ON SECOND MAXIMAL SUBGROUPS OF FINITE GROUPS

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Abstract. A second maximal subgroup of a finite group G is a subgroup H that is maximal in every maximal overgroup of H in G. We prove that every maximal subgroup of a prime-index subgroup of a finite group is a second maximal subgroup.

1. Introduction

Let G be a finite group and let H be a proper subgroup of G. We say that H is a *weak second maximal* subgroup if it is a maximal subgroup of a maximal subgroup of G. Properties of weak second maximal subgroups and their influence on the structure of the ambient group have been studied since the 1950s. For example, a theorem of Huppert [8] states that G is supersolvable if every weak second maximal subgroup is normal. And work of Janko [9] shows that A_5 and $\mathrm{SL}_2(5)$ are the only nonsolvable groups with the property that every weak second maximal subgroup is nilpotent.

The term weak second maximal subgroup was first coined by Flavell in [7]. In addition, he refers to a proper subgroup H of G as a *second maximal* subgroup if it is maximal in *every* maximal overgroup of H in G. For example, if we take the subgroup $H = \langle (1,2) \rangle$ of $G = S_4$, then H is a weak second maximal subgroup because it is maximal in $\langle (1,2), (1,3) \rangle$, but it is not a second maximal subgroup since it is not maximal in $\langle (1,2), (1,3)(2,4) \rangle$. We note that some authors use the terms 2-*maximal* and *strictly* 2-*maximal* in place of weak second maximal and second maximal, respectively (see [11], for example).

Given a proper subgroup H of G, it is natural to consider the number of maximal subgroups of G containing H, which we denote by $m(G,H)$, and there is an extensive literature in this direction. In [7], for example, Flavell reveals a relationship between $m(G,H)$ and the indices of maximal overgroups of a second maximal subgroup H of G. Indeed, [7, Theorem A] establishes the upper bound

$$m(G,H) \leqslant 1 + \max\{|G:M| \,:\, M \text{ is a maximal subgroup of } G \text{ containing } H\}$$

for every second maximal subgroup H of a finite group G, together with a detailed description of the pairs (G,H) for which equality holds (H/H_G is solvable in each case, where H_G denotes the core of H in G). In the same paper, Flavell asked whether or not the same upper bound holds when H is a *weak* second maximal subgroup. For solvable groups, this was answered in the affirmative by Meng and Guo [14].

In this paper, we seek to shed new light on the following natural problem: when does a weak second maximal subgroup H of G have the stronger second maximal subgroup property? In particular, we would like to understand how this is related to the index of a maximal overgroup of H. A recent result in this direction due to Konovalova, Monakhov and Sokhor [11] states that if H is a weak second maximal subgroup of G and M is a maximal overgroup of H, then H is a second maximal subgroup if M is normal in G, or if G is p-solvable and $|G:M| = p$ for some prime p. Our main theorem shows that the p-solvable hypothesis is not needed.

Theorem A. *Let G be a finite group and let H be a maximal subgroup of a prime-index subgroup of G. Then H is a second maximal subgroup of G.*

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In order to discuss possible extensions of Theorem A, it will be convenient to introduce the following notation. For each positive integer $n \geqslant 2$, let $(\mathcal{P}_n)$ be the following assertion:

> *For any finite group G with a maximal subgroup M of index n, every maximal subgroup of M is a second maximal subgroup of G.*

So Theorem A states that $(\mathcal{P}_n)$ holds for every prime n. And the previous example with $G = S_4$, $M = \langle (1,2),(1,3)\rangle$ and $H = \langle (1,2)\rangle$ shows that $(\mathcal{P}_4)$ is false.

In fact, it is straightforward to show that $(\mathcal{P}_n)$ is false for every composite even integer and every composite prime power:

(a) Suppose $n = 2m \geqslant 4$ is even and let $G = S_n$, $M = S_{n-1}$ and $K = S_m \wr S_2$, so M and K are maximal in G and $|G : M| = n$. Then we can take a subgroup $H = S_m \times S_{m-1} < S_m \times S_m < K$ that is maximal in M, so H is a weak second maximal subgroup, but it is not a second maximal subgroup.

(b) Let $n = p^d$ be a prime power with $d \geqslant 2$. Let $V = \mathbb{F}_p^d$ and set $G = \mathrm{AGL}(V) = V{:}\mathrm{GL}(V)$, $M = \mathrm{GL}(V)$ and $K = V{:}H$, where H is the stabilizer in M of a 1-dimensional subspace of V. Then the irreducibility of M on V implies that M is maximal in G with $|G : M| = |V| = n$, and K is maximal in G since H is maximal in M. However, H acts reducibly on V since $d \geqslant 2$ and thus H is not maximal in K.

The problem for odd composite integers $n \geqslant 15$ that are not prime powers is more subtle. In the statement of the following result, we set

$$\mathcal{A} = \{n \in \mathbb{N} : S_n \text{ and } A_n \text{ are the only primitive subgroups of } S_n\}.$$

With the aid of Magma [3], it is easy to check that

$$\{n \in \mathcal{A} : n \leqslant 100\} = \{34, 39, 46, 51, 58, 69, 70, 75, 76, 86, 87, 88, 92, 93, 94, 95, 96, 99\}.$$

Theorem B. *Let G be a finite group and let H be a maximal subgroup of a maximal subgroup M of G such that $|G : M| = n \in \mathcal{A}$ is odd. Then H is a second maximal subgroup of G.*

So this shows that $(\mathcal{P}_n)$ is true for every odd integer $n \in \mathcal{A}$. Now if we define

$$\delta(\mathcal{A}) = \lim_{n\to\infty} \frac{|\mathcal{A} \cap \{1,\ldots,n\}|}{n}$$

to be the natural density of $\mathcal{A}$ as a subset of $\mathbb{N}$, then a well known theorem of Cameron, Neumann and Teague [6] states that $\delta(\mathcal{A}) = 1$. This means that S_n and A_n are the only primitive subgroups of S_n for almost every positive integer n. And the same conclusion holds if we only consider the set of odd positive integers. So Theorem B implies that $(\mathcal{P}_n)$ is true for almost every odd positive integer. We refer the reader to Remark 3.2 for further discussion in this direction.

Our proof of Theorem A uses a result of Meng and Guo [13, Lemma 1] to reduce the problem to a question concerning transitive permutation groups of prime degree. The result we need in this setting is stated as Theorem 2.1 in Section 2 and we will use it to establish Theorem A by eliminating the existence of a minimal counterexample. So the main bulk of the argument concerns the proof of Theorem 2.1. By a classical result of Burnside, every transitive permutation of prime degree is either 2-transitive or solvable (and hence affine), and it is easy to read off the complete list of cases by appealing to the classification of the finite 2-transitive groups. Note that by invoking the latter result, our argument depends on the Classification of Finite Simple Groups. The case where G is an almost simple group with socle $\mathrm{PSL}_d(q)$ and degree $(q^d-1)/(q-1)$ requires the most effort.

We will also use [13, Lemma 1] in our proof of Theorem B to show that if $(\mathcal{P}_n)$ fails for a given positive integer n, then this must be witnessed by a primitive subgroup $G \leqslant S_n$. And then it just remains to show that there are no such examples when n is odd and $G = S_n$ or A_n, which is how we proceed.

Finally, we anticipate that Theorems A and B will be useful for studying the so-called *WSM-groups*. These are the finite groups for which every weak second maximal subgroup is second maximal. The solvable WSM-groups have been studied by Meng and Guo in [13], but a complete classification remains out of reach. This is also related to Problem 19.54 in the Kourovka Notebook [12], which asks for a description of the chief factors of a WSM-group.

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2. Proof of Theorem A

In this section, we will prove Theorem A. As we will see, the key ingredient is the following result concerning transitive permutation groups of prime degree.

Theorem 2.1. *Let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group of prime degree with point stabilizer G_α and let M be a maximal subgroup of G. If M_α is maximal in G_α, then it is also maximal in M.*

As in Theorem 2.1, let $G \leqslant \mathrm{Sym}(\Omega)$ be a transitive permutation group with point stabilizer $H = G_\alpha$ and prime degree r. Since every transitive group of prime degree is primitive, it follows that H is a core-free maximal subgroup of G. By a classical theorem of Burnside, G is either solvable or 2-transitive, so by inspecting the list of 2-transitive permutation groups (see [5, Chapter 7] for example), which relies on the Classification of Finite Simple Groups, we deduce that one of the following holds (also see [10, Theorem 3]):

(a) $C_r \trianglelefteqslant G \leqslant \mathrm{AGL}_1(r)$.
(b) $G = A_r$ or S_r with $r \geqslant 5$.
(c) $(G, r) = (\mathrm{PSL}_2(11), 11)$, $(\mathrm{M}_{11}, 11)$ or $(\mathrm{M}_{23}, 23)$.
(d) $\mathrm{PSL}_d(q) \trianglelefteqslant G \leqslant \mathrm{P\Gamma L}_d(q)$ with $d \geqslant 2$ and $r = (q^d-1)/(q-1)$.

Let M be a maximal subgroup of G and assume that M_α is maximal in $H = G_\alpha$. Our aim is to show that M_α is also maximal in M. With this goal in mind, let us record two basic observations:

(i) Clearly, if M is transitive on Ω then $|M : M_\alpha| = r$ and thus M_α is maximal in M. Therefore, we may assume that M has two or more orbits on Ω.
(ii) Let α^M be the M-orbit of α. Since we are assuming M_α is maximal in G_α, it follows that $|\alpha^M| \geqslant 2$ and our goal is to show that M acts primitively on α^M.

We will now consider cases (a)-(d) in turn.

Proposition 2.2. *The conclusion to Theorem 2.1 holds in case (a).*

Proof. Here $G = N{:}H \leqslant \mathrm{AGL}_1(r)$ is a primitive affine group with socle $N = C_r$ and point stabilizer $H = G_\alpha \leqslant C_{r-1}$. In particular, G is a Frobenius group with kernel N and complement H. As explained above in (i), we may assume M is intransitive on Ω, which in turns means that $|M|$ is indivisible by r (otherwise M contains the unique Sylow r-subgroup N, which acts transitively on Ω). Therefore, Hall's theorem implies that $M \leqslant H^x < G$ for some $x \in N$, so the maximality of M in G forces $M = H^x$.

If $H^x = H$, then $M = H$ and thus $M_\alpha = G_\alpha$, a contradiction. Therefore, since G is a Frobenius group, we have $M_\alpha = H \cap H^x = 1$ and so the maximality of M_α in H forces $|H|$ to be a prime. In particular, M has prime order and we conclude that $M_\alpha = 1$ is maximal in M, as required. □

Proposition 2.3. *The conclusion to Theorem 2.1 holds in case (b).*

Proof. Here $G = S_r$ or A_r with $r \geqslant 5$. Given a subset Δ of Ω, let S_Δ be the symmetric group on Δ. Since we may assume M is intransitive on Ω, it follows that

$$M = (S_{\Delta_1} \times S_{\Delta_2}) \cap G,$$

where the Δ_i are nonempty and $\Omega = \Delta_1 \cup \Delta_2$ is a partition. Without loss of generality, we may assume that $\alpha \in \Delta_1$, in which case $\Delta_1 = \alpha^M$ has at least two elements.

Clearly, if $G = S_r$, or if $G = A_r$ and $|\Delta_1| \geqslant 3$, then M acts primitively on Δ_1 and we conclude that M_α is maximal in M. And similarly, if $G = A_r$ and $|\Delta_1| = 2$, say $\Delta_1 = \{\alpha, \alpha'\}$, then $|\Delta_2| = r - 2 \geqslant 3$ and we can choose distinct points $\beta, \beta' \in \Delta_2$ so that M contains the double transposition $(\alpha, \alpha')(\beta, \beta')$. So once again, M acts primitively on Δ_1 and the result follows. □

Proposition 2.4. *The conclusion to Theorem 2.1 holds in case (c).*

Proof. It is straightforward to use Magma [3] to handle these special cases.

To do this, we first construct G as a permutation group on $\Omega = G/H$ and we construct a set of representatives of the conjugacy classes of maximal subgroups of G. For each representative M, we calculate the orbits $\Omega_1, \ldots, \Omega_t$ of M on Ω. Then for each i, we check that if M acts imprimitively on Ω_i then $|\Omega_i|$ does not coincide with the index of a maximal subgroup of H. This allows us to conclude that M acts primitively on α^M whenever M_α is a maximal subgroup of G_α, as required. □

To complete the proof of Theorem 2.1, it just remains to consider case (d) above. Here G is an almost simple group with socle $T = \mathrm{PSL}_d(q)$, where $q = p^f$ for some prime p. Since we are assuming

$$r = \frac{q^d - 1}{q - 1}$$

is a prime, it follows that d is a prime. Moreover, $\gcd(d, q-1) = 1$. Indeed, if d divides $q-1$, then d divides r and thus $d = 1 + q + \cdots + q^{d-1}$, which is absurd. Therefore, $T = \mathrm{SL}_d(q)$ and we have $G = T{:}\langle\varphi\rangle$, where φ is a field automorphism of T of order e and $e \geqslant 1$ is a divisor of f.

In order to handle this case, we need to introduce some additional notation, and we require a preliminary lemma.

Let $V = \mathbb{F}_q^d$ be the natural module for $T = \mathrm{SL}_d(q)$ and note that we may identify Ω with the set $\mathbb{P}(V)$ of 1-dimensional subspaces of V. Fix a 1-space $\alpha \in \Omega$ and observe that $T_\alpha = U{:}L$ is a maximal parabolic subgroup of T, with unipotent radical U and Levi subgroup $L = \mathrm{GL}_{d-1}(q)$. Here U is an elementary abelian p-group of order q^{d-1}, which we may identify with the natural module for L. In particular, we have $H = G_\alpha = U{:}(L{:}\langle\varphi\rangle)$ and U is the unique minimal normal subgroup of H.

Visibly, L preserves a decomposition $V = \alpha \oplus W$, where W has dimension $d-1$. In terms of this decomposition, we can write

$$U = \{u_f \,:\, f \in \mathrm{Hom}_{\mathbb{F}_q}(W, \alpha)\}, \quad L = \{\ell_g \,:\, g \in \mathrm{GL}(W)\} \tag{1}$$

as subgroups of $\mathrm{SL}(V)$, where

$$u_f(a + w) = a + w + f(w), \quad \ell_g(a + w) = (\det g)^{-1}a + g(w)$$

for all $a \in \alpha$, $w \in W$.

Let T_W be the setwise stabilizer in T of the hyperplane W; this is a maximal parabolic subgroup of T with Levi decomposition $T_W = Q{:}L$. For a concrete description, we can take

$$Q = \{v_h \,:\, h \in \mathrm{Hom}_{\mathbb{F}_q}(\alpha, W)\}, \quad v_h(a + w) = a + w + h(a) \tag{2}$$

for all $a \in \alpha$, $w \in W$. Note that T_α and T_W are conjugate subgroups in $\mathrm{Aut}(T)$ with the property $T_\alpha \cap T_W = L$.

Lemma 2.5. *In terms of the above notation, the following statements hold:*

(i) *If J is a complement of U in T_α, then $J = L^x$ for some $x \in U$.*

(ii) *If $d \geqslant 3$ and S is a subgroup of T containing L, then $S = L$, T_α, T_W or T.*

(iii) *The unipotent radical Q of T_W acts regularly on the set $\Omega \setminus \mathbb{P}(W)$.*

Proof. First consider (i). We need to show that the first cohomology group $H^1(L,U)$ is trivial. For $d=2$, this is clear since $|U|=q$ and $|L|=q-1$ are coprime. Now assume $d \geqslant 3$. If $q \geqslant 3$ then $Z(L)$ contains a nontrivial element acting as a scalar on U, which forces $H^1(L,U)=0$ (see [15, Proposition 2.7(b)], for example, which is sometimes referred to as Sah's Lemma). Finally, if $q=2$ then $L=\mathrm{SL}_{d-1}(2)$ and the main theorem of [2] gives $H^1(L,U)=0$ (note that $H^1(L,U) \neq 0$ when $(d,q)=(4,2)$, but this case does not arise since d is a prime in the setting we are interested in).

Next let us turn to (ii), so $d \geqslant 3$. First observe that $T_\alpha = U{:}L$ and $T_W = Q{:}L$, where L acts irreducibly on U and Q. Therefore, L is maximal in both T_α and T_W.

Let $S \neq L$ be an overgroup of L in T. If S acts reducibly on V, then $L < S \leqslant T_X < T$ for some subspace X of V. But α and W are the only proper nonzero subspaces of V preserved by L, so $X \in \{\alpha, W\}$ and then the maximality of L in T_X forces $S = T_X$. Now assume S is irreducible on V. By appealing to Aschbacher's theorem on the subgroup structure of finite classical groups (see [1]), it is straightforward to show that L is not contained in a proper irreducible subgroup of T, so $S=T$ and the result follows. Alternatively, if S is irreducible then there exists an element $x \in S \setminus (T_\alpha \cup T_W)$ and we can show that $T = \langle L, x\rangle$, which means that $S=T$.

Finally, let us consider (iii). Fix a 1-space $\beta = \langle a+w\rangle$ in $\Omega \setminus \mathbb{P}(W)$, where $0 \neq a \in \alpha$ and $w \in W$. Working with the explicit description of Q in (2), let $1 \neq v_h \in Q$ and note that $0 \neq h \in \mathrm{Hom}_{\mathbb{F}_q}(\alpha, W)$. Then

$$v_h(a+w) = a + (w + h(a)) \notin \beta,$$

so $Q_\beta = 1$ and the result follows since $|Q| = |\Omega \setminus \mathbb{P}(W)| = q^{d-1}$. □

Proposition 2.6. *The conclusion to Theorem 2.1 holds in case (d).*

Proof. Let M be a maximal subgroup of G such that M_α is maximal in $H = G_\alpha$. Set $J = M \cap T$ and note that $J \trianglelefteqslant M$. Recall that we may as well assume M acts intransitively on Ω, so the transitivity of T (as a nontrivial normal subgroup of the primitive group G) implies that M is core-free and thus $G = TM$. In particular, J is a proper subgroup of T. There are now two cases to consider.

Case 1. $U \leqslant M_\alpha$

First assume M_α contains U, which is the unipotent radical of T_α. Then

$$U \leqslant M_\alpha \cap T \leqslant M \cap T = J$$

and thus J contains the unipotent radical of a parabolic subgroup of T. So by the main theorem of [16], it follows that J is contained in a maximal parabolic subgroup T_X of T, where X is a proper nonzero subspace of V.

For each $g \in M$ we have $J = J^g \leqslant T_{X^g}$ since $J \trianglelefteqslant M$. It follows that $J \leqslant T_Y$, where

$$Y = \bigcap_{g \in M} X^g.$$

We claim that $\alpha \subseteq Y$, so Y is nonzero.

Seeking a contradiction, suppose $\alpha \nsubseteq X^g$ for some $g \in M$. Then by considering the decomposition $V = \alpha \oplus W$, there exists a vector $v = a+w \in X^g$ with $a \in \alpha$ and $0 \neq w \in W$. Fix $f \in \mathrm{Hom}_{\mathbb{F}_q}(W, \alpha)$ such that $f(w) \neq 0$, so we have $\alpha = \langle f(w)\rangle$. Since X^g is J-invariant it is also U-invariant, so by working with the description of U in (1) we see that

$$f(w) = u_f(v) - v \in X^g$$

and thus $\alpha \subseteq X^g$, a contradiction.

Therefore, Y is a proper nonzero M-invariant k-space containing α, so $M \leqslant G_Y < G$ and the maximality of M implies that $M = G_Y$ and thus $\alpha^M \subseteq \mathbb{P}(Y)$. If $Y = \alpha$ then $M = H$

and $M_\alpha = H$, which is a contradiction (since we are assuming M_α is maximal in $H = G_\alpha$). Therefore, α is a proper subspace of Y and thus $2 \leqslant k < d$.

Finally, let us observe that the parabolic subgroup T_Y induces the full linear group $\mathrm{GL}(Y)$ on Y. Therefore, since $T_Y \leqslant G_Y = M$, it follows that $\alpha^M = \mathbb{P}(Y)$. Moreover, the action of M on α^M is 2-transitive and therefore primitive. This completes the proof in Case 1.

Case 2. $U \not\leqslant M_\alpha$

To complete the proof, let us assume $U \not\leqslant M_\alpha$. By the maximality of M_α in G_α we have $G_\alpha = UM_\alpha$. Since $U \cap M_\alpha$ is normalized by M_α and centralized by the abelian group U, we deduce that $U \cap M_\alpha \trianglelefteqslant G_\alpha$. Therefore, $U \cap M_\alpha = 1$ since U is a minimal normal subgroup of G_α, whence M_α is a complement to U in G_α. In addition,

$$T_\alpha = G_\alpha \cap T = (UM_\alpha) \cap T = U(M_\alpha \cap T)$$

and thus Lemma 2.5(i) implies that $M_\alpha \cap T$ and L are U-conjugate. Therefore, without loss of generality, we may assume that $M_\alpha \cap T = L$. We now divide the remainder of the argument into two subcases according to d.

Case 2(a). $d \geqslant 3$

Suppose $d \geqslant 3$. Since $J = M \cap T$ contains $L = M_\alpha \cap T$, Lemma 2.5(ii) implies that $J = L$, T_α, T_W or T. Of course, if $J = T$ then M is transitive on Ω, contrary to our earlier assumption. Next suppose $J \leqslant T_\alpha$, so J fixes α. Since $J \trianglelefteqslant M$ we have $J = J^g$ for all $g \in M$, so J fixes every point in α^M. Since $d \geqslant 3$, it is easy to see that $\mathrm{Fix}_\Omega(L) = \{\alpha\}$, so

$$\alpha^M \subseteq \mathrm{Fix}_\Omega(J) \subseteq \mathrm{Fix}_\Omega(L) = \{\alpha\}$$

and thus $\alpha^M = \{\alpha\}$. This means that $M \leqslant G_\alpha$ and hence $M_\alpha = G_\alpha$, which is a contradiction.

To complete the argument in Case 2(a), we may assume $J = T_W = Q{:}L$ as above. Since L acts transitively on $\mathbb{P}(W)$ and Q is transitive on $\Omega \setminus \mathbb{P}(W)$, it follows that J has exactly two orbits on Ω, namely $\mathbb{P}(W)$ and $\Omega \setminus \mathbb{P}(W)$, which must coincide with the orbits of M on Ω since we are assuming M is intransitive. In particular, we have

$$|M : M_\alpha| = |\Omega \setminus \mathbb{P}(W)| = q^{d-1}.$$

Since M stabilizes W we have $M \leqslant G_W$ and thus $M = G_W$ by the maximality of M in G. By Lemma 2.5(iii), Q acts regularly on $\Omega \setminus \mathbb{P}(W)$, so $Q_\alpha = 1$ and thus $Q \cap M_\alpha = 1$. It follows that $M = G_W = Q{:}M_\alpha$ with $Q \trianglelefteqslant M$. Finally, since M_α acts irreducibly on Q, we conclude that M_α is a maximal subgroup of M, as required.

Case 2(b). $d = 2$

Finally, let us assume $d = 2$, in which case $r = q + 1$ is a Fermat prime and $q = 2^{2^m}$ with $m \geqslant 1$. If $q = 4$ then $T \cong A_5$ and we have already handled this possibility in Proposition 2.3. So we may assume $q \geqslant 16$. Note that $T_\alpha = U{:}L$ is a Borel subgroup with $|U| = q$, $L = C_{q-1}$ and $\mathrm{Fix}_\Omega(L) = \{\alpha, W\}$. In addition, it is clear that T_α and T_W are the only Borel subgroups of T containing L, corresponding to the two points fixed by L.

In fact, by inspecting Dickson's list of the maximal subgroups of T (see [4, Tables 8.1, 8.2], for example), we observe that L has exactly three maximal overgroups in T, namely T_α, T_W and $N_L(T) = L.2 = D_{2(q-1)}$. For example, L is not contained in a subfield subgroup $\mathrm{SL}_2(q_0)$ with $q = q_0^2$ since the latter does not contain an element of order $q - 1$. Since $J = M \cap T$ contains L, it follows that $J \leqslant T_\alpha$, T_W or $N_T(L)$.

First assume $J \not\leqslant N_T(L)$, so $L < J \leqslant T_\gamma$ for some $\gamma \in \{\alpha, W\}$ and the maximality of L in T_γ forces $J = T_\gamma$. Since $J \trianglelefteqslant M$ and $\mathrm{Fix}_\Omega(T_\gamma) = \{\gamma\}$, we deduce that $M \leqslant G_\gamma$ and so the maximality of M implies that $M = G_\gamma$. If $\gamma = \alpha$, then $M_\alpha = G_\alpha$ and we reach a contradiction. Therefore, $\gamma = W$ and we have $M = Q{:}M_\alpha$, where Q is the unipotent radical of $M = G_W$. And since M_α acts irreducibly on Q, we conclude that M_α is maximal in M.

Finally, let us assume $L \leqslant J \leqslant N_T(L)$, so $J = L$ or $N_T(L) = L.2$. In both cases, L is characteristic in J (recall that $q-1$ is odd). Since $J \trianglelefteqslant M$ we deduce that $M \leqslant N_G(L)$ and thus $M = N_G(L)$ by maximality. Then M acts transitively on $\mathrm{Fix}_\Omega(L) = \{\alpha, W\}$ and $|M : M_\alpha| = 2$. In particular, M_α is maximal in M and the proof of the proposition is complete. □

This completes the proof of Theorem 2.1 and we will now use it to prove Theorem A. This relies on the following result, which is [13, Lemma 1]. In the statement, we write H_G and M_G for the cores of H and M, respectively (so H_G is the largest normal subgroup of G contained in H).

Lemma 2.7. *Let G be a finite group and let H be a proper subgroup of G. Suppose M and K are maximal overgroups of H in G such that H is maximal in M, but not maximal in K. Then $H = M \cap K$ and $H_G = M_G$.*

We are now ready to prove Theorem A.

Proof of Theorem A. Suppose G is a counterexample of minimal order, so G has subgroups H, M and K, where M and K are maximal overgroups of H in G with $|G : M|$ a prime, and H is maximal in M, but not in K. Then $M \neq K$ and $H = M \cap K$. In addition, the minimality of $|G|$ implies that H is core-free (otherwise we could pass to the quotient G/H_G), so M is also core-free by Lemma 2.7.

Setting $\Omega = G/M$, we can now view $G \leqslant \mathrm{Sym}(\Omega)$ as a transitive permutation group of prime degree with point stabilizer $G_\alpha = M$. Since $H = M \cap K = K_\alpha$ is maximal in M by hypothesis, Theorem 2.1 now implies that H is maximal in K. This final contradiction completes the proof of the theorem. □

3. Proof of Theorem B

In this final section, we will prove Theorem B. To do this, let n be a positive integer and suppose the assertion $(\mathcal{P}_n)$ is false. So this means that there exists a finite group G with a proper subgroup H contained in maximal subgroups M and K of G with the property that $|G : M| = n$ and H is maximal in M but not in K. By passing to G/H_G, we may assume that $H_G = 1$ and then Lemma 2.7 implies that M is core-free. So now we can view $G \leqslant \mathrm{Sym}(\Omega)$ as a primitive permutation group on $\Omega = G/M$ with point stabilizer M.

In other words, we have shown that if $(\mathcal{P}_n)$ is false, then this must be witnessed by a primitive subgroup $G \leqslant S_n$. For n odd, the next result rules out S_n and A_n as possible witnesses.

Proposition 3.1. *Let $n \geqslant 5$ be an odd integer and let $G = S_n$ or A_n in its natural action of degree n. Fix a point stabilizer $M = G_\alpha = S_{n-1} \cap G$ and let H be a maximal subgroup of M. Then H is a second maximal subgroup of G.*

Proof. Let K be a maximal subgroup of G containing H. We need to prove that H is maximal in K. We may as well assume that $K \neq M$, which implies that $H = M \cap K = K_\alpha$. Clearly, if K acts primitively on $\Omega = \{1, \ldots, n\}$, then K_α is maximal in K. Therefore, we may assume that K is either intransitive or imprimitive on Ω.

First assume K is intransitive and let Δ be the K-orbit containing α. Set $\Gamma = \Omega \setminus \Delta$ and note that $|\Delta| \geqslant 2$ and $K = G_\Delta$. Suppose $G = S_n$, so $K = S_\Delta \times S_\Gamma$ and $K_\alpha = (S_\Delta)_\alpha \times S_\Gamma$. Since $(S_\Delta)_\alpha$ is maximal in S_Δ, we deduce that K_α is maximal in K. Now assume $G = A_n$. If $|\Gamma| = 1$ then $K_\alpha = A_{n-2}$ is a maximal subgroup of $K = A_{n-1}$. Now suppose $|\Gamma| \geqslant 2$. Here the projection map $\pi : K \to S_\Delta$ is surjective and so the maximality of $(S_\Delta)_\alpha$ in S_Δ implies that the preimage $K_\alpha = \pi^{-1}((S_\Delta)_\alpha)$ is maximal in K.

Finally, suppose K is transitive and imprimitive. Here K is the stabilizer of a partition Π of Ω into b sets of size a, where $n = ab$ and $a, b \geqslant 3$ are odd (since n is odd). In particular,

$K = (S_a \wr S_b) \cap G$. Let $\Gamma = \Gamma_1$ be the part in Π containing α and let

$$J = (G_\alpha)_{\Gamma \setminus \{\alpha\}}$$

be the setwise stabilizer of $\Gamma \setminus \{\alpha\}$ in G_α. Since every element of K_α stabilizes Γ, it follows that $K_\alpha \leqslant J < M$ and we note that J is a proper subgroup of $M = G_\alpha$ since $\Gamma \setminus \{\alpha\}$ is a nonempty proper subset of $\Omega \setminus \{\alpha\}$. Since $a, b \geqslant 3$, there exist distinct points r, s, t, u such that $r, s \in \Gamma \setminus \{\alpha\}$ and α, u, v are contained in three distinct parts of Π. Then the double transposition $(r,s)(t,u)$ is in J, but it does not preserve Π, so it is not in K_α and we conclude that $H = K_\alpha < J < M$. But this is incompatible with the maximality of H in M, so this case does not arise and the proof of the proposition is complete. □

Proof of Theorem B. Fix an odd integer $n \in \mathcal{A}$ and recall that if $(\mathcal{P}_n)$ is false, then this must be witnessed by a primitive subgroup of S_n. By definition of $\mathcal{A}$, the only primitive subgroups are S_n and A_n, but neither group is a witness by Proposition 3.1. Therefore, we conclude that $(\mathcal{P}_n)$ is true. □

Remark 3.2. In view of Theorem B, we know that $(\mathcal{P}_n)$ is true for every odd integer $n \in \mathcal{A}$. And by Theorem A, we also know that $(\mathcal{P}_n)$ holds for every odd prime n. However, it remains an open problem to completely determine the set of integers n such that $(\mathcal{P}_n)$ holds.

As explained above, to prove that $(\mathcal{P}_n)$ holds for a given integer n, it suffices to show that the conclusion holds for every primitive subgroup $G \leqslant S_n$, where we take $M = G_\alpha$ to be the stabilizer of a point $\alpha \in \{1, \ldots, n\}$. Let us consider the first three composite odd integers $n \notin \mathcal{A}$ that are not prime powers, so $n \in \{15, 21, 33\}$.

(a) For $n = 15$ we can consider the primitive subgroup $G = S_6$, so $M = G_\alpha = S_2 \wr S_3$. With the aid of Magma, we can show that M has a maximal subgroup $H = S_2 \times S_3$, which is contained in a maximal subgroup of $K = S_3 \wr S_2$. Since K is maximal in G, we conclude that $(\mathcal{P}_{15})$ is false.

(b) Next suppose $n = 21$. Here we take $G = S_7$ with $M = G_\alpha = S_5 \times S_2$, which allows us to identify $\Omega = \{1, \ldots, 21\}$ with the set of 2-element subsets of $\{1, \ldots, 7\}$. If we choose a 2-set β disjoint from α, then $H = M_\beta = (S_3 \times S_2) \times S_2$ is maximal in M and we see that H is also contained in a maximal subgroup $K = S_3 \times S_4$ of G. But we have $H < S_3 \times D_8 < K$, so H is not maximal in K and thus $(\mathcal{P}_{21})$ is false.

(c) Finally, suppose $n = 33$. Here $\mathrm{PSL}_2(32)$ and $\mathrm{P\Gamma L}_2(32) = \mathrm{PSL}_2(32).5$ are the only primitive subgroups of S_{33} that do not contain A_{33}. So we may write $G = \mathrm{PSL}_2(32).k$ with $k \in \{1, 5\}$ and we note that $M = G_\alpha = 2^5{:}(31{:}k)$ is a Borel subgroup of G. Using Magma, it is straightforward to check that every maximal subgroup of M is a second maximal subgroup of G. It follows that $(\mathcal{P}_{33})$ holds, even though $33 \notin \mathcal{A}$ is not a prime power.

We can also show that $(\mathcal{P}_n)$ is false for $n \in \{35, 45, 55, 65, 85, 91\}$ and true for $n \in \{57, 63\}$. Let us also observe that it is easy to generalize the above construction in (b) to show that $(\mathcal{P}_n)$ is false for every integer $n = m(m-1)/2$ with $m \equiv 3 \pmod 4$ and $m \geqslant 7$.

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