2E_6(2)$ has a maximal subgroup $H = \mathrm{Fi}_{22}$ (this case was handled in Lemma 7.5) and we find that the `2A`-involutions in H embed in G as long root elements.

7.1. **Non-generic subgroups.** In this section we handle the non-generic subgroups arising in part (i) of Theorem 7.2, where $S \notin \mathrm{Lie}(p)$.

Lemma 7.10. *The conclusion to Theorem 7.1 holds if $S = \mathrm{Alt}_n$.*

Proof. By the main theorem of [28], we may assume $n \in \{6, 7\}$, so $|H|$ is at most $7! = a_1$ and we note that $|x^G| > q^{16} = b_1$ for all nontrivial $x \in G$. This gives $\mathcal{Q}(G,H) < a_1^2/b_1$, which is less than 1 for all $q \geqslant 3$ (and it is less than q^{-1} for $q \geqslant 4$). Finally, if $q = 2$ then $G = E_8(2)$ is the only option (see [28, Theorem 1]) and $|x^G| > 2^{58}$ for all $1 \neq x \in G$. The result follows as before. □

Lemma 7.11. *The conclusion to Theorem 7.1 holds if S is a sporadic simple group.*

Proof. The possibilities for G_0 and S are recorded in [54, Table 10.2], which is further refined in [60, Theorem 8] to give the list of cases recorded in Table 8. Recall that we may assume G_0 is one of the groups in (18). In each case, we claim that $b(G,H) = 2$.

The cases with $S \in \{\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{J}_1, \mathrm{Ru}, \mathrm{HS}\}$ are very straightforward; we have $|H| \leqslant |\mathrm{Aut}(S)| = a_1$ and by applying Proposition 2.11 we deduce that $|x^G| > f(q) = b_1$ for all $x \in G$ of prime order, where $f(q) = q^{34}$ if $S = \mathrm{M}_{22}$, Ru or HS, otherwise $f(q) = (q-1)q^{21}$. One checks that this gives $\mathcal{Q}(G,H) < a_1^2/b_1 < q^{-1}$ for all q satisfying the restrictions on p in Table 8. The case $S = \mathrm{Th}$ is also straightforward. Here $\bar{G} = E_8$ and $p = 3$, so Proposition 7.8

2A	3510	A_1	3A	3294720	A_5T_1
2B	1216215	A_1^2	3B	25625600	A_2^3
2C	36486450	A_1^3	3C	461260800	D_4T_2
2D	61776	τ	3D	3690086400	A_2^3
2E	19459440	τ'			
2F	22239360	τ'			

TABLE 9. Elements of order 2 and 3 in $\mathrm{Fi}_{22}.2 < E_6(4).2$

implies that there are no long root elements in H. In particular, if $x \in H$ has prime order, then $|x^G| > q^{92} = b_1$ and we deduce that $\mathcal{Q}(G,H) < a_1^2/b_1 < q^{-1}$, where $a_1 = |S|$.

If $S = \mathrm{J}_2$ then the same argument reduces the problem to the cases $G_0 = E_6^{\epsilon}(2), E_7(2)$, but by inspecting [3, 37, 77] we see that none of these groups have a maximal subgroup with socle J_2. Similarly, if $S = \mathrm{J}_3$ then we may assume $G_0 = E_6^{\epsilon}(4)$. Let $x \in G$ be an element of prime order r. If $r = 2$ then $|x^G| > (4-1)4^{21} = b_1$ and we note that $i_2(H) \leqslant i_2(S.2) = 46683 = a_1$. On the other hand, if $r > 2$ then $|x^G| > (4-1)4^{31} = b_2$. Setting $a_2 = |S|$, it follows that $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2 < 1$.

Finally, let us assume $S = \mathrm{Fi}_{22}$. Here $\bar{G} = E_6$, $p = 2$ and we may assume $q \geqslant 4$. Let $x \in G$ be an element of prime order r. If $r = 2$ then $|x^G| > (q-1)q^{21} = b_1$ and we note that $i_2(H) \leqslant i_2(\mathrm{Fi}_{22}.2) = 79466751 = a_1$. For $r > 2$ we have $|x^G| > (q-1)q^{31} = b_2$ and it follows that $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2$, where $a_2 = |S|$. This yields $\mathcal{Q}(G,H) < q^{-1}$ if $q \geqslant 8$.

The case $q = 4$ requires special attention. First observe that $|{}^2E_6(4)|$ is indivisible by 11, so $G_0 = E_6(4)$ is the only option. Let V be the adjoint module for G_0 and note that S acts irreducibly on V (see [60, p.27]). In fact, V is the unique 78-dimensional irreducible module for $S.2$ (over $\mathbb{F}_4$) and we can use MAGMA to compute the action on V of a set of conjugacy class representatives in H (see [21, Lemma 2.11] for the details).

Let $x \in H$ be an element of prime order r. First assume $r \in \{2,3\}$. If $r = 2$ and $x \in S$, then we compute the Jordan form of x on V and we identify x^G by inspecting [40, Table 6]. There are 3 classes of involutions in $S.2 \setminus S$ and we note that $G_0.2$ has two classes of involutory graph automorphisms, represented by τ and τ', where $C_{G_0}(\tau) = F_4(4)$. As in previous cases, we can identify the corresponding G-class of each involution in $S.2 \setminus S$ by computing the Jordan form on V. Indeed, if x has Jordan form (J_2^{26}, J_1^{26}) then x is conjugate to τ, whereas the graph automorphisms in the other class have Jordan form (J_2^{36}, J_1^{6}) on V. Finally, if $r = 3$ then $\dim C_V(x) = \dim C_{\bar{G}}(x)$ and this uniquely determines $C_{\bar{G}}(x)$. The results are summarised in Table 9. (Here we use the notation from [26] for the classes in $\mathrm{Fi}_{22}.2$, while the unipotent classes in G_0 are labelled as in [57]. For elements of order 3, we give the structure of $C_{\bar{G}}(x)^0$.)

In each case, it is easy to determine a lower bound on $|x^G|$ and we deduce that the contribution to $\mathcal{Q}(G,H)$ from elements of order 2 or 3 is less than $\sum_{i=1}^{8} a_i^2/b_i$, where

$$a_1 = 3510,\ a_2 = 1216215,\ a_3 = 36486450,\ a_4 = 61776,$$

$$a_5 = 41698800,\ a_6 = 3294720,\ a_7 = 3715712000,\ a_8 = 461260800$$

and

$$b_1 = 3.4^{21},\ b_2 = 3.4^{31},\ b_3 = \frac{1}{2}4^{40},\ b_4 = \frac{1}{6}4^{26},\ b_5 = b_6 = \frac{1}{6}4^{42},\ b_7 = \frac{1}{6}4^{54},\ b_8 = \frac{1}{6}4^{48}.$$

Finally, if $r \geqslant 5$ then we find that $\dim C_V(x) = \dim C_{\bar{G}}(x) \leqslant 18$ and thus $|x^G| > \frac{1}{6}4^{60} = b_9$. By setting $a_9 = |S|$, we conclude that

$$\mathcal{Q}(G,H) < \sum_{i=1}^{9} a_i^2/b_i < 1$$

and thus $b(G,H) = 2$. □

Lemma 7.12. *The conclusion to Theorem 7.1 holds if $S \notin \mathrm{Lie}(p)$ is a simple group of Lie type.*

Proof. The possibilities for S are recorded in [54, Tables 10.3 and 10.4] and we may assume G_0 is one of the groups in (18). In each case, set

$$a_1 = i_2(\mathrm{Aut}(S)),\ \ a_2 = |\mathrm{Aut}(S)|,\ \ b_1 = \left\{ \begin{array}{ll} \ell_1 & \text{if } p=2 \\ \min\{\ell_3,\ell_5\} & \text{if } p>2 \end{array}\right.,\ b_2 = \left\{ \begin{array}{ll} \ell_4 & \text{if } p=2 \\ \ell_2 & \text{if } p>2 \end{array}\right.$$

where the ℓ_i are defined in Table 4. Then by applying Propositions 2.11 and 7.8, we deduce that

$$\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2.$$

It is routine to check that this upper bound is less than 1 unless (G_0,S) is one of the following:

$$(F_4(3), {}^3D_4(2)),\ (F_4(5), {}^3D_4(2)),\ (E_6^\epsilon(3), {}^3D_4(2)),\ (E_6^\epsilon(3), {}^2F_4(2)'),\ (E_6^\epsilon(4), \Omega_7(3)).$$

First assume $(G_0,S) = (F_4(5), {}^3D_4(2))$. Here $b_1 = 5^{16}$ and since $|H|$ is indivisible by 5 we can take $b_2 = \ell_4 = 4.5^{29}$ as a lower bound on $|x^G|$ for all $x \in H$ of odd prime order. One checks that $a_1^2/b_1 + a_2^2/b_2 < 1$.

Next assume $G_0 = E_6^\epsilon(3)$ and $S = {}^3D_4(2)$ or ${}^2F_4(2)'$. Let $x \in H$ be an element of prime order r. If $r = 2$ then $|x^G| > 2.3^{25} = b_1$. Similarly, if $r = 3$ then $|x^G| > 2.3^{31} = b_2$ since H does not contain long root elements by Proposition 7.8. Now assume $r \geqslant 5$. Since r divides $|Z(C_{G_0}(x))|$ it follows that $C_{\bar{G}}(x)^0 \neq D_5T_1$ or A_5T_1, whence $|x^G| > \frac{1}{6}3^{48} = b_3$ and we conclude that

$$\mathcal{Q}(G,H) < \sum_{i=1}^{3} a_i^2/b_i < 1,$$

where $a_1 = i_2(\mathrm{Aut}(S))$, $a_2 = i_3(\mathrm{Aut}(S))$ and $a_3 = |\mathrm{Aut}(S)|$.

Now assume $(G_0,S) = (F_4(3), {}^3D_4(2))$, so $G = F_4(3)$ and $H = {}^3D_4(2)$ or ${}^3D_4(2).3$. Here we proceed as we did in the proof of Lemma 7.11 for the case $S = \mathrm{Fi}_{22}$ with $G_0 = E_6(4)$. First, with the aid of Magma, we observe that ${}^3D_4(2).3$ has a unique 52-dimensional irreducible module V over $\mathbb{F}_3$, which we may identify with the adjoint module for G_0 (as noted in [60, Table 6.36], S acts irreducibly on V). Suppose $x \in {}^3D_4(2).3$ has prime order r. If $r = 3$, then we can compute the Jordan form of x on V and use [40, Table 4] to determine the $\bar{G}$-class of x up to one of two possibilities. Similarly, if $r \in \{2,7,13\}$ then we can compute $\dim C_V(x) = \dim C_{\bar{G}}(x)$, which yields a lower bound on $|x^G|$. See [21, Lemma 2.11] for further details on the computation.

For example, suppose $r = 3$ and x is in the H-class labelled `3A` in [26], so $|x^H| = 139776$. Then we calculate that x has Jordan form (J_3^{15}, J_1^7) on V, which implies that x is either in the $\bar{G}$-class labelled A_2 or $\tilde{A}_2$. In particular, $|x^G| > \frac{1}{4}3^{30}$. Similarly, if $r = 7$ then $\dim C_V(x) = 10$ and we deduce that $|x^G| > \frac{1}{2}3^{42}$.

In this way, by considering each H-class of prime order elements in turn, we obtain an upper bound on $\mathcal{Q}(G,H)$ which allows us to conclude that $\mathcal{Q}(G,H) < 1$. We leave the reader to check this details.

Finally, let us assume $(G_0,S) = (E_6^\epsilon(4), \Omega_7(3))$. By Lagrange's Theorem, we see that $\epsilon = +$ is the only possibility. Now $\mathrm{SO}_7(3)$ has a unique 78-dimensional irreducible module V over $\mathbb{F}_4$, which we identify with the adjoint module for G_0. We can now proceed as in the previous case, using MAGMA to compute $\dim C_V(x)$ for each $x \in H$ of prime order (see [21, Lemma 2.11]). As before, this information translates into a lower bound on $|x^G|$ and this allows us to determine an upper bound on $\mathcal{Q}(G,H)$. In this way, one checks that $\mathcal{Q}(G,H) < 1$ and the result follows. □

7.2. **Generic subgroups.** To complete the proof of Theorem 7.1, we may assume G_0 is one of the groups in (18) and S is in $\mathrm{Lie}(p)$, as in part (ii) of Theorem 7.2. In view of Craven's theorem (see Theorem 7.3), there are very few possibilities for G_0 and S and it is a straightforward exercise to verify Theorem 7.1 in these cases.

Lemma 7.13. *The conclusion to Theorem 7.1 holds if $S \in \mathrm{Lie}(p)$.*

Proof. By Theorem 7.3, we have $G_0 = E_8(q)$ or $E_7(q)$. First assume $G_0 = E_8(q)$. By inspecting the possibilities for S in Theorem 7.3 and by applying Proposition 2.14, we deduce that

$$i_2(\mathrm{Aut}(S)) \leqslant 2s(s+1) = a_1, \;\; |\mathrm{Aut}(S)| \leqslant |\mathrm{Aut}(\mathrm{L}_2(3^7))| = a_2,$$

where $s = 2621$. Setting $b_1 = q^{58}$ and $b_2 = q^{92}$, we deduce that $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2 < 1$ and thus $b(G,H) = 2$. Similarly, if $G_0 = E_7(q)$ then $|\mathrm{Aut}(S)| \leqslant 31200 = a_1$ and we have $|x^G| > q^{34} = b_1$ for all nontrivial $x \in G$, whence $\mathcal{Q}(G,H) < a_1^2/b_1 < 1$ and the result follows. □

This completes the proof of Theorem 7.1.

8. Twisted groups

In this final section of the paper, we complete the proof of Theorem 1 by handling the remaining almost simple primitive groups with socle

$$G_0 \in \{{}^3D_4(q), {}^2F_4(q)', {}^2G_2(q)' \,(q \geqslant 27), {}^2B_2(q)\}. \tag{19}$$

Our main result is the following.

Theorem 8.1. *If G_0 is one of the groups in* (19)*, then G is not extremely primitive.*

Let G be an almost simple primitive group with point stabilizer H and socle G_0 as in (19). Recall that we handled the special cases

$$G_0 \in \{{}^2B_2(8), {}^2B_2(32), {}^2F_4(2)', {}^3D_4(2)\}$$

in Theorem 2.15, so for the remainder of this section we will assume G_0 is not one of these groups. Furthermore, in view of Theorems 3.2 and 4.1, we may assume that H is neither a parabolic nor a maximal rank subgroup of G. Then by inspecting [63] and [6, Tables 8.16, 8.43 and 8.51], it follows that either

(a) H is a subfield subgroup; or
(b) $G_0 = {}^3D_4(q)$ and $H_0 = H \cap G_0$ is either $G_2(q)$ or $\mathrm{PGL}_3^{\epsilon}(q)$ with $q \equiv \epsilon \pmod 3$.

First we handle the subfield subgroups in (a).

Lemma 8.2. *If $G_0 = {}^3D_4(q)$ and H is a subfield subgroup, then G is not extremely primitive.*

Proof. We proceed as in the proof of Proposition 6.6. Write $q = q_0^k$, where $k \neq 3$ is a prime and set $\bar{G} = D_4$. Fix a Steinberg endomorphism $\psi = \sigma\tau$ of $\bar{G}$, where σ is a standard Frobenius morphism of $\bar{G}$ corresponding to the map $\lambda \mapsto \lambda^{q_0}$ on $\mathbb{F}_q$ and τ is the standard triality graph automorphism of $\bar{G}$. Then $H_0 = \bar{G}_{\psi} = {}^3D_4(q_0)$ and $G_0 = \bar{G}_{\psi^k} = {}^3D_4(q)$.

Let α_0 be the highest root in the root system of $\bar{G}$ and let X_α be the root subgroup of $\bar{G}$ corresponding to the root α. Consider the ψ-stable maximal rank subgroup $\bar{M} = \bar{X}\bar{Y}$ of $\bar{G}$, where $\bar{X} = \langle X_{\pm\alpha_0}\rangle$ and $\bar{Y} = C_{\bar{G}}(\bar{X})^0$. Then $\bar{M}$ is of type A_1^4 and we set

$$M = \bar{M}_{\psi} = (\mathrm{SL}_2(q_0) \circ Y).d = d.(\mathrm{L}_2(q_0) \times \mathrm{L}_2(q_0^3)).d,$$

where $Y = (\bar{Y}_\psi)'$ and $d = (2, q-1)$. By [50], M is a maximal subgroup of H_0 and we focus our attention on the subgroup $K = T \circ Y \leqslant M$, where $T = Z_{q_0+1}$ is a maximal torus of $\mathrm{SL}_2(q_0)$.

Then by arguing as in the proof of Lemma 6.5, using [35] for information on the maximal subgroups of H_0, we deduce that M is the unique maximal overgroup of K in H_0. Writing $G = G_0.A$ and $H = H_0.A$, we can now repeat the argument in the proof of Proposition 6.6 to show that G is not extremely primitive, noting that K and M are A-stable. □

Lemma 8.3. *Suppose* $G_0 \in \{{}^2F_4(q), {}^2G_2(q), {}^2B_2(q)\}$ *and* H *is a subfield subgroup of* G. *Then* $b(G,H) = 2$.

Proof. In view of [17, Propositions 4.38 and 4.40], we may assume $G_0 = {}^2F_4(q)$. Let H be a subfield subgroup of G with $H_0 = H \cap G_0 = {}^2F_4(q_0)$, where $q = q_0^k$ for some odd prime k.

First assume $k \geqslant 5$, so $q \geqslant 32$. If $x \in G_0$ is an involution of type u_1 in the notation of [71, Table II], then

$$|x^G \cap H| = (q_0^3+1)(q_0^2-1)(q_0^6+1) < q^{11/5} = a_1, \;\; |x^G| > (q-1)q^{10} = b_1.$$

For all other nontrivial elements in G, we have $|x^G| > (q-1)q^{13} = b_2$ and we note that $|H| < \log_2 q . q^{26/5} = a_2$. It follows that $\mathcal{Q}(G,H) < a_1^2/b_1 + a_2^2/b_2$ and one checks that this upper bound is less than q^{-1} for all $q \geqslant 32$.

Now suppose $k = 3$. Let $x \in G$ be an element of prime order r. First assume $r = 2$, so x is conjugate to u_1 or u_2 in the notation of [71]. As above, if $x = u_1$ then $|x^G \cap H| < q^{11/3} = a_1$ and $|x^G| > (q-1)q^{10} = b_1$. Similarly, if $x = u_2$ then $|x^G \cap H| < q^{14/3} = a_2$ and $|x^G| > (q-1)q^{13} = b_2$. Next assume x is semisimple. Both H_0 and G_0 have a unique conjugacy class of elements of order 3 (represented by the element t_4 in [71, Table IV]) and we get $|x^G \cap H| < q^6 = a_3$ and $|x^G| > (q-1)q^{17} = b_3$. For $r \geqslant 5$, we have $|x^G| > \frac{1}{3}q^{20} = b_4$ and we note that $|H_0| < q^{26/3} = a_4$. Finally, suppose $x \in G$ is a field automorphism. If $r = 3$ then $|x^G| > \frac{1}{2}q^{52/3} = b_5$ and we observe that H contains precisely $2(i_3({}^2F_4(q_0)) + 1) < 2q^6 = a_5$ field automorphisms of order 3. And for $r \geqslant 5$, we get $|x^G| > \frac{1}{2}q^{104/5} = b_6$ and we note that $|H| < \log_2 q . q^{26/3} = a_6$. We conclude that

$$\mathcal{Q}(G,H) < \sum_{i=1}^{6} a_i^2/b_i < q^{-1}$$

and the result follows. □

Finally, let us turn to the two remaining cases with $G_0 = {}^3D_4(q)$.

Lemma 8.4. *If* $G_0 = {}^3D_4(q)$ *and* $H_0 = G_2(q)$, *then* G *is not extremely primitive.*

Proof. Here H_0 is the centralizer in G_0 of a triality graph automorphism. Therefore, if G contains a graph automorphism then $\mathrm{soc}(H)$ will be a direct product of non-isomorphic simple groups and thus G is not extremely primitive by Lemma 2.1(v). This allows us to assume that $G = G_0.A$ and $H = H_0.A$, where A is a group of field automorphisms.

Through the work of Cooperstein [27] and Kleidman [34], the maximal subgroups of H_0 are known for all q. In particular, we note that H_0 has a maximal subgroup $M = K.2$, where $K = \mathrm{SL}_3(q)$ and the outer involution acts as a graph automorphism on K. Clearly, M is the unique maximal overgroup of K in H_0. By inspecting [35], we see that

$$N_{G_0}(K) = N_{G_0}(L) = (K \circ L).(3, q^2+q+1).2$$

is a maximal subgroup of G_0, where $L = Z_{q^2+q+1}$. Here the outer involution induces a graph automorphism on K and inverts L. Write $L = \langle g \rangle$ and note that $L = \langle g^{-2} \rangle$ and $g \in C_{G_0}(K)$. Seeking a contradiction, suppose g normalizes M and choose an element $x \in M \setminus K$. Then $g^x = g^{-1}$, so $[g,x] = g^{-2} \in M$ and thus $L < K = M'$ since L has odd order. But $K \cap L = Z(K) = Z_{(3,q-1)}$ and so we have reached a contradiction. Therefore, g

does not normalize M and the result now follows from Lemma 2.4, noting that K and M are both A-stable. □

Lemma 8.5. *If* $G_0 = {}^3D_4(q)$ *and* $H_0 = \mathrm{PGL}_3^{\epsilon}(q)$*, where* $q \equiv \epsilon \pmod 3$*, then* $b(G,H) = 2$.

Proof. Here $H_0 = C_{G_0}(\tau)$, where τ is a triality graph automorphism of G_0. Since $q \neq 3$ and we handled the case $q = 2$ in Theorem 2.15, we may assume $q \geqslant 4$. We claim that $\mathcal{Q}(G,H) < 1$.

Let $x \in H$ be an element of prime order r and write $q = p^f$ with p prime. Let U and V be the natural modules for H_0 and $\bar{G} = D_4$, respectively, where $\bar{G}$ is the ambient simple algebraic group. Note that the embedding of H_0 in G_0 arises from the embedding of H_0 in $\mathrm{P}\Omega_8^+(q^3)$ through the action of H_0 on its adjoint module. In particular, we can work with the adjoint representation to determine the Jordan form on V for each $x \in H_0$.

First assume $x \in H_0$ and $r = p$. If $p = 2$ then H_0 has a unique class of involutions and we calculate that x has Jordan form (J_2^4) on V, so x is contained in the G_0-class labelled $3A_1$ in [73, Section 0.5]. In particular,

$$|x^G \cap H| = i_2(H_0) = (q+\epsilon)(q^3-\epsilon) = a_1, \;\; |x^G| = q^2(q^6-1)(q^8+q^4+1) = b_1.$$

Now assume $p \neq 2$ (so $p \geqslant 5$). If x has Jordan form (J_2, J_1) on U, then x acts on V as (J_3, J_2^2, J_1), which implies that x is in the G_0-class $3A_1$. Similarly, if x has Jordan form (J_3) on U, then it acts as (J_5, J_3) on V, which places x in the G_0-class labelled $D_4(a_1)$. In both cases, this allows us to compute $|x^G \cap H|$ and $|x^G|$ precisely and we conclude that the total contribution to $\mathcal{Q}(G,H)$ from unipotent elements is at most $a_1^2/b_1 + a_2^2/b_2$, where

$$a_2 = q(q^2-1)(q^3-\epsilon), \;\; b_2 = q^6(q^2-1)(q^6-1)(q^8+q^4+1).$$

Next assume $x \in H_0$ is semisimple. If $r = 2$ then

$$|x^G \cap H| = i_2(H_0) = q^2(q^2+\epsilon q+1) = a_3, \;\; |x^G| = i_2(G_0) = q^8(q^8+q^4+1) = b_3.$$

If $r > 2$, then $|x^G| > (q-1)q^{17} = b_4$ and we note that $|H_0| < q^8 = a_4$.

Next suppose $x \in G$ is a graph automorphism. Here $|x^G| > q^{14} = b_5$ and the total number of graph automorphisms in H is equal to

$$2(i_3(\mathrm{PGL}_3^{\epsilon}(q))+1) = 4q^2(q^4+2q^2+3\epsilon q+2)+2 = a_5.$$

Finally, suppose $x \in G$ is a field automorphism of order r. If $r = 2$, then

$$|x^G \cap H| \leqslant \frac{|\mathrm{PGU}_3(q)|}{|\mathrm{SL}_2(q)|} = q^2(q^3+1) = a_6, \;\; |x^G| > \frac{1}{2}q^{14} = b_6.$$

Similarly, if $r \geqslant 3$ then $|x^G| > \frac{1}{2}q^{56/3} = b_7$ and we note that H contains fewer than $\log_2 q.q^8 = a_7$ field automorphisms of G_0.

Putting these estimates together, we conclude that

$$\mathcal{Q}(G,H) < \sum_{i=1}^{7} a_i^2/b_i,$$

which is less than 1 for all $q \geqslant 4$ (and it is less than q^{-1} for $q \geqslant 11$). □

This completes the proof of Theorem 8.1. By combining this result with Theorems 2.15, 3.2, 4.1, 5.2, 5.3, 6.1 and 7.1, we conclude that the proof of Theorem 1 is complete.

References

[1] M. Aschbacher and G.M. Seitz, *Involutions in Chevalley groups over fields of even order*, Nagoya Math. J. **63** (1976), 1–91.

[2] H. Azad, M. Barry and G.M. Seitz, *On the structure of parabolic subgroups*, Comm. Algebra **18** (1990), 551–562.

[3] J. Ballantyne, C. Bates and P. Rowley, *The maximal subgroups of* $E_7(2)$, LMS J. Comput. Math. **18** (2015), 323–371.

[4] W. Bosma, J. Cannon and C. Playoust, *The* MAGMA *algebra system I: The user language*, J. Symbolic Comput. **24** (1997), 235–265.

[5] N. Bourbaki, *Groupes et Algebrès de Lie* (*Chapters 4,5 and 6*), Hermann, Paris, 1968.

[6] J.N. Bray, D.F. Holt and C.M. Roney-Dougal, *The Maximal Subgroups of the Low-dimensional Finite Classical Groups*, London Math. Soc. Lecture Note Series, vol. 407, Cambridge University Press, 2013.

[7] R. Brauer, *Number theoretical investigations on groups of finite order*, Proc. Intern. Symposium on Algebraic Number Theory, Tokyo and Nikko, 1956, pp. 55–62.

[8] T. Breuer, *The GAP Character Table Library, Version 1.2.1*, GAP package, http://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib, 2012.

[9] T.C. Burness, *On base sizes for actions of finite classical groups*, J. London Math. Soc. **75** (2007), 545–562.

[10] T.C. Burness, *On base sizes for almost simple primitive groups*, J. Algebra **516** (2018), 38–74.

[11] T.C. Burness, *Base sizes for primitive groups with soluble stabilisers*, submitted (arxiv:2006.10510)

[12] T.C. Burness, M. Garonzi and A. Lucchini, *On the minimal dimension of a finite simple group.* With an appendix by T.C. Burness and R.M. Guralnick. J. Combin. Theory Ser. A **171** (2020), 105175, 32 pp.

[13] T.C. Burness, M. Garonzi and A. Lucchini, *Finite groups, minimal bases and the intersection number*, submitted (arxiv:2009.10137)

[14] T.C. Burness and M. Giudici, *Classical groups, derangements and primes*, Australian Mathematical Society Lecture Series, vol. 25, Cambridge University Press, Cambridge, 2016.

[15] T.C. Burness, R.M. Guralnick and J. Saxl, *On base sizes for symmetric groups*, Bull. Lond. Math. Soc. **43** (2011), 386–391.

[16] T.C. Burness, R.M. Guralnick and J. Saxl, *On base sizes for algebraic groups*, J. Eur. Math. Soc. (JEMS) **19** (2017), 2269–2341.

[17] T.C. Burness, M.W. Liebeck and A. Shalev, *Base sizes for simple groups and a conjecture of Cameron*, Proc. London Math. Soc. **98** (2009), 116–162.

[18] T.C. Burness, E.A. O'Brien and R.A. Wilson, *Base sizes for sporadic simple groups*, Israel J. Math. **177** (2010), 307–333.

[19] T.C. Burness, C.E. Praeger and Á. Seress, *Extremely primitive classical groups*, J. Pure Appl. Algebra **216** (2012), 1580–1610.

[20] T.C. Burness, C.E. Praeger and Á. Seress, *Extremely primitive sporadic and alternating groups*, Bull. London Math. Soc. **44** (2012), 1147–1154.

[21] T.C. Burness and A.R. Thomas, *Computations concerning the classification of extremely primitive groups*, available at http://seis.bristol.ac.uk/~tb13602/epcomp.pdf.

[22] T.C. Burness and A.R. Thomas, *A note on extremely primitive affine groups*, Arch. Math., to appear.

[23] T.C. Burness and A.R. Thomas, *On the involution fixity of exceptional groups of Lie type*, Internat. J. Algebra Comput. **28** (2018), 411–466.

[24] R.W. Carter, *Simple groups of Lie type*, Pure and Applied Mathematics, vol. 28. John Wiley & Sons, London-New York-Sydney, 1972.

[25] A. M. Cohen, M. W. Liebeck, J. Saxl and G. M. Seitz, *The local maximal subgroups of exceptional groups of Lie type*, Proc. London Math. Soc. **64** (1992), 21–48.

[26] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson, *Atlas of Finite Groups*, Oxford University Press, 1985.

[27] B.N. Cooperstein, *Maximal subgroups of* $G_2(2^n)$, J. Algebra **70** (1981), 23–36.

[28] D.A. Craven, *Alternating subgroups of exceptional groups of Lie type*, Proc. Lond. Math. Soc. **115** (2017), 449–501.

[29] D.A. Craven, *On medium-rank Lie primitive and maximal subgroups of exceptional groups of Lie type*, submitted.

[30] D.A. Craven, *Maximal* PSL_2 *subgroups of exceptional groups of Lie type*, Mem. Amer. Math. Soc., to appear.

[31] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.11.0, 2020, (http://www.gap-system.org).

[32] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups, Number 3*, Mathematical Surveys and Monographs, vol. 40, Amer. Math. Soc., 1998.

[33] R.M. Guralnick, *Cyclic quotients of transitive groups*, J. Algebra **234** (2000), 507–532.

[34] P.B. Kleidman, *The maximal subgroups of the Chevalley groups* $G_2(q)$ *with* q *odd, the Ree groups* ${}^2G_2(q)$, *and their automorphism groups*, J. Algebra **117** (1988), 30–71.

[35] P. B. Kleidman, *The maximal subgroups of the Steinberg triality groups* ${}^3D_4(q)$ *and of their automorphism groups*, J. Algebra **115** (1988), 182–199.

[36] P. B. Kleidman and M. W. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, London Math. Soc. Lecture Note Series, vol. 129, Cambridge University Press, 1990.

[37] P. B. Kleidman and R. A. Wilson, *The maximal subgroups of* $E_6(2)$ *and* $\mathrm{Aut}(E_6(2))$, Proc. London Math. Soc. **60** (1990), 266–294.

[38] R. Lawther, *Some coset actions in* $G_2(q)$, Proc. London Math. Soc. **61** (1990), 1–17.

[39] R. Lawther, *Folding actions*, Bull. London Math. Soc. **25** (1993), 132–144.
[40] R. Lawther, *Jordan block sizes of unipotent elements in exceptional algebraic groups*, Comm. Algebra **23** (1995), 4125–4156.
[41] R. Lawther, *Unipotent classes in maximal subgroups of exceptional algebraic groups*, J. Algebra **322** (2009), 270–293.
[42] R. Lawther, *Sublattices generated by root differences*, J. Algebra **412** (2014), 255–263.
[43] R. Lawther, M.W. Liebeck and G.M. Seitz, *Fixed point spaces in actions of exceptional algebraic groups*, Pacific J. Math. **205** (2002), 339–391.
[44] R. Lawther, M.W. Liebeck and G.M. Seitz, *Fixed point ratios in actions of finite exceptional groups of Lie type*, Pacific J. Math. **205** (2002), 393–464.
[45] R. Lawther, M.W. Liebeck and G.M. Seitz, *Outer unipotent classes in automorphism groups of simple algebraic groups*, Proc. Lond. Math. Soc. **109** (2014), 553–595.
[46] M. Lee, *Regular orbits of quasisimple linear groups I*, submitted (arxiv:1911.05785)
[47] M. Lee, *Regular orbits of quasisimple linear groups II*, in preparation.
[48] M. Lee, *Regular orbits of quasisimple linear groups III*, in preparation.
[49] M.W. Liebeck, B.M.S. Martin and A. Shalev, *On conjugacy classes of maximal subgroups of finite simple groups, and a related zeta function*, Duke Math. J. **128** (2005), 541–557.
[50] M.W. Liebeck, J. Saxl and G.M. Seitz, *Subgroups of maximal rank in finite exceptional groups of Lie type*, Proc. London Math. Soc. **65** (1992), 297–325.
[51] M.W. Liebeck and G.M. Seitz, *Maximal subgroups of exceptional groups of Lie type, finite and algebraic*, Geom. Dedicata **36** (1990), 353–387.
[52] M.W. Liebeck and G.M. Seitz, *Subgroups generated by root elements in groups of Lie type*, Annals of Math. **139** (1994), 293–361.
[53] M.W. Liebeck and G.M. Seitz, *Reductive subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc. **580**, 1996.
[54] M.W. Liebeck and G.M. Seitz, *On finite subgroups of exceptional algebraic groups*, J. reine angew. Math. **515** (1999), 25–72.
[55] M.W. Liebeck and G.M. Seitz, *A survey of of maximal subgroups of exceptional groups of Lie type*, in Groups, combinatorics & geometry (Durham, 2001), 139–146, World Sci. Publ., River Edge, NJ, 2003.
[56] M. W. Liebeck and G. M. Seitz, *The maximal subgroups of positive dimension in exceptional algebraic groups*, Mem. Amer. Math. Soc. **802**, 2004.
[57] M.W. Liebeck and G.M. Seitz, *Unipotent and nilpotent classes in simple algebraic groups and Lie algebras*, Mathematical Surveys and Monographs, vol. 180, Amer. Math. Soc., 2012.
[58] M.W. Liebeck and A. Shalev, *Maximal subgroups of symmetric groups*, J. Combin. Theory Ser. A **75** (1996), 341–352.
[59] M.W. Liebeck and A. Shalev, *Simple groups, permutation groups, and probability*, J. Amer. Math. Soc. **12** (1999), 497–520.
[60] A.J. Litterick, *On non-generic finite subgroups of exceptional algebraic groups*, Mem. Amer. Math. Soc. **253** (2018), no. 1207, v+156 pp.
[61] F. Lübeck, *Centralisers and numbers of semisimple classes in exceptional groups of Lie type*, `http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/CentSSClasses`
[62] F. Lübeck, *Computation of Kazhdan-Lusztig polynomials and some applications to finite groups*, Trans. Amer. Math. Soc. **373** (2020), 2331–2347.
[63] G. Malle, *The maximal subgroups of* ${}^2F_4(q^2)$, J. Algebra **139** (1991), 52–69.
[64] G. Malle and D.M. Testerman, *Linear algebraic groups and finite groups of Lie type*, Cambridge Studies in Advanced Mathematics, 133. Cambridge University Press, Cambridge, 2011.
[65] A. Mann, C.E. Praeger and Á. Seress, *Extremely primitive groups*, Groups Geom. Dyn. **1** (2007), 623–660.
[66] W.A. Manning, *Simply transitive primitive groups*, Trans. Amer. Math. Soc. **29** (1927), 815–825.
[67] S.P. Norton and R.A. Wilson, *The maximal subgroups of* $F_4(2)$ *and its automorphism group*, Comm. Algebra **17** (1989), 2809–2824.
[68] D.V. Pasechnik and C.E. Praeger, *On transitive permutation groups with primitive subconstituents*, Bull. London Math. Soc. **31** (1999), 257–268.
[69] P. Schmid, *The solution of the* $k(GV)$ *problem*, ICP Advanced Texts in Mathematics, vol. 4. Imperial College Press, London, 2007.
[70] K. Shinoda, *The conjugacy classes of Chevalley groups of type* (F_4) *over finite fields of characteristic* 2, J. Fac. Sci. Univ. Tokyo **21** (1974), 133–159.
[71] K. Shinoda, *The conjugacy classes of the finite Ree groups of type* (F_4), J. Fac. Sci. Univ. Tokyo **22** (1975), 1–15.
[72] K. Shinoda, *A characterization of odd order extensions of the Ree groups* ${}^2F_4(q)$, J. Fac. Sci. Univ. Tokyo **22** (1975), 79–102.
[73] N. Spaltenstein, *Caractères unipotents de* ${}^3D_4(\mathbb{F}_q)$, Comment. Math. Helv. **57** (1982), 676–691.
[74] M. Suzuki, *On a class of doubly transitive groups*, Annals of Math. **75** (1962), 105–145.

[75] A.R. Thomas, *The irreducible subgroups of the exceptional algebraic groups*, Mem. Amer. Math. Soc., to appear.
[76] H.N. Ward, *On Ree's series of simple groups*, Trans. Amer. Math. Soc. **121** (1966), 62–89.
[77] R.A. Wilson, *Maximal subgroups of* $^2E_6(2)$ *and its automorphism groups*, preprint (arxiv:1801.08374)
[78] R.A. Wilson et al., *A World-Wide-Web Atlas of finite group representations*, `http://brauer.maths.qmul.ac.uk/Atlas/v3/`

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