Irreducible subgroups of classical algebraic groups

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Introduction

G	-	group
H	-	subgroup of G
K	-	field (algebraically closed, $\mathrm{char}(K) = p \geqslant 0$)
V	-	irreducible KG-module ($\dim V > 1$)

Definition. (G, H, V) is an **irreducible triple** if $V|_H$ is irreducible

Problem. Given G and K, determine all the irreducible triples (G, H, V)

Symmetric groups: $G = S_n$

If $p = 0$ then $V = S^\lambda$ is a Specht module, where λ is a partition of n.

Saxl, 1987: Determined all irreducible triples (S_n, H, S^λ) when $p = 0$

e.g. $\lambda = (n-1, 1)$: H is 2-transitive

e.g. $\lambda = (n-2, 2)$: $(n, H) = (9, \Gamma\mathrm{L}_2(8))$, $(11, \mathrm{M}_{11}), \ldots$

Brundan & Kleshchev, 2001: A classification for $p > 3$

Partial results when $p = 2, 3$...

Algebraic groups

An **algebraic group** over K is an affine variety $G \subseteq K^m$ with a compatible group structure.

e.g. $G = \mathrm{SL}_n(K) = \{A \in \mathrm{M}_n(K) \mid \det(A) - 1 = 0\} \subset K^{n^2}$

G inherits the **Zariski topology** from K^m.

G contains a unique maximal closed connected subgroup, denoted G^0, which is normal and has finite index; G is **connected** iff $G = G^0$.

The **simple algebraic groups** (no proper nontrivial closed connected normal subgroup):

Classical: $\mathrm{SL}_n(K)$, $\mathrm{SO}_{2n+1}(K)$, $\mathrm{Sp}_{2n}(K)$, $\mathrm{SO}_{2n}(K)$

Exceptional: $E_8(K)$, $E_7(K)$, $E_6(K)$, $F_4(K)$, $G_2(K)$

Irreducible triples for simple algebraic groups

K - algebraically closed field of characteristic $p \geqslant 0$

G - simple algebraic group over K

H - closed subgroup of G

V - nontrivial irreducible KG-module

Dynkin, 1957: H connected, $p = 0$

e.g. $G = \mathrm{SL}_{2n} = \mathrm{SL}(W)$, $H = \mathrm{Sp}_{2n}$, $V = S^k(W)$

e.g. $G = \mathrm{SL}_{27} = \mathrm{SL}(W)$, $H = E_6$, $V = \Lambda^4(W)$

e.g. $G = E_6$, $H = G_2$, $V = V_{27}$

Irreducible triples for simple algebraic groups

K	-	algebraically closed field of characteristic $p \geqslant 0$
G	-	simple algebraic group over K
H	-	closed subgroup of G
V	-	nontrivial irreducible KG-module

Dynkin, 1957: H connected, $p = 0$

Seitz, 1987: G classical, H connected, $p > 0$

Testerman, 1988: G exceptional, H connected, $p > 0$

Ghandour, 2010: G exceptional, H disconnected and infinite

Our problem. Determine the irreducible triples (G, H, V), where G is classical and H is disconnected and infinite

An application: Subgroup structure

Irreducible triples arise naturally in the study of **maximal subgroups** of classical groups:

Question. Let H be a simple group and let $\varphi : H \to \mathrm{SL}(V)$ be an irreducible representation.

Is $\varphi(H)$ a maximal subgroup of $\mathrm{SL}(V)$ (or $\mathrm{Sp}(V)$, $\mathrm{SO}(V)$)?

If $\varphi(H)$ is non-maximal, say

$$\varphi(H) < G < \mathrm{SL}(V),$$

then $(G, \varphi(H), V)$ is an irreducible triple.

Some related work

1. Ford (1996): Determined the irreducible triples (G, H, V), where

- G is classical; H is disconnected and infinite
- H^0 is **simple**
- The composition factors of $V|_{H^0}$ are p-**restricted**.

2. Guralnick & Tiep (2008): The triples (G, H, V), where

$G = \mathrm{SL}(W)$, $V = S^k(W)$, $k \geqslant 4$ and H is **any** closed subgroup.

3. Liebeck, Seitz & Testerman (2014): Investigating the triples (G, H, V), where G is simple, H is connected and $V|_H$ is **multiplicity-free**.

A two-step strategy

Our problem. Determine the irreducible triples (G, H, V), where G is classical and H is disconnected and infinite

Step 1. $H < G$ is maximal

Step 2. $H < M < G \implies (G, M, V)$ is an irreducible triple determined in Step 1 (M disconnected) or by Seitz (M connected)

Main ingredients.

- Chevalley's theory of highest weight representations
- Clifford theory
- Liebeck-Seitz subgroup structure theorem for classical groups

Weights and representations

Let G be a simple algebraic group over K.

$$\begin{aligned}
T \quad &- \quad \text{maximal torus of } G \ (T \cong (K^*)^n) \\
X(T) = \mathrm{Hom}(T, K^*) \quad &- \quad \text{character group of } T \ (X(T) \cong \mathbb{Z}^n) \\
\{\alpha_1, \ldots, \alpha_n\} \subset X(T) \otimes_{\mathbb{Z}} \mathbb{R} \quad &- \quad \text{simple roots} \\
\{\lambda_1, \ldots, \lambda_n\} \subset X(T) \otimes_{\mathbb{Z}} \mathbb{R} \quad &- \quad \text{fundamental dominant weights}
\end{aligned}$$

Let V be a finite-dimensional KG-module. Then

$$V = \bigoplus_{\mu \in X(T)} V_\mu, \quad V_\mu = \{v \in V \mid t \cdot v = \mu(t)v \text{ for all } t \in T\}$$

and $\mu \in X(T)$ is a **weight** of V if $V_\mu \neq 0$.

An example: $G = \mathrm{SL}_{n+1}(K)$

Maximal torus: $T = \{\mathrm{diag}(t_1, \ldots, t_{n+1}) \mid t_i \in K^*\}$

Lie algebra: $\mathrm{Lie}(G) = \{A \in \mathrm{M}_{n+1}(K) \mid \mathrm{Tr}(A) = 0\}$

Roots: $\mathrm{Lie}(G)$ is a KG-module, via $x \cdot A = xAx^{-1}$. The roots of G are the non-zero weights for this module.

e.g. If $t = \mathrm{diag}(t_1, \ldots, t_{n+1}) \in T$, $A = E_{i,j} \in \mathrm{Lie}(G)$ (so $i \neq j$), then

$$t \cdot A = tAt^{-1} = t_i t_j^{-1} A$$

so the map $\alpha : t \mapsto t_i t_j^{-1}$ is a root.

Simple roots: $\{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i(t) = t_i t_{i+1}^{-1}$

Weights and representations

$\{\alpha_1, \dots, \alpha_n\} \subset X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ - simple roots
$\{\lambda_1, \dots, \lambda_n\} \subset X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ - fundamental dominant weights

Let V be a finite-dimensional irreducible KG-module.

Theorem. There is a unique weight $\lambda = \sum_i a_i \lambda_i$ of V (with $a_i \in \mathbb{N}_0$) such that all weights of V are of the form

$$\lambda - \sum_i c_i \alpha_i \text{ with } c_i \in \mathbb{N}_0$$

We call λ the **highest weight** of V, and we write $V = L_G(\lambda)$.

Weights and representations

Theorem. There is a one-to-one correspondence

$$\lambda = \sum_{i=1}^{n} a_i \lambda_i \longleftrightarrow L_G(\lambda)$$

between dominant weights and finite-dimensional irreducible KG-modules.

Example. Suppose $G = \mathrm{SL}(W) = \mathrm{SL}_{n+1}(K)$. Then

$$L_G(\lambda_1) = W,\ L_G(\lambda_k) = \Lambda^k(W),\ L_G(k\lambda_1) = S^k(W),\ L_G(\lambda_n) = W^*$$

Definition. If $p > 0$ and $\lambda = \sum_i a_i \lambda_i$, then $V = L_G(\lambda)$ is p-**restricted** if $0 \leqslant a_i < p$ for all i.

Clifford theory

Theorem (Clifford, 1937) Let G be a group, K a field and let V be an irreducible KG-module. Let N be a normal subgroup of finite index.

- $V|_N$ is completely reducible, so $V|_N = V_1 \oplus \cdots \oplus V_t$ and the V_i are irreducible KN-modules
- G/N transitively permutes the V_i

Let G be a simple algebraic group, $V = L_G(\lambda)$ an irreducible KG-module, H a closed subgroup. Suppose $V|_H$ is irreducible.

- $V|_{H^0}$ **irreducible:** Read off (G, H, V) from Seitz's theorem.
- $V|_{H^0}$ **reducible:** Apply Clifford's theorem to $V|_H$, using the transitivity of H/H^0 to severely restrict the possibilities for λ.

Strategy

Our problem. Determine the irreducible triples (G, H, V), where G is classical and H is disconnected and infinite.

Step 1. $H < G$ is maximal

Step 2. $H < M < G \implies (G, M, V)$ is an irreducible triple determined in Step 1 (M disconnected) or by Seitz (M connected)

Maximal subgroups of classical groups

Let $G = \mathrm{Cl}(W)$ be a simple classical algebraic group. We define five collections of infinite closed subgroups of G:

$\mathcal{C}_1$	Stabilizers of subspaces of W
$\mathcal{C}_2$	Stabilizers of orthogonal decompositions $W = \bigoplus_i W_i$
$\mathcal{C}_3$	Stabilizers of totally singular decompositions $W = W_1 \oplus W_2$
$\mathcal{C}_4$	Stabilizers of tensor product decompositions $W = \bigotimes_i W_i$
$\mathcal{C}_6$	Classical subgroups (stabilizers of forms on W)

Theorem (Liebeck & Seitz, 1998) Let H be an infinite closed subgroup of $G = \mathrm{Cl}(W)$. Then one of the following holds:

- H is contained in a member of $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4 \cup \mathcal{C}_6$
- H^0 is simple (modulo scalars) and W is an irreducible KH^0-module

Results: Geometric subgroups

Theorem (B, Ghandour & Testerman, 2013) Let $G = \mathrm{SL}_{n+1}(K)$ and let H be an infinite disconnected geometric maximal subgroup of G.

Let $V = L_G(\lambda)$ be a p-restricted irreducible KG-module.

Then $V|_H$ is irreducible iff (H, λ) is one of the following:

Collection	Type of H	λ	Conditions
$\mathcal{C}_2$	$\bigoplus_{i=1}^{t} \mathrm{GL}_\ell(K).S_t$	$\lambda_1,\ \lambda_n$	$\ell \geqslant 1,\ t \geqslant 2$
		$\lambda_2, \ldots, \lambda_{n-1}$	$\ell = 1,\ t \geqslant 2$
$\mathcal{C}_4$	$\bigotimes_{i=1}^{t} \mathrm{GL}_\ell(K).S_t$	$\lambda_1,\ \lambda_n$	$\ell \geqslant 3,\ t \geqslant 2$
		$\lambda_2,\ \lambda_{n-1}$	$\ell \geqslant 3,\ t = 2,\ p \neq 2$
$\mathcal{C}_6$	$\mathrm{SO}_{n+1}(K).2$	$\lambda_1, \ldots, \lambda_n$	$n \geqslant 3$ odd, $p \neq 2$

Geometric subgroups: The exotic examples

- **Ford's example:** $G = \mathrm{SO}_{2n+1}(K)$, $H = \mathrm{SO}_{2n}(K).2$, $p \neq 2$ and

$$\lambda = a_1\lambda_1 + \cdots + a_{n-1}\lambda_{n-1} + \lambda_n, \text{ where}$$

  (i) if $a_i, a_j \neq 0$, where $i < j < n$ and $a_k = 0$ for all $i < k < j$, then $a_i + a_j \equiv i - j \pmod{p}$;

  (ii) if $i < n$ is maximal such that $a_i \neq 0$, then $2a_i \equiv -2(n-i) - 1 \pmod{p}$.

- $G = \mathrm{Sp}_{2n}(K)$, $H = (\mathrm{Sp}_n(K) \times \mathrm{Sp}_n(K)).S_2$ and

$$\lambda = \lambda_{n-1} + a\lambda_n$$

where $0 \leqslant a < p$ and $2a + 3 \equiv 0 \pmod{p}$.

Results: Non-geometric subgroups

Theorem (B, Ghandour, Marion & Testerman, 2013) Let $G = \mathrm{Cl}(W)$ be a simple classical algebraic group and let H be an infinite disconnected non-geometric maximal subgroup of G. Write $W = L_{H^0}(\delta)$.

Let $V = L_G(\lambda)$ be p-restricted and assume $V \neq W, W^*$.

Then $V|_H$ is irreducible iff (G, H, λ) is one of the following:

G	H	λ	δ	Conditions
Sp_{20}	$\mathrm{SL}_6.2$	λ_3	δ_3	$p \neq 2, 3$
Sp_{20}	$\mathrm{SL}_6.2$	λ_2	δ_3	$p \neq 2$
SO_7	$\mathrm{SL}_3.2$	$2\lambda_1$	$\delta_1 + \delta_2$	$p = 3$
SO_{14}	$\mathrm{SL}_4.2$	λ_6, λ_7	$\delta_1 + \delta_3$	$p = 2$
SO_{26}	$\mathrm{SO}_8.S_3$	λ_{12}, λ_{13}	δ_2	$p = 2$

Methods

1. **Geometric subgroups:** A combinatorial analysis of weight restrictions

- H^0 is reductive and we can describe the embedding of H in G in terms of the **root subgroups** of G

- Compute the restriction of T-weights to a maximal torus $S \leqslant H^0$

- Identify a T-weight μ of $V = L_G(\lambda)$ such that $\mu|_S$ is the highest weight of a composition factor of $V|_{H^0}$

- If $\mu|_S$ and $\lambda|_S$ are non-conjugate (under the action of H/H^0) then $V|_H$ is reducible (by Clifford theory)

Methods

2. Non-geometric subgroups: Parabolic embeddings & induction

- $X = H^0 \in \{A_m, D_m, E_6\}$ and $W = L_X(\delta)$, where $G = \mathrm{Cl}(W)$
- $B_X = U_X T_X \rightsquigarrow$ parabolic subgroup of G stabilising the flag

$$W > [W, U_X] > [[W, U_X], U_X] > \cdots > 0$$

  whose quotients are sums of specific T_X-weight spaces of W.

- If $H = X.2$ then some quotient is 2-dimensional $\rightsquigarrow$ severe restrictions on δ, and partial information on the coefficients of λ
- $P_X = Q_X L_X \rightsquigarrow P = QL$, where $L' = L_1 \cdots L_r$ is semisimple and $V/[V, Q] = M_1 \otimes \cdots \otimes M_r$ is an irreducible KL'-module.

  Further conditions on λ via the projections $\pi_i : L'_X \to L_i$...

An example

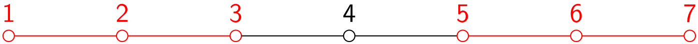
G - $\mathrm{SL}(W) = \mathrm{SL}_8 = \langle U_{\pm\alpha_i} \mid 1 \leqslant i \leqslant 7 \rangle$

H - Stabilizer of a decomposition $W = W_1 \oplus W_2$ with $\dim W_i = 4$

S - maximal torus of $X = [H^0, H^0]$

$$H^0 = \left\{ \left(\begin{array}{c|c} x & \\ \hline & y \end{array} \right) : x, y \in \mathrm{GL}_4 \right\} \cap G$$

$$X = \langle U_{\pm\alpha_1}, U_{\pm\alpha_2}, U_{\pm\alpha_3} \rangle \times \langle U_{\pm\alpha_5}, U_{\pm\alpha_6}, U_{\pm\alpha_7} \rangle \cong \mathrm{SL}_4 \times \mathrm{SL}_4$$



If $\chi = \sum_{i=1}^{7} b_i \lambda_i$ then

$\chi|_S$:

Set $V = L_G(\lambda)$ with $\lambda = \sum_{i=1}^{7} a_i \lambda_i$, and assume $V|_H$ is irreducible.

By Seitz, $V|_{H^0}$ is reducible, so Clifford theory implies that

$$V|_{H^0} = V_1 \oplus V_2$$

where $V_1|_X = L_X(\lambda|_S)$, $V_2|_X = L_X(\mu|_S)$ and

$$\lambda|_S : \quad \overset{a_1}{\circ}\text{———}\overset{a_2}{\circ}\text{———}\overset{a_3}{\circ} \qquad \overset{a_5}{\circ}\text{———}\overset{a_6}{\circ}\text{———}\overset{a_7}{\circ}$$

$$\mu|_S : \quad \overset{a_5}{\circ}\text{———}\overset{a_6}{\circ}\text{———}\overset{a_7}{\circ} \qquad \overset{a_1}{\circ}\text{———}\overset{a_2}{\circ}\text{———}\overset{a_3}{\circ}$$

Step 1. $a_4 \neq 0 \implies \chi = \lambda - \alpha_4$ is a weight of V, and $\chi|_S$ is the highest weight of a composition factor of $V|_{H^0}$.

Since $\chi = \lambda - \alpha_4 = \lambda + \lambda_3 - 2\lambda_4 + \lambda_5$ we have

$$\begin{array}{lccccccc}
\lambda|_S : & a_1 & a_2 & a_3 & \quad & a_5 & a_6 & a_7 \\
\mu|_S : & a_5 & a_6 & a_7 & \quad & a_1 & a_2 & a_3 \\
\chi|_S : & a_1 & a_2 & a_3 + 1 & \quad & a_5 + 1 & a_6 & a_7
\end{array}$$

so $\chi|_S$ is not equal to $\lambda|_S$ nor $\mu|_S$. This is a contradiction, so $a_4 = 0$.

Step 2. $a_4 = 0$, $a_3 \neq 0 \implies \nu|_S = (\lambda - \alpha_3 - \alpha_4)|_S$ is the highest weight of a composition factor of $V|_{H^0}$. But

$$\begin{array}{lccccccc}
\nu|_S : & a_1 & a_2 + 1 & a_3 - 1 & \quad & a_5 + 1 & a_6 & a_7
\end{array}$$

In this way, we reduce to the case $\lambda = a_1\lambda_1 + a_7\lambda_7$.

Step 3. $\lambda = a_1\lambda_1 + a_7\lambda_7$ and $a_1 \neq 0$. Here

$$\chi = \lambda - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 = \lambda - \lambda_1 - \lambda_4 + \lambda_5$$

is the highest weight of a composition factor of $V|_{H^0}$ and we have

$$\begin{array}{lcc}
\lambda|_S : & \overset{a_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ} & \overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{a_7}{\circ} \\
\mu|_S : & \overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{a_7}{\circ} & \overset{a_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ} \\
\chi|_S : & \overset{a_1-1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ} & \overset{1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{a_7}{\circ}
\end{array}$$

Hence $(a_1, a_7) = (1, 0)$, so $\lambda = \lambda_1$ and $V = W$.

Similarly, if $a_7 \neq 0$ then $(a_1, a_7) = (0, 1)$ and $V = W^*$.

Results: Non-maximal subgroups

Theorem (B & Testerman, 2014) Let $G = \mathrm{Cl}(W)$ be a simple classical algebraic group and let H be an infinite disconnected subgroup of G, which is non-maximal.

Let $V = L_G(\lambda)$ be p-restricted and $V \neq W, W^*$. Assume $V|_H$ is irreducible and $V|_{H^0}$ is reducible. Then one of the following holds:

- $G = \mathrm{SL}_{n+1}$, $\lambda = \lambda_k$ $(1 < k \leqslant (n+1)/2)$, $H = T.Y$ and $Y < S_{n+1}$ is k-transitive.
- $G = \mathrm{SO}(W)$ is an orthogonal group, H preserves an orthogonal direct sum decomposition of W, and V is a spin module.
- $\operatorname{rank} G \leqslant 8$ and (G, H, V) is known.

Results: Irreducible chains

Let $G = Cl(W)$ be a simple classical algebraic group and let V be an irreducible KG-module. An **irreducible chain** is a sequence of closed positive-dimensional subgroups

$$H_\ell < H_{\ell-1} < \cdots < H_2 < H_1 = G$$

such that each $V|_{H_i}$ is irreducible.

Let $\ell(G, V)$ be the length of the longest such chain.

Theorem (B & Testerman, 2014) One of the following holds:

- $V = W, W^*$
- $G = \mathrm{SO}(W)$ and V is a spin module
- $G = \mathrm{SL}(W)$ and $V = \Lambda^2(W), \Lambda^3(W), \Lambda^2(W)^*$ or $\Lambda^3(W)^*$
- $\ell(G, V) \leqslant 5$

Example. $C_1^3.3 < C_1^3.S_3 < D_4 < D_4.2 < C_4$, with $p = 2$, $V = V_{C_4}(\lambda_3)$