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Fixed point spaces in actions of classical algebraic groups

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Abstract. Let G be a simple classical algebraic group over an algebraically closed field K of characteristic $p \geqslant 0$, and let H be a maximal closed non-subspace subgroup of G. Given such a pair (G, H), we obtain a close to best possible upper bound for the ratio $\dim(x^G \cap H)/\dim x^G$, where $x \in G$ is a semisimple or unipotent element of prime order. We apply this result to the study of fixed point spaces.

1 Introduction

Let G be a simple classical algebraic group over an algebraically closed field K of arbitrary characteristic $p \geqslant 0$. If G has natural module V, we write $G = \mathrm{Cl}(V)$. Following [19], we say that a maximal closed subgroup H of G is a *subspace subgroup* if it is reducible on V, or if it is an orthogonal group on V embedded in a symplectic group with $p = 2$.

The major motivation for this paper arises from a result of Liebeck and Shalev concerning finite almost simple classical groups [19, Theorem ($\star$)]. This result states that there exists a constant $\delta > 0$ such that, if X is any finite almost simple classical group, M is a maximal subgroup of X which is not a subspace subgroup, and $x \in X$ is an element of prime order, then

$$|x^X \cap M| < |x^X|^{1-\delta}. \tag{1}$$

In [19], the authors apply Theorem ($\star$) to a number of problems concerning finite simple groups and finite permutation groups. Among other things, they use the result to obtain lower bounds for fixed point ratios of primitive actions of classical groups, to prove the Cameron–Kantor base conjecture, and to prove a major part of the Guralnick–Thompson genus conjecture. Theorem ($\star$) of [19] is an existence result, and offers no information on the value of δ in (1); as a consequence, the applications in [19] also involve undetermined constants. It is very desirable to strengthen these results with explicit constants.

In this paper, we obtain a result analogous to [19, Theorem ($\star$)] for algebraic groups. If x is a prime order semisimple or unipotent element of a simple classical algebraic group G, then in Theorem 1 below we find an explicit $\imath > 0$ with the property that for all non-subspace maximal closed subgroups H of G,

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leqslant 1 - \imath.$$

In general $\imath = 1/2$, with some explicit exceptions. We then apply this result to the study of fixed point spaces $C_\Omega(x)$, where $\Omega = G/J$ is a coset variety, $x \in G$ and

$$C_\Omega(x) = \{\omega \in \Omega : \omega x = \omega\}.$$

We obtain a number of corollaries concerning lower bounds for the codimension of $C_\Omega(x)$ in Ω.

In future work, we plan to use Theorem 1 and its corollaries to strengthen [19, Theorem $(\star)$] for finite groups by providing an explicit constant $\delta > 0$ in (1). This will yield explicit bounds in the several applications described above.

Theorem 1. *If $G = \mathrm{Cl}(V)$ is a simple classical algebraic group and H is a maximal closed subgroup of G which is not a subspace subgroup, and $x \in G$ is a non-scalar semisimple or unipotent element of prime order, then*

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leqslant \frac{1}{2} + \varepsilon,$$

where $\varepsilon = 0$ or $(G, H^\circ, \varepsilon)$ is given in Table 1.

Table 1

G	p	H°	ε
SL_{2n}	arbitrary	Sp_{2n}	$1/2n$
Sp_{2n}	arbitrary	Sp_n^2	$1/(2n+2)$
SO_{2n}	arbitrary	GL_n	$1/(2n-2)$
SO_7	$\neq 2$	G_2	$1/4$
Sp_6	2	G_2	$1/4$
SO_8	$\neq 2$	SO_7	$1/3$
SO_8	2	Sp_6	$1/3$

Remark 1. Note that for (G, H°) as listed in the last four rows of Table 1, H° is irreducibly embedded in G.

Remark 2. The bounds listed in Table 1 are best possible. Examples for which we have equality are given in Table 2. We also record some examples where an upper bound of $1/2$ is sharp. Here the matrix notation is taken from [5] (see Section 2 for a description). In Lemmas 7.6 and 7.7, we demonstrate the sharpness of the upper bounds recorded in 1 for the irreducible embeddings.

Table 2

G	H°	p	x	$\dim x^{H^\circ}/\dim x^G$
SL_{2n}	Sp_{2n}	arbitrary	$[\lambda I_n, \lambda^{-1} I_n]^\diamond$	$1/2 + 1/2n$
		$\neq 2$	$[J_2^n]$	$1/2 + 1/2n$
Sp_{2n}	Sp_n^2	arbitrary	$[\lambda I_{n/2}, \lambda^{-1} I_{n/2}, \lambda I_{n/2}, \lambda^{-1} I_{n/2}]^\diamond$	$1/2 + 1/(2n+2)$
		$\neq 2$	$[J_2^n]$	$1/2 + 1/(2n+2)$
SO_{2n}	GL_n	arbitrary	$[\lambda I_{n/2}, \lambda^{-1} I_{n/2}, \lambda I_{n/2}, \lambda^{-1} I_{n/2}]^\diamond$	$1/2 + 1/(2n-2)$
(n even)		arbitrary	$[J_2^n]$	$1/2 + 1/(2n-2)$
SL_{2n}	SO_{2n}	$\neq 2$	$[-I_n, I_n]$	$1/2$
(n even)				
SL_{2n+1}	SO_{2n+1}	$\neq 2$	$\pm[-I_{n+1}, I_n]$	$1/2$
SL_{2n}	$GL_n^2 \cap SL_{2n}$	arbitrary	$[A, A]^\dagger$	$1/2$
Sp_{2n}	GL_n	$\neq 2$	$[-I_2, I_{n-2}, -I_2, I_{n-2}]^\diamond$	$1/2$
SO_{2n}	SO_n^2	$\neq 2$	$[-I_2, I_{n-2}, -I_2, I_{n-2}]$	$1/2$

† $A \in SL_n$ is any semisimple or unipotent matrix.

Recall that if X is a finite group acting transitively on a set Λ, then the *fixed point ratio* of an element $x \in X$ is defined to be the proportion of points fixed by x. Bounds on fixed point ratios for actions of finite groups of Lie type have been obtained and applied in a number of papers (see [7], [8], [9], [11] for example). Now, if G is an algebraic group, $x \in G$ and Ω is a homogeneous G-space, then the codimension

$$f(x, \Omega) = \dim \Omega - \dim C_\Omega(x)$$

provides a natural algebraic group analogue of the notion of fixed point ratio. In [16], the authors obtain close to best possible lower bounds for $f(x, \Omega)$, where x is an arbitrary element of an exceptional simple algebraic group G. Now from [16, (1.14)] we have

$$f(x, G/H) = \dim x^G - \dim(x^G \cap H),$$

so in a similar spirit to [16], we may use Theorem 1 to obtain lower bounds for $f(x, G/H)$, when G is classical.

Corollary 1. *Let G be a simple classical algebraic group over an algebraically closed field of characteristic $p \geqslant 0$, and let $\Omega = G/J$, where the closed subgroup J lies in a maximal non-subspace subgroup H of G. Then for an arbitrary non-scalar element $x \in G$,*

$$f(x, \Omega) \geqslant \left(\tfrac{1}{2} - \varepsilon\right) \dim x^G,$$

where $\varepsilon \geqslant 0$ is as given in Theorem 1 for the pair (G, H°).

Given $x \in G = \mathrm{Cl}(V)$, let $v(x)$ denote the codimension of the largest eigenspace of x on V. In Proposition 2.9, we obtain bounds on $\dim x^G$ in terms of $v(x)$. As a result, we have the following corollary which is in a similar spirit to the results of Gluck and Magaard in [9, §§1.3, 1.4] on finite groups of classical Lie type, where the authors obtain upper bounds for certain fixed point ratios of a unipotent element g in terms of $v(g)$.

Corollary 2. *With (G, Ω, ε) as in Corollary 1, let n denote the dimension of the natural G-module. Set $t = 1$ if $G = \mathrm{SL}_n$, and $t = 0$ otherwise. If $x \in G$ and $v(x) = s$ then*

$$f(x, \Omega) \geqslant \left(\tfrac{1}{2} - \varepsilon\right) M,$$

where

$$M = \max\left((1+t)s(n-s-1), \frac{n}{2-t}(s-1)\right).$$

If G is a simple classical algebraic group and $x \in G$, then it is well known that $\dim x^G \geqslant 2r$, where r denotes the rank of G. Therefore we also have the following immediate corollary.

Corollary 3. *With (G, Ω, ε) as above, and $r = \mathrm{rank}\, G$, for an arbitrary non-scalar element $x \in G$ we have*

$$f(x, \Omega) \geqslant r(1 - 2\varepsilon).$$

The structure of the paper is as follows. In Section 2 we introduce a number of preliminary results taken from the literature which we will need to prove Theorem 1. In [18, Theorem 1], Liebeck and Seitz classify the maximal closed subgroups of a simple classical algebraic group $G = \mathrm{Cl}(V)$. They define six families $\mathscr{C}_1, \dots, \mathscr{C}_6$ of maximal closed subgroups, and they show that for every closed subgroup H either H is contained in a member of some $\mathscr{C}_i$, or modulo scalars, H is quasisimple and $E(H)$ is irreducible on V. For the purpose of proving Theorem 1, we may ignore the classes $\mathscr{C}_1$ and $\mathscr{C}_5$ for they consist of subspace subgroups and finite subgroups respectively. In Sections 3–6 we prove that the conclusion of Theorem 1 is true when the maximal closed subgroup is a member of one of the classes $\mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4$ and $\mathscr{C}_6$ respectively (see Section 2 for a description of these classes). In Section 7 we complete the proof of Theorem 1 by dealing with the case where H is not a member of any of the classes $\mathscr{C}_i$. Here H° is simple and irreducibly embedded in G and our proof relies on recent work of Lübeck [20] and Guralnick–Saxl [12] on the irreducible representations of simple algebraic groups. Finally, in Section 8 we use Theorem 1 to derive lower bounds for the codimension of $C_\Omega(x)$ in Ω, and prove Corollary 1.

2 Preliminary results

In this section we introduce some notation and results which we shall need for the proof of Theorem 1.

Following [5], we denote by $[M_1, \dots, M_n]$ a block diagonal matrix with matrices $M_1, \dots, M_n$ down the diagonal. We use $[J_i^m]$ to denote a block diagonal matrix with m unipotent Jordan i-blocks down the diagonal. If $\{e_1, f_1, \dots, e_l, f_l\}$ denotes a standard symplectic or orthogonal basis of the natural G-module (as described in [14, §2.5]) then unless otherwise stated, all symplectic or orthogonal matrices will be written with respect to this specific basis ordering. However, it will also be necessary from time to time to consider the ordering $\{e_1, \dots, e_l, f_1, \dots, f_l\}$, and any matrix A written with respect to this ordering will be denoted by $[A]^{\diamondsuit}$.

In this preliminary section, we are primarily concerned with obtaining bounds on $\dim x^H$, where H is a reductive group, and x is semisimple or unipotent. The most basic result in this direction is the following well-known proposition. We can say much more when H° is semisimple and x is an involution, as the subsequent result demonstrates.

Proposition 2.1 ([13, (1.6)]). *If H is a connected reductive algebraic group and $x \in H$, then* $\dim C_H(x) \geqslant \operatorname{rank} H$.

Proposition 2.2 ([16, (1.5)]). *Let H be an algebraic group with H° semisimple. If $x \in H$ is an involution, then*

$$\dim C_{H^\circ}(x) \geqslant |\Sigma^+(H^\circ)|,$$

where $\Sigma^+(H^\circ)$ denotes the set of positive roots in the associated root system of H°.

If G is a simple classical algebraic group, and $x \in G$ is a semisimple element then we can easily calculate $\dim C_G(x)$ from knowledge of the eigenvalues of x. The case where x is unipotent is not as straightforward and to calculate unipotent class dimensions we shall make much use of the following proposition. The classification of involution classes in $G = \mathrm{SO}_n$ when $p = 2$ is given in [2] and we use the notation therein for the class representatives.

Proposition 2.3 ([16, (1.10)]). *Let G be a simple classical algebraic group over an algebraically closed field of characteristic $p \geqslant 0$, and let u be a non-identity unipotent element in G. Suppose that for each i the Jordan canonical form for u has precisely n_i Jordan blocks of size i.*

(i) *If $G = \mathrm{SL}_n$ then*

$$\dim C_G(u) = 2\sum_{i<j} i n_i n_j + \sum_i i n_i^2 - 1.$$

(ii) *If $G = \mathrm{Sp}_{2n}$ with $p \neq 2$, then n_i is even whenever i is odd, and*

$$\dim C_G(u) = \sum_{i<j} i n_i n_j + \frac{1}{2}\sum_i i n_i^2 + \frac{1}{2}\sum_{i\,\mathrm{odd}} n_i.$$

(iii) *If $G = \mathrm{SO}_n$ with $p \neq 2$, then n_i is even whenever i is even, and*

$$\dim C_G(u) = \sum_{i<j} i n_i n_j + \frac{1}{2}\sum_i i n_i^2 - \frac{1}{2}\sum_{i\,\mathrm{odd}} n_i.$$

(iv) *If $G = \mathrm{SO}_n$ with $p = 2$ and $m = \lfloor n/2 \rfloor$ then the conjugacy classes of involutions in G are represented by elements a_{m-k}, c_{m-k} (with $0 \leqslant k \leqslant m$ and $m-k$ even), and if $n = 2m+1$ there is a further class b_{m-k} (with $0 \leqslant k \leqslant m$ and $m-k$ odd), where each of $a_{m-k}, b_{m-k}, c_{m-k}$ has $m-k$ Jordan 2-blocks and the rest of size 1. If $n = 2m+1$ then*

$$\dim C_G(a_{m-k}) = m^2 + m + k^2, \quad \dim C_G(b_{m-k}) = \dim C_G(c_{m-k}) = m^2 + k^2 + k;$$

and if $n = 2m$ then

$$\dim C_G(a_{m-k}) = m^2 + k^2 - k, \quad \dim C_G(c_{m-k}) = m^2 - m + k^2.$$

Proof. Parts (i), (ii) and (iii) follow from [22, pp. 34–39], and (iv) follows from [2, §§7, 8].

It is well known that in good characteristic the unipotent conjugacy classes in a simple classical algebraic group G are parametrized by a subset $\mathscr{S}$ of the set of all partitions of $n = \dim V$, where V is the natural G-module (see [13, §7]). For example, if $G = \mathrm{SL}_n$, then the unipotent conjugacy classes are in 1–1 correspondence with the set of all partitions of n, whereas if $G = \mathrm{Sp}_n$, then only those partitions of n where odd parts occur with even multiplicity correspond in a 1–1 fashion with the unipotent classes in G. In all cases the correspondence is given by

$$(n^{a_n}, \ldots, 1^{a_1}) \perp n \leftrightarrow [J_n^{a_n}, \ldots, J_1^{a_1}]^G \tag{2}$$

In a simple algebraic group G, conjugacy classes are closed precisely when their elements are semisimple (see [13, Proposition 1.7]). Let $\mathscr{U}$ denote the collection of unipotent elements in G. It is well known that $\mathscr{U}$ is a closed, irreducible subset of G having codimension equal to the rank of G. Hence, if $u \in \mathscr{U}$, then the closure of the class u^G lies in $\mathscr{U}$. This gives rise to a natural partial ordering on the collection of unipotent classes; if $u, v \in \mathscr{U}$ then we write $u^G \leqslant v^G$ if u lies in the closure of v^G.

Lemma 2.4 ([13, (7.19)]). *Let G be a simple classical algebraic group over an algebraically closed field K of characteristic $p \geqslant 0$. If $G \neq A_n$ assume that $p \neq 2$. Let n denote the dimension of the natural G-module and let u_1^G and u_2^G be two unipotent conjugacy classes in G, with corresponding partitions $\lambda, \mu \perp n$. Then*

$$u_1^G \leqslant u_2^G \quad \textit{if and only if } \lambda \leqslant \mu,$$

where the partial order on the set of all partitions of n is the usual dominance ordering.

Definition 2.5. Let $G = \mathrm{Cl}(V)$ be a simple classical algebraic group, and let $n = \dim V$. The *associated partition* $\lambda(x) \perp n$ of a semisimple or unipotent element $x \in G$ is defined as follows. If x is unipotent then we define $\lambda(x)$ to be the partition in the correspondence labelled (2). If $p \neq 2$ then $\dim x^G$ is completely determined by $\lambda(x)$, and this remains true in arbitrary characteristic if $G = \mathrm{SL}_n$ (see Proposition 2.3). Now suppose that $x \in G$ is semisimple. The natural G-module V decomposes into a direct sum of eigenspaces under the action of x. The parts of $\lambda(x)$ are defined to be the dimensions of these eigenspaces. If $x \in G = \mathrm{SL}_n$ is semisimple and $\lambda(x) = (\lambda_1, \ldots, \lambda_r) \perp n$ then $C_G(x)^\circ = (\prod_i \mathrm{GL}_{\lambda_i}) \cap G$. If G preserves a non-zero form on V then we need to distinguish between non-degenerate and totally singular eigenspaces. Under the action of a semisimple element $x \in G$ we have

$$V = W_1 \oplus W_{-1} \oplus (U_1 \oplus U_1') \oplus \cdots \oplus (U_s \oplus U_s'),$$

where $W_{\pm 1}$ denote the non-degenerate eigenspaces corresponding to the eigenvalues ± 1, and every other eigenspace is totally singular and occurs in a pair (U_i, U_i'), where $\dim U_i = \dim U_i' = \lambda_i$ and $U_i \oplus U_i'$ is non-degenerate. If $a = \dim W_1$ and $b = \dim W_{-1}$ then we define $\lambda(x)$ to be the partition $(a, b, \lambda_1^2, \ldots, \lambda_s^2) \perp n$, where we set $b = 0$ if $p = 2$. If $G = \mathrm{Sp}_n$ then $C_G(x)^\circ = \mathrm{Sp}_a \times \mathrm{Sp}_b \times \prod_i \mathrm{GL}_{\lambda_i}$. Similarly if $G = \mathrm{SO}_n$.

Definition 2.6. Given $x \in G = \mathrm{Cl}(V)$, define

$$\nu(x) = \min\{\dim[V, \lambda x] : \lambda \in K^*\}.$$

Observe that $\nu(x) > 0$ if x is non-scalar, and note in general that $\nu(x)$ is the codimension of the largest eigenspace of x with respect to the natural action of G on V.

Remark 2.7. In view of Proposition 2.3, note that if G is of type B_n or D_n with $p \neq 2$ and $x \in G$ is unipotent, then $\nu(x)$ must be even. Similarly, if $G = C_n$ or D_n and $x \in G$ is semisimple, then $\nu(x) = s$ is even if $s < n$.

If $x \in G = \mathrm{Cl}(V)$ is semisimple or unipotent, it is possible to derive upper and lower bounds for $\dim x^G$ as functions of $\nu(x)$ and $\dim V$. Before we state and prove this result, we first prove a useful corollary to Lemma 2.4.

Corollary 2.8. *Let $H = \mathrm{Cl}(W)$ be a simple classical algebraic group embedded in a simple algebraic group $G = \mathrm{Cl}(V)$ and suppose that $p \neq 2$ if $G \neq \mathrm{SL}(V)$. Let $x \in H$ be unipotent and let $y \in H$ be a long root element. Then with respect to the actions of x and y on V, we have $\nu(x) \geqslant \nu(y)$.*

Proof. Let $\dim W = d$ and $\dim V = n$, and let $\lambda \perp d$ denote the associated partition of $y \in H$. Since $y \in H$ is a long root element, it follows that $\lambda = (2, 1^{d-2})$ if $H = \mathrm{SL}_d$ or Sp_d, and otherwise $\lambda = (2^2, 1^{d-4})$. In any case, we have $\lambda \leqslant \pi$ for all valid partitions $\pi \perp d$ (i.e. those which correspond to unipotent classes in H), where $\leqslant$ denotes the dominance ordering on partitions of d. Hence Lemma 2.4 implies that

$y \in \overline{x^H} \subset \overline{x^G}$, and thus $y^G \leqslant x^G$ as unipotent classes in G. So by a further application of Lemma 2.4, if y^G and x^G correspond respectively to the partitions $\tilde{\lambda}, \tilde{\mu} \perp n$, then $\tilde{\lambda} \leqslant \tilde{\mu}$. Now if k (resp. l) denotes the number of non-zero parts of $\tilde{\lambda}$ (resp. $\tilde{\mu}$), then $\tilde{\lambda} \leqslant \tilde{\mu}$ implies that $l \leqslant k$. Since $v(y) = n - k$ and $v(x) = n - l$, we deduce that $v(x) \geqslant v(y)$.

The following is a slight improvement of [19, (3.4)].

Proposition 2.9. *Let G be a simple classical algebraic group, and suppose that $x \in G$ is a semisimple or unipotent element such that $v(x) = s$. In the case when $p = 2$ and x is unipotent assume in addition that x is an involution. Then*

$$f(s) \leqslant \dim x^G \leqslant g(s),$$

where $f(s)$ and $g(s)$ are recorded in Table 3.

Table 3

G	$f(s)$	$g(s)$
SL_n	$\max(2s(n-s), ns)$	$s(2n-s-1)$
Sp_{2n}	$\max(s(2n-s), ns)$	$\frac{1}{2}(4ns - s^2 + 1)$
SO_{2n}	$\max(s(2n-s-1), n(s-1))$	$\frac{1}{2}(4ns - s^2 - 2s)$
SO_{2n+1}	$\max\big(s(2n-s), \frac{1}{2}(2ns + s - 2n - 1)\big)$	$\frac{1}{2}(4ns - s^2 + 1)$

Proof. The stated values for $f(s)$ when $G = SL_n$ and $G = Sp_{2n}$ follow from [19, Lemma 3.4]. Here the authors derive upper and lower bounds for $|x^{G_\sigma}|$, where G_σ is the fixed point subgroup of a Frobenius morphism σ of G, and $x \in G_\sigma$ is a prime order semisimple or unipotent element. The proof of these bounds given in [19, (3.4)] easily translates to the corresponding algebraic groups, and furthermore, it remains valid for our more general hypothesis on the order of the element x.

The corresponding values for the lower bounds $f(s)$ stated in [19, (3.4)] for the orthogonal groups are slightly inaccurate. For example, suppose that $G = SO_{2n}$, $p \neq 2$ and $x = [J_2^s, I_{2n-2s}] \in G$, so that $v(x) = s \leqslant n$ is even. Then [19, (3.4)] implies that $\dim x^G \geqslant \max(s(2n-s), ns) = s(2n-s)$, but using Proposition 2.3 we calculate that $\dim x^G = s(2n-s-1)$. Following the approach of [19, (3.4)], the proof of the corrected bound $f(s)$ stated in Table 3 for $G = SO_{2n}$ goes as follows. First note that

$$\max(s(2n-s-1), n(s-1)) = \begin{cases} s(2n-s-1), & \text{if } s \leqslant n \\ n(s-1), & \text{otherwise.} \end{cases}$$

Suppose that x is semisimple and $s < n$, in which case s must be even, and $SO_{2n-s} \leqslant C_G(x)$. It follows that $\dim C_G(x) \leqslant \dim SO_{2n-s} + \dim SO_s$, and thus $\dim x^G \geqslant 2ns - s^2 > s(2n-s-1)$. If $s = n$, the largest possible centralizer is GL_n, and so $\dim x^G \geqslant n(n-1) = n(2n-n-1)$. Now since $\dim \mathrm{GL}_m \geqslant 2\dim \mathrm{SO}_m$ for all m, it follows that if $s > n$, then

$$\dim C_G(x) \leqslant \dim \mathrm{GL}_{2n-s}^{2n/(4n-2s)} = 2n^2 - ns,$$

and so $\dim x^G \geqslant n(s-1)$ as required.

Now assume that x is unipotent and $p \neq 2$. Suppose that the Jordan decomposition of x has precisely n_i Jordan blocks of size i, so that $\sum in_i = 2n$ and $\sum n_i = 2n-s$. From Proposition 2.3, we have $\dim x^G = 2n^2 - n - g$, where

$$g = \sum_{i<j} in_in_j + \frac{1}{2}\sum in_i^2 - \frac{1}{2}\sum_{i\,\mathrm{odd}} n_i.$$

We have

$$\begin{aligned} n(2n-s) &= \frac{1}{2}\left(\sum in_i\right)\left(\sum n_i\right) \\ &= \sum_{i<j} in_in_j + \frac{1}{2}\sum in_i^2 + \frac{1}{2}\sum_{i<j}(j-i)n_in_j \geqslant g, \end{aligned}$$

since the last term is non-negative. Hence, $\dim x^G \geqslant n(s-1)$. Similarly we can show that $\dim x^G \geqslant s(2n-s-1)$. Finally, if $p = 2$, then by hypothesis x is an involution and hence $s \leqslant n$. Furthermore, x must be G-conjugate to $[J_2^s, I_{2n-2s}]$, and so using Proposition 2.3 (iv) we see that $\dim x^G \geqslant 2ns - s^2 - s$. The proof of the lower bound for $G = \mathrm{SO}_{2n+1}$ is similar.

The values for $g(s)$ stated in Table 3 are a slight improvement on those given in [19, (3.4)]. The proof in each case is straightforward. For example, suppose that $G = \mathrm{SO}_{2n}$ and $x \in G$ is semisimple, with $v(x) = s$. Here we have $g(s) = 2ns - s^2/2 - s$. Now if s is even (which must be the case if $s < n$) then clearly we have

$$\dim C_G(x) \geqslant \dim \mathrm{SO}_{2n-s} + \dim T_{s/2},$$

so that $\dim x^G \leqslant g(s)$ (here T_i denotes an i-dimensional torus). If $s = n$ is odd then $C_G(x)^\circ = \mathrm{GL}_n$ and thus $\dim x^G = n^2 - n < g(n)$. If $s > n$ is odd, then

$$\dim C_G(x) \geqslant \dim \mathrm{GL}_{2n-s} + \dim T_{s-n},$$

so that $\dim x^G \leqslant 4ns - 2n^2 - s^2 - s$ which is less than $g(s)$ for $n < s \leqslant 2n-1$. Using Lemma 2.4, we see that when $p \neq 2$ the largest unipotent class x^G such that $v(x) = s$ is given by $[J_{s+1}, I_{2n-s-1}]^G$. This class has dimension $g(s)$. By hypothesis, if $p = 2$ we need only consider involution classes. From Proposition 2.3 (iv) we see that the larger unipotent class of involutions has dimension $2ns - s^2$. This is at most $g(s)$ since $s > 0$ is even. The stated values for $g(s)$ for the other types of G are just as easily verified.

Remark 2.10. The bounds recorded in Table 3 are useful in general arguments, although not surprisingly, we can obtain better bounds given an explicit pair (G, s). If x

is unipotent we can appeal to Proposition 2.3 and Lemma 2.4, and for x semisimple, calculating the largest and smallest possible centralizers of x in G is quite straightforward. We illustrate this with an example.

Suppose that $G = \mathrm{SO}_{20}$ and $\nu(x) = 12$. If x is semisimple, then in arbitrary characteristic we clearly have

$$\dim(\mathrm{SO}_8 \times T_6) \leqslant \dim C_G(x) \leqslant \dim(\mathrm{GL}_8 \times \mathrm{SO}_4),$$

and so $120 \leqslant \dim x^G \leqslant 156$. If x is unipotent, we are interested in the 8-part partitions of 20, where even parts occur with even multiplicity. With respect to the dominance ordering on partitions, the least and greatest such partitions are $(3^4, 2^4)$ and $(13, 1^7)$ respectively. If we assume that $p \neq 2$, then using Proposition 2.3 (iii) and Lemma 2.4, we obtain exactly the same bounds for $\dim x^G$ as in the semisimple case. This may be compared with the bound arising from Proposition 2.9, namely $110 \leqslant \dim x^G \leqslant 174$.

In Sections 3–6 we shall make much use of the $\mathscr{C}_i$ notation of [18]. For the reader's convenience, we briefly define the four collections of maximal non-subspace subgroups H of positive dimension of a simple algebraic group $G = \mathrm{Cl}(V)$.

Class $\mathscr{C}_2$. Stabilizers of orthogonal decompositions. Here $H = G_{\{V_1,\dots,V_t\}}$, where $V = \bigoplus_{i=1}^t V_i$, $t > 1$ and the subspaces V_i are mutually orthogonal and isometric.

Class $\mathscr{C}_3$. Stabilizers of totally singular decompositions. Here we have $G = \mathrm{Sp}(V)$ or $\mathrm{SO}(V)$ and $H = G_{\{W,W'\}}$, where $V = W \oplus W'$ and W, W' are maximal totally singular subspaces. Note that if $G = \mathrm{SO}(V)$ and $\dim V \equiv 2 \pmod 4$ then H is not maximal.

Class $\mathscr{C}_4$. Tensor product subgroups. In this case either $V = V_1 \otimes V_2$ with $\dim V_i > 1$ and $H = N_G(\mathrm{Cl}(V_1) \circ \mathrm{Cl}(V_2))$ acting naturally on the tensor product, or $V = \bigotimes_{i=1}^k V_i$ with $k > 1$, the V_i mutually isometric and $H = N_G(\prod \mathrm{Cl}(V_i))$, again acting naturally. See Section 5 for details of which classical subgroups appear as factors.

Class $\mathscr{C}_6$. Classical subgroups. These are the subgroups $N_G(\mathrm{Sp}(V))$ and $N_G(\mathrm{SO}(V))$ in $G = \mathrm{SL}(V)$.

Lemma 2.11. *Let $G = \mathrm{Cl}(V)$ be a simple classical algebraic group, and let H be a maximal closed subgroup in one of the collections $\mathscr{C}_i$, $i = 2, 3, 4$ or 6. Let $x \in G$ be a semisimple or unipotent element. Then $x^G \cap H^\circ$ is a finite union of distinct H°-classes.*

Proof. In each case H is a reductive group, so that H° has only finitely many distinct unipotent conjugacy classes and the result is immediate if x is unipotent. If x is semisimple then the result follows from [16, (1.3)].

3 Proof of Theorem 1 for $H \in \mathscr{C}_2$

Let $G = \mathrm{Cl}(V)$ be a simple classical algebraic group, and let $H = (\mathrm{Cl}_m \wr S_k) \cap G$ be a maximal closed subgroup in the collection $\mathscr{C}_2$. In Lemmas 3.1 and 3.2 we suppose that our given semisimple or unipotent element $x \in G$ satisfies

$$\dim(x^G \cap H) = \dim(x^G \cap H^\circ).$$

Under this hypothesis, Lemma 3.1 quickly reduces the problem to the cases when $k = 2, 3$, which are dealt with in Lemma 3.2. Finally, in Lemma 3.3 we complete the proof by considering the case when $\dim(x^G \cap H) \neq \dim(x^G \cap H^\circ)$.

Lemma 3.1. *Let $G = \mathrm{Cl}(V)$ be a simple classical algebraic group, and let $H = (\mathrm{Cl}_m \wr S_k) \cap G$ be a maximal closed subgroup in the collection $\mathscr{C}_2$. Let $x \in G$ be a semisimple or unipotent element, and assume that x is an involution if $p = 2$ and x is unipotent. If x satisfies $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$ then $\dim(x^G \cap H)/\dim x^G \leqslant 1/2$ for all $k \geqslant 4$.*

Proof. From Lemma 2.11 we may assume that $x \in H^\circ$ and $\dim(x^G \cap H) = \dim x^{H^\circ}$. Since $x \in H^\circ$, x fixes a decomposition $V = V_1 \oplus \cdots \oplus V_k$, where $\dim V_i = m$ for each i. Let $s = \nu(x)$ and $s_i = \nu_{V_i}(x_i)$, where x_i denotes the restriction of x to V_i. Clearly we have $\sum_i s_i \leqslant s$.

Suppose that $G = \mathrm{SL}_{mk}$, so that $\mathrm{Cl}_m = \mathrm{GL}_m$. From Proposition 2.9 we deduce that $\dim x_i^{\mathrm{GL}_m} \leqslant 2s_i(m-1)$, and so $\dim x^{H^\circ} \leqslant 2s(m-1)$. A further application of 2.9 yields that $\dim x^G \geqslant mks$, and thus $\dim x^{H^\circ}/\dim x^G \leqslant 1/2$ if $k \geqslant 4$.

Similarly, if $G = \mathrm{Sp}_{mk}$ then $\dim x_i^{\mathrm{Sp}_m} \leqslant s_i m$ and $\dim x^G \geqslant mks/2$. The case when $G = \mathrm{SO}_{mk}$ is entirely similar.

Lemma 3.2. *Let $G = \mathrm{Cl}(V)$ be a simple classical algebraic group, and let $H = (\mathrm{Cl}_m \wr S_k) \cap G$ be a maximal closed subgroup in the collection $\mathscr{C}_2$. Let $x \in G$ be a semisimple or unipotent element such that $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$, and assume that x is an involution if x is unipotent and $p = 2$. Then the conclusion of Theorem 1 is true for all such elements.*

Proof. Write $V = V_1 \oplus \cdots \oplus V_k$, where $\dim V_i = m$. With reference to Lemma 2.11, we may assume that $x \in H^\circ$ and $\dim(x^G \cap H) = \dim x^{H^\circ}$, and from Lemma 3.1, we may assume that $k = 2$ or 3. Let x_i denote the restriction of x to V_i and let $\pi_i \perp m$ be the associated partition to x_i with respect to the action on V_i (see Definition 2.5). Then $\dim C_{H^\circ}(x)$ is completely determined by the partitions $\{\pi_i\}$, unless $p = 2$, x is unipotent and G is Sp_{km} or SO_{km}. To be precise, $\dim C_{H^\circ}(x) = \sum_i \dim C_{\mathrm{Cl}_m}(x_i)$. Furthermore, if x is unipotent, then $\dim C_G(x)$ is determined by the partition $\pi = (\pi_1, \ldots, \pi_k) \perp km$ and we use Proposition 2.3 to compute dimensions.

If x is semisimple, then $\dim C_G(x)$ is not determined (in general) by the associated partitions $\{\pi_i\}$. However we can quite easily compute a sharp upper bound. For example, suppose that $k = 2$ and $x \in \mathrm{Sp}_{2m}$ is semisimple with associated partitions

$$\pi_1 = (2a, 2b, \lambda_1^2, \ldots, \lambda_r^2), \quad \pi_2 = (2c, 2d, \mu_1^2, \ldots, \mu_t^2) \perp m, \tag{3}$$

where in accordance with Definition 2.5 the first two parts of each partition denote the dimensions of the non-degenerate eigenspaces of x_i, and $b = d = 0$ if $p = 2$. We may assume without loss of generality that $\lambda_1 \geqslant \cdots \geqslant \lambda_r$, $\mu_1 \geqslant \cdots \geqslant \mu_t$, $a \geqslant b$, $c \geqslant d$ and $r \geqslant t$. If $r > t$ then set $\mu_i = 0$ for $t < i \leqslant r$. It is clear that

$$\mathrm{Sp}_{2(a+c)} \times \mathrm{Sp}_{2(b+d)} \times \mathrm{GL}_{\lambda_1+\mu_1} \times \cdots \times \mathrm{GL}_{\lambda_r+\mu_r}$$

is the largest possible centralizer of x in G.

In general, given a semisimple or unipotent element $x \in H^\circ$, let $\dim x^G \geqslant \beta(x)$ be the sharp lower bound derived from the associated partitions $\{\pi_i\}$ (we have equality if x is unipotent). We then define

$$\alpha(x) = (1+2\varepsilon)\beta(x) - 2\dim x^{H^\circ}, \tag{4}$$

where $\varepsilon = \varepsilon(G, H^\circ)$ is given in the statement of Theorem 1. If $\alpha(x) \geqslant 0$ for each possible pair of partitions π_1 and π_2, then

$$\frac{\dim x^{H^\circ}}{\dim x^G} \leqslant \frac{1}{2} + \varepsilon$$

as required.

Let $G = \mathrm{SL}_{2m}$ and $H = (\mathrm{GL}_m \wr S_2) \cap G$, so that $\varepsilon = 0$. Let $x \in H^\circ$ be semisimple, with associated partitions $\pi_1 = (\lambda_1, \ldots, \lambda_r)$ and $\pi_2 = (\mu_1, \ldots, \mu_t)$. We may assume that each π_i has decreasing parts and that $r \geqslant t$. If $r > t$ then set $\mu_i = 0$ for $t < i \leqslant r$. One can easily verify that

$$\beta(x) = 4m^2 - \sum_i \lambda_i^2 - \sum_i \mu_i^2 - 2\sum_i \lambda_i \mu_i,$$

and thus $\alpha(x) = \sum_{i=1}^r (\lambda_i - \mu_i)^2 \geqslant 0$. If $x \in H^\circ$ is unipotent with $\pi_i \perp m$ given by

$$\pi_1 = (m^{a_m}, \ldots, 1^{a_1}), \quad \pi_2 = (m^{b_m}, \ldots, 1^{b_1}), \tag{5}$$

then

$$\dim x^G = 4m^2 - 2\sum_{i<j} i(a_i+b_i)(a_j+b_j) - \sum_i i(a_i+b_i)^2,$$

and

$$\alpha(x) = \sum_{i=1}^m \Big(\sum_{j=i}^m (a_j - b_j)\Big)^2 \geqslant 0.$$

Now suppose that $G = \mathrm{Sp}_{2m}$ and $H = (\mathrm{Sp}_m \wr S_2) \cap G$. In this case we have $\varepsilon = 1/(2m+2)$. Let $m = 2l$. If $x \in H^\circ$ is semisimple with associated partitions π_1 and

π_2 labelled as in (3) (with the corresponding assumptions on parts) then one calculates that

$$\alpha(x) = 2l^2 + 2l(a-c)^2 + 2l(b-d)^2 + l(a+b+c+d) \\ + l\sum_i(\lambda_i - \mu_i)^2 - 2\sum_i \lambda_i\mu_i - 4ac - 4bd.$$

Hence $\alpha(x) \geqslant 0$ since $l = a + b + \sum_i \lambda_i = c + d + \sum_i \mu_i$. If $p \neq 2$ and $x \in H^\circ$ is unipotent with the $\{\pi_i\}$ labelled as in (5), then

$$\alpha(x) = (l+1)\sum_{i=1}^{m}\Big(\sum_{j=i}^{m}(a_j - b_j)\Big)^2 + 4l^2 - \sum_i i(a_i^2 + b_i^2) \\ - 2\sum_{i<j} i(a_ia_j + b_ib_j) + l\sum_{i\,\mathrm{odd}}(a_i + b_i).$$

where $m = 2l$. Since $2l = \sum_i ia_i = \sum_i ib_i$, we deduce that $\alpha(x) \geqslant 0$. The case when $G = \mathrm{SO}_{2m}$ (with $p \neq 2$ if x is unipotent) is entirely similar and we leave it to the reader.

From Proposition 2.3 (iv) we deduce that if $G = \mathrm{Sp}_{2m}$ or SO_{2m} and $x \in H^\circ$ is a unipotent involution (so that $p = 2$) then we need only consider the elements $[a_{l-r}, a_{l-s}] = [a_{m-(r+s)}]$ and $[c_{l-r}, c_{l-s}] = [c_{m-(r+s)}]$ for all r and s in the range $1 < r$, $s < l$, with $l - r$ and $l - s$ even, where $l = \lfloor m/2 \rfloor$ and in the notation of [2], a_i and c_i are unipotent involution class representatives in Cl_m. This is very straightforward. For example, suppose that

$$G = \mathrm{SO}_{2(2l+1)} \quad \text{and} \quad x = [c_{l-r}, c_{l-s}] = [c_{2l+1-(r+s+1)}].$$

Then from Proposition 2.3 (iv) we compute that

$$\alpha(x) = 12l + (r-s)^2 - 2r - 2s \geqslant 0.$$

Similarly if $k = 3$.

If $k = 3$ then in each case (G, H°), one easily deduces that $\alpha(x) \geqslant 0$ for all semisimple elements $x \in H^\circ$. The arguments are entirely similar to those given previously for the case when $k = 2$.

If $G = \mathrm{SL}_{3m}$, $H = (\mathrm{GL}_m \wr S_3) \cap G$, and $x \in H^\circ$ is unipotent, then with π_1 and π_2 as in (5), and $\pi_3 = (m^{c_m}, \dots, 1^{c_1}) \perp m$, we have

$$\alpha(x) = \dim x^{H^\circ} + \sum_{i=1}^{m}\Big(\sum_{j=i}^{m}(a_j - b_j)\Big)^2 + \sum_{i=1}^{m}\Big(\sum_{j=i}^{m}(a_j - c_j)\Big)^2 + \sum_{i=1}^{m}\Big(\sum_{j=i}^{m}(b_j - c_j)\Big)^2,$$

so that $\alpha(x) \geqslant 0$.

With the same labelling of the associated partitions $\{\pi_i\}$, if $x \in \mathrm{Sp}_{3m}$ is unipotent and $p \neq 2$ we have

$$\alpha(x) = \dim x^{H^\circ} + \sum_{i=1}^{m}\Big(\sum_{j=i}^{m}(a_j - b_j)\Big)^2 + \sum_{i=1}^{m}\Big(\sum_{j=i}^{m}(a_j - c_j)\Big)^2 + \sum_{i=1}^{m}\Big(\sum_{j=i}^{m}(b_j - c_j)\Big)^2 - 6l + \frac{1}{2}\sum_{i\,\mathrm{odd}}(a_i + b_i + c_i),$$

where $m = 2l$. Then $\alpha(x) \geqslant 0$ since $\dim x^{H^\circ} \geqslant 6l$. Similarly if $x \in \mathrm{SO}_{3m}$ is unipotent and $p \neq 2$.

Lemma 3.3. *Let G be a simple classical algebraic group, and let $H = (\mathrm{Cl}_m \wr S_k) \cap G$ be a maximal closed subgroup in the collection $\mathscr{C}_2$. Let $x \in G$ be a semisimple or unipotent element of prime order. Then the conclusion of Theorem* 1 *is true.*

Proof. In view of Lemma 3.2, we may assume that $\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi)$, where $1 \neq \pi \in S_k$. We follow closely the proof of [19, Lemma 4.5].

Replacing x by a suitable G-conjugate, we may assume that $x \in H^\circ\pi$, i.e. x is the image of $(b_1, \dots, b_k)\pi$, where $b_i \in \mathrm{Cl}_m$ for each i. Let r denote the (prime) order of x, hence of π and each b_i, and suppose that π comprises exactly h r-cycles and f fixed points. From the proof of [19, Lemma 4.5] we have the following important facts:

($\diamond$) x is H°-conjugate to $b\pi$, where $b = (1, \dots, 1, b_{hr+1}, \dots, b_k)$,
($\star$) $\nu(x) \geqslant mh(r-1)$.

We may assume that $\dim(x^G \cap H) = \dim x^{H^\circ}$. This follows immediately from the fact that $x^G \cap H^\circ\pi$ is a finite union of H°-classes. To see this, note that the reductive algebraic group Cl_m only has finitely many distinct classes of semisimple or unipotent elements of prime order r, and so the claim follows from ($\diamond$).

Suppose that $H = (\mathrm{GL}_m \wr S_k) \cap \mathrm{SL}_{mk}$, so that $\dim H = km^2 - 1$. Using the fact that $\nu(x) \geqslant mh(r-1)$, we see that Proposition 2.9 implies that $\dim x^G \geqslant m^2kh(r-1)$. So

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leqslant \frac{km^2 - 1}{km^2h(r-1)} < \frac{1}{2},$$

unless $(h, r) = (1, 2)$. If $(h, r) = (1, 2)$ then x is an involution and from ($\diamond$), without any loss of generality, we may take x to be the image of $b(12)$, where $b = (I_m, I_m, b_3, \dots, b_k)$ and the $b_i \in \mathrm{GL}_m$ are involutions. One easily checks that if $t \in \mathrm{GL}_m$ is an involution then $\dim t^{\mathrm{GL}_m} \leqslant m^2/2$. Hence

$$\dim C_{H^\circ}(x) = \dim C_{\mathrm{GL}_m^k}(x) \geqslant m^2 + \tfrac{1}{2}m^2(k-2),$$

so that $\dim(x^G \cap H) = \dim x^{H^\circ} \leqslant km^2/2$. Since $\nu(x) \geqslant m$, Proposition 2.9 implies that $\dim x^G \geqslant km^2$ and thus $\dim(x^G \cap H)/\dim x^G \leqslant 1/2$ as required.

Now consider $H = (\mathrm{Sp}_m \wr S_k) \cap \mathrm{Sp}_{mk}$. Using ($\star$) with Proposition 2.9, and the bound $\dim(x^G \cap H) \leqslant \dim H$, we quickly reduce to the cases $(h, r) = (1, 2), (2, 2)$ and $(1, 3)$. Let $m = 2l$.

Suppose that $(h, r) = (1, 3)$. Following [19, (3.4)], we may assume that

$$x = (I_{2l}, I_{2l}, I_{2l}, b_4, \ldots, b_k)(123),$$

where each $b_i \in \mathrm{Sp}_{2l}$ is an element of order 3. We claim that $\dim C_{\mathrm{Sp}_{2l}}(b_i) \geqslant 2l^2/3$ for each i. If x (and thus each b_i) is semisimple of order 3 then $C_{\mathrm{Sp}_{2l}}(b_i) = \mathrm{Sp}_{2j} \times \mathrm{GL}_{l-j}$, for some $0 \leqslant j < l$. Let $\varphi(j) = \dim(\mathrm{Sp}_{2j} \times \mathrm{GL}_{l-j})$. One easily checks that for $0 \leqslant j < l$, we have $\varphi(j) \geqslant 2l^2/3 + l/3 - 1/12 > 2l^2/3$. Now suppose that x is unipotent, so that by hypothesis we must have $p = 3$. We may assume that $l > 1$. Write $2l = 3a + b$, where $0 \leqslant b < 3$. Since the Jordan form of b_i cannot have a Jordan block of size $j > 3$, it follows from Lemma 2.4 that the largest unipotent class in Sp_{2l} of elements of order 3 is represented by $y = [J_3^a, J_b]$. We calculate using Proposition 2.3 (ii) that $\dim C_{\mathrm{Sp}_{2l}}(y) \geqslant 3a^2/2 + ab + a/2 + b/2$. Since $2l^2/3 = 3a^2/2 + b^2/2 + ab$, and $0 \leqslant b \leqslant 2$, we deduce that $\dim C_{\mathrm{Sp}_{2l}}(y) \geqslant 2l^2/3$ as claimed.

From the claim it follows that

$$\dim x^{H^\circ} \leqslant \dim H - \dim \mathrm{Sp}_{2l} - \tfrac{2}{3}l^2(k-3) = \tfrac{4}{3}kl^2 + lk - l.$$

From Proposition 2.9 we have $\dim x^G \geqslant 4l^2k$ since $v(x) \geqslant 4l$ from $(\star)$. When $l \geqslant 2$, these bounds are sufficient to yield $\dim(x^G \cap H)/\dim x^G < 1/2$. If $t \in \mathrm{Sp}_2$ has order 3, then it is clear that $\dim C_{\mathrm{Sp}_2}(t) = 1$, so that $\dim(x^G \cap H) \leqslant 2k$. This is sufficient since we have $\dim x^G \geqslant 4k$.

Assume now that $(h, r) = (2, 2)$; in this case we have $k \geqslant 4$ and x is an involution. Without loss of generality, we may take

$$x = (I_{2l}, I_{2l}, I_{2l}, I_{2l}, b_5, \ldots, b_k)(12)(34),$$

where each $b_i \in \mathrm{Sp}_{2l}$ is an involution. Now from Proposition 2.2 we have $\dim C_{\mathrm{Sp}_{2l}}(b_i) \geqslant l^2$ for each i, and hence

$$\dim x^{H^\circ} \leqslant \dim H - 2\dim \mathrm{Sp}_{2l} - (k-4)l^2.$$

Since $(h, r) = (2, 2)$, it follows from $(\star)$ that $v(x) \geqslant 4l$, and thus $\dim x^G \geqslant 4l^2k$ (by Proposition 2.9). This is sufficient to imply that $\dim(x^G \cap H)/\dim x^G < 1/2$ for all possible l, k.

This leaves us to deal with the case $(h, r) = (1, 2)$. Following the same procedure as in the previous case, we deduce that $\dim(x^G \cap H) \leqslant kl^2 + kl - l$. From Proposition 2.9 we have $\dim x^G \geqslant 4l^2(k-1)$ since $v(x) \geqslant 2l$. Using these bounds, we deduce that $\dim(x^G \cap H)/\dim x^G \leqslant 1/2$ unless $k = 2$ or $(k, l) = (3, 1)$. Consider the latter case. As usual, we may assume that $x = (I_2, I_2, b)(12)$, where $b \in \mathrm{Sp}_2$ satisfies $b^2 = I_2$. If x is semisimple then b must be scalar, so that $\dim(x^G \cap H) \leqslant 3$ and $\dim x^G = 8$. If b is a unipotent involution then $\dim C_{H^\circ}(x) = 4$ and x is G-conjugate to $[J_2^3]$. So $\dim(x^G \cap H) \leqslant 5$ and $\dim x^G = 12$, which yields a ratio of $5/12 < 1/2$.

Finally, note that if $k = 2$ then from the statement of Theorem 1 we have $\varepsilon = 1/4l$. We may take $x = (I_{2l}, I_{2l})(12)$, so that $\dim x^G = 4l^2$ (note that x is an a_{2l} involution if $p = 2$). Since $C_{H^\circ}(x) = \mathrm{Sp}_{2l}$, it follows that

$$\frac{\dim(x^G \cap H)}{\dim x^G} \leqslant \frac{2l^2+l}{4l^2} = \frac{1}{2} + \frac{1}{4l},$$

as required.

The final case to consider is $H = (\mathrm{SO}_m \wr S_k) \cap \mathrm{SO}_{mk}$. Using the bounds $\dim(x^G \cap H) \leqslant \dim H = \frac{1}{2}(km^2 - km)$ and $\dim x^G \geqslant \frac{1}{2}mk(mh(r-1)-1)$ (via ($\star$) and Proposition 2.9), we deduce that $\dim(x^G \cap H)/\dim H \leqslant 1/2$ unless $(h,r) = (1,2)$. Suppose that $m = 2l$. Now, if $t \in \mathrm{SO}_{2l}$ is an involution, then from Proposition 2.2 we have $\dim t^{\mathrm{SO}_{2l}} \leqslant l^2$. Hence

$$\dim(x^G \cap H) \leqslant (k-1)(2l^2 - l) - (k-2)l^2 = kl^2 - kl + l,$$

and since $\dim x^G \geqslant 2lk(2l-1)$, this implies that $\dim(x^G \cap H)/\dim H < 1/2$. Similarly if $m = 2l+1$.

4 Proof of Theorem 1 for $H \in \mathscr{C}_3$

As described in Section 2, here $G = \mathrm{Sp}_{2n}$ or SO_{2n}, and $H = \mathrm{Stab}_G\{U, W\}$, where U and W are maximal totally singular subspaces of the natural G-module. Hence $H^\circ \cong \mathrm{GL}_n$ and $|H : H^\circ| = 2$. In fact,

$$H^\circ = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-T} \end{pmatrix} : A \in \mathrm{GL}_n \right\} \cong \mathrm{GL}_n \tag{6}$$

Lemma 4.1. *If $G = \mathrm{Sp}_{2n}$ and $H \in \mathscr{C}_3$ then the conclusion of Theorem* 1 *is true.*

Proof. Note that if $p = 2$ then H is not maximal in G since $H < O_{2n} < G$, and so by hypothesis, we may assume that $p \neq 2$. We begin by assuming that $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. From Lemma 2.11, we may assume without loss of generality that $x \in H^\circ$ and $\dim(x^G \cap H) = \dim x^{H^\circ}$. We follow a similar approach to that in the proof of Lemma 3.2. Given $x \in H^\circ = \mathrm{GL}_n$ semisimple or unipotent, the associated partition $\lambda \perp n$ derived from the action of x on the natural GL_n-module completely determines $\dim C_{H^\circ}(x)$. From λ we obtain a sharp lower bound $\dim x^G \geqslant \beta(x)$ and we define $\alpha(x)$ as in equation (4).

Suppose that $x \in H^\circ = \mathrm{GL}_n$ is semisimple, and let $\lambda = (\lambda_1, \ldots, \lambda_k) \perp n$ be the associated partition. We may assume that $\lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant 0$ and that k is even. Clearly $C_{H^\circ}(x)^\circ = \prod_i \mathrm{GL}_{\lambda_i}$, and so $\dim x^{H^\circ} = n^2 - \sum_{i=1}^k \lambda_i^2$. There are two possible candidates for the centralizer $C_G(x)^\circ$ of maximal dimension, namely

(i) $\mathrm{Sp}_{2\lambda_1} \times \mathrm{Sp}_{2\lambda_2} \times \mathrm{GL}_{\lambda_3+\lambda_4} \times \cdots \times \mathrm{GL}_{\lambda_{k-1}+\lambda_k}$,

(ii) $\mathrm{Sp}_{2\lambda_1} \times \mathrm{GL}_{\lambda_2+\lambda_3} \times \cdots \times \mathrm{GL}_{\lambda_{k-2}+\lambda_{k-1}} \times \mathrm{GL}_{\lambda_k}$.

In (i) we calculate that $\alpha(x) = \sum_{i=2}^{k/2} (\lambda_{2i-1} - \lambda_{2i})^2 + n - \lambda_1 - \lambda_2 \geqslant 0$. Similarly for (ii).

If $x \in H^\circ$ is unipotent, let $\lambda = (n^{a_n}, \ldots, 1^{a_1}) \perp n$ be the associated partition. It is a basic fact of linear algebra that if $A \in \mathrm{GL}_n$ is unipotent then A and A^{-T} are GL_n-conjugate. Hence in view of the isomorphism (6), considering x as an element of Sp_{2n} the associated partition is $(n^{2a_n}, \dots, 1^{2a_1}) \perp 2n$ and thus $\dim x^G$ is completely determined by λ since $p \neq 2$. From Proposition 2.3 we deduce that $\alpha(x) = n - \sum_{i\,\mathrm{odd}} a_i \geqslant 0$.

Now suppose that $\dim(x^G \cap H) \neq \dim(x^G \cap H^\circ)$. We can assume that

$$x \in H - H^\circ = H^\circ \tau \quad \text{where } \tau = \begin{pmatrix} & I_n \\ -I_n & \end{pmatrix} \in \mathrm{Sp}_{2n},$$

with respect to the basis ordering $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ of a standard symplectic basis of the natural Sp_{2n}-module. Furthermore, we may assume that x is an involution since we are only concerned with elements of prime order. It is clear that τ induces an involutory graph automorphism of $A_{n-1} < H^\circ$.

If $n = 2m+1$ then from [16, (1.4)] we know that there exists a unique H°-class of involutions in $H^\circ\tau$, and $C_{A_{n-1}}(x) = B_m$ for any x in this class. Since τ is an involution in the adjoint group PSp_{2n}, we may take $x = \tau$. Now τ does not centralize the T_1 torus in $H^\circ = T_1 A_{n-1}$, and so

$$C_{H^\circ}(\tau) = B_m \quad \text{and} \quad \dim(\tau^G \cap H) = \dim \tau^{H^\circ} = 2m^2 + 3m + 1.$$

Since $C_G(\tau) \cong \mathrm{GL}_n$, we have $\dim \tau^G = 4m^2 + 6m + 2$ and

$$\dim(\tau^G \cap H)/\dim \tau^G = 1/2.$$

Now suppose that $n = 2m$. From [16, (1.4)] we know that there are precisely two distinct H°-conjugacy classes of involutions in $H^\circ\tau$, and so we may assume that $\dim(x^G \cap H) = \dim x^{H^\circ}$. As before we have $C_G(\tau) = \mathrm{GL}_n$ and $C_{H^\circ}(\tau) = \mathrm{SO}_n$, and thus $\dim \tau^{H^\circ}/\dim \tau^G = 1/2$. For a representative of the other class of involutions in $H^\circ\tau$, consider the element $\omega \in H^\circ\tau$, where the action of ω on a standard symplectic basis of the natural Sp_{2n}-module is given by

$$\begin{aligned} e_i &\mapsto f_{2m+1-i} && (1 \leqslant i \leqslant m), \\ e_i &\mapsto -f_{2m+1-i} && (m+1 \leqslant i \leqslant 2m), \\ f_i &\mapsto -e_{2m+1-i} && (1 \leqslant i \leqslant m), \\ f_i &\mapsto e_{2m+1-i} && (m+1 \leqslant i \leqslant 2m). \end{aligned}$$

Since $\omega^2 = I_{2n}$ and $\tau^2 = -I_{2n}$, we conclude that ω is not G-conjugate to τ. Clearly we have $C_G(\omega) = \mathrm{Sp}_n^2$, so that $\dim \omega^G = n^2$. A straightforward direct calculation yields that $C_{A_{n-1}}(\omega) = \mathrm{Sp}_n$, and since ω does not centralize the T_1 torus, we deduce that $\dim \omega^{H^\circ} = n^2/2 - n/2$, so that $\dim(\omega^G \cap H)/\dim \omega^G = 1/2 - 1/2n$.

Lemma 4.2. *If $G = \mathrm{SO}_{2n}$ and $H \in \mathscr{C}_3$ then the conclusion of Theorem* 1 *is true.*

Proof. Recall from the statement of Theorem 1 that we have $\varepsilon = 1/(2n-2)$ in this case. We need only consider the case $n = 2m$, since otherwise H is not maximal in G. We adopt the same approach as for Lemma 4.1.

Suppose that $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. We may assume that $x \in H^\circ$ and $\dim(x^G \cap H) = \dim x^{H^\circ}$. If x is semisimple, let $\lambda = (\lambda_1, \ldots, \lambda_k) \perp n$ be the associated partition with $\lambda_1 \geqslant \cdots \geqslant \lambda_k$ and k even. Then $\dim x^{H^\circ} = n^2 - \sum_i \lambda_i^2$ and there are three possible candidates for the centralizer $C_G(x)^\circ$ of largest dimension:

(i) $\mathrm{SO}_{2\lambda_1} \times \mathrm{SO}_{2\lambda_2} \times \mathrm{GL}_{\lambda_3+\lambda_4} \times \cdots \times \mathrm{GL}_{\lambda_{k-1}+\lambda_k}$;

(ii) $\mathrm{SO}_{2\lambda_1} \times \mathrm{GL}_{\lambda_2+\lambda_3} \times \cdots \times \mathrm{GL}_{\lambda_{k-2}+\lambda_{k-1}} \times \mathrm{GL}_{\lambda_k}$;

(iii) $\mathrm{GL}_{\lambda_1+\lambda_2} \times \mathrm{GL}_{\lambda_3+\lambda_4} \times \cdots \times \mathrm{GL}_{\lambda_{k-1}+\lambda_k}$;

here case (i) is only possible if $p \neq 2$.

For each possibility, we calculate $\beta(x)$ and derive $\alpha(x)$ as before. We then show that $\alpha(x) \geqslant 0$ for all partitions $\lambda \perp n$. The approach in each case is similar, and so we shall only deal with (i), and leave (ii) and (iii) to the reader. In (i) we have

$$\beta(x) = 2n^2 - n - \lambda_1^2 - \lambda_2^2 + \lambda_1 + \lambda_2 - \sum_{i=1}^{k} \lambda_i^2 - 2\sum_{i=2}^{k/2} \lambda_{2i-1}\lambda_{2i},$$

and

$$\alpha(x) = n\sum_{i=2}^{k/2} (\lambda_{2i-1} - \lambda_{2i})^2 + n^2 + n(\lambda_1 + \lambda_2) - 2\sum_{i=1}^{k} \lambda_i^2.$$

Since $n = \sum_i \lambda_i$, we deduce that $\alpha(x) \geqslant 0$.

Now suppose that $x \in H^\circ$ is unipotent and $p \neq 2$. As in Lemma 4.1, if $\lambda = (n^{a_n}, \ldots, 1^{a_1}) \perp n$ is the associated partition of x with respect to the natural GL_n-module then $\mu = (n^{2a_n}, \ldots, 1^{2a_1}) \perp 2n$ is the partition of x as an element of SO_{2n}. Via Proposition 2.3(iii) we calculate that

$$\alpha(x) = \dim x^{H^\circ} - \frac{1}{2}n\Big(n - \sum_{i\,\mathrm{odd}} a_i\Big).$$

Suppose that $\nu(x) = s$, so that $n - s = \sum_i a_i$. Then by Proposition 2.9 we have

$$\dim x^{H^\circ} \geqslant ns = n\Big(n - \sum_i a_i\Big).$$

Now $n - \sum_i a_i \geqslant n/2 - \sum_{i\,\mathrm{odd}} a_i/2$ since $n = \sum_i i a_i$, and hence $\alpha(x) \geqslant 0$ as required.

If $p = 2$, we proceed as in Lemma 4.1, using Proposition 2.3 to calculate $\dim x^{H^\circ}$ and $\dim x^G$ for each unipotent involution $x \in H^\circ$. If x has associated partition

$(2^r, 1^{n-2r}) \perp n$, then $\dim x^{H^\circ} = 2nr - 2r^2$ and $\dim x^G = 4nr - 4r^2 - 2r$ since $x \in G$ is an involution of type a_{2r}. Hence

$$\frac{\dim x^{H^\circ}}{\dim x^G} = \frac{1}{2} + \frac{1}{4n - 4r - 2} \leqslant \frac{1}{2} + \frac{1}{2n-2},$$

since $n \geqslant 2r$.

Now suppose that $\dim(x^G \cap H) \neq \dim(x^G \cap H^\circ)$. As in Lemma 4.1, we may choose $x \in H^\circ\tau$, where

$$\tau = \begin{pmatrix} & I_n \\ I_n & \end{pmatrix} \in \mathrm{SO}_{2n},$$

with respect to the basis ordering $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of a standard orthogonal basis of the natural SO_{2n}-module. As before, τ induces an involutory graph automorphism of A_{n-1}, and since $n = 2m$, there are precisely two distinct involution classes in $H^\circ\tau$. We have $\tau^2 = I_{2n}$ and as before, when $p \neq 2$, we calculate that $C_G(\tau) = \mathrm{SO}_n^2$ and $C_{H^\circ}(\tau) = \mathrm{SO}_n$, so that

$$\frac{\dim \tau^{H^\circ}}{\dim \tau^G} = \frac{1}{2} + \frac{1}{2n} < \frac{1}{2} + \varepsilon.$$

We can represent the other class by the element $\theta \in H^\circ\tau$ defined as follows:

$$\begin{aligned}
e_i &\mapsto -f_{2m+1-i} && (1 \leqslant i \leqslant m),\\
e_i &\mapsto f_{2m+1-i} && (m+1 \leqslant i \leqslant 2m),\\
f_i &\mapsto -e_{2m+1-i} && (1 \leqslant i \leqslant m),\\
f_i &\mapsto e_{2m+1-i} && (m+1 \leqslant i \leqslant 2m).
\end{aligned}$$

Since $\theta^2 = -I_{2n}$, θ does indeed lie in the other involution class. Now $C_G(\theta) = \mathrm{GL}_n$ and $C_{H^\circ}(\theta) = \mathrm{Sp}_n$, and thus $\dim \theta^{H^\circ}/\dim \theta^G = 1/2$. Finally, suppose that $p = 2$. In this case, τ and θ are c_{2m} and a_{2m} unipotent involutions respectively, and $C_{H^\circ}(\tau) = C_{\mathrm{Sp}_{2m}}(t)$ and $C_{H^\circ}(\theta) = \mathrm{Sp}_{2m}$, where $t \in \mathrm{Sp}_{2m}$ is a long root involution. We conclude that $\dim \tau^{H^\circ}/\dim \tau^G = 1/2 + 1/2n$ and $\dim \theta^{H^\circ}/\dim \theta^G = 1/2$.

5 Proof of Theorem 1 for $H \in \mathscr{C}_4$

Lemma 5.1. *If* $G = \mathrm{SL}_n$ *and* $H = N_G(\mathrm{SL}_a \otimes \mathrm{SL}_b)$ (*where* $n = ab$ *and* $a, b \geqslant 2$) *then the conclusion of Theorem* 1 *is true.*

Proof. Clearly it is sufficient to prove that Theorem 1 holds when $G = \mathrm{PSL}_{ab}$ and $H = N_G(\mathrm{PSL}_a \times \mathrm{PSL}_b)$. We may assume that $a \geqslant b$. Given $h \in H$, define $\varphi_h \in \mathrm{Aut}(\mathrm{PSL}_a \times \mathrm{PSL}_b)$ by

$$\varphi_h(x, y) = h^{-1}(x, y)h \quad \text{for all } x \in \mathrm{PSL}_a,\ y \in \mathrm{PSL}_b.$$

Since each factor is simple, it follows that φ_h either fixes each factor, or interchanges the factors. Of course, φ_h can only interchange (algebraically) isomorphic factors, and so we have

$$H = \begin{cases} H^\circ, & \text{if } a \neq b \\ H^\circ \cup H^\circ\tau, & \text{if } a = b \end{cases}$$

where $H^\circ = \mathrm{PSL}_a \times \mathrm{PSL}_b$, and τ interchanges factors, i.e. $\varphi_\tau(x, y) = (y, x)$.

Let $x \in G$ be a semisimple or unipotent element, and assume to begin with that $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. From Lemma 2.11 we may assume that $x \in H^\circ$ and $\dim(x^G \cap H) = \dim x^{H^\circ}$. Let $\nu(x) = s$. If $s \geqslant n/2$ then we deduce from Proposition 2.3 that $\dim x^G \geqslant n^2/2$. Since $\dim x^{H^\circ} \leqslant a^2 - a + b^2 - b$ and $n = ab$, we conclude that $\dim x^{H^\circ}/\dim x^G \leqslant 1/2$ for all $a, b \geqslant 2$. If $s < n/2$ we follow closely the method of Liebeck and Shalev in [19, Lemma 4.3]. Write $x = (x_1, x_2) \in \mathrm{PSL}_a \times \mathrm{PSL}_b$. Let $\nu(x) = s$ and $\nu(x_i) = s_i$, with respect to the obvious natural modules. From [19, Lemma 3.7] we have $s \geqslant \max(as_2, bs_1)$ and using the upper bound stated in [19, (3.4)], we have

$$\dim x^{H^\circ} \leqslant \max_{s_1 \leqslant s/b,\, s_2 \leqslant s/a} \{2as_1 - s_1^2 + 2bs_2 - s_2^2\} \leqslant \frac{2as}{b} - \frac{s^2}{b^2} + \frac{2bs}{a} - \frac{s^2}{a^2},$$

since the maximum is clearly attained when $s_1 = s/b$ and $s_2 = s/a$. From Proposition 2.9, we have $\dim x^G \geqslant 2s(n - s)$, and using these bounds we calculate that $\dim x^{H^\circ}/\dim x^G \leqslant 1/2$ if $g(a, n, s) \geqslant 0$, where

$$g(a, n, s) = s(n^2(1 - a^2) + a^4) + n^3(a^2 - 2) - 2na^4.$$

Since $n^2(1 - a^2) + a^4 < 0$ and $s < n/2$, it follows that $g(a, n, s) \geqslant g\big(a, n, \frac{1}{2}(n - 1)\big)$, and it is easily checked that $g\big(a, n, \frac{1}{2}(n - 1)\big) \geqslant 0$ for all possible (a, n) (with $n \leqslant a^2$ since we assume that $a \geqslant b$), unless $(a, n) = (2, 4)$. However we can exclude this case since if $(a, n) = (2, 4)$ and $s \leqslant 1$ then [19, (3.7)] implies that $s_1 = s_2 = 0$, which in turn implies that x is scalar.

Now suppose that $\dim(x^G \cap H) \neq \dim(x^G \cap H^\circ)$, so that $a = b$ and we may assume that $x \in H^\circ\tau$. The action of τ on a basis of $V = V_a \otimes V_a$ is given by

$$v_i \otimes v_j \mapsto v_j \otimes v_i,$$

unless $p \neq 2$ and $\frac{1}{2}a(a - 1)$ is odd, where in order to ensure that τ has determinant 1 we take $v_i \otimes v_j \mapsto \lambda v_j \otimes v_i$, where $\lambda \in K$ satisfies $\lambda^{a^2} = -1$. Since τ is an involution and we are only interested in elements of prime order, we need only consider involutions in $H^\circ\tau$. From the proof of [19, Lemma 4.5] (cf. ($\diamondsuit$) from the proof of Lemma 3.3) we deduce that τ^{H° is the unique H°-class of involutions in $H^\circ\tau$. Clearly there exists a basis for $V_a \otimes V_a$ with respect to which τ has the matrix $[A^r, I_a]$, where $2r = a^2 - a$ and

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Hence $\dim \tau^G = \frac{1}{2}(a^4 - a^2)$. Since $(g,h)^\tau = (h,g)$, it follows that $C_{H^\circ}(\tau) \cong \mathrm{PSL}_a$, and so $\dim \tau^{H^\circ} = a^2 - 1$. Hence $\dim \tau^{H^\circ}/\dim \tau^G \leqslant 2/a^2 \leqslant 1/2$, since $a \geqslant 2$.

Lemma 5.2. *If* $G = \mathrm{Sp}_{2n}$ *and* $H = N_G(\mathrm{Sp}_{2a} \otimes \mathrm{SO}_b)$ (*where* $p \neq 2$, $n = ab$, $a \geqslant 1$ *and* $b \geqslant 2$) *then the conclusion of Theorem* 1 *is true.*

Proof. As before, we may assume that G is adjoint, so that

$$H = N_G(\mathrm{PSp}_{2a} \times \mathrm{PSO}_b) = \mathrm{PSp}_{2a} \times \mathrm{PSO}_b.$$

If $x \in G$ is semisimple or unipotent, then from Lemma 2.11 we may assume that $\dim(x^G \cap H) = \dim x^H$. We follow the approach of Lemma 5.1. Let $v(x) = s$. If $s \geqslant n$ then $\dim x^G \geqslant n^2$, and since $\dim x^H \leqslant 2a^2 + b^2/2 - b + 1/2$, we deduce that $\dim x^H/\dim x^G \leqslant 1/2$ if

$$n^2(a^2-1) + 2na - a^2(4a^2+1) \geqslant 0.$$

This clearly holds if $n \geqslant 3a$. If $b = 2$, then $\dim x^H \leqslant 2a^2$ and $\dim x^G \geqslant n^2 = 4a^2$ which is sufficient.

Now suppose that $v(x) = s < n$, so that $\dim x^G \geqslant s(2n - s)$. Write $x = (x_1, x_2) \in H$ and $v(x_i) = s_i$. Then from [19, (3.7)] and [19, (3.4)] we have $s \geqslant \max(2as_2, bs_1)$ and

$$\dim x^H \leqslant \max_{s_1 \leqslant s/b,\, s_2 \leqslant s/2a} \left\{ \frac{1}{2}s_1(4a - s_1 + 1) + \frac{1}{2}s_2(2b - s_2 + 1) \right\}.$$

The maximum is realized when s_1 and s_2 are as large as possible, and we calculate that $\dim x^H/\dim x^G \leqslant 1/2$ if $h(a,n,s) \geqslant 0$, where

$$h(a,n,s) = s(n^2(1-4a^2) + 4a^4) + 4n^3(2a^2-1) - 2n^2a - 4na^3(4a+1).$$

Then $h(a,n,s) \geqslant h(a,n,n-1) \geqslant 0$ unless $(a,n) \in \{(1,2),(1,3)\}$. We can exclude the case $(a,n) = (1,2)$ since x is assumed to be non-scalar. If $(a,n) = (1,3)$ we must have $s = 2$, $s_1 = 0$ and $s_2 = 1$, and thus Proposition 2.9 implies that $\dim x^H \leqslant 2$ and $\dim x^G \geqslant 8$.

Lemma 5.3. *Suppose that* $H = N_G(\mathrm{SL}_2 \otimes \mathrm{SL}_2 \otimes \mathrm{SL}_2)$, *where* $G = \mathrm{Sp}_8$ *if* $p \neq 2$ *and* $G = \mathrm{SO}_8$ *if* $p = 2$. *Then the conclusion of Theorem* 1 *is true.*

Proof. We may assume that G is adjoint. We have $H/H^\circ \cong S_3$, where S_3 acts naturally on $H^\circ = \mathrm{PSL}_2^3$ by permuting factors. Let $\pi \in S_3$ be such that $\dim(x^G \cap H) = \dim(x^G \cap H^\circ\pi)$; thus we may assume that $x \in H^\circ\pi$.

Suppose that $\pi = 1$. Then from Lemma 2.11, we may assume that $\dim(x^G \cap H) = \dim x^{H^\circ}$. It is clear that $\dim x^{H^\circ} \leqslant 6$, and from repeated application of [19, (3.7)], we see that $v(x) \geqslant 4$, and hence from Proposition 2.9 we deduce that $\dim x^G$ is at least 16 if $p \neq 2$ and at least 12 if $p = 2$.

Now assume that π is a transposition, so that we need only consider involutions. Suppose that $\pi = (12) \in S_3$, so that if $(g_1, g_2, g_3) \in H^\circ$ we have $(g_1, g_2, g_3)^\pi = (g_2, g_1, g_3)$, and $\dim \pi^{H^\circ} = 3$. The action of π on a basis of $V_2 \otimes V_2 \otimes V_2$ is given by

$$v_i \otimes v_j \otimes v_k \mapsto v_j \otimes v_i \otimes v_k,$$

and so it is clear that π is G-conjugate to $[-I_2, I_6]$ if $p \neq 2$, and to a_2 otherwise. Hence $\dim \pi^G = 12$ if $p \neq 2$ and $\dim \pi^G = 10$ if $p = 2$. Now we easily calculate that

$$\{(g, g^{-1}, z)\pi : g, z \in \mathrm{PSL}_2,\ z^2 = 1\}$$

is the complete set of involutions in $H^\circ\pi$, and that $\pi^{H^\circ} = \{(g, g^{-1}, 1)\pi : g \in \mathrm{PSL}_2\}$. If $z \in \mathrm{PSL}_2$ is an involution, then

$$(1, 1, z)\pi^{(1,g,h)} = (g, g^{-1}, h^{-1}zh)\pi.$$

Since there exists a unique class of involutions in PSL_2, we deduce that there are precisely two H°-classes of involutions in $H^\circ\pi$ with representatives π and $(1, 1, z)\pi$, where $z \in \mathrm{PSL}_2$ is an involution. Clearly $C_{H^\circ}((1, 1, z)\pi) \cong \mathrm{PSL}_2 \times C_{\mathrm{PSL}_2}(z)$, so that $\dim((1, 1, z)\pi)^{H^\circ} = 5$. We also calculate that $(1, 1, z)\pi$ is G-conjugate to $[-iI_4, iI_4]$ if $p \neq 2$, and to a_4 if $p = 2$, where $i \in K$ satisfies $i^2 = -1$. Hence $\dim((1, 1, z)\pi)^G$ is equal to 20 if $p \neq 2$ and to 12 if $p = 2$.

Finally, suppose that $x \in H^\circ\omega$, where $\omega = (123) \in S_3$, so that

$$(g_1, g_2, g_3)^\omega = (g_3, g_1, g_2),$$

and ω acts on a basis of $V_2 \otimes V_2 \otimes V_2$ by

$$v_i \otimes v_j \otimes v_k \mapsto v_k \otimes v_i \otimes v_j.$$

Clearly $\dim \omega^{H^\circ} = 6$, and as before we calculate that $\dim \omega^G = 22$ if $p \neq 2$ and $\dim \omega^G = 18$ if $p = 2$. From the proof of [19, Lemma 4.5] it follows that ω^{H° is the unique H°-class of elements of order 3 in $H^\circ\omega$, and we are done.

Lemma 5.4. *The conclusion of Theorem* 1 *is true in the remaining* $\mathscr{C}_4$ *cases.*

Proof. In each case, we may assume that G is adjoint. Let $x \in H$ be semisimple or unipotent of prime order. If $G = \mathrm{PSL}_{a^t}$ and $H = N_G(\mathrm{PSL}_a^t)$ (where $t \geqslant 3$) then $\dim(x^G \cap H) \leqslant \dim H = t(a^2 - 1)$ and Proposition 2.9 implies that

$$\dim x^G \geqslant 2(a^t - 1).$$

This is sufficient to imply that $\dim(x^G \cap H)/\dim x^G \leqslant 1/2$, unless $(a, t) = (2, 3)$. However, we may ignore this case since H is not maximal in G (see Lemma 5.3). The case when $G = \mathrm{PSO}_{a^t}$ and $H = N_G(\mathrm{PSO}_a^t)$ (where $a \neq 2, 4$, $t \geqslant 3$ and $p \neq 2$) is just as easy. Similarly for $G = \mathrm{PSp}_{2^t a^t}$ and $H = N_G(\mathrm{PSp}_{2a}^t)$ (with $p \neq 2$ and $t \geqslant 3$ odd); we quickly reduce to the case $(a, t) = (1, 3)$, which has been dealt with in Lemma 5.3.

The argument for $G = \mathrm{PSO}_{2^t a^t}$ and $H = N_G(\mathrm{PSp}_{2a}^t)$ (where $t \geqslant 3$ is even, or $p = 2$) is again similar. We have $v(x) \geqslant 2$ for all prime order semisimple and unipotent elements $x \in H$ (cf. Remark 2.7), and applying Proposition 2.9, we are left to deal with the case $(a, t) = (1, 3)$ for which we have Lemma 5.3.

If $G = \mathrm{PSO}_{4ab}$ and $H = N_G(\mathrm{PSp}_{2a} \times \mathrm{PSp}_{2b})$ then H is connected if $a \neq b$, or if $a = b$ is odd. If $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$ then the proof is similar to Lemma 5.2. If $a = b$ is even, then $H = H^\circ \cup H^\circ\tau$, where $\tau \in G$ interchanges the factors. As in Lemma 5.1, there is a unique class of involutions in the coset $H^\circ\tau$. Since τ is G-conjugate to $[-I_{2a^2-a}, I_{2a^2+a}]$ if $p \neq 2$, and to c_{2a^2-a} if $p = 2$, we deduce that $\dim \tau^G = 4a^4 - a^2$. Clearly $C_{H^\circ}(\tau) \cong \mathrm{Sp}_{2a}$, and an upper bound of $1/2$ follows immediately. The case when $G = \mathrm{PSO}_{ab}$, $H = N_G(\mathrm{PSO}_a \times \mathrm{PSO}_b)$ is similar.

6 Proof of Theorem 1 for $H \in \mathscr{C}_6$

In this section, we deal with the classical subgroups $N_G(\mathrm{Sp}_{2n})$ and $N_G(\mathrm{SO}_{2n})$ in $G = \mathrm{SL}_{2n}$. Notice that we exclude the case when $G = \mathrm{Sp}_{2n}$, $H = N_G(\mathrm{SO}_{2n})$ and $p = 2$ since this is a subspace subgroup (see Section 1). Clearly it is sufficient to prove that Theorem 1 holds when $H \in \mathscr{C}_6$ under the assumption that G is adjoint. We have $N_{\mathrm{PSL}_{2n}}(\mathrm{PSp}_{2n}) = \mathrm{PSp}_{2n}$ and $N_{\mathrm{PSL}_{2n+1}}(\mathrm{SO}_{2n+1}) = \mathrm{SO}_{2n+1}$ since B_n and C_n fail to admit any non-trivial graph automorphisms. However, PSO_{2n} does admit an involutory graph automorphism τ which interchanges the two conjugacy classes of maximal parabolic subgroups with Levi factors A_{n-1}. Since $\tau \in \mathrm{GL}_n$ has determinant -1, we have $N_{\mathrm{PSL}_{2n}}(\mathrm{PSO}_{2n}) = \mathrm{PSO}_{2n}$ if $p \neq 2$. If $p = 2$ then $N_{\mathrm{PSL}_{2n}}(\mathrm{PSO}_{2n})$ is a maximal closed subgroup of PSp_{2n}.

Lemma 6.1. *If H is a maximal closed non-subspace subgroup in the collection $\mathscr{C}_6$, then the conclusion to Theorem* 1 *is true.*

Proof. By hypothesis, H is a simple algebraic group. If $x \in G$ is semisimple or unipotent then by Lemma 2.11 we may assume that $x \in H$ and $\dim(x^G \cap H) = \dim x^H$. The proof is very straightforward: the associated partition λ completely determines $\dim x^H$ and $\dim x^G$, and we show that

$$\alpha(x) = (1 + 2\varepsilon)\dim x^G - 2\dim x^H \geqslant 0,$$

for all possible associated partitions λ, where $\varepsilon = \varepsilon(G, H^\circ)$ is given in the statement of Theorem 1.

If $(G, H) = (\mathrm{PSL}_{2n}, \mathrm{PSp}_{2n})$ then $\varepsilon = 1/2n$ in this case. Let $x \in H$ be semisimple with associated partition $\lambda = (2a, 2b, \lambda_1^2, \ldots, \lambda_k^2) \perp 2n$ in accordance with Definition 2.5. Then

$$\alpha(x) = 2n^2 - 4a^2 - 4b^2 - 2\sum_i \lambda_i^2 + 2na + 2nb \geqslant 0,$$

since $n = a + b + \sum_i \lambda_i$. If $p \neq 2$ and x is unipotent with partition given by $\lambda = (2n^{a_{2n}}, \ldots, 1^{a_1}) \perp 2n$, then

$$\alpha(x) = \dim x^G - n\Big(2n - \sum_{i\,\mathrm{odd}} a_i\Big).$$

Now if $\nu(x) = s$, then $2n - s = \sum_i a_i$. From Proposition 2.9 we have

$$\dim x^G \geqslant 2ns = n\Big(4n - 2\sum_i a_i\Big) \quad \text{and} \quad 4n - 2\sum_i a_i \geqslant 2n - \sum_{i\,\mathrm{odd}} a_i$$

since $2n = \sum_i ia_i$. Hence $\alpha(x) \geqslant 0$.

If x is unipotent and $p = 2$, then by hypothesis, x is an involution and therefore has associated partition $\lambda = (2^r, 1^{2n-2r}) \perp 2n$ for some $r > 0$. Using Proposition 2.3, we calculate that $\dim x^G = 4nr - 2r^2$, and $\dim x^H = 2nr - r^2$ if x is of type a_r, and $2nr - r^2 + r$ otherwise. Clearly in each case we have $\dim x^H/\dim x^G \leqslant 1/2 + 1/2n$.

The proof when $(G, H) = (\mathrm{PSL}_n, N_G(\mathrm{PSO}_n))$ is entirely similar and we leave it to the reader.

7 Proof of Theorem 1 for $H \notin \mathscr{C}_i$

Let $G = \mathrm{Cl}(V)$ be a simple classical algebraic group. According to [18, Theorem 1], if $H < G$ is a closed subgroup which is not contained in a member of one of the collections $\mathscr{C}_i$ $(1 \leqslant i \leqslant 6)$, then $E(H)$ is simple and acts irreducibly on V, i.e. we have an irreducible embedding $\varphi : E(H) \to G$. Since $\dim(x^G \cap H) = 0$ for all $x \in G$ if H is finite, we may assume that $E(H) = H^\circ$ is a connected simple algebraic group of positive dimension.

Let $\lambda = a_1\lambda_1 + \cdots + a_r\lambda_r$ denote the highest weight of the irreducible embedding φ and write $V = M(\lambda)$. Here $r = \mathrm{rank}\, H^\circ$ and the λ_i are the fundamental dominant weights corresponding to a fixed fundamental system of roots. We follow Bourbaki [3] in labelling the Dynkin diagram of H°. Now H is connected unless there exists some element $t \in N_G(H^\circ)$ inducing a non-trivial graph automorphism τ of H°. If $M(\lambda)^\tau$ denotes the corresponding 'twisted' irreducible H°-module, then from [14, (5.4.2) (ii)] we know that $M(\lambda)^\tau \cong M(\tau(\lambda))$, so that H is connected unless $\tau(\lambda) = \lambda$ for some non-trivial graph automorphism τ of H°.

Let $x \in H$ be semisimple or unipotent of prime order. The following lemma provides us with an upper bound for $\dim(x^G \cap H)$.

Lemma 7.1. *Let (G, H) be as above. If $x \in H$ is a semisimple or unipotent element of prime order, then* $\dim(x^G \cap H) \leqslant \dim H - \mathrm{rank}\, H^\circ$.

Proof. From Lemma 2.11, we know that $x^G \cap H^\circ$ is a finite union of H°-classes. If τ is a non-trivial graph automorphism of H° then it is well known that $x^G \cap H^\circ\tau$ is a finite union of H°-classes since x has prime order. So, replacing x by a suitable G-conjugate, we may assume that $\dim(x^G \cap H) = \dim x^{H^\circ}$. If $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$ then we may assume that $x \in H^\circ$ and the conclusion of the lemma follows from Proposition 2.1. Otherwise, $x \in H^\circ\tau$, and from [16, (1.4)], [10, Table 4.3.3] and [2, (8.7)] we deduce that $\dim C_{H^\circ}(x) > \mathrm{rank}\, H^\circ$ for all prime order elements $x \in H^\circ\tau$.

In order to derive a lower bound for $\dim x^G$, we may appeal to [12, Theorem 8.3] since $H < \mathrm{Cl}(V) = G$ is an irreducible subgroup. This result states that if $x \in H$ and $n = \dim V$ then

$$\nu(x) > \max\left(\tfrac{1}{2}\sqrt{n}, 2\right),$$

with the exception of a small number of cases (G, H). Using Proposition 2.9, this provides us with a lower bound for $\dim x^G$. The relevant exceptions to [12, (8.3)] are the following irreducible embeddings $H^\circ \xrightarrow{\lambda} G$:

(a) $\mathrm{Sp}_6 \xrightarrow{\lambda_3} \mathrm{SO}_8$ $(p = 2)$

(b) $\mathrm{SO}_7 \xrightarrow{\lambda_3} \mathrm{SO}_8$ $(p \neq 2)$

(c) $\mathrm{SL}_2 \xrightarrow{4\lambda_1} \mathrm{SO}_5$ $(p \neq 2, 3)$

(d) $\mathrm{SL}_2 \xrightarrow{3\lambda_1} \mathrm{Sp}_4$ $(p \neq 2, 3)$

(e) $\mathrm{SL}_3 \xrightarrow{2\lambda_1} \mathrm{SL}_6$ $(p \neq 2)$

(f) $G_2 \xrightarrow{\lambda_1} \mathrm{SO}_7$ $(p \neq 2)$

(g) $G_2 \xrightarrow{\lambda_1} \mathrm{Sp}_6$ $(p = 2)$

where λ denotes the highest weight of the irreducible embedding.

For now, we shall assume that φ is not one of these exceptional cases. Therefore we can say that $\nu(x) \geqslant 3$ for all $x \in H$, and from Proposition 2.9 we deduce that $\dim x^G \geqslant f(G, n)$, for some function of the dimension n of the natural G-module. For example, if $G = \mathrm{Sp}_n$ then $f(G, n) = 3n - 9$. For a fixed H let $N(H) \in \mathbb{Z}$ be minimal such that $n \geqslant N(H)$ implies that

$$\frac{\dim H - \operatorname{rank} H^\circ}{f(G, n)} \leqslant \frac{1}{2}.$$

Using Lemma 7.1, we observe that $n \geqslant N(H)$ implies that

$$\dim(x^G \cap H)/\dim x^G \leqslant 1/2.$$

For example, suppose that $H^\circ = \mathrm{SO}_{10}$ and $G = \mathrm{Sp}_n$. Then $\dim H - \operatorname{rank} H^\circ = 40$ and $40/(3n - 9) \leqslant 1/2$ if and only if $n \geqslant 89/3$. So in this case, $N(H) = 30$. In general, if $G = \mathrm{Sp}_n$ and H° is a simple algebraic group, we easily calculate that $N(H) = \lceil M \rceil$, where M is given in Table 4.

Table 4
$G = \mathrm{Sp}_n$

H°	M
SL_d	$2d^2/3 - 2d/3 + 3$
Sp_{2d}	$4d^2/3 + 3$
SO_{2d}	$4d^2/3 - 4d/3 + 3$
SO_{2d+1}	$4d^2/3 + 3$
E_6	51
E_7	87
E_8	163
F_4	35
G_2	11

In the case of $G = \mathrm{Sp}_n$, this leaves us to deal with a finite collection of irreducible representations $\varphi : H^\circ \to \mathrm{Sp}_n$, where $n < N(H)$. Detailed information on the small degree irreducible representations of simple algebraic groups can be found in [20]. Careful consideration of [20, Tables A.6–48] and [4, Table 2], together with [20, Theorem 1], yields the following complete list of (self-dual) irreducible representations $H^\circ \to \mathrm{Sp}_n$, where $n < N(H)$, with $N(H)$ as given in Table 4:

(h)	$\mathrm{SL}_6 \xrightarrow{\lambda_3} \mathrm{Sp}_{20}$	$(p \neq 2)$	(k)	$\mathrm{SO}_{11} \xrightarrow{\lambda_5} \mathrm{Sp}_{32}$	$(p \neq 2)$
(i)	$\mathrm{Sp}_6 \xrightarrow{\lambda_3} \mathrm{Sp}_{14}$	$(p \neq 2)$	(l)	$E_7 \xrightarrow{\lambda_7} \mathrm{Sp}_{56}$	$(p \neq 2)$,
(j)	$\mathrm{SO}_{12} \xrightarrow{\lambda_5, \lambda_6} \mathrm{Sp}_{32}$	$(p \neq 2)$			

in addition to the previous exceptional cases labelled (d) and (g).

Applying the same procedure to the other classical groups, we conclude that we are left to deal with the following additional irreducible representations.

(m)	$\mathrm{SO}_{10} \xrightarrow{\lambda_4, \lambda_5} \mathrm{SL}_{16}$	(p arbitrary)	(u)	$E_7 \xrightarrow{\lambda_7} \mathrm{SO}_{56}$	$(p = 2)$
(n)	$\mathrm{SL}_6 \xrightarrow{\lambda_3} \mathrm{SO}_{20}$	$(p = 2)$	(v)	$F_4 \xrightarrow{\lambda_1} \mathrm{SO}_{26}$	$(p = 2)$
(o)	$\mathrm{Sp}_6 \xrightarrow{\lambda_2} \mathrm{SO}_{14}$	$(p \neq 3)$	(w)	$F_4 \xrightarrow{\lambda_4} \mathrm{SO}_{26}$	$(p \neq 3)$
(p)	$\mathrm{Sp}_8 \xrightarrow{\lambda_4} \mathrm{SO}_{16}$	$(p = 2)$	(x)	$\mathrm{SL}_3 \xrightarrow{\lambda_1+\lambda_2} \mathrm{SO}_7$	$(p = 3)$
(q)	$\mathrm{Sp}_{10} \xrightarrow{\lambda_5} \mathrm{SO}_{32}$	$(p = 2)$	(y)	$\mathrm{Sp}_6 \xrightarrow{\lambda_2} \mathrm{SO}_{13}$	$(p = 3)$
(r)	$\mathrm{SO}_{12} \xrightarrow{\lambda_5, \lambda_6} \mathrm{SO}_{32}$	$(p = 2)$	(z)	$F_4 \xrightarrow{\lambda_4} \mathrm{SO}_{25}$	$(p = 3)$
(s)	$\mathrm{SO}_7 \xrightarrow{\lambda_2} \mathrm{SO}_{14}$	$(p = 2)$	(z′)	$G_2 \xrightarrow{\lambda_2} \mathrm{SO}_7$	$(p = 3)$.
(t)	$\mathrm{SO}_9 \xrightarrow{\lambda_4} \mathrm{SO}_{16}$	$(p \neq 2)$			

As previously remarked, $H = N_G(H^\circ)$ is connected unless H° admits a non-trivial graph automorphism which fixes the highest weight of the irreducible embedding $\varphi : H^\circ \to G$. Using this criterion, one can easily verify that the only embeddings (a)–(z′) where H is not connected are those labelled (h), (n) and (x).

The embeddings (o), (q), (s) and (n) are particularly easy to deal with. In each case, $G = \mathrm{SO}_{2m}$ (for some m) and [12, (8.3)] implies that $\nu(\varphi(x)) \geqslant 4$ for all $x \in H$ (see Remark 2.7). The desired conclusion now follows immediately from Proposition 2.9 and Lemma 7.1. Similarly for (x), if $x \in H$ is unipotent, then $\nu(\varphi(x)) \geqslant 4$ (see Remark 2.7) and thus $\dim \varphi(x)^{\mathrm{SO}_7} \geqslant 12$ (the smallest class corresponds to the partition $(3, 2^2)$). If $x \in H$ is semisimple, then we also have $\dim \varphi(x)^G \geqslant 12$ since $\nu(\varphi(x)) \geqslant 3$ and the largest possible centralizer is $\mathrm{SO}_4 \times \mathrm{SO}_3$. This is sufficient since from Lemma 7.1 we have $\dim(x^G \cap H) \leqslant 6$.

In cases (c), (d), (e) and (h), we can easily calculate directly with the representation φ, and deduce in each case that we have an upper bound of $1/2$. The values (α, β) recorded in the following table are easily verified. In each case, $\nu(\varphi(x)) \geqslant \alpha$ for all non-scalar semisimple elements $x \in H$. Similarly for β and non-trivial unipotent elements in H.

Embedding	(c)	(d)	(e)	(h)
(α, β)	$(2,4)$	$(2,3)$	$(2,3)$	$(8,6)$

Note that for (h), we have $v(\varphi(u)) = 6$ if $u = [J_2, I_4]$, and so from Corollary 2.8 we have $v(\varphi(y)) \geqslant 6$ for all unipotent elements $y \in H$. Also referring to (h), if $\tau \in H$ denotes the 'inverse transpose' involutory graph automorphism of $H^\circ = A_5$, then from [16, (1.4)] we know that there are precisely two distinct A_5-classes of involutions in the coset $A_5\tau$, with representatives τ and ω say. It is easy to check that up to Sp_{20}-conjugacy, the action of τ is given by $[-iI_{10}, iI_{10}]$, while ω acts as $[-I_{10}, I_{10}]$.

Lemma 7.2. *The conclusion to Theorem* 1 *is true for the embeddings* (k), (j), (r), (m) *and* (p).

Proof. In each case, φ is a spin representation. We have the following table of lower bounds for $v(\varphi(x))$, and we use the (α, β) notation as before.

Embedding	(k)	(j)	(r)	(m)	(p)
(α, β)	$(10,8)$	$(10,8)$	$(10,8)$	$(5,4)$	$(8,4)$

Note that (k) is a restriction of (j), and (p) is a restriction of (m) (with $p = 2$). In order to justify the stated values of β, in view of Corollary 2.8, it is sufficient to show that $v(\varphi(u)) \geqslant \beta$ where $u \in H^\circ$ is a long root element and φ is either (j) or (m). Consider (j). Let $u = [J_2, I_4] \in A_5 < \mathrm{SO}_{12}$ be a long root element. From [17, Proposition 2.6], we have

$$M(\lambda_5) \downarrow A_5 = V_6 \oplus (V_6)^* \oplus \left(\bigwedge^3 V_6\right),$$

and

$$M(\lambda_6) \downarrow A_5 = 0^2 \oplus \left(\bigwedge^2 V_6\right) \oplus \left(\bigwedge^2 V_6\right)^*,$$

where V_6 is the natural A_5-module, and 0 denotes the trivial A_5-module. We then calculate that $v(\varphi(u)) = 8$ for both spin representations. Similarly in (m) we have

$$M(\lambda_4) \downarrow A_4 = M(\lambda_5) \downarrow A_4 = 0 \oplus \left(\bigwedge^2 V_5\right) \oplus \left(\bigwedge^4 V_5\right),$$

and thus $v(\varphi(u)) = 4$, where $u \in \mathrm{SO}_{10}$ is a long root element. For (k), (j), (r) and (m) we deduce immediately from Proposition 2.9 and Lemma 7.1 that Theorem 1 holds for unipotent elements. In (p), we have $p = 2$ and so by hypothesis we need only consider unipotent involutions. Hence $\dim x^{\mathrm{Sp}_8} \leqslant 20$ (with equality if x is a c_4 involution) and since $v(\varphi(x)) \geqslant 4$, from 2.9 we conclude that $\dim \varphi(x)^{\mathrm{SO}_{16}} \geqslant 44$.

Consider (j) and let $x \in \mathrm{SO}_{12}$ be semisimple. Since a graph automorphism of SO_{12} interchanges the weights λ_5 and λ_6, we need only consider $\lambda = \lambda_5$. Without loss of generality, let $x = [\mu_1, \dots, \mu_6] \in \mathrm{GL}_6 < \mathrm{SO}_{12}$. Then $v(\varphi(x)) = v(\psi(x))$, where $\psi : \mathrm{GL}_6 \to \mathrm{GL}_{32}$ is the representation afforded by the GL_6-module

$$0 \oplus 0' \oplus \left(\bigwedge^2 V_6\right) \oplus \left(\bigwedge^4 V_6\right)$$

and where $0'$ is the 1-dimensional module $v \mapsto (\det x)v$ (see [14, p. 196]). Let $\theta : \mathrm{GL}_6 \to \mathrm{GL}_{15}$ be the representation afforded by the module $\bigwedge^2 V_6$. One easily verifies that $v(\theta(x)) \geqslant 5$ for all non-scalar semisimple elements $x \in \mathrm{GL}_6$. Similarly for the representation afforded by $\bigwedge^4 V_6$. So if $x \in \mathrm{GL}_6$ is non-scalar semisimple then $v(\psi(x)) \geqslant 10$. Suppose that $x = [\mu I_6] \in \mathrm{GL}_6$. Then $\psi(x) = [1, \mu^6, \mu^2 I_{15}, \mu^4 I_{15}]$ and clearly $v(\psi(x)) \geqslant 10$ unless $\mu^2 = 1$. But $\mu^2 = 1$ implies that $x \in \mathrm{SO}_{12}$ is scalar, so that we may indeed take $\alpha = 10$ for (j), and hence for (k) and (r) too. This is sufficient to deduce the desired conclusion via Lemma 7.1 and Proposition 2.9.

Similarly, we have $\alpha = 5$ for (m), and the required conclusion follows in the usual manner. For (p), we may choose $x \in \mathrm{GL}_4 < \mathrm{GL}_5 < \mathrm{SO}_{10}$ and in the same way we calculate that $v(\varphi(x)) \geqslant 8$. In fact, one easily checks that $v(\varphi(x)) \geqslant 10$, unless $x = [\mu, I_3]$, in which case $C_G(\varphi(x))^\circ = \mathrm{GL}_8$, or $x = [\mu, \mu, I_2]$ where

$$C_G(\varphi(x))^\circ = \mathrm{SO}_8 \times \mathrm{GL}_4$$

(where $\mu \neq 1$). If $v(\varphi(x)) \geqslant 10$ then $\dim \varphi(x)^G \geqslant 78$ (since the largest possible centralizer is $\mathrm{GL}_6 \times \mathrm{SO}_4$) which is sufficient since $\dim x^{\mathrm{Sp}_8} \leqslant 32$ for all $x \in \mathrm{Sp}_8$. Otherwise, $(\dim x^{\mathrm{Sp}_8}, \dim \varphi(x)^G) = (14, 56)$ or $(22, 76)$.

Consider the representation labelled (y): this is the embedding $\varphi : \mathrm{Sp}_6 \to \mathrm{SO}_{13}$ and $p = 3$. The Sp_6-module which affords φ is a section of $\bigwedge^2 V_6$. If $x \in \mathrm{Sp}_6$ is unipotent then we can assume that x has order 3, so that $\dim x^{\mathrm{Sp}_6} \leqslant \dim u^{\mathrm{Sp}_6} = 14$, where $u = [J_3^2]$. This is sufficient since $v(\varphi(x)) \geqslant 4$ (by [12, (8.3)] and Remark 2.7) and so from Proposition 2.9 we have $\dim \varphi(x)^{\mathrm{SO}_{13}} \geqslant 32$. From the proof of Lemma 7.2 we have $v(\varphi(x)) \geqslant 5$ for all semisimple elements $x \in \mathrm{Sp}_6$ (since $\alpha = 5$ for (m)). This implies that $\dim \varphi(x)^{\mathrm{SO}_{13}} \geqslant 40$ (the largest possible centralizer is $\mathrm{SO}_8 \times \mathrm{SO}_5$). From Lemma 7.1 we have $\dim(x^G \cap H) \leqslant 18$.

Lemma 7.3. *Theorem* 1 *holds for the irreducible embedding* (t).

Proof. This spin representation embeds SO_9 in SO_{16}. The argument when x is semisimple is similar to that for (p) in Lemma 7.2. We have $\dim x^{\mathrm{SO}_9} \leqslant 32$ and $v(\varphi(x)) \geqslant 8$. Hence $\dim \varphi(x)^G \geqslant 64$ unless $C_G(\varphi(x))^\circ = \mathrm{GL}_8$ (the next largest centralizer is SO_8^2). It is clear that we can only have a GL_8 centralizer if $x = [\mu, I_3] \in \mathrm{GL}_4 < \mathrm{SO}_9$, where $\mu \neq 1$. In this case, $\dim \varphi(x)^G = 56$ and $\dim x^{\mathrm{Sp}_8} = 14$.

Now if $u \in \mathrm{SO}_9$ is a long root element then from the proof of Lemma 7.2 we know that $v(\varphi(u)) = 4$. So from Proposition 2.9 and Corollary 2.8, it follows that $\dim \varphi(x)^G \geqslant 44$ for all unipotent elements $x \in \mathrm{SO}_9$. This leaves us to explicitly calculate with those unipotent elements whose conjugacy classes have dimension greater than 22. Up to conjugacy, we have the results recorded in Table 5, from which the conclusion of Theorem 1 follows immediately.

Table 5
Embedding (t), x unipotent

x	$\varphi(x)$	$\dim x^{\mathrm{SO}_9}$	$\dim \varphi(x)^{\mathrm{SO}_{16}}$
$[J_9]$	$[J_{11}, J_5]$	32	108
$[J_7, I_2]$	$[J_7^2, I_2]$	30	102
$[J_5, J_3, I_1]$	$[J_5^2, J_3^2]$	28	94
$[J_5, J_2^2]$	$[J_5, J_4^2, J_3]$	26	92
$[J_4^2, I_1]$	$[J_5, J_4^2, I_3]$	26	90
$[J_5, I_4]$	$[J_4^4]$	24	88
$[J_3^3]$	$[J_4^2, J_2^4]$, $p \neq 3$	24	80, $p \neq 3$
	$[J_3^4, J_2^2]$, $p = 3$		78, $p = 3$

The images $\varphi(x)$ may be calculated directly by restricting the spin representation of SO_{10}, or by appealing to some well-known results from representation theory. We sketch briefly the arguments involved. Here V_{16} denotes the SO_9 spin module $M(\lambda_4)$ in question, and $M(n)$ denotes an irreducible A_1-module of highest weight n. Note that the results of Table 5 remain valid if $p = 0$.

Let $x = [J_9] \in A_1 < B_4$, and assume that $p \geqslant 11$ (since x is assumed to have prime order). From [17, Proposition 2.13] and [1, Lemma 2.2] we have $V_{16} \downarrow A_1 = M(10) \oplus M(4)$, so that $[J_2]$ acts on V_{16} as $[J_{11}, J_5]$.

Suppose that $x = [J_7, I_2] \in D_4 < B_4$ ($p \geqslant 7$). We have $V_{16} \downarrow \mathrm{SO}_8 = M(\lambda_3) \oplus M(\lambda_4)$. The triality graph automorphism of D_4 acts as a permutation on the irreducible D_4 modules $M(\lambda_1), M(\lambda_3)$ and $M(\lambda_4)$. Since $[J_7, I_1]$ represents the unique unipotent SO_8-conjugacy class of dimension 24, it follows that x acts on V_{16} as $[J_7^2, I_2]$. Similarly for $x = [J_5, J_3, I_1]$.

The action of $x = [J_4^2, I_1] \in \mathrm{SO}_9$ is given by the action of $[J_4, I_1] \in A_4 < D_5$. From [17, (2.6)] we have $V_{16} \downarrow A_4 = M(\lambda_4) \oplus M(\lambda_2) \oplus 0$. We now calculate easily that up to conjugacy, $[J_4, I_1]$ acts on V_{16} as $[J_5, J_4^2, I_3]$ as claimed.

Both $x = [J_5, J_2^2]$ and $[J_5, I_4]$ lie in the subgroup $\mathrm{SO}_5 \times \mathrm{SO}_4 < \mathrm{SO}_9$. Since this subgroup is isomorphic to $\mathrm{Sp}_4 \times (\mathrm{SL}_2 \otimes \mathrm{SL}_2)$, and

$$V_{16} \downarrow \mathrm{Sp}_4 \times (\mathrm{SL}_2 \otimes \mathrm{SL}_2) = M(\lambda_1) \otimes (M(\lambda_1) \oplus M(\lambda_1)),$$

it is straightforward to verify the results of Table 5 in these cases.

Finally, suppose that $x = [J_3^3] \in \mathrm{SO}_9$. We have $p \geqslant 3$. Now, $x \in \mathrm{SO}_3^3 \cong A_1^3$ and

$$V_{16} \downarrow A_1^3 = (M(1) \otimes M(1) \otimes M(1))^2.$$

If $p > 3$ then one readily checks that

$$M(1) \otimes M(1) \otimes M(1) = M(3) \oplus M(1) \oplus M(1),$$

so that $[J_2]^3 \in A_1^3$ acts on V_{16} as $[J_4^2, J_2^4]$. If $p = 3$, then the A_1-modules $M(1)$ and $M(2)$ are both tilting, and $T(3) := M(2) \otimes M(1)$ is an indecomposable tilting A_1-module of highest weight 3 and $\dim C_{T(3)}(u) = 2$, where $u = [J_2] \in A_1$ (see [21, (2.3)]). From this we deduce that $[J_2]^3 \in A_1^3$ acts on $M(1) \otimes M(1) \otimes M(1) = T(3) \oplus (0 \otimes M(1))$ as $[J_3^2, J_2]$. Hence, x acts on V_{16} as $[J_3^4, J_2^2]$.

Lemma 7.4. *The conclusion to Theorem* 1 *is true for the embeddings* (l), (u), (v), (w) *and* (z).

Proof. In each case, $E(H) = H^\circ$ is an exceptional algebraic group. We have the following table, where (α, β) is defined as before.

Embedding	(l)	(u)	(w)	(z)
(α, β)	$(14, 12)$	$(14, 12)$	$(8, 6)$	$(7, 6)$

In each case, these lower bounds on $\nu(\varphi(x))$ are sufficient to imply that Theorem 1 holds in each case (via Proposition 2.9 and Lemma 7.1). So it suffices to prove the recorded values in the table. The values of β follow immediately from [15, Tables 3, 7]. To obtain the lower bounds for semisimple elements, we use some well-known results from representation theory. Consider (l) and (u), and let $x \in E_7$ be semisimple. Without loss of generality, we may assume that $x \in A_7 < E_7$. From [17, Proposition 2.3] we have

$$V_{56} \downarrow A_7 = \left(\bigwedge^2 V_8\right) \oplus \left(\bigwedge^2 V_8\right)^*,$$

where V_8 is the natural A_7-module. One easily checks that $\nu(\rho(y)) \geqslant 7$ for all semisimple $y \in \mathrm{SL}_8$, where $\rho : \mathrm{SL}_8 \xrightarrow{\bigwedge^2 V_8} \mathrm{SL}_{28}$. Hence $\nu(\varphi(x)) \geqslant 14$ if x is semisimple and φ is one of the embeddings (l) or (u). Now consider (w) and (z). If $x \in F_4$ is semisimple then we may assume that $x \in B_4 < D_5$. Let V_{27} denote the irreducible E_6-module $M(\lambda_1)$. Then from [17, Table 8.7] we have

$$V_{27} \downarrow D_5 = M(\lambda_1) \oplus M(\lambda_4) \oplus 0,$$

where 0 is the trivial D_5-module. Since $M_{F_4}(\lambda_4)$ is a section of V_{27}, it follows that $\nu(\varphi(x)) \geqslant \nu(x) + \nu(\rho(x))$, where $\nu(x)$ is with respect to the action of x on the natural D_5-module, and ρ is the spin representation $D_5 \xrightarrow{\lambda_4} \mathrm{SL}_{16}$. Now $\nu(x) \geqslant 2$ and from the proof of Lemma 7.2 we have $\nu(\rho(x)) \geqslant 5$ and hence we may take $\alpha = 8$ for (w) (see Remark 2.7) and $\alpha = 7$ for (z).

The fact that Theorem 1 holds for the embedding labelled (v) is immediate from our work above with the representation (w). From [14, (5.4.2) (ii)] we have $M(\lambda_1) \cong M(\lambda_4)^\tau$, where $\tau : F_4 \to F_4$ is a graph automorphism.

Lemma 7.5. *Theorem* 1 *holds for the irreducible embedding* (i).

Proof. This is the embedding $\varphi : \mathrm{Sp}_6 \to \mathrm{Sp}_{14}$, with highest weight λ_3 and $p \neq 2$. If $x \in \mathrm{Sp}_6$ is semisimple then $\nu(\varphi(x)) \geqslant 4$ (see Remark 2.7) and Proposition 2.9 implies that $\dim \varphi(x)^{\mathrm{Sp}_{14}} \geqslant 40$ which is sufficient since $\dim x^{\mathrm{Sp}_6} \leqslant 18$. Now suppose that $x \in \mathrm{Sp}_6$ is unipotent. We have $\nu(\varphi(x)) \geqslant 3$ and thus $\dim \varphi(x)^{\mathrm{Sp}_{14}} \geqslant 33$. In view of Proposition 2.3, this just leaves us to deal with $x = [J_6] \in \mathrm{Sp}_6$.

Let $x = [J_6] \in \mathrm{Sp}_6$. Since x is assumed to have prime order, we must have $p \geqslant 7$. Let V_{14} denote the Sp_6-module in question, and as before, let $M(n)$ denote an irreducible A_1-module with high weight n. The element $u = [J_2] \in A_1$ acts on $M(5)$ (up to Sp_6-conjugacy) as x. By calculating the weights of $\bigwedge^3 M(5)$, we observe that

$$\bigwedge^3 V(5) \downarrow A_1 = 9/5/3,$$

that is, $\bigwedge^3 M(5) \downarrow A_1$ has the same composition factors as the A_1-module $W(9) + W(5) + W(3)$, where $W(n)$ denotes the Weyl module of A_1 of highest weight n (i.e. $W(n) = S^n V_2$, where V_2 is the natural A_1-module). Of course $W(n) = M(n)$ if $p > n$. Now from [1, Lemma 2.2], we have

$$\mathrm{Ext}_1^{A_1}(M(9), M(3)) = 0,$$

and hence it follows that

$$V_{14} \downarrow A_1 = W(9) \oplus W(3) \quad \text{if } p > 7.$$

Thus, up to conjugacy we have $\varphi(x) = [J_{10}, J_4] \in \mathrm{Sp}_{14}$ and $\dim \varphi(x)^{\mathrm{Sp}_{14}} = 94$ if $p > 7$. Of course we arrive at the same conclusion if we allow $p = 0$. Now suppose that $p = 7$. From [21, Lemma 2.2 (ii)] we know that $W(9) + W(5) + W(3)$ has composition factors $M(9), M(3)^2$ and $M(5)$. Since $p = 7$, from [1, Lemma 2.2] we have

$$\mathrm{Ext}_1^{A_1}(M(5), M(3)) = \mathrm{Ext}_1^{A_1}(M(9), M(5)) = 0.$$

Hence V_{14} is a direct summand of $\bigwedge^3 M(5)$. Now $p = 7$ and so $\bigwedge^3 M(5)$ is a direct summand of $M(5) \otimes M(5) \otimes M(5)$. Hence from [21, Lemmas 2.1, 2.3], we conclude that $V_{14} \downarrow A_1$ is tilting. From [21, Lemma 2.3 (d)], we deduce that $\nu(\varphi(x)) = 12$ and so up to conjugacy, $\varphi(x) = [J_7^2] \in \mathrm{Sp}_{14}$ if $p = 7$. This gives us $\dim \varphi(x)^{\mathrm{Sp}_{14}} = 90$.

Lemma 7.6. *Theorem* 1 *holds for the irreducible embeddings* (a) *and* (b).

Proof. These are both exceptions to the main statement of [12, (8.3)], and a full explicit calculation is required. In both cases, if $x \in H^\circ$ is semisimple then we may assume that $x = [\mu_1, \mu_2, \mu_3] \in \mathrm{GL}_3 < H^\circ$. Then $\dim \varphi(x)^{\mathrm{SO}_8} = \dim z^{\mathrm{SO}_8}$ where

$$\mu z = [1, \mu_1\mu_2\mu_3, \mu_1, \mu_2\mu_3, \mu_2, \mu_1\mu_3, \mu_3, \mu_1\mu_2] \in \mathrm{SO}_8,$$

and $\mu^2 = \mu_1\mu_2\mu_3$. It is then straightforward to check that for both (a) and (b)

$$\max\left(\frac{\dim x^{H^\circ}}{\dim \varphi(x)^{\mathrm{SO}_8}}\right) = \frac{5}{6},$$

where the maximum is taken over all non-scalar semisimple elements in H°. This maximum is realized for both embeddings if $x = [\mu_1, 1, 1] \in \mathrm{GL}_3 < H^\circ$, where $\mu_1 \neq 1$.

The results for unipotent elements are given in Tables 6 and 7 for (a) and (b) respectively. To produce Table 7, we apply representation-theoretic arguments similar to those used in the construction of Table 5. For Table 6, we first calculate the image of each unipotent involution class of Sp_6 in SO_8 under the isomorphism $\mathrm{Sp}_6 \cong \mathrm{Stab}_{\mathrm{SO}_8}(U)$, where U is a 1-dimensional non-singular subspace of the natural SO_8-module. We find that $b_1, c_2 \mapsto c_2$, $a_2 \mapsto a_2$ and $b_3 \mapsto c_4$. From triality, we know that the a_2 and c_4 involution classes in SO_8 are fixed by each spin representation of D_4, and we easily calculate directly that the c_2 class in SO_8 is mapped via a spin representation to the a_4 class.

Since an upper bound of 1/2 fails to hold for both of these irreducible embeddings, they are recorded in Table 1. Note that in Table 6 we need only consider involutory unipotent elements since $p = 2$.

Table 6
Embedding (a), x unipotent

x	$\varphi(x)$	$\dim x^{\mathrm{Sp}_6}$	$\dim \varphi(x)^{\mathrm{SO}_8}$
b_1	a_4	6	12
a_2	a_2	8	10
c_2	a_4	10	12
b_3	c_4	12	16

Table 7
Embedding (b), x unipotent

x	$\varphi(x)$	$\dim x^{\mathrm{SO}_7}$	$\dim \varphi(x)^{\mathrm{SO}_8}$
$[J_7]$	$[J_7, I_1]$	18	24
$[J_5, I_2]$	$[J_4^2]$	16	20
$[J_3^2, I_1]$	$[J_3^2, I_2]$	14	18
$[J_3, J_2^2]$	$[J_3, J_2^2, I_1]$	12	16
$[J_3, I_4]$	$[J_2^4]$	10	12
$[J_2^2, I_3]$	$[J_2^2, I_4]$	8	10

From Tables 6 and 7, we conclude that we have a sharp upper bound of 5/6 with respect to the irreducible embeddings (a) and (b). This upper bound is realized for both semisimple and unipotent elements. This is listed in Table 1.

Lemma 7.7. *The conclusion of Theorem* 1 *holds for each of the irreducible embeddings* (a)–(z$'$).

Proof. It remains to show that Theorem 1 holds for (f), (g) and (z$'$). The fact that Theorem 1 holds for (z$'$) will follow immediately from our work with (f). This is easily seen by considering the composition of φ with τ, where $\tau : G_2 \to G_2$ is a graph automorphism. From [14, (5.4.2) (ii)] we have $M(\lambda_1)^\tau \cong M(\lambda_2)$, and since τ is an algebraic automorphism, all conjugacy class dimensions are preserved.

Recall that both (f) and (g) are exceptional cases of [12, (8.3)]. Suppose that $x \in G_2$ is unipotent, and assume for now that $p \neq 2$. Using the standard labelling of the unipotent classes in G_2, we have the following table, where x denotes an arbitrary element of each unipotent conjugacy class. The values for $\dim x^{G_2}$ are taken from [6, p. 401] and [16, (1.7)], and the stated values for $\dim \varphi(x)^{\mathrm{SO}_7}$ are derived from [15, Table 1].

Table 8
Embedding (f), x unipotent

x class	p	$\dim x^{G_2}$	$\dim \varphi(x)^{\mathrm{SO}_7}$
A_1	arbitrary	6	8
$\tilde{A}_1$	3	6	12
	$\neq 2, 3$	8	12
$\tilde{A}_1^{(3)}$	3	8	12
$G_2(a_1)$	arbitrary	10	14

From Table 8 we immediately conclude that if $x \in G_2$ is unipotent then $\dim(x^G \cap H)/\dim x^G \leqslant 3/4$, with equality possible. This case is recorded in Table 1. Similarly, if $p = 2$ and $x \in G_2$ is unipotent, then as always we may assume that x is an involution, i.e. x is in either the A_1 or the $\tilde{A}_1$ unipotent class. If x lies in the latter class then $\dim x^{G_2} = 8$ and $\dim \varphi(x)^{\mathrm{Sp}_6} = 12$ since $\varphi(x)$ is Sp_6-conjugate to b_3 (see [15, Table 1]). Otherwise, x is a long root element and we may embed x in A_2 (where the A_2 is generated by long root subgroups). We have $\dim x^{G_2} = 6$ and

$$V_6 \downarrow A_2 = V_3 \oplus (V_3)^*, \tag{7}$$

where V_3 is the natural A_2-module, and so it follows that $\varphi(x)$ is Sp_6-conjugate to a_2, so that $\dim \varphi(x)^{\mathrm{Sp}_6} = 8$. Again, this embedding is listed in Table 1.

If $x \in G_2$ is semisimple then we may choose $x \in A_2 < G_2$, where A_2 is generated by long root subgroups. In arbitrary characteristic, we claim that

$$\frac{\dim x^{G_2}}{\dim \varphi(x)^{H^\circ}} \leqslant \frac{5}{7}$$

(note that this is slightly better than the upper bound recorded in Table 1). If $p = 2$ then (7) implies that $\dim \varphi(x)^{\mathrm{Sp}_6} \geqslant 14$, with equality if and only if $x = [\mu, \mu^{-1}, 1] \in A_1 < A_2$, where $\mu \neq 1$. In this case we have $C_{G_2}(x) = A_1 T_1$, so that $\dim x^{G_2}/\dim \varphi(x)^{\mathrm{Sp}_6} = 5/7$. If $\dim \varphi(x)^{\mathrm{Sp}_6} = 16$, then $x = [\mu, \mu, \mu^{-2}] \in A_2$, and $\dim x^{G_2} \leqslant 10$. Otherwise, $\dim \varphi(x)^{\mathrm{Sp}_6} = 18$ and $\dim x^{G_2} \leqslant 12$. The case when $p \neq 2$ is very similar, using the fact that

$$V_7 \downarrow A_2 = V_3 \oplus (V_3)^* \oplus 0.$$

In this case, note that $\dim \varphi(x)^{\mathrm{SO}_7} \geqslant 12$, with equality if and only if $x = [-I_2, 1] \in A_2$. Since this element is an involution, we have $C_{G_2}(x) = A_1^2$ in this case.

This completes the proof of Theorem 1 in the case where the maximal closed subgroup H is not a member of one of the classes $\mathscr{C}_i$. In view of our work in Sections 3, 4, 5 and 6, the proof of Theorem 1 is complete.

8 Fixed point spaces

With the proof of Theorem 1 now complete, we are in a position to apply the result to the study of fixed point spaces. Recall the general situation; G is a simple algebraic group over an algebraically closed field K of characteristic $p \geqslant 0$, and we are interested in obtaining lower bounds for the codimension of the fixed point space $C_\Omega(x)$, where $x \in G$ and Ω is an algebraic variety on which G acts transitively and morphically. Following [16], we denote this codimension by $f(x, \Omega)$. Of course, if $\omega \in \Omega$ and $H = G_\omega$, then the action of G on Ω is equivalent to the usual action of G on the coset variety G/H, and so without loss of generality we may assume that $\Omega = G/H$, where H is a closed subgroup.

In order to deduce such lower bounds from Theorem 1, we apply a well-known result ([16, (1.14)]) which states that for $x \in H$,

$$f(x, \Omega) = \dim x^G - \dim(x^G \cap H). \tag{8}$$

For the purpose of deriving lower bounds $f(x, \Omega)$, we may of course assume that x is semisimple or unipotent since $C_\Omega(x) = C_\Omega(x_s) \cap C_\Omega(x_u)$, where $x = x_s x_u = x_u x_s$ is the Jordan decomposition of x. Furthermore, it follows immediately from (8) that we may assume that G acts primitively on Ω.

With these assumptions, the following result is immediate from equation (8) and Theorem 1.

Lemma 8.1. *Let G be a simple classical algebraic group over an algebraically closed field of characteristic $p \geqslant 0$, and let $\Omega = G/J$, where J is a closed subgroup of G lying* *in some maximal closed non-subspace subgroup H of G. If $x \in G$ is an element of prime order, then*

$$f(x, \Omega) \geqslant \left(\tfrac{1}{2} - \varepsilon\right) \dim x^G,$$

where $\varepsilon \geqslant 0$ is as defined in Theorem 1 *for the pair* (G, H°).

In order to prove Corollary 1 (from which Corollaries 2 and 3 follow immediately) we just need to show that Lemma 8.1 extends to arbitrary elements of G. Since $C_\Omega(x) \subset C_\Omega(x^n)$ for all $n \in \mathbb{N}$, we see that the result extends immediately to arbitrary elements of finite order.

Now suppose that $x \in G$ is semisimple and has infinite order. As previously remarked, we may as well assume that $\Omega = G/H$, with H a maximal closed non-subspace subgroup. Replacing x by a suitable G-conjugate, we may assume that $x \in H$. It is clear from the classification of maximal closed subgroups ([18, Theorem 1]) that all maximal non-subspace subgroups are reductive, and so by replacing x by some finite power we may assume that $x \in T \leqslant H^\circ$, where T is a maximal torus of H°. Let L denote the closure in G of the subgroup $\langle x \rangle$, so that L is a closed subgroup of T, having positive dimension. It is clear that if x fixes $\omega \in \Omega$ then so does every element of L. So in particular, for the purpose of obtaining a lower bound $f(x, \Omega)$, we may replace x by any element of prime order of the subtorus $L^\circ \leqslant T$.

Finally, suppose that $p = 0$, so that all unipotent elements of G have infinite order. If $x \in G$ is unipotent, then for any closed subgroup H of G we have $x^G \cap H \subset H^\circ$ because there are no unipotent elements in the set $H - H^\circ$ (since H/H° is finite). It is straightforward to check that the arguments in Sections 3, 4, 5 and 6 make no assumption on the order of x in the case where $x \in G$ is unipotent and $\dim(x^G \cap H) = \dim(x^G \cap H^\circ)$. Similarly, one easily verifies that the work in Section 7 remains valid for unipotent elements if $p = 0$.

Combining the above results, we conclude that Lemma 8.1 holds for arbitrary elements of G, and the proof of Corollary 1 is complete.

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