EXTREMELY PRIMITIVE SPORADIC AND ALTERNATING GROUPS

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Abstract. A non-regular primitive permutation group is said to be extremely primitive if a point stabilizer acts primitively on each of its orbits. By a theorem of Mann and the second and third authors, every finite extremely primitive group is either almost simple or of affine type. In a recent paper we classified the extremely primitive almost simple classical groups, and in this note we determine the examples with a sporadic or alternating socle. We obtain two infinite families for A_n (or S_n); they comprise the natural 2-primitive action of n points, plus the action on partitions of $\{1,\dots,n\}$ into subsets of size $n/2$ (with $n/2$ odd). There are 20 examples for sporadic groups, including the rank 6 representation of Co_2 on the cosets of McL.

1. Introduction

A non-regular primitive permutation group G on a set Ω is said to be *extremely primitive* if a point stabilizer $H = G_\alpha$ acts primitively on each of its orbits. Equivalently, G is extremely primitive if $H \cap H^x$ is a maximal subgroup of H for all $x \in G \setminus H$. Moreover, by an old theorem of Manning [13], if G is extremely primitive on Ω then G_α is faithful on each of its orbits in $\Omega \setminus \{\alpha\}$, so $H \cap H^x$ is also core-free in H. For example, every 2-primitive group G is extremely primitive, and the finite groups with this property can be determined via the classification of finite simple groups.

By a theorem of Mann and the second and third authors [12, Theorem 1.1], every finite extremely primitive group is either almost simple or of affine type, and the affine examples are known up to a finite number of possibilities. In [7], we classified the extremely primitive almost simple classical groups, and the purpose of this note is to determine the almost simple examples with a sporadic or alternating socle.

Theorem 1. *Let G be a finite almost simple primitive permutation group, with point stabilizer H and socle G_0. Assume G_0 is a sporadic or alternating group. Then G is extremely primitive if and only if (G,H) is one of the cases listed in Table 1.*

Remark 1. In Table 1, we set $\alpha = |G : G_0|$ and we adopt the standard Atlas [8] notation for simple groups.

Corollary 1. *Let G, G_0 and H be defined as above. Then G is extremely primitive, but not 2-primitive, if and only if one of the following holds:*

(i) $G = S_n$ *or* A_n, $H = (S_{n/2} \wr S_2) \cap G$, $n \equiv 2 \pmod 4$ *and* $n \geq 10$*;*

(ii) $(G_0,H) = (\mathrm{J}_2, \mathrm{U}_3(3).\alpha), (\mathrm{HS}, \mathrm{M}_{22}.\alpha), (\mathrm{Suz}, G_2(4).\alpha), (\mathrm{McL}, \mathrm{U}_4(3).\alpha), (\mathrm{Ru}, {}^2F_4(2).2),$ $(\mathrm{Co}_2, \mathrm{U}_6(2).2)$ *or* $(\mathrm{Co}_2, \mathrm{McL})$.

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G_0	H	Rank	Conditions
A_n	$N_G((S_{n/2} \wr S_2) \cap G)$	$(n+2)/4$	$n \equiv 2 \pmod 4$
A_n	$N_G(A_{n-1})$	2	$G \leqslant S_n$
A_n	$N_G(D_{10})$	2	$n = 5$
M_{11}	S_6	2	
M_{11}	$\mathrm{L}_2(11)$	2	
M_{12}	M_{11}	2	$G = G_0$
M_{22}	$\mathrm{L}_3(4).\alpha$	2	
M_{23}	M_{22}	2	
M_{24}	M_{23}	2	
J_2	$\mathrm{U}_3(3).\alpha$	3	
HS	$\mathrm{M}_{22}.\alpha$	3	
HS	$\mathrm{U}_3(5)$	2	$G = G_0$
Suz	$G_2(4).\alpha$	3	
McL	$\mathrm{U}_4(3).\alpha$	3	
Ru	${}^2F_4(2).2$	3	
Co_2	$\mathrm{U}_6(2).2$	3	
Co_2	McL	6	
Co_3	McL.2	2	

TABLE 1. The extremely primitive sporadic and alternating groups

In order to complete the classification of the almost simple extremely primitive groups it remains to determine the examples with socle an exceptional group of Lie type. These groups will be the subject of a future paper.

Our proof of Theorem 1 uses some recent work on bases for almost simple primitive groups. Recall that if G is a permutation group on Ω then a subset of Ω is a *base* if its pointwise stabilizer in G is trivial; the minimal size of a base is denoted by $b(G)$. Now, if $b(G) = 2$ then H has a regular orbit on Ω, so G is not extremely primitive. In [5, 6] (see also [10, 15]), the primitive almost simple groups G with $b(G) = 2$ and socle a sporadic or alternating group are determined. This observation greatly simplifies the proof of Theorem 1 since we can immediately eliminate the cases with $b(G) = 2$. For instance, if $G = S_n$ or A_n and a point stabilizer H acts primitively on $\{1, \ldots, n\}$ then [5, Theorem 1] states that $b(G) = 2$ for all $n > 12$, so in this situation we may assume $n \leq 12$ and the remaining cases are easily handled.

In addition to the main theorem of [6] on base sizes, in our analysis of sporadic groups we make use of the data recorded in the Web Atlas [17] on permutation representations and maximal subgroups of sporadic groups. In many cases, this information allows us to construct G and H as suitable permutation groups (using GAP [9] or MAGMA [1]) and then quickly determine whether or not G is extremely primitive. For some of the larger sporadic groups, this approach is not practical (due to the large degrees of the relevant permutation representations). However, by applying an easy lemma (see Lemma 2.2) we can essentially reduce the problem to a handful of cases (G, H) in which the corresponding permutation character 1_H^G is *multiplicity-free*, which means that all irreducible constituents of 1_H^G occur with multiplicity 1. The multiplicity-free actions of almost simple sporadic groups have been classified (see [3]), and a great deal of information is known about these actions. In particular, all the subdegrees in such actions have been computed and it is straightforward to decide whether or not G is extremely primitive.

This note is organized as follows. In Section 2 we record a couple of preliminary results which will be useful in the proof of Theorem 1. The proof itself is given in Sections 3 and Sections 4.

2. Preliminaries

Let G be a primitive permutation group on a finite set Ω with point stabilizer H. A subset B of Ω is a *base* for G if the pointwise stabilizer of B in G is trivial; we write $b(G)$ for the minimal size of a base for G. Determining $b(G)$ is an interesting problem, with important applications in computational group theory (see [16, Chapter 4], for example).

Bases for almost simple primitive groups with a sporadic or alternating socle are studied in [5, 6, 10], and the cases with $b(G) = 2$ have been completely determined (see also [15] for two exceptional cases involving the Baby Monster sporadic group). Clearly, if $b(G) = 2$ then $H \cap H^x = 1$ for some $x \in G$, and thus G is not extremely primitive (note that a maximal subgroup of an almost simple group cannot be of prime order). This observation, combined with the main theorems of [5] and [6], plays a key role in the proof of Theorem 1.

Lemma 2.1. *Let G be an almost simple permutation group, and let $b(G)$ be the minimal size of a base for G. If $b(G) = 2$ then G is not extremely primitive.*

The next lemma provides four conditions on the point stabilizer H, each of which implies that G is not extremely primitive. Here $F(H)$ and $\mathrm{Soc}(H)$ denote the Fitting subgroup and socle of H, respectively.

Lemma 2.2. *Suppose $|H|$ is composite and one of the following conditions hold:*

(i) $Z(H) \neq 1$.

(ii) *$F(H)$ is not elementary abelian.*

(iii) *$F(H)$ is an elementary abelian group Z_p^e, but $|\Omega| - 1$ is indivisible by p^e.*

(iv) *$\mathrm{Soc}(H)$ is not a product of isomorphic simple groups.*

Then G is not extremely primitive.

Proof. By Manning's theorem [13], if G is extremely primitive then $H = G_\alpha$ is faithful on each of its orbits in $\Omega \setminus \{\alpha\}$. It is a basic fact that the socle of a primitive permutation group is a product of isomorphic simple groups, so (iv) is immediate. See [7, Lemma 2.2] for parts (i), (ii) and (iii). $\square$

3. Sporadic groups

In this section we assume G is an almost simple primitive permutation group with socle G_0 and point stabilizer H, where G_0 is a sporadic simple group.

Proposition 3.1. *If G is extremely primitive then (G, H) is one of the cases listed in* [6, Tables 1 and 2]*, and either H is almost simple, or* $(G, H) = (\mathrm{M}_{11}, \mathrm{M}_9{:}2)$*,* $(\mathrm{J}_2, 3.A_6.2)$ *or* $(\mathrm{J}_2.2, 3.A_6.2^2)$*.*

Proof. By Lemma 2.1 we have $b(G) > 2$, so by the main theorems of [6] and [14], either (G, H) is one of the cases listed in [6, Tables 1 and 2], or $(G, H) = (\mathbb{B}, 2^{2+10+20}.(\mathrm{M}_{22}{:}2 \times S_3))$. In the latter case, the Fitting subgroup $F(H) = 2^{2+10+20}$ is nonabelian, so Lemma 2.2(ii) eliminates this possibility.

We now inspect each case (G, H) in [6, Tables 1 and 2] with the property that H is not an almost simple group. We claim that at least one of the four conditions on H

in the statement of Lemma 2.2 is satisfied, unless $(G,H) = (\mathrm{M}_{11}, \mathrm{M}_9{:}2)$, $(\mathrm{J}_2, 3.A_6.2)$ or $(\mathrm{J}_2.2, 3.A_6.2^2)$.

Clearly, if (G,H) is one of the following cases

$$\begin{array}{lll} (\mathrm{M}_{12}{:}2, (2\times 2\times A_5).2) & (\mathrm{Fi}_{23}, O_7(3)\times S_3) & (\mathrm{Co}_1, (A_4\times G_2(4)){:}2) \\ (\mathrm{Fi}_{24}', (3\times O_8^+(3){:}3){:}2) & (\mathrm{Fi}_{24}, S_3\times O_8^+(3){:}S_3) & (\mathbb{B}, (2^2\times F_4(2)){:}2) \end{array}$$

then $\mathrm{Soc}(H)$ is not a product of isomorphic simple groups, so Condition (iv) of Lemma 2.2 is satisfied. In each of the remaining cases, we claim that at least one of the Conditions (i) – (iii) holds. We present this information in Table 2; for a line representing a group G, the subgroups in the column labelled "Condition (x)" are those for which Lemma 2.2(x) holds.

This is a straightforward verification. The claim is clear if $G \in \{\mathrm{M}_{11}, \mathrm{J}_1, \mathrm{J}_3, \mathrm{O'N}\}$, so assume otherwise. If H is a subgroup in the second column of Table 2 then H has a normal subgroup of order 2, so $Z(H) \neq 1$ and thus Condition (i) is satisfied. Clearly, if H is in the third column then $F(H)$ is not elementary abelian, so Condition (ii) holds, and for H in the fourth column, it is easy to check that $F(H) \neq 1$ is elementary abelian and $|G:H|-1$ is indivisible by $|F(H)|$ (see [8]). □

Proposition 3.2. *Suppose* $G \notin \{\mathrm{Fi}_{23}, \mathrm{Co}_1, \mathrm{O'N}, \mathrm{HN}, \mathrm{HN.2}, \mathrm{Fi}_{24}', \mathrm{Fi}_{24}, \mathrm{Ly}, \mathrm{Th}, \mathbb{B}\}$. *Then the conclusion to Theorem 1 holds.*

Proof. By the previous proposition, we immediately deduce that Theorem 1 holds if $G = \mathrm{O'N}.2, \mathrm{J}_4$ or $\mathbb{M}$, so we will assume otherwise. An explicit permutation representation of G on $n < 5000$ points is given in the Web Atlas [17] (see [6, Table 3] for the precise value of n) and using MAGMA [1] (in particular the command MaximalSubgroups) we can construct G and H as subgroups of S_n. For each case (G,H) appearing in Table 1 we use the MAGMA command CosetAction to construct G as a permutation group on the set of right cosets of H in G and we quickly deduce that G is extremely primitive. In each of the remaining cases, it is easy to find an element $x \in G$ (by random search) such that $H\cap H^x$ is non-maximal (here it is convenient to use the MAGMA command IsMaximal to deduce non-maximality), which proves that G is not extremely primitive in these cases. □

To complete the proof of Theorem 1 for sporadic groups, it remains to show that none of the following cases (G,H) are extremely primitive (see [6, Table 2]):

$$\begin{array}{lllll} (\mathrm{Fi}_{23}, O_8^+(3){:}S_3) & (\mathrm{Fi}_{23}, \mathrm{S}_8(2)) & (\mathrm{Ly}, G_2(5)) & (\mathrm{Co}_1, \mathrm{Co}_2) & (\mathrm{Co}_1, \mathrm{Co}_3) \\ (\mathrm{Co}_1, \mathrm{U}_6(2){:}S_3) & (\mathrm{HN}, A_{12}) & (\mathrm{HN}, \mathrm{U}_3(8){:}3) & (\mathrm{HN.2}, S_{12}) & (\mathrm{HN.2}, \mathrm{U}_3(8){:}6) \\ (\mathrm{O'N}, \mathrm{L}_3(7){:}2) & (\mathrm{Th}, {}^3D_4(2){:}3) & (\mathrm{Fi}_{24}', \mathrm{Fi}_{23}) & (\mathrm{Fi}_{24}', O_{10}^-(2)) & (\mathrm{Fi}_{24}, O_{10}^-(2){:}2) \\ (\mathbb{B}, \mathrm{Fi}_{23}) & (\mathbb{B}, \mathrm{Th}) & & & \end{array} \tag{1}$$

Proposition 3.3. *Suppose* $(G,H) = (\mathrm{Co}_1, \mathrm{U}_6(2){:}S_3)$ *or* $(\mathbb{B}, \mathrm{Th})$. *Then G is not extremely primitive.*

Proof. First consider the case $G = \mathrm{Co}_1$ and $H = \mathrm{U}_6(2){:}S_3$. The Web Atlas [17] gives a permutation representation of G on 98280 points, together with generators for H in terms of the given generators of G. As before, using MAGMA, we can construct G and H in terms of this representation, and it is easy to find an element $x \in G$ such that $H\cap H^x$ is a non-maximal subgroup of H.

Now assume $(G,H) = (\mathbb{B}, \mathrm{Th})$. The character tables of G and H are available in the GAP Character Table Library [2], together with the fusion of H-classes in G. Given this information, it is easy to calculate the permutation character 1_H^G, and by computing $\langle 1_H^G, 1_H^G\rangle$ we deduce that the rank of G is 34. The smallest maximal subgroup of H has order 120 (see [17], for example), so the largest primitive permutation representation of H

G	Condition (i)	Condition (ii)	Condition (iii)
M_{12}	$2 \times S_5$	$2^{1+4}{:}S_3, 4^2{:}D_{12}$	$3^2{:}2.S_4$
$\mathrm{M}_{12}.2$		$2^{1+4}{:}S_3.2, 4^2{:}D_{12}.2, 3^{1+2}{:}D_8$	
M_{22}			$2^4{:}A_6, 2^4{:}S_5, 2^3{:}\mathrm{L}_3(2)$
$\mathrm{M}_{22}.2$	$2^3{:}\mathrm{L}_3(2) \times 2$		$2^4{:}S_6, 2^5{:}S_5$
M_{23}			$2^4{:}A_7, 2^4{:}(3 \times A_5){:}2$
M_{24}			$2^4.A_8, 2^6{:}3.S_6,$ $2^6{:}(\mathrm{L}_3(2) \times S_3)$
J_2		$2^{1+4}.A_5, 2^{2+4}{:}(3 \times S_3)$	$A_4 \times A_5, A_5 \times D_{10}$
$\mathrm{J}_2.2$	$\mathrm{L}_3(2){:}2 \times 2$	$2^{1+4}.A_5.2, 2^{2+4}{:}(3 \times S_3).2$	$(A_4 \times A_5){:}2, (A_5 \times D_{10}).2$
$\mathrm{J}_3.2$			$(3 \times \mathrm{M}_{10}){:}2$
HS	$4.2^4.S_5$	$4^3{:}\mathrm{L}_3(2)$	$2^4.S_6$
HS.2	$S_8 \times 2, (2 \times A_6.2.2).2$	$4^3.(2 \times \mathrm{L}_3(2)), 2^{1+6}.S_5$	$2^5.S_6$
McL	$2.A_8$	$3^{1+4}{:}2.S_5$	$3^4{:}\mathrm{M}_{10}, 2^4.A_7$
McL.2	$2.S_8, \mathrm{M}_{11} \times 2$	$3^{1+4}{:}4.S_5$	$3^4{:}(\mathrm{M}_{10} \times 2)$
Co_3	$2.\mathrm{S}_6(2)$	$3^{1+4}{:}4.S_6$	$3^5{:}(2 \times \mathrm{M}_{11}), 2^4.A_8$
Co_2		$2^{1+8}{:}\mathrm{S}_6(2), (2^4 \times 2^{1+6}).A_8,$ $2^{4+10}.(S_5 \times S_3)$	$2^{10}{:}\mathrm{M}_{22}{:}2$
He			$2^2.\mathrm{L}_3(4).S_3, 2^6{:}3.S_6$
He.2			$2^2.\mathrm{L}_3(4).D_{12}$
Suz		$2^{1+6}.\mathrm{U}_4(2), 2^{4+6}{:}3.A_6,$ $2^{2+8}{:}(A_5 \times S_3)$	$3.\mathrm{U}_4(3){:}2, 3^5.\mathrm{M}_{11},$ $(A_4 \times \mathrm{L}_3(4)){:}2$
Suz.2	$\mathrm{M}_{12}{:}2 \times 2, \mathrm{J}_2{:}2 \times 2$	$2^{1+6}.\mathrm{U}_4(2).2, 2^{4+6}{:}3.S_6,$ $2^{2+8}{:}(S_5 \times S_3)$	$3.\mathrm{U}_4(3).2.2, 3^5.(\mathrm{M}_{11} \times 2),$ $(A_4 \times \mathrm{L}_3(4){:}2){:}2$
Fi_{22}	$2.\mathrm{U}_6(2)$	$(2 \times 2^{1+8}){:}(\mathrm{U}_4(2){:}2),$ $2^{5+8}{:}(S_3 \times A_6),$ $3^{1+6}{:}2^{3+4}{:}3^2{:}2$	$2^{10}{:}\mathrm{M}_{22}, 2^6{:}\mathrm{S}_6(2),$ $\mathrm{U}_4(3){:}2 \times S_3$
$\mathrm{Fi}_{22}.2$	$2.\mathrm{U}_6(2).2,$ $O_8^+(2){:}S_3 \times 2$	$(2 \times 2^{1+8}{:}\mathrm{U}_4(2){:}2){:}2,$ $2^{5+8}{:}(S_3 \times S_6),$ $3^{1+6}{:}2^{3+4}{:}3^2.2.2$	$2^{10}{:}\mathrm{M}_{22}{:}2, 2^7{:}\mathrm{S}_6(2),$ $\mathrm{U}_4(3).2.2 \times S_3,$ $3^5{:}(2 \times \mathrm{U}_4(2){:}2)$
Ru		$2^{3+8}{:}\mathrm{L}_3(2), 2^{1+4+6}.S_5$	$2^6.\mathrm{U}_3(3).2, (2^2 \times \mathrm{Sz}(8)){:}3$
Fi_{23}	$2.\mathrm{Fi}_{22}, 2^2.\mathrm{U}_6(2).2$	$3^{1+8}.2^{1+6}.3^{1+2}.2S_4$	$2^{11}.\mathrm{M}_{23}$
J_4		$2^{1+12}.3.\mathrm{M}_{22}{:}2$	$2^{11}{:}\mathrm{M}_{24}, 2^{10}{:}\mathrm{L}_5(2)$
Ly			$3.\mathrm{McL}{:}2$
Co_1		$2^{1+8}.O_8^+(2), 2^{2+12}{:}(A_8 \times S_3),$ $2^{4+12}.(S_3 \times 3.S_6)$	$3.\mathrm{Suz}{:}2, 2^{11}{:}\mathrm{M}_{24}$
HN	$2.\mathrm{HS}.2$		
HN.2	$4.\mathrm{HS}.2$		
$\mathrm{O'N}.2$	$4.\mathrm{L}_3(4).2.2$		
Th			$2^5.\mathrm{L}_5(2)$
Fi_{24}'	$2.\mathrm{Fi}_{22}{:}2$	$3^{1+10}{:}\mathrm{U}_5(2){:}2$	$3^7.O_7(3), 2^{11}.\mathrm{M}_{24}$
Fi_{24}	$\mathrm{Fi}_{23} \times 2, (2 \times 2.\mathrm{Fi}_{22}){:}2,$ $(2 \times 2^2.\mathrm{U}_6(2)){:}S_3$	$3^{1+10}{:}(\mathrm{U}_5(2){:}2 \times 2)$	$3^7.O_7(3){:}2, 2^{12}.\mathrm{M}_{24}$
$\mathbb{B}$	$2.{}^2E_6(2){:}2$	$2^{1+22}.\mathrm{Co}_2, 2^{9+16}.\mathrm{S}_8(2)$	
$\mathbb{M}$	$2.\mathbb{B}$		

TABLE 2. Some non-extremely primitive sporadic groups

has degree $|H|/120$. Since $|G:H|-1 > 33|H|/120$ we conclude that G is not extremely primitive. $\square$

Proposition 3.4. *The conclusion to Theorem 1 for sporadic groups holds in each of the remaining cases.*

Proof. According to [3], the action of G on Ω is *multiplicity-free* in each of the remaining 15 cases to be considered. By definition, this means that every irreducible constituent of the corresponding permutation character 1_H^G occurs with multiplicity 1. The multiplicity-free actions of almost simple sporadic groups are classified in [3], and a great deal of information on these actions is given in [4]. In particular, all the subdegrees have been calculated and we can combine this information with the known orders of the maximal subgroups of H (see [17], for example, or take a suitable permutation representation of H and use MAGMA to compute the index of every maximal subgroup of H). Note that the maximal subgroups of $O_8^+(3){:}S_3$ are determined in [11].

In this way we quickly deduce that none of the remaining possibilities give rise to an extremely primitive group. Indeed, in each case there is at least one nontrivial subdegree which does not coincide with the index of a maximal subgroup of H; the smallest subdegrees with this property are as follows (here we adopt the same ordering of cases used in (1) above):

109200	107100	19530	4600	257600
–	16632	1539	5040	1539
6384	17199	275264	1570800	1570800
412896	–			

For example, if $(G,H) = (\mathbb{B}, \mathrm{Fi}_{23})$ then the smallest nontrivial subdegree is 412896 (see [4], and also [14, Table 2]), but it is easy to check that this is not the index of a maximal subgroup of H. $\square$

4. Alternating groups

Let G be an almost simple primitive permutation group with socle $G_0 = A_n$ and point stabilizer H. If $n \leq 12$ then the conclusion to Theorem 1 can be easily verified using MAGMA [1], so for the remainder we will assume $n > 12$.

Proposition 4.1. *If $n > 12$ and H is a primitive subgroup of S_n then G is not extremely primitive.*

Proof. The main theorem of [5] implies that $b(G) = 2$. Now apply Lemma 2.1. $\square$

Proposition 4.2. *If H is an intransitive subgroup of type $S_k \times S_{n-k}$ with $1 \leq k < n/2$, then G is extremely primitive if and only if $k = 1$.*

Proof. Clearly, if $k=1$ then G is 2-primitive so let us assume $k>1$. If $k>2$ then the socle of H is not a product of isomorphic simple groups, so G is not extremely primitive. The same argument applies if $k=2$ and $G = S_n$. Finally, suppose $k=2$ and $G = A_n$, so $H = S_{n-2}$. Take $H = G_S$, where $S = \{1,2\}$, and set $S' = \{2,3\}$. Then

$$H_{S'} = H_{1,2,3} = A_{n-3} < A_{n-2} = H_{1,2} < H,$$

so G is not extremely primitive. $\square$

To complete the proof of Theorem 1 we may assume H is the G-stabilizer of a partition of $\{1,\ldots,n\}$ into n/k subsets of size k, where $2 \leq k \leq n/2$.

Proposition 4.3. *If H is an imprimitive subgroup of type $S_k \wr S_{n/k}$ with $2 \leq k \leq n/2$, then G is extremely primitive if and only if $k = n/2$ and $n \equiv 2 \pmod 4$.*

Proof. Set $\ell = n/k$ and take H to be the G-stabilizer of the partition $P = \{P_0, \ldots, P_{\ell-1}\}$, where $P_i = \{ik+1, \ldots, (i+1)k\}$. Define a new partition $P' = \{P'_0, \ldots, P'_{\ell-1}\}$, where $P'_0 = \{2, \ldots, k+1\}$, $P'_1 = \{1, k+2, \ldots, 2k\}$ and $P'_i = P_i$ for all $i > 1$. If $k = 2$ then $H_{P'}$ induces the Klein four-group on $P_0 \cup P_1$, while $H_{\{P_0,P_1\}}$ induces D_8, so $H_{P'} < H_{\{P_0,P_1\}} < H$ and thus G is not extremely primitive. Similarly, if $k \geq 3$ then $H_{P'} = ((S_{k-1} \wr S_2) \times (S_k \wr S_{\ell-2})) \cap G$, which is a proper subgroup of $H_{\{P_0,P_1\}}$ if $\ell \geq 3$.

Finally, suppose $\ell = 2$. If $P'' = \{P''_0, P''_1\}$ is a partition then the H-orbit of P'' is precisely the set of partitions $\{A, B\}$ with $|P_0 \cap A| = |P_0 \cap P''_0|$, so the rank of G is $\lfloor n/4 \rfloor + 1$. First assume $n/2$ is even. We define a new partition $P'' = \{P''_0, P''_1\}$, where

$$P''_0 = \{1, 2, \ldots, n/4, 3n/4+1, 3n/4+2, \ldots n\}, \quad P''_1 = \{n/4+1, \ldots, 3n/4\}.$$

Then

$$H_{P''} = ((S_{n/4} \times S_{n/4} \times S_{n/4} \times S_{n/4}).2^2) \cap G < ((S_{n/4} \wr S_2) \wr S_2) \cap G < H$$

and thus G is not extremely primitive. Finally, let us assume $n/2$ is odd. Suppose $P'' = \{P''_0, P''_1\}$ is a partition, where $|P_0 \cap P''_0| = t$. Then $H_{P''} = ((S_t \times S_{n/2-t}) \wr S_2) \cap G$ is a maximal subgroup of H for $0 < t < n/2$, whence G is extremely primitive and we record this case in Table 1. □

This completes the proof of Theorem 1.

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