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ON BASE SIZES FOR ACTIONS OF FINITE CLASSICAL GROUPS

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Abstract

Let G be a finite almost simple classical group and let Ω be a faithful primitive non-standard G-set. A base for G is a subset $B \subseteq \Omega$ whose pointwise stabilizer is trivial; we write $b(G)$ for the minimal size of a base for G. A well-known conjecture of Cameron and Kantor asserts that there exists an absolute constant c such that $b(G) \leqslant c$ for all such groups G, and the existence of such an undetermined constant has been established by Liebeck and Shalev. In this paper we prove that either $b(G) \leqslant 4$ or $G = \mathrm{U}_6(2).2$, $G_\omega = \mathrm{U}_4(3).2^2$ and $b(G) = 5$. The proof is probabilistic, using bounds on fixed point ratios.

1. *Introduction*

A *base* for a finite permutation group G on a set Ω is a subset $B \subseteq \Omega$ whose pointwise stabilizer is trivial. Bases are of interest in several different fields. For example, they play an important rôle in computational group theory because each element of G is uniquely determined by its action on B and can therefore be stored as a $|B|$-tuple rather than a $|\Omega|$-tuple. In this respect, small bases are of particular interest. We write $b(G)$ for the minimal size of a base for G. Determining $b(G)$ is a fundamental problem in permutation group theory.

In this paper we obtain upper bounds on base sizes for primitive actions of almost simple classical groups. A main motivation comes from a conjecture of Cameron and Kantor [**8**, **9**] which was settled in the affirmative by Liebeck and Shalev in [**22**, 1.3]. The conjecture concerns *non-standard* actions of finite almost simple groups.

Definition 1. Let G be a finite almost simple group with socle G_0. Then a transitive action of G on a set Ω is said to be *standard* if one of the following holds:

(i) $G_0 = A_n$ and Ω is an orbit of subsets or partitions of $\{1, \ldots, n\}$;

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(ii) G is a classical group in a subspace action.

Naturally, an action of G which is equivalent to a standard action of an isomorphic group is also said to be a standard action of G. Non-standard actions are defined accordingly. Roughly speaking, if G is a classical group with natural module V then a transitive action of G is a *subspace action* if G permutes subspaces of V, or pairs of subspaces of complementary dimension (see Definition 2).

The Cameron-Kantor Conjecture asserts that there exists an absolute constant c such that $b(G) \leqslant c$ for any almost simple primitive permutation group G in a faithful non-standard action. In general, the orders of the groups in (i) and (ii) are not bounded from above by a fixed polynomial function of their degree and thus $b(G)$ is unbounded for standard actions.

The case $G_0 = A_n$ was settled by Cameron and Kantor [**9**]. With an elementary counting argument they showed that almost every pair of points in a non-standard G-set form a base for G as $|G|$ tends to infinity. In recent work [**14**], Guralnick and Saxl have proved that $b(G) = 2$ if $n > 12$, and it quickly follows that $b(G) \leqslant 3$ for all n. Combined with work of James [**16**], the primitive actions of alternating and symmetric groups which admit a base of size two have been completely determined.

For classical groups, the existence of an absolute constant c which satisfies the conclusion of the Cameron-Kantor Conjecture is established in [**23**, 1.3] but the proof does not yield an explicit value. However, some recent progress has been made in this direction. In [**23**, 1.11] Liebeck and Shalev show that if G is a classical group in a non-subspace action then $b(G) \leqslant 3$ as $|G|$ tends to infinity. Although this relies on the additional hypothesis that the dimension of the natural module is greater than fifteen, this asymptotic result is essentially best possible because it is easy to see that there exist non-standard primitive actions of classical groups of arbitrarily large rank with $b(G) > 2$. In addition, some non-asymptotic results have been obtained in specific cases. For example, the action of $\mathrm{PGL}_{2n}(q)$ on the cosets of $\mathrm{PGSp}_{2n}(q)$ is studied in [**12**] where it is shown that $b(G) = 3$ if $n > 3$. Precise base size results for partition actions of linear groups can be found in [**17**].

The main feature of Theorem 1 below is an explicit bound on $b(G)$ which is valid for *all* non-standard actions of finite almost simple classical groups. Here $H = G_\omega$

denotes the point stabilizer in G of an arbitrary element ω from the permutation domain.

THEOREM 1. *Let G be a finite almost simple classical group in a faithful primitive non-standard action. Then either $b(G) \leqslant 4$ or $G = \mathrm{U}_6(2).2$, $H = \mathrm{U}_4(3).2^2$ and $b(G) = 5$.*

REMARK 1. In the statement of Theorem 1, we omit any action of G which is equivalent to a standard action of an isomorphic group. A list of these standard actions is presented in Table 1.

In the forthcoming paper [**7**] we consider base sizes for primitive actions of exceptional groups. Our main result states that if G is a finite almost simple primitive exceptional group of Lie type then $b(G) \leqslant 6$ (see [**7**] for a more detailed statement). In view of Theorem 1, we get the following corollary.

COROLLARY 1. *If G is a finite almost simple group of Lie type in a faithful primitive non-standard action then $b(G) \leqslant 6$.*

The proof of the Cameron-Kantor Conjecture for classical groups relies on an existence result [**22**, Theorem $(\star)$]. This states that there exists an absolute constant $\epsilon > 0$ so that if G is any finite almost simple classical group in a primitive non-subspace action then

$$\mathrm{fpr}(x, \Omega) < |x^G|^{-\epsilon} \tag{1.1}$$

for all elements $x \in G$ of prime order, where $\mathrm{fpr}(x, \Omega)$ denotes the *fixed point ratio of x*, i.e. the proportion of points in the permutation domain Ω which are fixed by x. The connection to bases arises from the following observation: if $Q(G, c)$ denotes the probability that a randomly chosen c-tuple in Ω does not form a base for G then

$$Q(G, c) \leqslant \sum_{i=1}^{k} |x_i^G| \cdot \mathrm{fpr}(x_i, \Omega)^c, \tag{1.2}$$

where $x_1, \ldots, x_k$ represent the distinct G-classes of elements of prime order in G. Of course, G admits a base of size c if and only if $Q(G, c) < 1$. In the proof of [**22**, 1.3] it is shown that $b(G) \leqslant 11\epsilon^{-1}$, an undetermined bound since [**22**, Theorem ($\star$)] is strictly an existence result.

The main theorem of [**3**] states that (1.1) holds with $\epsilon = 1/2 - 1/n - \iota$, where n is a natural number associated to each almost simple classical group and $\iota \geqslant 0$ is a known constant depending on G and Ω with the property that $\iota \to 0$ as $n \to \infty$ (the precise values of ι are listed in [**3**, Table 1]). In general, n is simply the dimension of the natural G-module - see Remark 2.

In applying (1.1) and (1.2), the zeta-type function

$$\eta_G(t) = \sum_{C \in \mathscr{C}} |C|^{-t}$$

arises naturally in the proof of [**22**, 1.3], where $t \in \mathbb{R}$ and $\mathscr{C}$ denotes the set of conjugacy classes in G of elements of prime order. Evidently there exists a real number $T_G \in (0,1)$ such that $\eta_G(T_G) = 1$ and applying [**3**, Theorem 1] we deduce that G admits a base of size c if

$$T_G < c(1/2 - 1/n - \iota) - 1. \tag{1.3}$$

In this way we can bound $b(G)$ by bounding the function T_G. Indeed, Proposition 2.3 states that $T_G < 1/3$ if $n \geqslant 6$ and Theorem 1 quickly follows, with the exception of a short list of cases. For these remaining cases we either establish a stronger upper bound for T_G and obtain the desired result via (1.3), or we derive an explicit upper bound $Q(G,c) \leqslant F(c,q)$ via (1.2) with the property that $F(c,q) < 1$ for all possible values of q. Here the computer package GAP [**11**] is a useful tool when dealing with groups of small rank over small fields. In proving Theorem 1 we provide detailed results in the case where $G_0 = \mathrm{PSp}_4(q)'$ or $\mathrm{PSL}_n^{\epsilon}(q)$ for $n \leqslant 5$ (see Proposition 4.1 and Tables 5 and 6).

Layout. This paper is organized as follows. In the next section we present some preliminary results which we require for the proof of Theorem 1. In particular, we show that G admits a base a size of size c if (1.3) holds (Proposition 2.2) and we prove that $T_G < 1/3$ if $n \geqslant 6$ (Proposition 2.3). In Section 3 we establish Theorem 1 for $n \geqslant 6$; the remaining small rank cases are considered in Section 4.

Notation. Our notation and terminology for classical groups is standard (see [**20**] and [**3**] for example). In particular, we write $\mathrm{PSL}_n^{+}(q) = \mathrm{PSL}_n(q) = \mathrm{L}_n(q)$

and $\mathrm{PSL}_n^-(q) = \mathrm{PSU}_n(q) = \mathrm{U}_n(q)$. We adopt the terminology of [**13**, 2.5.13] for the various automorphisms of simple groups of Lie type. Further, G' denotes the derived subgroup of a group G; G^m is the direct product of m copies of G and $H.G$ denotes an (arbitrary) extension of a group H by G. We write n for a cyclic group of order n, while p^m denotes an elementary abelian p-group of order p^m for a prime p and D_n is the dihedral group of order n. If $\mathbb{F}$ is a field then $\bar{\mathbb{F}}$ will denote the algebraic closure of $\mathbb{F}$. Also, we write (a, b) for the highest common factor of the integers a and b, while $\delta_{i,j}$ is the usual Kronecker delta. We use $\log n$ to denote $\log_e n$ and write $\mathbb{F}_q = \mathrm{GF}(q)$ for the field of q elements.

2. *Preliminary results*

Let G be a finite almost simple classical group over $\mathbb{F}_q$, where $q = p^f$ and p is prime, with socle G_0 and natural module V. In studying actions of classical groups, it is natural to distinguish between those actions which permute subspaces of the natural module and those which do not. In this paper we are interested in bases for *non-subspace actions*. This notion was introduced by Liebeck and Shalev in [**22**].

Definition 2. A subgroup H of G not containing G_0 is a *subspace subgroup* if for each maximal subgroup M of G_0 containing $H \cap G_0$ one of the following holds:

(i) M is the stabilizer in G_0 of a proper non-zero subspace U of V, where U is totally singular, non-degenerate, or, if G_0 is orthogonal and $p = 2$, a non-singular 1-space (U can be any subspace if $G_0 = \mathrm{PSL}(V)$);

(ii) $M = \mathrm{O}_{2m}^{\pm}(q)$ if $(G_0, p) = (\mathrm{Sp}_{2m}(q)', 2)$.

A faithful transitive action of G on a set Ω is a *subspace action* if the point stabilizer G_ω of an element $\omega \in \Omega$ is a subspace subgroup of G. Non-subspace subgroups and actions are defined accordingly.

Recall the definition of a *standard action* from the Introduction (see Definition 1). Evidently, every subspace action of G is standard, but the converse does not hold because a non-subspace action of G may be equivalent to a standard action of an isomorphic group. The primitive standard actions that arise in this way are listed in Table 1. Here H is the stabilizer in G of an element in the permutation domain; the *type of H* provides an approximate group-theoretic structure for $H \cap \mathrm{PGL}(V)$, where

V is the natural G_0-module (this notation is consistent with [**20**]). An *l-subspace* is an l-dimensional subspace of the natural module for the relevant classical group isomorphic to G_0; an *$\epsilon 4$-subspace* for $\Omega_5(q)$ is a non-degenerate 4-subspace whose stabilizer in $\Omega_5(q)$ is an orthogonal group of type $\mathrm{O}_4^{\epsilon}(q)$.

G_0	type of H	conditions	equivalent action
$\mathrm{P}\Omega_8^+(q)$	$\Omega_7(q)$	H irreducible	$\mathrm{P}\Omega_8^+(q)$ on non-singular 1-subspaces
$\mathrm{PSL}_4^{\epsilon}(q)$	$\mathrm{Sp}_4(q)$		$\mathrm{P}\Omega_6^{\epsilon}(q)$ on non-singular 1-subspaces
$\mathrm{L}_4(q)$	$\mathrm{GL}_2(q^2)$	$q=2$	A_8 on 3-element subsets of $\{1,\ldots,8\}$
$\mathrm{L}_4(q)$	A_7	$q=2$	A_8 on 1-element subsets of $\{1,\ldots,8\}$
$\mathrm{PSp}_4(q)'$	$\mathrm{Sp}_2(q)\wr S_2$		$\Omega_5(q)$ on $+$ 4-subspaces
	$\mathrm{Sp}_2(q^2)$		$\Omega_5(q)$ on $-$ 4-subspaces
	$2^4.\mathrm{O}_4^-(2)$	$q=3$	$\mathrm{U}_4(2)$ on totally singular 2-subspaces
$\mathrm{L}_2(q)$	$\mathrm{GL}_2(q_0)$	$q=q_0^2$	$\Omega_4^-(q_0)$ on non-singular 1-subspaces
	$2^2.\mathrm{O}_2^-(2)$	$q=5$	A_5 on 4-element subsets of $\{1,\ldots,5\}$
	A_5	$q=9$	A_6 on 5-element subsets of $\{1,\ldots,6\}$

TABLE 1. *Some standard actions*

The main theorem on the subgroup structure of finite classical groups is due to Aschbacher. In [**1**], eight collections of subgroups of G are defined, labelled $\mathscr{C}_i$ for $1 \leqslant i \leqslant 8$, and in general it is shown that if H is a maximal subgroup of G not containing G_0 then either H is contained in one of these $\mathscr{C}_i$ collections or it belongs to a family of almost simple groups which act irreducibly on the natural G-module V (see [**20**] for detailed information on these subgroup collections). Due to the existence of certain outer automorphisms, a small additional collection arises when G_0 is $\mathrm{Sp}_4(q)'$ (q even) or $\mathrm{P}\Omega_8^+(q)$ (see [**1**, §14] and [**18**, §4]). Roughly speaking, a maximal subgroup is non-subspace unless it is a member of the collection $\mathscr{C}_1$ or is a particular example of a subgroup in $\mathscr{C}_8$, where we follow [**20**] in labelling the various $\mathscr{C}_i$ collections.

In [**3**, **4**, **5**, **6**] we studied *fixed point ratios* for non-subspace actions of finite classical groups. Let Ω be a G-set and recall that the fixed point ratio of $x \in G$, which we denote by $\mathrm{fpr}(x,\Omega)$, is the proportion of points in Ω which are fixed by x. If Ω is a transitive G-set then it is easy to see that

$$\mathrm{fpr}(x,\Omega) = \frac{|x^G \cap H|}{|x^G|}, \tag{2.1}$$

where $H = G_\omega$ for some $\omega \in \Omega$. In [**22**], Liebeck and Shalev prove that there exists an absolute constant $\epsilon > 0$ so that if G is any finite almost simple classical group in a primitive non-subspace action then $\mathrm{fpr}(x,\Omega) < |x^G|^{-\epsilon}$ for all elements $x \in G$ of prime order. This result plays a major rôle in their proof of the Cameron-Kantor base size conjecture (see Section 1) and finds a wide range of other interesting applications in [**22**]. The main theorem of [**3**] states that the Liebeck-Shalev result holds with an explicit constant $\epsilon \approx 1/2$.

THEOREM 2.1 ([**3**, Theorem 1]). *Let G be a finite almost simple classical group over $\mathbb{F}_q$ and let Ω be a faithful primitive non-subspace G-set. Then*

$$\mathrm{fpr}(x,\Omega) < |x^G|^{-\frac{1}{2}+\frac{1}{n}+\iota}$$

for all $x \in G$ of prime order, where $\iota = \iota(G,H) \geqslant 0$ is a known constant.

REMARK 2. In most cases we have $\iota = 0$; the precise values are listed in [**3**, Table 1]. The integer n in the statement of Theorem 2.1 is defined as follows: if $G_0 = \mathrm{Sp}_4(2)'$ or $\mathrm{SL}_3(2)$ then $n = 2$, otherwise n is the minimal dimension of a non-trivial irreducible $K\widehat{G}_0$-module, where $\widehat{G}_0$ is a covering group of G_0 and $K = \bar{\mathbb{F}}_q$, i.e. n is the dimension of the natural module for G_0, whenever this is well-defined.

Next we associate a "zeta function" to G which encodes the sizes of the conjugacy classes of elements of prime order.

DEFINITION 3. Let $\mathscr{C}$ be the set of conjugacy classes of elements of prime order in G. Define

$$\eta_G(t) = \sum_{C \in \mathscr{C}} |C|^{-t}$$

for $t \in \mathbb{R}$, and fix $T_G \in (0,1)$ such that $\eta_G(T_G) = 1$.

REMARK 3. We refer the reader to [**23**, 1.10] for results on the asymptotic behaviour of $\eta_G(t)$. Note that in summing over *all* conjugacy classes, the definition of $\eta_G(t)$ in [**23**] differs slightly from ours.

PROPOSITION 2.2. *If Ω is a faithful primitive non-subspace G-set and $T_G < c(1/2 - 1/n - \iota) - 1$ then $b(G) \leqslant c$.*

Proof. We follow the proof of [**23**, 1.11]. Let $x_1, \ldots, x_k$ be representatives for the G-classes of elements of prime order in G and for $c \geqslant 1$ let $Q(G,c)$ be the probability that a randomly chosen c-tuple of points in Ω is not a base for G. Evidently, G admits a base of size c if and only if $Q(G,c) < 1$. Now, a c-tuple in Ω fails to be a base if and only if it is fixed by an element $x \in G$ of prime order. The probability that a random c-tuple is fixed by x is precisely $\mathrm{fpr}(x,\Omega)^c$, and since fixed point ratios are constant on conjugacy classes (see (2.1)) we deduce that

$$Q(G,c) \leqslant \sum_{x \in \mathscr{P}} \mathrm{fpr}(x,\Omega)^c = \sum_{i=1}^{k} |x_i^G| \cdot \mathrm{fpr}(x_i,\Omega)^c, \tag{2.2}$$

where $\mathscr{P}$ is the set of elements of prime order in G. Hence Theorem 2.1 gives

$$Q(G,c) \leqslant \sum_{i=1}^{k} |x_i^G|^{c(-1/2+1/n+\iota)+1} \leqslant \eta_G(c(1/2-1/n-\iota)-1)$$

and the desired result follows since $\eta_G(t) < 1$ for all $t > T_G$. $\square$

Definition 4. For $x \in \mathrm{PGL}(V)$ let $\hat{x}$ be a pre-image of x in $\mathrm{GL}(V)$ and define

$$\nu(x) := \min\{\dim[\bar{V}, \lambda\hat{x}] : \lambda \in K^*\},$$

where $\bar{V} = V \otimes K$ and $K = \bar{\mathbb{F}}_q$. Note that $\nu(x)$ is equal to the codimension of the largest eigenspace of $\hat{x}$ on $\bar{V}$ and thus $\nu(x) > 0$ if $x \neq 1$.

The next proposition allows us to effectively apply Proposition 2.2.

Proposition 2.3. *If $n \geqslant 6$ then $T_G < 1/3$.*

Proof. Let $\bar{G}$ be a simple classical algebraic group over $\bar{\mathbb{F}}_q$ of adjoint type which admits a Frobenius morphism σ such that the finite subgroup $\bar{G}_\sigma := \{g \in \bar{G} : g^\sigma = g\}$ has socle G_0. In the terminology of [**13**], $\bar{G}_\sigma$ is the group of inner-diagonal automorphisms of G_0. For now, let us assume $G_0 = \mathrm{PSp}_n(q)$. For $1 \leqslant s \leqslant n-1$, let $k_{s,u}$ (respectively, $k_{s,s}$) denote the number of distinct G-classes of unipotent (respectively semisimple) elements $x \in G$ of prime order with $\nu(x) = s$, and write $c(s)$ for the size of the smallest G-class of elements $x \in G$ of prime order with $\nu(x) = s$. In addition, let π be the set of distinct prime divisors of $\log_p q$ and for $r \in \pi$ let $\mathscr{F}_r$ be the set of G-classes of field automorphisms of order r. Then

$$\eta_G(t) \leqslant \sum_{r \in \pi} \sum_{C \in \mathscr{F}_r} |C|^{-t} + \sum_{s=1}^{n-1} (k_{s,u} + k_{s,s}).c(s)^{-t}.$$

Set $d=(2,q-1)$. We claim that the following bounds hold:

(i) $|\pi| \leqslant \log(\log_p q+2)$ and $|\mathscr{F}_r| \leqslant d(\log_p q-1)$ for all $r \in \pi$;
(ii) $|C| > \frac{1}{2d}q^{\frac{1}{4}n(n+1)}$ for all $C \in \mathscr{F}_r$;
(iii) $k_{s,u} < d.2^{s+\sqrt{s}}$;
(iv) $k_{s,s} < q^{\frac{1}{2}(s+1)}.\frac{1}{4}n(n+2)\log q$;
(v) $c(s) > \frac{1}{4d}(q+1)^{-1}q^{\max(s(n-s),\frac{1}{2}ns)+1} = d(s)$.

First consider (i). If $N \in \mathbb{N}$ then N has fewer than $\log(N+2)$ distinct prime divisors, hence $|\pi| \leqslant \log(\log_p q+2)$. If $r \in \pi$ then $|\mathscr{F}_r| \leqslant d(r-1)$ (see [**13**, 4.9.1]) and hence (i) follows since r divides $\log_p q$. Part (ii) is immediate from [**4**, 3.48] and we observe that (i) and (ii) imply that

$$\sum_{r\in\pi}\sum_{C\in\mathscr{F}_r}|C|^{-t} < d(\log_p q-1).\log(\log_p q+2).\left(\frac{1}{2d}q^{\frac{1}{4}n(n+1)}\right)^{-t} = \Gamma_1(n,q,t).$$

According to [**4**, (9)], there are fewer than $2^{s+\sqrt{s}}$ distinct $\bar{G}_\sigma$-classes of unipotent elements x with $\nu(x)=s$ and thus (iii) follows since each $\bar{G}_\sigma$-class can split into at most $d=|\bar{G}_\sigma : G_0|$ distinct G-classes. Next consider (iv). Fix a prime $r \neq p$ which divides $|\bar{G}_\sigma|$. Then according to [**4**, 3.40], there are at most $q^{(s+1)/2}$ distinct G-classes of elements $x \in G$ of order r with $\nu(x)=s$ and thus it suffices to show that there are fewer than $\frac{1}{4}n(n+2)\log q$ possibilities for r. Since $r \neq p$, it follows that r divides $\prod_{i=1}^{n/2}(q^{2i}-1) < q^{n(n+2)/4}$ and thus (iv) holds since any integer $N>6$ has fewer than $\log N$ distinct prime divisors. Finally, the bound in (v) follows at once from [**4**, 3.38] and we conclude that $\sum_s (k_{s,u}+k_{s,s}).c(s)^{-t} < \Gamma_2(n,q,t)$, where

$$\Gamma_2(n,q,t) = \sum_{s=1}^{n-1}\left[\left(d.2^{s+\sqrt{s}}+q^{\frac{1}{2}(s+1)}.\frac{1}{4}n(n+2)\log q\right).\left(\frac{1}{4d}(q+1)^{-1}q^{\alpha}\right)^{-t}\right]$$

and $\alpha = \max(s(n-s),\frac{1}{2}ns)+1$, hence $\eta_G(t) < \Gamma_1(n,q,t)+\Gamma_2(n,q,t)$. For $n \geqslant 8$, one can check that $\Gamma_1(n,q,1/3)+\Gamma_2(n,q,1/3) < 1$ with the exception of a short list of cases for which both n and q are small (there are no exceptions if $n>30$ or $q>43$). Here we can calculate $\rho(n,q)$ precisely, where $\rho(n,q)+1$ is the number of distinct prime divisors of $|\mathrm{Sp}_n(q)|$, and thus replace the term $\frac{1}{4}n(n+2)\log q$ by $\rho(n,q)$ in the above bound (iv) for $k_{s,s}$. Then the subsequent bound for $\eta_G(t)$ is sufficient with a much shorter list of exceptions; these cases can be checked by hand with the aid of [**4**, §§3.2-3.5]. For example, if $(n,q)=(8,2)$ then $G=\mathrm{Sp}_8(2)$ and $T_G < .228$ since the possibilities for $|C|$ are as follows, where C is a conjugacy class in G containing elements of prime order r.

r	$\lvert C\rvert$
2	255, 5355, 16065, 64260, 321300, 963900
3	10880, 609280, 3655680, 12185600
5	13160448, 157925376
7	1128038400
17	2786918400 (2 classes)

To complete the proof in the case $G_0 = \mathrm{PSp}_n(q)$, let us assume $n=6$. Here the more accurate bound

$$\begin{aligned}\eta_G(t) < 2\left(\frac{1}{2}(q^6-1)\right)^{-t} &+ (2+\frac{q}{2}\log(q^2-1)).\left(\frac{1}{4}(q+1)^{-1}q^{11}\right)^{-t} \\ &+ ((q^2+1)(q^6-1))^{-t} + (6+\frac{q}{2}\log(q^2-1)).\left(\frac{1}{4}(q+1)^{-1}q^{13}\right)^{-t} \\ &+ q^2\log(q^2+1)\left(\frac{1}{8}q^{14}\right)^{-t} + \frac{1}{3}q^3\log(q^6-1).\left(\frac{1}{4}q^{18}\right)^{-t} \\ &+ 2(\log_p q - 1).\log(\log_p q+2).\left(\frac{1}{4}q^{\frac{21}{2}}\right)^{-t}\end{aligned}$$

is sufficient unless $q=2$ or 3. These two cases can be checked directly; we find that $T_G < .307$ when $q=2$, while $T_G < .240$ if $q=3$.

A very similar argument applies when $G_0 = \mathrm{PSL}^{\epsilon}_n(q)$ or $\mathrm{P\Omega}^{\epsilon}_n(q)$; we leave the details to the reader (see [**4**, §3.5] for the required information on graph and graph-field automorphisms). □

REMARK 4. In certain cases we need a slightly stronger upper bound on T_G. More precisely, we require $T_G < 1/5$ if $G_0 = \mathrm{PSL}^{\epsilon}_{10}(q)$, and $T_G < 4/15$ if $G_0 = \mathrm{P\Omega}^{\epsilon}_n(q)$ and $n \geqslant 12$ is even. These bounds can be established by arguing as in the proof of Proposition 2.3.

The next two basic lemmas will be useful in the proof of Theorem 1.

LEMMA 2.4. *Let G be a transitive permutation group on a finite set Ω and write $H = G_\omega$ for some $\omega \in \Omega$. Let $x_1,\ldots,x_m$ represent distinct G-classes such that $\sum_i |x_i^G \cap H| \leqslant A$ and $|x_i^G| \geqslant B$ for all $1 \leqslant i \leqslant m$. Then for $c \in \mathbb{N}$ we have $\sum_i |x_i^G|\cdot \mathrm{fpr}(x_i,\Omega)^c \leqslant B(A/B)^c$.*

Proof. Without loss we may assume $|x_i^G| \geqslant |x_{i+1}^G|$. For $1 \leqslant i \leqslant m-1$ set $a_i = |x_i^G \cap H|$ and $b_i = |x_i^G| - B$. Then in view of (2.1) we have

$$\sum_{i=1}^{m} |x_i^G| \cdot \mathrm{fpr}(x_i,\Omega)^c \leqslant B\left(\frac{A-\sum_i a_i}{B}\right)^c + \sum_i (B+b_i)\left(\frac{a_i}{B+b_i}\right)^c$$
$$\leqslant B^{1-c}\left((A-\sum_i a_i)^c + \sum_i a_i^c\right)$$

and the claim follows. □

Lemma 2.5. *Let G be a permutation group on a finite set Ω. Then $b(G) \geqslant (\log|G|)/(\log|\Omega|)$.*

Proof. If $B \subseteq \Omega$ is a base then each element of G is uniquely determined by its action on B, whence $|G| \leqslant |\Omega|^{|B|}$ and the result follows. □

3. *Proof of Theorem 1, Part I: $n \geqslant 6$*

We begin the proof of Theorem 1 by assuming $n \geqslant 6$. For the reader's convenience, we partition the proof into three sections, according to the type of G_0.

3.1. $G_0 = \mathrm{PSp}_n(q)$

Lemma 3.1. *Suppose $G_0 = \mathrm{Sp}_6(q)$ and H is an irreducible subgroup of type $\mathrm{G}_2(q)$, where q is even. Then $b(G) = 4$.*

Proof. Here $G = G_0.\langle\phi\rangle$ and $H = G_2(q).\langle\phi\rangle$, where ϕ is a (possibly trivial) field automorphism of G_0. Let $\rho : \Omega_8^+(q) \to \Omega_8^+(q)$ be an irreducible spin representation and view G_0 as the stabilizer in $\Omega_8^+(q)$ of a 1-dimensional non-singular subspace of the natural module $\tilde{V}$ for $\Omega_8^+(q)$. Then the action of G on Ω is equivalent to the action of $\rho(G_0).\langle\phi\rangle$ on the set of 1-dimensional non-singular subspaces of $\tilde{V}$ and so it suffices to show that there exist non-singular vectors v_1, v_2, v_3, v_4 in $\tilde{V}$ such that $\bigcap_i G_{\langle v_i\rangle} = \{1\}$, and that the intersection of any three such stabilizers is non-trivial. In the proof we adopt the notation of [**2**] for labelling representatives of unipotent classes of involutions in orthogonal groups.

Fix non-singular vectors v_1, v_2, v_3, v_4 which span a -4-subspace W of $\tilde{V}$, so W is a 4-dimensional non-degenerate subspace and the restriction of the corresponding quadratic form on $\tilde{V}$ to W has defect -1. With a careful choice of the v_i, we may assume that the subspaces $\langle v_i \rangle$ are not simultaneously fixed by any field automorphism of G_0. We claim that $\bigcap_i G_{\langle v_i \rangle} = \{1\}$. Seeking a contradiction, suppose $x \in \bigcap_i G_{\langle v_i \rangle}$ has prime order r. Then $x \in G_0$ and Lagrange's Theorem implies that r divides $|\Omega_4^-(q)| = q^2(q^4-1)$ since x fixes the decomposition $\tilde{V} = W \oplus W^{\perp}$ and acts trivially on W. If $r=2$ then the proof of [**6**, 2.7] implies that x acts on $W^{\perp}$ as an a_2-involution, a contradiction since there are no such involutions in $\mathrm{O}_4^-(q)$. Now assume r is odd. Let $i \geqslant 1$ be minimal such that r divides q^i-1, so $i \in \{1,2,4\}$. We can rule out $i=4$ since $G_{0\langle v_1 \rangle} = G_2(q)$ and q^2+1 does not divide $|G_2(q)|$. If $i \in \{1,2\}$ then [**4**, 3.29] implies that $\nu(x) = 2$ (with respect to $\tilde{V}$) but there are no such elements in $\rho(G_0)$ (see the proof of [**6**, 2.7]). We conclude that $b(G) \leqslant 4$.

It remains to show that if v_1, v_2, v_3 are any non-singular vectors which span a 3-dimensional subspace U of $\tilde{V}$ then $\bigcap_i G_{\langle v_i \rangle}$ is non-trivial. If $x \in G_{0\langle v_1 \rangle} = G_2(q)$ is a long root involution then x lies in the a_2-class of G (see [**6**, 2.7, 2.13]) and thus some conjugate y of x fixes the decomposition $\tilde{V} = U \oplus U^{\perp}$, acting trivially on U and as an a_2-involution on $U^{\perp}$. We conclude that $y \in \bigcap_i G_{\langle v_i \rangle}$. □

LEMMA 3.2. *Suppose $G_0 = \mathrm{PSp}_8(q)$ and H is of type $\mathrm{Sp}_4(q) \wr S_2$ or $\mathrm{Sp}_4(q^2)$. Then $b(G) = 3$.*

Proof. If $q < 4$ then the claim is easily checked using GAP so we will assume $q \geqslant 4$. Let $\bar{G}$ be a simple algebraic group over $\bar{\mathbb{F}}_q$ which admits a Frobenius morphism σ such that $\bar{G}_\sigma$ has socle G_0. Let $x_1, \ldots, x_k$ represent the G-classes of elements of prime order in H and let $Q(G,c)$ be the probability that a randomly chosen c-tuple of points in Ω does not form a base. Now Lemma 2.5 implies that $b(G) \geqslant 3$ and according to (2.1) and (2.2) we have

$$Q(G,c) \leqslant \widehat{Q}(G,c) := \sum_{i=1}^{k} |x_i^G| \cdot \mathrm{fpr}(x_i,\Omega)^c = \sum_{i=1}^{k} |x_i^G| \cdot (|x_i^G \cap H|/|x_i^G|)^c. \quad (3.1)$$

(Note that $\mathrm{fpr}(x,\Omega) = 0$ if $x^G \cap H$ is empty, so we need only sum over G-classes of elements in H.) Therefore, it suffices to show that there exists a function $F(3,q)$ such that $\widehat{Q}(G,3) \leqslant F(3,q) < 1$ for all $q \geqslant 4$.

To avoid unnecessary repetition, we will assume H is of type $\mathrm{Sp}_4(q) \wr S_2$; a very similar argument applies when H is of type $\mathrm{Sp}_4(q^2)$. To define a suitable function F

we inspect the proof of [**5**, 2.8] (see the proof of [**5**, 3.3] when H is of type $\mathrm{Sp}_4(q^2)$). First suppose $x \in H \cap \mathrm{PGL}(V)$ is a semisimple element of odd prime order r and $i \geqslant 1$ is minimal such that r divides $q^i - 1$, so $i \in \{1, 2, 4\}$. Since x is semisimple we have $|x^G| \geqslant |x^{\bar{G}_\sigma}|$ by [**13**, 4.2.2(j)]. We consider each possibility for i in turn. If $i = 4$ then r divides q^2+1 and there are at most three possible types for $C_G(x)$, namely $\mathrm{GU}_2(q^2)$, $\mathrm{Sp}_4(q) \times \mathrm{GU}_1(q^2)$ and $\mathrm{GU}_1(q^2)^2$ (see [**4**, 3.30], for example). Now, if $C_G(x)$ is of type $\mathrm{GU}_2(q^2)$ then

$$|x^{\bar{G}_\sigma} \cap H| = \left(\frac{|\mathrm{Sp}_4(q)|}{|\mathrm{GU}_1(q^2)|}\right)^2 = q^8(q^2-1)^4,$$

$$|x^{\bar{G}_\sigma}| = \frac{|\mathrm{Sp}_8(q)|}{|\mathrm{GU}_2(q^2)|} = q^{14}(q^2-1)^2(q^4+1)(q^6-1)$$

and there are precisely $\frac{1}{4}(r-1) \leqslant \frac{1}{4}q^2$ distinct $\bar{G}_\sigma$-classes of such elements x. We also note that there are fewer than $\log(q^2+1)$ distinct odd prime divisors of q^2+1. By considering the other two possibilities for $C_G(x)$ we deduce that the contribution to $\widehat{Q}(G,3)$ from elements of odd prime order dividing q^2+1 is at most $\sum_{j=1}^{3} n_j b_j (a_j/b_j)^3$, where

j	n_j	a_j	b_j
1	$\frac{1}{4}q^2\log(q^2+1)$	$q^8(q^2-1)^4$	$q^{14}(q^2-1)^2(q^4+1)(q^6-1)$
2	$\frac{1}{4}q^2\log(q^2+1)$	$2q^4(q^2-1)^2$	$q^{12}(q^2-1)(q^4+1)(q^6-1)$
3	$\frac{1}{32}q^2(q^2-4)\log(q^2+1)$	$2q^8(q^2-1)^4$	$q^{16}(q^2-1)^3(q^4+1)(q^6-1)$

The contribution from elements of odd prime order dividing q^2-1 can be estimated in a similar fashion. For example, if $C_G(x)$ is of type $\mathrm{GL}_2^\epsilon(q)^2$ then r divides $q-\epsilon$,

$$|x^{\bar{G}_\sigma} \cap H| = 2\left(\frac{|\mathrm{Sp}_4(q)|}{|\mathrm{Sp}_2(q)||\mathrm{GL}_1^\epsilon(q)|}\right)^2 + \left(\frac{|\mathrm{Sp}_4(q)|}{|\mathrm{GL}_1^\epsilon(q)^2|}\right)^2 = a_4, \quad |x^{\bar{G}_\sigma}| = \frac{|\mathrm{Sp}_8(q)|}{|\mathrm{GL}_2^\epsilon(q)^2|} = b_4$$

and there are precisely $\binom{(r-1)/2}{2} \leqslant \frac{1}{8}(q-\epsilon-1)(q-\epsilon-3)$ distinct $\bar{G}_\sigma$-classes of such elements for a given fixed prime r. Since there are fewer than $\log(q-\epsilon)$ odd prime divisors of $q-\epsilon$ we conclude that the contribution to $\widehat{Q}(G,3)$ from these elements is at most $n_4 b_4 (a_4/b_4)^3$, where

$$n_4 = \frac{1}{8}(q-\epsilon-1)(q-\epsilon-3).\log(q-\epsilon).$$

We leave the reader to consider the other possibilities for $C_G(x)$.

Next we consider unipotent elements. Applying Lemma 2.4 and [**4**, 3.18, 3.20] we calculate that the contribution to $\widehat{Q}(G,3)$ from the unipotent elements of odd order p is at most $\sum_{j=1}^{6} d_j (c_j/d_j)^3$, where the terms c_j and d_j are defined as follows:

j	λ	c_j	d_j
1	$(2,1^6)$	$2(q^4-1)$	$\frac{1}{2}(q^8-1)$
2	$(2^2,1^4)$	$2q^2(q^4-1)+(q^4-1)^2$	$\frac{1}{2}q(q^2-q+1)(q^3-1)(q^8-1)$
3	$(2^3,1^2)$	$2q^2(q^4-1)^2$	$\frac{1}{2}q^2(q^2+1)(q^6-1)(q^8-1)$
4	(2^4)	$q^4(q^4-1)^2$	$\frac{1}{2}q^4(q^2-1)(q^6-1)(q^8-1)$
5	$(4,1^4)$	$2q^2(q^2-1)(q^4-1)$	$\frac{1}{2}q^6(q^6-1)(q^8-1)$
6	(4^2)	$q^4(q^2-1)^2(q^4-1)^2$	$\frac{1}{2}q^9(q-1)(q^4-1)(q^6-1)(q^8-1)$

Here the second column lists the possible partitions of 8 which correspond to the Jordan normal form on V of a unipotent element $x \in H$. The entries in the table are easily verified. For example, if $\lambda = (2^2, 1^4)$ then we can write λ as a sum of two partitions of 4 in essentially two distinct ways, namely $(2^2) \oplus (1^4)$ and $(2, 1^2) \oplus (2, 1^2)$, and applying [**4**, 3.18] we deduce that

$$|x^G \cap H| \leqslant 2\frac{|\mathrm{Sp}_4(q)|}{|\mathrm{SO}_2^+(q)|q^3} + \left(\frac{|\mathrm{Sp}_4(q)|}{|\mathrm{Sp}_2(q)|q^3}\right)^2 = c_2, \;\; |x^G| \geqslant \frac{|\mathrm{Sp}_8(q)|}{|\mathrm{Sp}_4(q)||\mathrm{O}_2^-(q)|q^{11}} = d_2.$$

In the same way, we can estimate the contribution from semisimple and unipotent involutions.

Finally, suppose x is a field automorphism of prime order r. Then r divides $\log_p q$ and the proof of [**5**, 2.8] gives $|x^{\bar{G}_\sigma} \cap H| < 4q^{20\left(1-\frac{1}{r}\right)}$ and $|x^G| > \frac{1}{2d}q^{36\left(1-\frac{1}{r}\right)} = f(r,q)$, where $d = (2, q-1)$. Therefore $\mathrm{fpr}(x, \Omega) < 8dq^{-16\left(1-\frac{1}{r}\right)} = g(r,q)$ and so Lemma 2.4 implies that the contribution to $\widehat{Q}(G,3)$ from field automorphisms is less than

$$\sum_{r\in\pi}(r-1).h(r,q) < h(2,q) + 2h(3,q) + 4h(5,q) + \log_p q.q^{36}g(7,q)^3,$$

where $h(r,q) = f(r,q)g(r,q)^3$ and π is the set of distinct prime divisors of $\log_p q$.

In this way we obtain a function $F(3,q)$ which satisfies $\widehat{Q}(G,3) \leqslant F(3,q) < 1$ for all $q \geqslant 4$. We conclude that $b(G) = 3$. □

Proposition 3.3. *If $G_0 = \mathrm{PSp}_n(q)$ and $n \geqslant 6$ then $b(G) \leqslant 4$.*

Proof. Applying Propositions 2.2 and 2.3 we see that $b(G) \leqslant c$ if

$$c(1/2 - 1/n - \iota) - 1 \geqslant 1/3. \tag{3.2}$$

In particular, if $\iota = 0$ then $b(G) \leqslant 4$ for all $n \geqslant 6$. Next assume $\iota \in \{1/n, 1/(n+2)\}$, in which case $n \equiv 0\,(4)$ and H is of type $\mathrm{Sp}_{n/2}(q) \wr S_2$ or $\mathrm{Sp}_{n/2}(q^2)$ (see [**3**, Table 1]). Here (3.2) implies that $b(G) \leqslant 4$ if $n \geqslant 12$, while Lemma 3.2 applies if $n = 8$.

According to [**3**, Table 1] it remains to deal with the following cases:

	G_0	type of H	ι	$b(G)$
(i)	$\mathrm{Sp}_8(2)$	A_{10}	.062	3
(ii)	$\mathrm{Sp}_6(q)$	$G_2(q),\ p=2$	.084	4
(iii)	$\mathrm{Sp}_6(2)$	$\mathrm{U}_3(3)$	.054	4

In (i) and (iii) we calculate using GAP [**11**]. For (ii) see Lemma 3.1. □

3.2. $G_0 = \mathrm{P}\Omega_n^{\epsilon}(q)$

Now suppose G_0 is an orthogonal group. Note that we may assume $n \geqslant 7$ and that q is odd if n is odd.

LEMMA 3.4. *Suppose $G_0 = \mathrm{P}\Omega_8^+(q)$ and H is of type $\mathrm{GL}_4^{\epsilon'}(q)$, or $G_0 = \mathrm{P}\Omega_{10}^{\epsilon}(q)$ and H is of type $\mathrm{GL}_5^{\epsilon'}(q)$. Then $b(G) \in \{3,4\}$.*

Proof. First note that Lemma 2.5 gives $b(G) \geqslant 3$. For the upper bound we proceed as in the proof of Lemma 3.2: close inspection of the proof of [**5**, 3.3] yields a function $F(4,q)$ which satisfies $\widehat{Q}(G,4) \leqslant F(4,q) < 1$ for all values of q. The details are left to the reader. □

PROPOSITION 3.5. *If $G_0 = \mathrm{P}\Omega_n^{\epsilon}(q)$ and $n \geqslant 7$ then $b(G) \leqslant 4$.*

Proof. If $\iota = 0$ then (3.2) implies that $b(G) \leqslant 4$ so assume $\iota > 0$. If H is of type $\mathrm{GL}_{n/2}^{\epsilon'}(q)$ then $\iota = 1/(n-2)$ and the conclusion $b(G) \leqslant 4$ follows via Proposition 2.2 and Remark 4 if $n \geqslant 12$; the cases $n \in \{8,10\}$ were considered in Lemma 3.4 (note that we may assume $\epsilon = +$ if $n = 8$ (see [**20**, Table 3.5.F], for example)). According to [**3**, Table 1] it remains to deal with the cases listed in the table below (we omit the irreducible embedding $\Omega_7(q) < \mathrm{P}\Omega_8^+(q)$ since the corresponding action is equivalent to a subspace action - see Table 1).

	G_0	type of H	ι	$b(G)$
(i)	$\Omega_{10}^-(2)$	A_{12}	.087	$3+\alpha$
(ii)	$\mathrm{P}\Omega_8^+(3)$	$\Omega_8^+(2)$	.081	$3+\alpha$
(iii)	$\Omega_8^+(2)$	$\mathrm{O}_4^-(2) \wr S_2$	.001	3
(iv)	$\Omega_8^+(2)$	A_9	.124	4
(v)	$\Omega_7(q)$	$G_2(q),\ p>2$	.108	4
(vi)	$\Omega_7(3)$	$\mathrm{Sp}_6(2)$	.065	3

The results for (i)-(iv) and (vi) can be checked using GAP [**11**]; here $\alpha = 0$ if $G = G_0$, otherwise $\alpha = 1$.

For (v) we argue as in the proof of Lemma 3.1. Here $G = G_0.\langle\phi\rangle$ and $H = G_2(q).\langle\phi\rangle$, where ϕ is a (possibly trivial) field automorphism of G_0; the action of G on Ω is equivalent to the action of $\rho(G_0).\langle\phi\rangle$ on the set of 1-dimensional non-singular subspaces of the natural module $\tilde{V}$ for J, where ρ is an irreducible spin representation of $J = \mathrm{P}\Omega_8^+(q)$. Suppose $x \in \bigcap_{i=1}^4 G_{\langle v_i\rangle}$ has prime order r, where the v_i are non-singular vectors which span a $-$4-subspace W of $\tilde{V}$ and are chosen so that no field automorphism of G_0 simultaneously fixes each of the subspaces $\langle v_i\rangle$. Then x fixes the decomposition $\tilde{V} = W \oplus W^{\perp}$ and acts trivially on W. For $r > 2$ we get a contradiction precisely as in the proof of Lemma 3.1; if $r = 2$ then [**4**, 3.55] implies that $C_J(x)$ is of type $\mathrm{GL}_4^{\epsilon}(q)$, $\mathrm{O}_4^+(q)^2$ or $\mathrm{O}_4^+(q^2)$, and this contradicts the assumption that x fixes the above decomposition and acts trivially on W. We conclude that $b(G) \leqslant 4$. To see that equality holds, let v_1, v_2, v_3 be any non-singular vectors which span a 3-dimensional subspace U of $\tilde{V}$ and let $x \in G_{0\langle v_1\rangle} = G_2(q)$ be a long root element. Then x has order p and the proof of [**6**, 2.13] implies that x has Jordan form $[J_2^2, I_4]$ on $\tilde{V}$. In particular, some conjugate y of x fixes the decomposition $\tilde{V} = U \oplus U^{\perp}$ and acts trivially on U. This implies that $y \in \bigcap_i G_{\langle v_i\rangle}$ and thus $b(G) > 3$. □

3.3. $G_0 = \mathrm{PSL}_n^{\epsilon}(q)$

We begin with the exceptional case referred to in the statement of Theorem 1.

LEMMA 3.6. *Suppose $G_0 = \mathrm{U}_6(2)$ and H is an irreducible subgroup of type $\mathrm{U}_4(3)$. Then $b(G) = 5$ if $G = G_0.2$, otherwise $b(G) = 4$.*

Proof. One can construct H in GAP and the desired result quickly follows. □

LEMMA 3.7. *Suppose $G_0 = \mathrm{PSL}_n^{\epsilon}(q)$ and H is of type $\mathrm{Sp}_n(q)$, where $n \in \{6, 8\}$. Then $b(G) \in \{3, 4\}$.*

Proof. First observe that Lemma 2.5 implies that $b(G) \geqslant 3$. If $(n, q) = (6, 2)$ then using GAP [**11**] we calculate that $b(G) = 3 + \alpha$, where $\alpha = 1$ if $G = \mathrm{U}_6(2).2$, otherwise $\alpha = 0$. For $(n, q) \neq (6, 2)$ we proceed as in the proof of Lemma 3.2; by inspecting the proof of [**4**, 8.1] we derive a function $F(4, q)$ such that $\widehat{Q}(G, 4) \leqslant F(4, q) < 1$ for all values of q. □

Proposition 3.8. *If $G_0 = \mathrm{PSL}^{\epsilon}_n(q)$ and $n \geqslant 6$ then $b(G) \leqslant 4$.*

Proof. As before, we quickly reduce to the case $\iota > 0$. If H is of type $\mathrm{Sp}_n(q)$ then $\iota = 1/n$ and Lemma 2.5 implies that $b(G) \geqslant 3$. Now, if $n \geqslant 12$ then (3.2) holds with $c = 4$; the case $n = 10$ follows from Proposition 2.2 since $T_G < 4/15$ (see Remark 4), and the cases $n = 6$ and 8 are handled in Lemma 3.7. According to [**3**, Table 1], it remains to deal with the case where $G_0 = \mathrm{U}_6(2)$ and H is an irreducible subgroup of type $\mathrm{U}_4(3)$. This is the content of Lemma 3.6 above. □

4. *Proof of Theorem 1, Part II: $n < 6$*

In this final section we complete the proof of Theorem 1 by dealing with the remaining groups of small rank. Here Theorem 2.1 is less useful; we proceed by considering each maximal non-subspace subgroup of G in turn. As before, we use the proof of [**3**, Theorem 1] in [**4**, **5**, **6**] to derive bounds on fixed point ratios; the computer package GAP is a useful tool when the underlying field is small.

Our detailed results are recorded in Tables 5 and 6. The maximal non-subspace subgroups are determined in [**19**] and can be verified using a combination of [**1**], [**15**] and [**24**]; as before, we exclude any subgroup for which the corresponding primitive action is equivalent to a standard action of an isomorphic group (see Table 1). The third column lists certain necessary, but not necessarily sufficient, conditions for H to be maximal in G. In the column headed '$b(G)$' we record the possibilities for $b(G)$. Where more than one possibility is listed, we mean that $b(G)$ always takes one of the listed values, but we do not claim that each possibility is attainable.

The next proposition follows from the results in Tables 5 and 6 and completes the proof of Theorem 1. Here we adopt the notation of [**10**] for labelling representatives of G_0-classes of involutions when $G_0 = \mathrm{U}_4(3)$.

Proposition 4.1. *Let G be a finite almost simple group with socle G_0, where $G_0 = \mathrm{PSp}_4(q)'$ or $\mathrm{PSL}^{\epsilon}_n(q)$ for $n \leqslant 5$. If Ω is a faithful primitive non-standard G-set then either $b(G) \leqslant 3$ or $b(G) = 4$ and (G, Ω) is listed in Table 2.*

The cases in Table 2 can be checked using GAP. For the remainder, we use the proof of [**3**, Theorem 1] and the bounds on fixed point ratios therein to define a function $F(c,q)$ such that $\widehat{Q}(G,c) \leqslant F(c,q) < 1$ (see (3.1)), with $c = 2$ or 3 as required. For small values of q, in many cases precise base size calculations are possible using GAP. Below we provide some details when $G_0 = \mathrm{PSp}_4(q)'$ and $\mathrm{L}_2(q)$. (Note that the case $G_0 = \mathrm{L}_2(q)$ was not considered in [**3**]-[**6**] since Theorem 2.1 is trivial in this case.)

4.1. $G_0 = \mathrm{PSp}_4(q)'$

Let G be a finite almost simple group with socle $G_0 = \mathrm{PSp}_4(q)'$. We use the symbol $(\leftrightarrow)$ to denote the additional hypothesis "$p = 2$ and G contains graph-field automorphisms". (Note that some authors favour the term *graph automorphism* here.)

Proposition 4.2. *Let G be a finite almost simple group with socle $G_0 = \mathrm{PSp}_4(q)'$ and let Ω be a faithful primitive non-standard G-set. Then the possibilities for $H = G_\omega$ are listed in Table 3.*

Proof. The maximal subgroups of G are listed in [**19**, Chapter 5] and [**1**, 14.2] (also see [**20**, Table 3.5.C]). The subgroups in Aschbacher's $\mathscr{C}_1$ collection can be omitted since they are subspace subgroups. In addition, each action corresponding to a subgroup of type $\mathrm{Sp}_2(q) \wr S_2$, $\mathrm{Sp}_2(q^2)$ or $2^4.\mathrm{O}_4^-(2)$ $(q = 3)$ is equivalent to a $\mathscr{C}_1$-action of an isomorphic classical group and so these subgroups are excluded too (see Table 1). The second column records necessary conditions for the relevant subgroup H to be maximal in G. □

Lemma 4.3. *If H is of type $2^4.\mathrm{O}_4^-(2)$ then $b(G) = 3$ if $G = \mathrm{PGSp}_4(5)$, otherwise $b(G) = 2$.*

G_0	type of H	remarks
$\mathrm{U}_4(3)$	$2^4.\mathrm{Sp}_4(3)$	$G = G_0.\langle 2B, 2D\rangle$
	$\mathrm{L}_3(4)$	
$\mathrm{Sp}_4(2)'$	$\mathrm{O}_2^-(2) \wr S_2$	$G = \mathrm{Aut}(G_0)$
$\mathrm{U}_3(3)$	$\mathrm{L}_3(2)$	$G = G_0.2$

Table 2. $n < 6$: *Exceptional cases with $b(G) = 4$*

Proof. Here $q = p > 3$ and $H \leqslant 2^4.\mathrm{O}_4^-(2) = \widetilde{H}$. For $q < 17$ the result follows via GAP so let us assume $q \geqslant 17$. Now the prime divisors of $|\widetilde{H}|$ are 2, 3 and 5, and the proof of [**4**, 6.6] gives $i_r(\widetilde{H}) \leqslant n_r$, where $n_5 = 384$, $n_3 = 80$ and $n_2 = 155$, and $i_r(\widetilde{H})$ is the number of elements of order r in $\widetilde{H}$. Therefore $\widehat{Q}(G,2) \leqslant (384^2 + 80^2).a_1^{-1} + 155^2.a_2^{-1}$, where

$$a_1 = \frac{|\mathrm{Sp}_4(q)|}{|\mathrm{GU}_2(q)|} = q^3(q-1)(q^2+1), \quad a_2 = \frac{|\mathrm{Sp}_4(q)|}{|\mathrm{Sp}_2(q^2)|2} = \frac{1}{2}q^2(q^2-1),$$

and thus $\widehat{Q}(G,2) < 1$ for all $q \geqslant 17$. □

Lemma 4.4. *If H is of type $Sz(q)$ then $b(G) = 3$.*

Proof. Here q is even, $\log_2 q > 1$ is odd and $H \cap \mathrm{PGL}(V) = Sz(q)$ is the centralizer in G_0 of an involutory graph-field automorphism, where V is the natural G_0-module. Now, if $G \neq G_0$ then

$$\frac{\log|G|}{\log|\Omega|} \geqslant \frac{\log(2q^4(q^2-1)(q^4-1))}{\log(q^2(q+1)(q^2-1))} > 2$$

for all $q \geqslant 8$ and thus Lemma 2.5 implies that $b(G) \geqslant 3$. The case $G = G_0$ is studied in [**21**]; in particular the subdegrees of G are listed in [**21**, Table 1] and we deduce that $b(G) \geqslant 3$. Therefore, it suffices to show that there exists a function $F(q)$ such that $\widehat{Q}(G,3) \leqslant F(q) < 1$ for all values of q.

Suppose $x \in H \cap \mathrm{PGL}(V)$ has prime order r. If $r = 2$ then x is G-conjugate to c_2 (see [**2**] and [**4**, 2.56]) and thus

$$|x^G \cap H| = (q^2+1)(q-1) = a_1, \quad |x^G| = (q^2-1)(q^4-1) = b_1.$$

type of H	conditions
$\mathrm{GL}_2(q).2$	q odd
$\mathrm{GU}_2(q)$	q odd
$\mathrm{Sp}_4(q_0)$	$q = q_0^k$, k prime
$2^4.\mathrm{O}_4^-(2)$	$q = p > 3$
$Sz(q)$	$p = 2$, $\log_2 q > 1$ odd
$\mathrm{L}_2(q)$	$p > 3$
A_6	$q = p > 3$
A_7	$q = 7$
$\mathrm{O}_2^\epsilon(q) \wr S_2$	$(\leftrightarrow)$ holds, $\epsilon = -$ if $q = 2$
$\mathrm{O}_2^-(q^2).2$	$(\leftrightarrow)$ holds

Table 3. $G_0 = \mathrm{PSp}_4(q)'$: *Maximal non-subspace subgroups*

Now assume $r > 2$. Then [**4**, 2.56] implies that $r \geqslant 5$, $\nu(x) = 3$ (see Definition 4) and $i \in \{1,4\}$, where $i \geqslant 1$ is minimal such that r divides $q^i - 1$. If $i = 1$ then $r \geqslant 7$ and [**4**, 2.56] gives

$$|x^{\mathrm{Sp}_4(q)} \cap H| \leqslant \frac{1}{r}|Sz(q)| = \frac{1}{r}a_2, \quad |x^{\mathrm{Sp}_4(q)}| = |\mathrm{Sp}_4(q) : \mathrm{GL}_1(q)^2| = b_2$$

and there are precisely $\frac{1}{2}(r-1)$ possibilities for x up to $\mathrm{Sp}_4(q)$-conjugacy. It follows that the contribution to $\widehat{Q}(G,3)$ from elements whose order divides $q-1$ is at most

$$b_2(a_2/b_2)^3\frac{1}{2}\sum_{r\in\Lambda}\frac{r-1}{r^3},$$

where Λ is the set of distinct prime divisors of $q-1$. Now

$$\frac{1}{2}\sum_{r\in\Lambda}\frac{r-1}{r^3} < \frac{1}{2}\sum_{m=3}^{\infty}\left(\frac{1}{2m+1}\right)^2 = \frac{1}{16}\pi^2 - \frac{1}{2} - \frac{1}{18} - \frac{1}{50} = n_2$$

and so the contribution here is less than $n_2b_2(a_2/b_2)^3$. Similarly, the elements of prime order dividing q^2+1 contribute less than $n_3b_3(a_3/b_3)^3$, where

$$a_3 = q^2(q^2+1)(q-1), \quad b_3 = q^4(q^2-1)^2, \quad n_3 = \frac{1}{32}\pi^2 - \frac{1}{4} - \frac{1}{36}.$$

Now, if $x \in G$ is an involutory graph-field automorphism then we may assume x centralizes $Sz(q)$ and thus

$$|x^G \cap H| = i_2(Sz(q)) + 1 = (q^2+1)(q-1)+1 = a_4, \quad |x^G| = q^2(q+1)(q^2-1) = b_4.$$

Finally, if x is a field automorphism of prime order r then

$$|x^G| = \frac{q^4(q^2-1)(q^4-1)}{q^{\frac{4}{r}}(q^{\frac{2}{r}}-1)(q^{\frac{4}{r}}-1)} = f(r,q), \quad \mathrm{fpr}(x,\Omega) = \frac{q^{\frac{2}{r}}(q^{\frac{1}{r}}+1)(q^{\frac{2}{r}}-1)}{q^2(q+1)(q^2-1)} = g(r,q)$$

and so the contribution here is at most

$$\sum_{r\in\Gamma}(r-1).h(r,q) < 2h(3,q) + \log_2 q.q^4(q^2-1)(q^4-1).g(5,q)^3 = F_1(q),$$

where $h(r,q) = f(r,q)g(r,q)^3$ and Γ is the set of distinct prime divisors of $\log_2 q$. We conclude that

$$\widehat{Q}(G,3) < \sum_{j=1}^{4} n_jb_j(a_j/b_j)^3 + F_1(q) = F(q)$$

(where $n_1 = n_4 = 1$) and one can check that $F(q) < 1$ for all $q \geqslant 8$. □

The remaining cases in Table 3 are dealt with in a similar fashion and our results are presented in Table 6.

4.2. $G_0 = \mathrm{L}_2(q)$

Finally, let us assume G is an almost simple group with socle $G_0 = \mathrm{L}_2(q)$. Note that we may assume $q > 4$ since $\mathrm{L}_2(2) \cong S_3$, $\mathrm{L}_2(3) \cong A_4$ and $\mathrm{L}_2(4) \cong \mathrm{L}_2(5)$. The maximal subgroups of G are well-known; those which correspond to faithful non-standard actions are listed in Table 4.

LEMMA 4.5. *If H is of type $\mathrm{GL}_1(q^2)$ then $b(G) \in \{2,3\}$.*

Proof. If $q = 5$ then a calculation with GAP gives $b(G) = 3$ if $G = \mathrm{PGL}_2(5)$, otherwise $b(G) = 2$. For the remainder let us assume $q \geqslant 7$. Let V denote the natural G_0-module and observe that $H \cap \mathrm{PGL}(V) \leqslant (q+1).2 \cong D_{2(q+1)}$. Define $\widehat{Q}(G,3)$ as in (3.1) and let $x \in H \cap \mathrm{PGL}(V)$ be an element of prime order r. If $r = p$ then $p = 2$ and we have $|x^G \cap H| = q+1 = a_1$ and $|x^G| = q^2 - 1 = b_1$. Next assume $r \neq p$. If $r = 2$ then there are two possibilities for $C_G(x)$: if $C_G(x)$ is of type $\mathrm{GL}_1(q)^2$ then $|x^G \cap H| = \frac{1}{2}(q+1) = a_2$ and $|x^G| = \frac{1}{2}q(q+1) = b_2$; otherwise $C_G(x)$ is of type $\mathrm{GL}_1(q^2)$ and we get $|x^G \cap H| = \frac{1}{2}(q+3) = a_3$ and $|x^G| = \frac{1}{2}q(q-1) = b_3$. If $r \neq p$ is odd then r divides $q+1$ and it is easy to see that $|x^{\tilde{G}} \cap H| = 2 = a_4$ and $|x^{\tilde{G}}| = q(q-1)$, where $\tilde{G} = \mathrm{PGL}_2(q)$. For such a prime r, there are precisely $\frac{1}{2}(r-1) \leqslant \frac{q}{2}$ possibilities for x up to $\tilde{G}$-conjugacy and we note that there are fewer than $\log(q+1)$ odd prime divisors of $q+1$. We conclude that the contribution to $\widehat{Q}(G,3)$ from $H \cap \mathrm{PGL}(V)$ is at most

$$F_1(3,q) = \sum_{i=1}^{3} b_i(a_i/b_i)^3 + n_4 b_4 (a_4/b_4)^3,$$

where $n_4 = \frac{q}{2}\log(q+1)$.

Finally, suppose $x \in H$ is a field automorphism of G_0 of prime order r. Here $q = q_0^r$ and we note that r is odd since $x \in H - \mathrm{PGL}(V)$. Now $|x^G \cap H| \leqslant q+1$

type of H	conditions
$\mathrm{GL}_1(q) \wr S_2$	$q > 5$
$\mathrm{GL}_1(q^2)$	
$\mathrm{GL}_2(q_0)$	$q = q_0^k$, $k > 2$ prime
$2^2.\mathrm{O}_2^-(2)$	$q = p > 5$
A_5	$q \equiv \pm 1\,(10),\ q \neq 9$

TABLE 4. *$G_0 = \mathrm{L}_2(q)$: Maximal non-subspace subgroups*

since $x^G \cap H \subseteq (q+1)x$, and we have

$$|x^G| \geqslant \frac{1}{d}|x^{\tilde{G}}| = \frac{1}{d}\frac{q(q^2-1)}{q^{\frac{1}{r}}(q^{\frac{2}{r}}-1)} = f(r,q),$$

where $d=(2,q-1)$. Therefore $\mathrm{fpr}(x,\Omega) \leqslant (q+1)f(r,q)^{-1} = g(r,q)$ and thus

$$\begin{aligned}\widehat{Q}(G,3) &\leqslant F_1(3,q) + \sum_{r\in\Lambda}(r-1).h(r,q)\\ &\leqslant F_1(3,q) + \alpha\left[2h(3,q)+4h(5,q)+\log_p q.q(q^2-1).g(7,q)^3\right] < 1\end{aligned}$$

for all $q \geqslant 7$, where $h(r,q) = f(r,q)g(r,q)^3$, Λ is the set of distinct odd primes which divide $\log_p q$ and $\alpha = 1$ if Λ is non-empty, otherwise $\alpha = 0$. □

A similar argument applies when H is of type $\mathrm{GL}_1(q) \wr S_2$ or $\mathrm{GL}_2(q_0)$ (note that in the former case we have $b(G) = 2$ if $G \leqslant \mathrm{PGL}_2(q)$ - see [**17**, 2.1]). It remains to consider the final two cases in Table 4.

Lemma 4.6. *If H is of type $2^2.\mathrm{O}_2^-(2)$ then $b(G) = 2 + \delta_{7,q}$.*

Proof. First observe that $H \cap \mathrm{PGL}(V) \leqslant S_4$. We will assume $q \geqslant 17$ as the cases with $q < 17$ are easily checked via GAP. Define $\widehat{Q}(G,c)$ as in (3.1) and let $x \in H$ be an element of prime order r. Then $|x^G| \geqslant 2^{-\delta_{2,r}}q(q-1)$ and thus

$$\widehat{Q}(G,2) \leqslant \frac{1}{2}q(q-1)\left(\frac{18}{q(q-1)}\right)^2 + q(q-1)\left(\frac{8}{q(q-1)}\right)^2 = F(2,q)$$

since $i_2(S_4) = 9$ and $i_3(S_4) = 8$, where $i_r(S_4)$ is the number of elements of order r in S_4. One can check that $F(2,q) < 1$ for all $q \geqslant 17$ and thus $b(G) = 2$. □

Lemma 4.7. *If H is of type A_5 then $b(G) = 2 + \delta_{11,q} + \delta_{19,q}$.*

Proof. Here $q \leqslant p^2$ and the maximality of H implies that $G = G_0$ and $H = A_5$ if $q = p$. Using GAP the result quickly follows when $q \leqslant 49$ (note that $q \neq 9$ - see Table 1) so we will assume $q > 49$. Now, if $x \in H \cap \mathrm{PGL}(V) = A_5$ has prime order r then $|x^G| \geqslant 2^{-\delta_{2,r}}q(q-1)$ and so Lemma 2.4 implies that the contribution to $\widehat{Q}(G,2)$ from $H \cap \mathrm{PGL}(V)$ is at most

$$F_1(2,q) = \frac{1}{2}q(q-1)\left(\frac{30}{q(q-1)}\right)^2 + q(q-1)\left(\frac{20}{q(q-1)}\right)^2 + 2q(q-1)\left(\frac{12}{q(q-1)}\right)^2$$

since $i_2(A_5) = 15$, $i_3(A_5) = 20$ and $i_5(A_5) = 12$. Finally, if $q = p^2$ and $x \in H - \mathrm{PGL}(V)$ is an involution then $|x^G \cap H| = 10$, $|x^G| \geqslant \frac{1}{2}q^{1/2}(q+1)$ and since

$q \geqslant 49$ we conclude that

$$\widehat{Q}(G,2) \leqslant F_1(G,2) + (\log_p q - 1).q^{\frac{1}{2}}(q+1).\left(\frac{20}{q^{1/2}(q+1)}\right)^2 < 1. \qquad \square$$

G_0	type of H	conditions	$b(G)$	remarks
$\mathrm{L}_5(q)$	$\mathrm{GL}_1(q) \wr S_5$	$q > 3$	2	see [**17**]
	$\mathrm{GL}_1(q^5)$		2	
	$\mathrm{GL}_5(q_0)$	$q = q_0^k$, k prime	$2, 3$	$b(G) = 2$ if $k > 2$
	$5^2.\mathrm{Sp}_2(5)$	$q = p \equiv 1\ (5)$	2	
	$\mathrm{O}_5(q)$	q odd	$2, 3$	
	$\mathrm{U}_5(q_0)$	$q = q_0^2$	$2, 3$	
	$\mathrm{L}_2(11)$	$q = p \equiv 1, 3, 4, 5, 9\ (11)$	2	
	M_{11}	$q = 3$	2	
$\mathrm{U}_5(q)$	$\mathrm{GU}_1(q) \wr S_5$		$2 + \delta_{2,q}$	
	$\mathrm{GU}_1(q^5)$		2	
	$\mathrm{GU}_5(q_0)$	$q = q_0^k$, $k > 2$ prime	2	
	$5^2.\mathrm{Sp}_2(5)$	$p \equiv 2, 3, 4\ (5)$, $q \equiv 4\ (5)$	2	
	$\mathrm{O}_5(q)$	q odd	$2, 3$	
	$\mathrm{L}_2(11)$	$q = p \equiv 2, 6, 7, 8, 10\ (11)$	2	
	$\mathrm{U}_4(2)$	$q = 5$	2	
$\mathrm{L}_4(q)$	$\mathrm{GL}_1(q) \wr S_4$	$q > 3$	2	see [**17**]
	$\mathrm{GL}_2(q) \wr S_2$	$q > 2$	$3 - \delta_{2,p}$	see [**17**]
	$\mathrm{GL}_2(q^2)$	$q > 2$	$2, 3$	$b(G) = 3$ if $q = 3$
	$\mathrm{GL}_4(q_0)$	$q = q_0^k$, k prime	$2, 3$	$b(G) = 2$ if $k > 2$
	$2^4.\mathrm{Sp}_4(2)$	$q = p \equiv 1\,(4)$	2	
	$\mathrm{O}_4^+(q)$	q odd	$2, 3$	
	$\mathrm{O}_4^-(q)$	q odd	$2, 3$	
	$\mathrm{U}_4(q_0)$	$q = q_0^2$	$2, 3$	$b(G) = 3$ if $q = 4$
	A_7	$q = p \equiv 1, 2, 4\,(7)$, $q > 2$	2	
	$\mathrm{U}_4(2)$	$q = p \equiv 1\,(6)$	2	
$\mathrm{U}_4(q)$	$\mathrm{GU}_1(q) \wr S_4$		$2, 3$	$b(G) = 2$ if $q > 3$
	$\mathrm{GU}_2(q) \wr S_2$		$2, 3$	
	$\mathrm{GL}_2(q^2)$		$2, 3$	
	$\mathrm{GU}_4(q_0)$	$q = q_0^k$, $k > 2$ prime	2	
	$2^4.\mathrm{Sp}_4(2)$	$q = p \equiv 3\,(4)$	$2, 3, 4$	$b(G) = 2$ if $q > 3$
	$\mathrm{O}_4^+(q)$	q odd	$2, 3$	
	$\mathrm{O}_4^-(q)$	q odd	$2, 3$	
	A_7	$q = p \equiv 3, 5, 6\,(7)$	$2 + \delta_{3,q}$	
	$\mathrm{L}_3(4)$	$q = 3$	4	
	$\mathrm{U}_4(2)$	$q = p \equiv 5\,(6)$	2	

TABLE 5. *Base sizes for non-standard actions, I*

G_0	type of H	conditions	$b(G)$	remarks
$\mathrm{PSp}_4(q)'$	$\mathrm{GL}_2(q).2$	q odd	$2,3$	
	$\mathrm{GU}_2(q)$	q odd	$2,3$	
	$\mathrm{Sp}_4(q_0)$	$q=q_0^k$, k prime	$2,3$	$b(G)=2$ if $k>2$
	$2^4.\mathrm{O}_4^-(2)$	$q=p>3$	$2,3$	$b(G)=3$ iff $G=\mathrm{PGSp}_4(5)$
	$Sz(q)$	$p=2,\ \log_2 q>1$ odd	3	
	$\mathrm{L}_2(q)$	$p>3$	2	
	A_6	$q=p>3$	2	
	A_7	$q=7$	2	
	$\mathrm{O}_2^{\epsilon}(q)\wr S_2$	$(\leftrightarrow)$ holds, $\epsilon=-$ if $q=2$	$2,3,4$	$b(G)=4$ iff $G=\mathrm{Sp}_4(2)'.2^2$
	$\mathrm{O}_2^-(q^2).2$	$(\leftrightarrow)$ holds	$2,3$	
$\mathrm{L}_3(q)$	$\mathrm{GL}_1(q)\wr S_3$	$q>3$	2	see [**17**]
	$\mathrm{GL}_1(q^3)$		$2,3$	$b(G)=2$ if $q>3$
	$\mathrm{GL}_3(q_0)$	$q=q_0^k$, k prime	$2,3$	$b(G)=2$ if $k>2$
	$3^2.\mathrm{Sp}_2(3)$	$q=p>5,\ p\equiv 1\,(3)$	2	
	$\mathrm{O}_3(q)$	q odd	$2,3$	
	$\mathrm{U}_3(q_0)$	$q=q_0^2$	$2,3$	
	A_6	$q=4$	3	
	$\mathrm{L}_3(2)$	$q=p\equiv 1,2,4\ (7)$	2	
$\mathrm{U}_3(q)$	$\mathrm{GU}_1(q)\wr S_3$		$2,3$	$b(G)=2+\delta_{3,q}+\delta_{4,q}$
$(q>2)$	$\mathrm{GU}_1(q^3)$		2	
	$\mathrm{GU}_3(q_0)$	$q=q_0^k$, $k>2$ prime	2	
	$3^2.\mathrm{Sp}_2(3)$	$q=p\equiv 2\,(3)$	2	
	$\mathrm{O}_3(q)$	q odd	$2,3$	
	A_6	$q=5$	3	
	A_7	$q=5$	4	
	$\mathrm{L}_3(2)$	$q=p\equiv 3,5,6\,(7)$	$2,3,4$	$b(G)=2$ if $q>5$
$\mathrm{L}_2(q)$	$\mathrm{GL}_1(q)\wr S_2$	$q>5$	$2,3$	
$(q>4)$	$\mathrm{GL}_1(q^2)$		$2,3$	
	$\mathrm{GL}_2(q_0)$	$q=q_0^k$, $k>2$ prime	2	
	$2^2.\mathrm{O}_2^-(2)$	$q=p>5$	$2+\delta_{7,q}$	**Std. action when $q=7$?**
	A_5	$q\equiv\pm1\,(10),\ q\neq 9$	$2,3$	$b(G)=2+\delta_{11,q}+\delta_{19,q}$

TABLE 6. *Base sizes for non-standard actions, II*

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