

ON BASE SIZES FOR ALGEBRAIC GROUPS

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ABSTRACT. Let G be a permutation group on a set Ω . A subset of Ω is a base for G if its pointwise stabilizer is trivial; the base size of G is the minimal cardinality of a base. In this paper we initiate the study of bases for algebraic groups defined over an algebraically closed field. In particular, we calculate the base size for all primitive actions of simple algebraic groups, obtaining the precise value in almost all cases. We also introduce and study two new base measures, which arise naturally in this setting. We give an application concerning the essential dimension of simple algebraic groups, and we establish several new results on base sizes for the corresponding finite groups of Lie type. The latter results are an important contribution to the classical study of bases for finite primitive permutation groups. We also indicate some connections with generic stabilizers for representations of simple algebraic groups.

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1. INTRODUCTION

Let G be a transitive permutation group on a set Ω with point stabilizer H . A subset of Ω is a *base* for G if its pointwise stabilizer in G is trivial. The *base size* of G , denoted by $b(G, H)$ (or just $b(G)$ if the context is clear), is the minimal size of a base for G . Equivalently, the base size is the smallest number b such that the intersection of some b conjugates of H in G is trivial.

Determining the base size of a given permutation group is a classical problem in permutation group theory, with a long tradition and many applications. For finite permutation

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groups, one of the earliest results is a theorem of Bochert [10] from 1889, which states that if G is a primitive permutation group of degree n not containing the alternating group A_n , then $b(G) \leq n/2$. The optimal bound in this situation was obtained by Liebeck [53], showing that $b(G) < 9 \log n$, unless $n = \binom{m}{k}^r$ and G is a subgroup of $S_m \wr S_r$ containing $(A_m)^r$, where A_m acts on k -element subsets of $\{1, \dots, m\}$. The proof of this result relies on the Classification of Finite Simple Groups. By imposing additional conditions on G it is possible to establish stronger bounds. For example, if G is a finite primitive solvable group then a theorem of Seress [69] states that $b(G) \leq 4$.

Bases arise naturally in several different contexts. For example, Bochert's result was motivated by the classical problem of bounding the order of a finite primitive permutation group, which attracted a lot of attention in the 19th century. Here $|G| \leq n^{b(G)}$, so an upper bound on the base size of G yields an upper bound on its order. In more recent years, bases have been used extensively in the computational study of finite permutation groups (see [70, Chapter 4] for further details), whence the problem of calculating base sizes has important practical applications. In the graph-theoretic literature, if Γ is a graph with automorphism group $G = \text{Aut}(\Gamma)$, then $b(G)$ is called the *fixing number* (also *determining number* or *rigidity index*) of Γ and this is a well-studied graph invariant (see [4] and the references therein). In a different direction, some classical problems in the representation theory of groups can also be stated in terms of bases. For instance, if H is a group and V is a faithful H -module, then H has a regular orbit on V if and only if the corresponding affine group $V \rtimes H \leq \text{AGL}(V)$ admits a base of size 2.

Recently, a number of papers have investigated bases for finite non-solvable permutation groups (see [5, 15, 16, 17, 18, 19, 20, 21, 30, 34, 41, 42, 44, 45], for example). One of the central motivations here comes from a conjecture of Cameron and Kantor [23] on finite almost simple primitive groups. The conjecture asserts that there exists an absolute constant c such that $b(G) \leq c$ for all such groups G , excluding a prescribed list of obvious exceptions involving the action of alternating and symmetric groups on subsets and partitions, and also the action of classical groups on subspaces of the natural module. This conjecture was proved by Liebeck and Shalev [60], using probabilistic methods, and more recently it has been shown that $c = 7$ is the best possible constant (see the sequence of papers [15, 16, 19, 20]). More precisely, confirming a conjecture of Cameron [22, p.122], it is known that $b(G) \leq 7$, with equality if and only if $G = M_{24}$ in its 5-transitive action on 24 points. Again, the proof uses probabilistic methods.

In this paper, we initiate the study of bases for infinite permutation groups. At this level, very little is known in general, with the exception of a few special cases. For example, in [34], Goldstein and Guralnick compute the base size for the action of the classical group $\text{PGL}_{2n}(k)$ on the set of cosets of the subgroup $\text{PGSp}_{2n}(k)$, for any field k . Bases for the action of $\text{PGL}_n(k)$ on subspace partitions of the natural module are studied by James [45]. In this paper we conduct a systematic study of bases for primitive actions of simple algebraic groups, motivated in part by the recent advances in our understanding of bases for finite groups of Lie type. Of course, in this context the aforementioned probabilistic methods are no longer available, so we need to develop a new approach and methodology.

As we will see, the unprecedented scope and precision of our results also sheds new light on the study of bases for finite primitive permutation groups. For instance, earlier work shows that there are infinitely many so-called *non-standard* finite primitive groups G with $b(G) = b$ for $2 \leq b \leq 5$, and a unique group with $b(G) > 6$, namely, the Mathieu group $G = M_{24}$ acting on 24 points (see [15, Definition 1.1] for the precise definition of a non-standard group). In this paper, we complete the picture via Theorem 13, which reveals that there are infinitely many with $b(G) = 6$ (only six examples were known previously). More generally, a major project is to determine the exact base size of every finite almost simple primitive group. In [17] and [18] we consider non-subspace actions of finite classical

groups (see [15, Definition 2.1]), and bases for finite exceptional groups will also be the subject of a future paper. In particular, our work is an important contribution to ongoing efforts to classify the finite primitive permutation groups with base size two.

Let G be a (closed) connected affine algebraic group over an algebraically closed field K of characteristic $p \geq 0$. Let Ω be a faithful transitive G -variety with point stabilizer H , so we may identify Ω with the coset variety G/H . We define three base-related measures that arise naturally in this context:

- (i) The *exact base size*, denoted $b(G, H)$, is the smallest integer c such that Ω contains c points with trivial pointwise stabilizer.
- (ii) The *connected base size*, denoted $b^0(G, H)$, is the smallest integer c such that Ω contains c points whose pointwise stabilizer has trivial connected component, i.e. the pointwise stabilizer is finite.
- (iii) The *generic base size*, denoted $b^1(G, H)$, is the smallest integer c such that the product variety $\Omega^c = \Omega \times \cdots \times \Omega$ (c factors) contains a non-empty open subvariety Λ and every c -tuple in Λ is a base for G .

Evidently, we have

$$b^0(G, H) \leq b(G, H) \leq b^1(G, H).$$

Our ultimate goal is to determine these base-related measures for all simple algebraic groups G and all closed maximal subgroups H of G (that is, for all primitive actions of simple algebraic groups). Indeed, we essentially achieve this goal by computing these quantities in almost every case. In the handful of exceptional cases, we give a very narrow range for the possible values. Note that if the context is clear, we will sometimes write $b(G)$, $b^0(G)$ and $b^1(G)$ for the three base measures defined above.

Remark 1. More generally, one can define $b^0(G, X)$, $b(G, X)$ and $b^1(G, X)$ for any affine algebraic group G and irreducible G -variety X . For instance, $b^0(G, X) = b$ (respectively, $b^1(G, X) = b$) if and only if b is minimal such that the product variety X^b contains a non-empty open subvariety X_0 with the property that the stabilizer of a point in X_0 is finite (respectively, trivial). In characteristic 0, these measures have been studied by various authors when G is semisimple and X is a KG -module (see Èlašvili [27, 28] and Popov [63, 64], for example). In addition, there is a connection between stabilizers of points on Grassmannians and stabilizers in certain tensor products of linear representations. We refer the reader to [65] for a very nice survey of results of this nature in characteristic 0. In particular, all cases of irreducible modules for simple algebraic groups where there is a regular orbit have been determined. This has recently been extended to positive characteristic by Guralnick and Lawther [39].

Remark 2. The connected base size is also related to the notion of *subgroup height* appearing in the geometric group theory literature. Following [33], an infinite subgroup H of a group G has *height* n , denoted $\mu(H) = n$, if there exists a collection of n distinct G -conjugates of H whose common intersection is infinite, but the intersection of any $n + 1$ distinct conjugates of H is finite. In particular, if H is core-free then $\Omega = G/H$ is a faithful transitive G -set and $\mu(H) = n$ if there exist n points in Ω whose pointwise stabilizer is infinite, but the stabilizer of any $n + 1$ points is finite. Evidently, if G is an algebraic group with point stabilizer H , then $b^0(G, H) \leq \mu(H) + 1$.

A simple algebraic group G is either classical or exceptional, and there is a dichotomy in our approach. The main theorem on the subgroup structure of classical algebraic groups is due to Aschbacher [1] (see also Liebeck and Seitz [56]). Roughly speaking, a maximal closed positive-dimensional subgroup H of G is either contained in one of five *natural*, or *geometric*, subgroup collections (denoted by $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ and \mathcal{C}_6 in [56]), or the

connected component H^0 is simple (modulo scalars) and acts irreducibly on the natural G -module V (we denote the latter collection by \mathcal{S}). The geometric collections include stabilizers of subspaces of V , and normalizers of appropriate direct sum and tensor product decompositions of V .

In stating our results for a classical group G , we make a distinction between the primitive actions of G in which a point stabilizer H acts reducibly on V , and those in which the stabilizer is irreducible. More precisely, we say that the action of G on Ω is a *subspace action* if one of the following holds:

- (i) Ω is an orbit of subspaces of V ; or
- (ii) the action of G on Ω is equivalent to the action of an isomorphic classical group L on an orbit of subspaces of the natural L -module.

The possibilities that arise in case (ii) are conveniently listed in Table 1. Here the ‘type of H ’ describes the approximate group-theoretic structure of H (this is consistent with the notation used in [56]). In addition, we use the abbreviations ‘n.s.’ and ‘n.d.’ to denote the terms ‘non-singular’ and ‘non-degenerate’, respectively.

It is worth noting that this is the first paper to systematically study the base size of classical groups (finite or infinite) in these natural subspace actions (see Theorem 4). As noted above (see Remark 1), some of our results for subspace actions of linear groups in characteristic 0 are related to earlier work of Èlašvili [27], Popov [63] and others.

G	Type of H	Conditions	Equivalent action
Sp_n	O_n	$p = 2, n \geq 4$	SO_{n+1} on n.s. 1-spaces
SO_8	$\mathrm{Sp}_4 \otimes \mathrm{Sp}_2$	$p \neq 2$	SO_8 on n.d. 3-spaces
SO_8	GL_4		SO_8 on n.d. 2-spaces
SO_8	SO_7	H irreducible, $p \neq 2$	SO_8 on n.d. 1-spaces
SO_8	Sp_6	H irreducible, $p = 2$	SO_8 on n.s. 1-spaces
SL_4	Sp_4		SO_6 on n.d. 1-spaces
Sp_4	$\mathrm{Sp}_2 \wr \mathrm{S}_2$	$p \neq 2$	SO_5 on n.d. 1-spaces
Sp_4	$\mathrm{Sp}_2 \wr \mathrm{S}_2$	$p = 2$	SO_5 on n.s. 1-spaces

TABLE 1. Some subspace actions

There is a similar description of the maximal subgroups of an exceptional algebraic group G , which is due to Liebeck and Seitz [57]. Essentially, a positive-dimensional maximal subgroup of G is either parabolic, or it is of the form $N_G(X)$ for a known reductive subgroup X . Once again, we will make a distinction between parabolic and non-parabolic subgroups.

In order to state our main results, we fix the following notation for the rest of the paper: let K be an algebraically closed field of characteristic $p \geq 0$, let G be a simple (and in particular connected) algebraic group over K and let Ω be a primitive G -variety with positive-dimensional point stabilizer H . We remark that in our results, we can take G to be any version of the simple algebraic group; the center will lie in the kernel of the action of G on Ω , and will be ignored in the statements. In addition, K can be any algebraically closed field of characteristic p (see the end of Section 2 for further comments on the underlying field).

In the statement of Theorem 1 we use the notation P_i to denote the standard maximal parabolic subgroup of G that corresponds to deleting the i -th node from the Dynkin diagram of G , in terms of the standard labelling (see [12, p.250]).

Theorem 1. *Let G be a simple algebraic group over an algebraically closed field and let Ω be a primitive G -variety with point stabilizer H . Assume G is not a classical group in a subspace action. Then $b^1(G, H) \leq 6$, with equality if and only if $(G, H) = (E_7, P_7)$, (E_6, P_1) or (E_6, P_6) .*

Theorem 2. *Let G be a simple classical algebraic group in a primitive non-subspace action with point stabilizer H . Then $b^1(G, H) \leq 4$, with equality if and only if $(G, H) = (\mathrm{SL}_6, \mathrm{Sp}_6)$, (SO_7, G_2) ($p \neq 2$) or (Sp_6, G_2) ($p = 2$).*

We note that Theorem 1 establishes a strong algebraic group analogue of Cameron's conjecture for finite almost simple primitive groups (strong in the sense that we are able to determine all the cases in which the generic base size is exactly 6). Also note that Theorem 2 can be viewed as an algebraic group version of the main theorem of [15].

Theorem 3. *Let G be a simple algebraic group over an algebraically closed field of characteristic $p \neq 2$ and let Ω be a primitive G -variety with point stabilizer H . Then $b(G, H) > 2$ if and only if one of the following holds:*

- (i) $\dim H > \frac{1}{2} \dim G$;
- (ii) $G = \mathrm{SO}_n$ and H is the stabilizer of a d -dimensional non-degenerate subspace of the natural G -module, where $n = 2d + \ell$ with $2 \leq \ell \leq d$ and $\ell^2 \leq n$;
- (iii) $G = \mathrm{SL}_n$ and H is of type $\mathrm{GL}_{n/2} \wr S_2$, where $n \geq 4$;
- (iv) $G = \mathrm{Sp}_6$ and H is of type $\mathrm{Sp}_2 \wr S_3$;
- (v) $G = E_6$ and $H = A_1 A_5$.

Remark 3. As a corollary of Theorem 3, note that if $b(G, H) > 2$ and $\dim H \leq \frac{1}{2} \dim G$, then either H contains a maximal torus of G , or $G = \mathrm{SO}_n$ (with n even) and H is the stabilizer of a d -dimensional non-degenerate subspace (d odd), with d satisfying the conditions in part (ii) of Theorem 3. In addition, in each of these cases we have

$$b^0(G, H) = b(G, H) = b^1(G, H) = 3.$$

We can also state a version of Theorem 3 when $p = 2$. Indeed, if we exclude the cases

$$(G, H) = (\mathrm{SO}_n, O_{n/2} \wr S_2) \text{ (} n/2 \text{ even)}, (E_7, A_7.2), (E_6, A_1 A_5), (G_2, A_1 \tilde{A}_1)$$

then $b(G, H) > 2$ if and only if we are in one of the cases (i) – (iv) in Theorem 3.

Theorems 1 – 3 follow immediately from the detailed results we present in Theorems 4 – 7 below. First assume G is a classical group in a primitive subspace action, so the point stabilizer H fixes a proper non-zero subspace U of the natural G -module V . Note that if G is a symplectic or orthogonal group then primitivity implies that either U is non-degenerate or totally singular with respect to the relevant underlying form on V , or G is orthogonal, $p = 2$ and U is a non-singular 1-space. Without loss of generality, we may assume that $\dim U \leq \frac{1}{2} \dim V$. Our main result on subspace actions is Theorem 4 below (in the statement of this result, $\delta_{i,j}$ denotes the familiar Kronecker delta).

Theorem 4. *Let G be a simple classical algebraic group in a primitive subspace action with point stabilizer $H = G_U$, where $d = \dim U$, $n = \dim V$ and $d \leq n/2$. Set $k = \lceil n/d \rceil$.*

- (i) *Suppose $G = \mathrm{SL}_n$ and $n \geq 2$. If d divides n then*

$$b^0(G, H) = b(G, H) = b^1(G, H) = k + \epsilon,$$

where

$$\epsilon = \begin{cases} 3 & \text{if } 1 < d = n/2 \\ 2 & \text{if } 1 < d < n/2 \\ 1 & \text{if } d = 1. \end{cases}$$

Otherwise, if d does not divide n then

$$k + 1 \leq b^0(G, H) = b(G, H) = b^1(G, H) \leq k + 2 + \delta_{3,k}.$$

(ii) Suppose $G = \mathrm{Sp}_n$ and $n \geq 4$. Then either

$$b^0(G, H) = b(G, H) = b^1(G, H) = k$$

or one of the following holds:

(a) $n = 6$, $d = 2$ and $b^0(G, H) = b(G, H) = b^1(G, H) = 4$;

(b) U is totally singular, $d = n/2$, $b^0(G, H) = b(G, H) = 4$ and $b^1(G, H) = 5 - \delta_{2,p}$;

(c) $H = O_n$, $p = 2$, $b^0(G, H) = b(G, H) = n$ and $b^1(G, H) = n + 1$.

(iii) Suppose $G = \mathrm{SO}_n$ and $n \geq 7$, with $p \neq 2$ if n is odd. Then either

$$b^0(G, H) = b(G, H) = b^1(G, H) = k$$

or one of the following holds:

(a) $n = (k - 1)d + 1$ (with $k \geq 4$ if U is totally singular), $b^0(G, H) = b(G, H) = k - 1$ and $b^1(G, H) = k - \epsilon$, where $\epsilon = 1$ if n is even, otherwise $\epsilon = 0$;

(b) U is totally singular, $d = n/2$, $n \neq 10$ and

$$b^0(G, H) = b(G, H) = b^1(G, H) = c(n),$$

where $c(8) = 7$, $c(12) = 6$ and $c(n) = 5$ for all $n \geq 14$;

(c) U is totally singular, $n = 10$, $d = 5$ and $5 \leq b^0(G, H) \leq b^1(G, H) \leq 6$;

(d) U is totally singular, $k = 3$ and $b^0(G, H) = b(G, H) = b^1(G, H) = 4 - \delta_{n,3d}$.

The next result deals with the non-subspace actions of classical groups.

Theorem 5. *Let G be a simple classical algebraic group in a primitive non-subspace action with point stabilizer H . Then one of the following holds:*

- (i) $b^0(G, H) = b(G, H) = b^1(G, H) = 2$;
- (ii) $b^0(G, H) = b(G, H) = b^1(G, H) = b > 2$ and (G, H, b) is recorded in Table 2;
- (iii) $b^0(G, H) = b(G, H) = 2$, $b^1(G, H) = 3$ and either $G = \mathrm{SL}_2$ and H is of type $\mathrm{GL}_1 \wr S_2$, or $p \neq 2$ and $(G, H) = (\mathrm{SL}_n, \mathrm{SO}_n)$, $(\mathrm{Sp}_n, \mathrm{GL}_{n/2})$ or $(\mathrm{SO}_n, O_{n/2} \wr S_2)$;
- (iv) $2 = b^0(G, H) \leq b(G, H) \leq b^1(G, H) = 3$, $p = 2$ and $(G, H) = (\mathrm{SO}_n, O_{n/2} \wr S_2)$, where $n \equiv 0 \pmod{4}$ and $n \geq 8$.

G	Type of H	Conditions	b
SL_n	$\mathrm{GL}_{n/2} \wr S_2$	$n \geq 4$	3
	Sp_n	$n = 6$	4
	Sp_n	$n \geq 8$	3
Sp_n	$\mathrm{Sp}_{n/2} \wr S_2$	$n \geq 8$	3
	$\mathrm{Sp}_{n/3} \wr S_3$	$n = 6$	3
	G_2	$(n, p) = (6, 2)$	4
SO_n	$\mathrm{GL}_{n/2}$	$n \geq 10$	3
	G_2	$n = 7, p \neq 2$	4

TABLE 2. Values of b in Theorem 5(ii)

In the next two theorems we present our results for parabolic and non-parabolic actions of exceptional algebraic groups, respectively.

Theorem 6. *Let G be a simple exceptional algebraic group and let $\Omega = G/H$, where $H = P_i$ is a maximal parabolic subgroup of G . Then*

$$c - \epsilon \leq b^0(G, H) \leq b(G, H) \leq b^1(G, H) \leq c,$$

where c is defined in Table 3. Here an asterisk indicates that $\epsilon = 1$, otherwise $\epsilon = 0$ and thus $b^0(G, H) = b(G, H) = b^1(G, H) = c$.

	$H = P_1$	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$G = E_8$	4	3	3	3	3	3	4	5
E_7	5	4	4	3	3	4	6	
E_6	6	5	4	4*	4	6		
F_4	5*	4*	4*	5*				
G_2	4*	4*						

TABLE 3. G exceptional, H parabolic

Theorem 7. *Let G be a simple exceptional algebraic group in a primitive non-parabolic action with point stabilizer H . Then one of the following holds:*

- (i) $b^0(G, H) = b(G, H) = b^1(G, H) = 2$;
- (ii) $b^0(G, H) = b(G, H) = b^1(G, H) = b > 2$ and (G, H, b) is recorded in Table 4;
- (iii) $b^0(G, H) = b(G, H) = 2$, $b^1(G, H) = 3$, $p \neq 2$ and

$$(G, H^0) = (E_8, D_8), (E_7, A_7), (E_6, C_4), (F_4, A_1C_3) \text{ or } (G_2, A_1\tilde{A}_1);$$

- (iv) $2 = b^0(G, H) \leq b(G, H) \leq b^1(G, H) \leq 3$, $p = 2$ and

$$(G, H^0) = (E_7, A_7), (E_6, A_1A_5) \text{ or } (G_2, A_1\tilde{A}_1).$$

G	H^0	Conditions	b
E_8	A_1E_7		3
E_7	A_1D_6		3
	T_1E_6		3
E_6	F_4		4
	D_5T_1		3
	A_1A_5	$p \neq 2$	3
F_4	B_4		4
	C_4	$p = 2$	4
	D_4		3
	\tilde{D}_4	$p = 2$	3
G_2	A_2		3
	\tilde{A}_2	$p = 3$	3

TABLE 4. Values of b in Theorem 7(ii)

We outline the idea behind the proof. Let G be an algebraic group over an algebraically closed field of characteristic $p \geq 0$, and let $\Omega = G/H$ be a faithful transitive G -variety, where H is a closed subgroup of G . Let $c \geq 2$ be an integer. The expression

$$\mathcal{Q}(G, c) = \frac{c}{c-1} \cdot \sup_{x \in \mathcal{P}} \left\{ \frac{\dim(x^G \cap H)}{\dim x^G} \right\}$$

will play a central role, where \mathcal{P} denotes the set of elements of prime order in H (including all nontrivial unipotent elements if $p = 0$) and x^G is the conjugacy class of x in G . In Theorem 2.13 we prove that if G is simple and H^0 is reductive, then $b^1(G, H) \leq c$ if $\mathcal{Q}(G, c) < 1$. This result is an essential tool in our analysis. Bounds on $\dim(x^G \cap H)$ in terms of $\dim x^G$ are obtained for classical groups in [13] (for H irreducible), and in [51] for exceptional groups, so we can compute good estimates for $\mathcal{Q}(G, c)$.

In order to obtain precise results, we require a lower bound on $b^0(G, H)$. By definition, if $b = b^0(G, H)$ then the product variety Ω^b contains a G -orbit of dimension $\dim G$, whence $\dim G \leq \dim \Omega^b = b \cdot \dim \Omega$ and thus

$$b^0(G, H) \geq \frac{\dim G}{\dim \Omega} = \frac{\dim G}{\dim G - \dim H}.$$

It turns out that this lower bound, combined with analysis of $\mathcal{Q}(G, c)$, is effective in most cases. However, we sometimes encounter problems if $\dim G / \dim \Omega = c - \epsilon$ for some integer c and small positive number ϵ (with $\epsilon < 1/10$, for example). Frequently, in such a situation, the usual analysis yields

$$c \leq b^0(G, H) \leq b^1(G, H) \leq c + 1$$

and thus further work is needed to determine the precise base size. The case $c = 2$ is particularly interesting because such a subgroup H often arises as the centralizer of an involution in $\text{Aut}(G)$ (at least when $p \neq 2$). Rather surprisingly, we find that the base size in this situation is determined by whether or not the relevant involution inverts a maximal torus of G .

Theorem 8. *Let G be a simple algebraic group of rank r over an algebraically closed field of characteristic $p \neq 2$. Let $H = C_G(\tau)$, where $\tau \in \text{Aut}(G)$ is an involution, and let $\Omega = G/H$ be the corresponding coset variety. If τ inverts a maximal torus of G then*

$$b^0(G, H) = b(G, H) = 2, \quad b^1(G, H) = 3,$$

otherwise $b^0(G, H) \geq 3$. More precisely, if τ inverts a maximal torus then the following hold:

- (i) H has a unique regular orbit on Ω .
- (ii) The generic 2-point stabilizer has order 2^r (more precisely is the 2-torsion subgroup of a maximal torus). That is, there exists a non-empty open subvariety U of $\Omega \times \Omega$ such that $|G_\alpha \cap G_\beta| = 2^r$ for all $(\alpha, \beta) \in U$.
- (iii) If $G < A \leq \text{Aut}(G)$ then A acts on Ω , $b^0(A, C_A(\tau)) = 2$ and $b(A, C_A(\tau)) = b^1(A, C_A(\tau)) = 3$.

As a special case of Theorem 8, we deduce that if $G = A_1$, $p \neq 2$ and $H = N_G(T)$ is the normalizer of a maximal torus of G , then the generic 2-point stabilizer has order 2. In general, if G is simple and H is the normalizer of a maximal torus then the generic 2-point stabilizer is trivial.

Theorem 9. *Let G be a simple algebraic group over an algebraically closed field and consider the action of G on $\Omega = G/H$, where H is the normalizer of a maximal torus of G . Then either $b^1(G, H) = 2$, or $G = A_1$ and the generic 2-point stabilizer has order 2.*

Theorem 9 answers a question posed by Zinovy Reichstein (personal communication), and the proof is an easy application of Theorem 2.13 (see Section 6 for the details). This result has the following corollary, where $\text{ed}(H)$ denotes the *essential dimension* of an algebraic group H . We refer the reader to Reichstein's ICM survey article [67] for further details and references.

Corollary 10. *Let G be a simple algebraic group of adjoint type over an algebraically closed field with G of rank $r \geq 2$. Let H be the normalizer of a maximal torus of G . Assume that the center of the Lie algebra $\text{Lie}(G)$ is trivial. Then*

$$\text{ed}(G) \leq \text{ed}(H) \leq \dim G - 2r.$$

Lemire [52] proved this in characteristic 0 (the first inequality is well known and follows from results of Springer – see [66, Proposition 4.3]). The point is that the action of H on G/H is generically free (essentially by Theorem 9) and is known to be versal [26, Example 7.3(c)], which gives the desired bound. A sketch proof is given in Section 6, and we refer the reader to [26, 67] for more details and generalities about essential dimension.

Let G be an algebraic group acting on an irreducible variety X . Suppose there is a non-empty open subvariety X_0 of X such that the stabilizer G_x has a certain property \mathcal{P} for all $x \in X_0$. In this situation, we say that a *generic stabilizer* in G has property \mathcal{P} . For example, notice that if there is at least one point $x \in X$ such that G_x is finite, then a generic stabilizer is finite (more generally, the generic stabilizers will have a fixed dimension). In particular, if $X = G/H$ is a faithful transitive G -variety, then $b^0(G, H) \leq b$ (respectively, $b^1(G, H) \leq b$) if and only if the generic stabilizer of a point in X^b is finite (respectively, trivial). In almost every case where G is simple and $X = G/H$ is primitive, we show that if $b^0(G, H) = b$ then the generic stabilizers of points in X^b form a single conjugacy class of subgroups of H .

Let G be a simple algebraic group over an algebraically closed field K and let V be a rational finite dimensional KG -module. In characteristic 0, the existence of a generic stabilizer is a nontrivial theorem under suitable hypotheses (that is, there is a non-empty open subvariety of V such that the stabilizer of each vector in this subvariety belongs to a fixed conjugacy class of subgroups). See for example [65, Chapter 7] and [68]. Rather less is known in the positive characteristic setting, although a recent theorem of Guralnick and Lawther [39] guarantees the existence of generic stabilizers when G is simple and V is irreducible. Suppose that H is the generic stabilizer in G of a vector in V (up to conjugacy), let c be a positive integer and let W be the direct sum of c copies of V . The G -stabilizer of a vector in an open subvariety of W is just the intersection of c G -conjugates of H . In particular, if $b^1(G, H) \leq c$ (in terms of the action of G on G/H) then this generic stabilizer is trivial. Similarly, if $b^0(G, H) \leq c$ then the generic stabilizer is finite (and if $b(G, H) \leq c$, then some stabilizer is trivial).

Remark 4. Related problems of this nature have been studied extensively in characteristic 0 (and hence also for large positive characteristic p). Indeed, work of Èlašvili [27] and Popov [63] provides a complete classification of the rational KG -modules V of a simple algebraic group G with the property that a generic stabilizer is finite (respectively, trivial). We refer the reader to [28, 64, 65] for further results (again, in characteristic 0) for semisimple algebraic groups G and irreducible KG -modules. We thank an anonymous referee for drawing our attention to this important earlier work.

The interesting situation where the generic stabilizer is finite, but nontrivial, also comes up extensively in recent work of Bhargava and coauthors (typically in characteristic 0, but the analogous results in positive characteristic should lead to results concerning function fields over finite fields) – see [6, 7, 8, 9].

Here we record some special cases that follow immediately from our earlier results.

Corollary 11. *Let G be a simple algebraic group over an algebraically closed field K of characteristic $p \geq 0$.*

- (i) *If $p \neq 2$, $G = \mathrm{SL}(U)$ and $W = \mathrm{Sym}^2(U) \oplus \mathrm{Sym}^2(U)$, then the generic stabilizer of a vector in W is a finite nontrivial elementary abelian 2-group.*
- (ii) *Let $G = E_6$ and $W = V \oplus V \oplus V \oplus V$, where V is an irreducible KG -module of dimension 27. Then the generic stabilizer of a vector in W is trivial.*
- (iii) *Let $G = G_2$ and $W = V \oplus V \oplus V$, where $p \neq 2$ and V is an irreducible KG -module of dimension 7. Then the generic stabilizer of a vector in W is trivial.*

Let us also give an example for a tensor product action. Suppose that G_1, G_2 are subgroups of $\mathrm{GL}(V)$, and consider the natural action of $G = G_1 \otimes G_2$ on $W = V \otimes V$. It is straightforward to see that the generic stabilizer of a vector in W is the intersection of generic conjugates of G_1 and G_2 . In particular, we obtain the following corollary:

Corollary 12. *Consider the natural action of $G = \mathrm{SO}(V) \otimes \mathrm{SO}(V)$ on $W = V \otimes V$. If $p \neq 2$ then the generic stabilizer of a vector in W is a finite nontrivial elementary abelian 2-group.*

Our results for algebraic groups have interesting consequences for the corresponding finite groups of Lie type. Let us briefly recall the general set-up. Let p be a prime, let G be a simple algebraic group over the algebraic closure \mathbb{F}_p of the prime field \mathbb{F}_p , and let σ be a Frobenius morphism of G such that the set of fixed points G_σ is a finite group of Lie type over \mathbb{F}_q , for some p -power q . If H is a closed positive-dimensional σ -stable subgroup of G then we can consider the action of G_σ on the set of cosets of H_σ in G_σ . We write $b(G_\sigma, H_\sigma)$ for the base size of G_σ in this action.

For a positive integer c , let $P(G_\sigma, c)$ be the probability that c randomly chosen points in G_σ/H_σ form a base for G_σ . We define the *asymptotic base size* of G_σ , denoted by $b^\infty(G_\sigma, H_\sigma)$, to be the smallest value of c such that $P(G_\sigma, c)$ tends to 1 as q tends to infinity. With this set-up, there are five base-related numbers to consider:

$$b(G, H), b^0(G, H), b^1(G, H), b(G_\sigma, H_\sigma), b^\infty(G_\sigma, H_\sigma).$$

In Section 2 of this paper we investigate various relations between these base measures. For example, in Proposition 2.7 we use the Lang-Weil estimates to prove that the asymptotic base size of G_σ coincides with the generic base size of G . We also show that $b^0(G, H) \leq b(G_\sigma, H_\sigma)$ if $q > 2$. In view of Theorem 2, the former observation implies that if G is a classical group in a suitable non-subspace action then $b^\infty(G_\sigma, H_\sigma) \leq 3$ if $\dim V > 7$, where V is the natural G -module. See [59, Theorem 1.11] for a similar result, requiring the stronger condition $\dim V > 15$. Similarly, if G is an exceptional algebraic group then using Theorems 6 and 7 we can compute the precise asymptotic base size $b^\infty(G_\sigma, H_\sigma)$ in almost all cases; this is a significant strengthening of the general estimate $b^\infty(G_\sigma, H_\sigma) \leq 6$ stated in [19, Theorem 2].

Recall that if G is a *non-standard* finite almost simple primitive permutation group with point stabilizer H (so G is not an alternating or symmetric group acting on subsets or partitions, nor a classical group in a subspace action) then $b(G) \leq 7$, with equality if and only if $G = \mathrm{M}_{24}$ in its 5-transitive action on 24 points. The main theorem of [19] reveals that there are infinitely many non-standard groups G with $b(G) = 5$, but it is not known whether or not there are infinitely many with $b(G) = 6$. Indeed, to date the only known examples (G, H) with $b(G, H) = 6$ are the following:

$$(E_6(2), P_1), (E_6(2), P_6), (\mathrm{M}_{23}, \mathrm{M}_{22}), (\mathrm{Co}_3, \mathrm{McL}.2), (\mathrm{Co}_2, \mathrm{U}_6(2).2), (\mathrm{Fi}_{22}.2, 2.\mathrm{U}_6(2).2).$$

Now, according to Theorem 6 we have $b^0(G, H) = 6$ if

$$(G, H) \in \{(E_6, P_1), (E_6, P_6), (E_7, P_7)\}.$$

Therefore, if $q > 2$ we deduce that $b(G_\sigma, H_\sigma) \geq 6$ for the corresponding primitive actions of $G_\sigma = E_6(q)$ and $E_7(q)$. Now the main theorem of [19] yields $b(G_\sigma, H_\sigma) \leq 6$, so $b(G_\sigma, H_\sigma) = 6$ for all $q > 2$ and we conclude that there are infinitely many non-standard primitive groups with base size 6 (see Remark 5.6).

Theorem 13. *There are infinitely many non-standard finite almost simple primitive permutation groups G with $b(G) = 6$.*

Finally, we make some remarks on the organisation of the paper. In Section 2 we present a number of preliminary results that we need for the proof of our main theorems. Two key results here are Proposition 2.5 and Theorem 2.13, which provide effective lower and upper bounds on the base measures $b^0(G, H)$ and $b^1(G, H)$, respectively. By considering the fixed points of a Frobenius morphism σ , we also investigate the connection between the base sizes of the algebraic group G and the corresponding finite group G_σ ; see Proposition 2.7. Next, in Section 3 we consider the special case where $H = C_G(\tau)$ for some involution $\tau \in \text{Aut}(G)$ (with $p \neq 2$), proving Theorem 8. The next two sections of the paper deal with the remaining primitive actions of classical and exceptional algebraic groups, respectively, and we complete the proofs of Theorems 4 – 7. In Section 4 we make a distinction between subspace and non-subspace actions of classical groups; subspace actions are handled in Section 4.1, and the remaining possibilities are considered in Section 4.2. Similarly, in Section 5 we distinguish between parabolic and non-parabolic actions of exceptional algebraic groups. Finally, in Section 6 we establish Theorem 9 and we sketch the proof of Corollary 10.

2. PRELIMINARIES

In this section we record a number of preliminary results that we will need in the proof of our main theorems. Throughout this section, unless stated otherwise, the terms ‘variety’ and ‘algebraic group’ refer respectively to an algebraic variety and an affine algebraic group defined over an algebraically closed field K of characteristic $p \geq 0$. We begin with two elementary results on fibers of morphisms. The first result is well known.

Lemma 2.1. *Let $\phi : X \rightarrow Y$ be a morphism of irreducible varieties. Then there exists a non-empty open subvariety U of $\overline{\phi(X)}$ such that each fiber $\phi^{-1}(u)$ has the same dimension for all $u \in U$. Moreover, if $u \in U$ then $\dim \phi^{-1}(u) \leq \dim \phi^{-1}(v)$ for all $v \in \phi(X)$.*

Lemma 2.2. *Let $\phi : X \rightarrow Y$ be a dominant morphism of irreducible varieties such that $\phi^{-1}(y)$ is non-empty and finite for some $y \in Y$. Then there exists a non-empty open subvariety U of Y , and a positive integer n , such that $|\phi^{-1}(u)| = n$ for all $u \in U$.*

Proof. This is also well known (see [36, Corollaire 4]), but we give a proof for completeness. First observe that $\dim X = \dim Y$ by Lemma 2.1, so $K(X)/K(Y)$ is a finite algebraic field extension, of degree δ say. Then by [71, Theorem 5.1.6(iii)], we can take n to be the separable degree of $[K(X) : K(Y)]$. \square

Lemma 2.3. *Let G be an algebraic group, let Ω be an irreducible G -variety and let $\Gamma = \Omega \times \cdots \times \Omega$ with $c \geq 1$ factors. Set*

$$\mu = \min \left\{ \dim \left(\bigcap_{i=1}^c G_{\alpha_i} \right) \mid (\alpha_1, \dots, \alpha_c) \in \Gamma \right\}.$$

- (i) The subset $\{(\alpha_1, \dots, \alpha_c) \in \Gamma \mid \dim(\bigcap_i G_{\alpha_i}) = \mu\}$ contains a non-empty open subvariety of Γ .
- (ii) If $\mu = 0$ then there exists a non-empty open subvariety U of Γ , and a positive integer n , such that $|\bigcap_i G_{\alpha_i}| \leq n$ for all $(\alpha_1, \dots, \alpha_c) \in U$.

Proof. We may assume G is connected. Consider the morphism of irreducible varieties $\phi : G \times \Gamma \rightarrow \Gamma \times \Gamma$ defined by

$$\phi : (g, \alpha_1, \dots, \alpha_c) \mapsto (g\alpha_1, \dots, g\alpha_c, \alpha_1, \dots, \alpha_c).$$

If $z = (g\alpha_1, \dots, g\alpha_c, \alpha_1, \dots, \alpha_c) \in \text{im}(\phi)$ then the fiber $\phi^{-1}(z)$ is isomorphic to $\bigcap_i G_{\alpha_i}$. Therefore Lemma 2.1 implies that there exists a non-empty open subvariety U of $\phi(G \times \Gamma)$ such that $\dim \phi^{-1}(z) = \mu$ for all $z \in U$. Part (i) now follows since ϕ maps onto the second Γ factor, and part (ii) follows immediately from Lemma 2.2. \square

Lemma 2.4. *Let G be an algebraic group and let X, Y be faithful irreducible G -varieties. Suppose there exists a non-empty open subvariety U of X such that G_u is finite for all $u \in U$. Then with respect to the induced action of G on $\Gamma = X \times Y$, there exists a non-empty open subvariety V of Γ such that G_v is trivial for all $v \in V$.*

Proof. Replacing X by a suitable non-empty open subvariety, we may assume that there is an integer n such that $|G_x| = n$ for all $x \in X$. Fix $x \in X$ and set $L = G_x$. For $y \in Y$, the G -stabilizer of $(x, y) \in \Gamma$ is $L \cap J$, where $J = G_y$. Suppose $z \in L$ is nontrivial and let $C_\Gamma(z)$ denote the set of fixed points of z on Γ . Then $C_\Gamma(z)$ is a proper closed subvariety of Γ (since G acts faithfully on X and Y), so the finite union $\bigcup_{1 \neq z \in L} C_\Gamma(z)$ is also contained in a proper closed subvariety of Γ . Therefore, for each $x \in X$, the set

$$\{(x, y) \mid y \in Y, G_x \cap G_y \neq 1\}$$

is contained in a proper closed subvariety of $\{x\} \times Y$.

Let $\pi : X \times Y \rightarrow X$ be the projection map and set $W = \{(x, y) \in \Gamma \mid G_x \cap G_y \neq 1\}$. For each $x \in X$ we have $\pi^{-1}(x) = \{x\} \times Y$, so by the above argument we deduce that $\dim(W \cap \pi^{-1}(x)) < \dim Y$. Therefore

$$\dim W \leq \dim X + \dim(W \cap \pi^{-1}(x)) < \dim X + \dim Y = \dim \Gamma,$$

hence $V = \Gamma \setminus \overline{W}$ is a non-empty open subvariety such that $G_v = 1$ for all $v \in V$. \square

Let G be a connected algebraic group and let Ω be a faithful transitive G -variety with point stabilizer H . Let $b(G, H)$ (or just $b(G)$ if the context is clear) denote the *base size* of the action of G on Ω , so $b(G, H)$ is the minimal integer c such that Ω contains c points with the property that their pointwise stabilizer in G is trivial. As advertised in the Introduction, we will also study two new base-related measures, which are defined as follows:

- (i) The *connected base size*, denoted $b^0(G, H)$, is the smallest integer c such that Ω contains c points whose pointwise stabilizer has trivial connected component, i.e. the pointwise stabilizer is finite.
- (ii) The *generic base size*, denoted $b^1(G, H)$, is the smallest integer c such that the product variety $\Omega^c = \Omega \times \dots \times \Omega$ (c factors) contains a non-empty open subvariety Λ and every c -tuple in Λ is a base for G .

From the definitions, it is clear that

$$b^0(G, H) \leq b(G, H) \leq b^1(G, H).$$

The next result records some additional properties of these base measures.

Proposition 2.5. *Let G be a connected algebraic group and let $\Omega = G/H$ be a transitive faithful G -variety. Then the following hold:*

- (i) $b^0(G, H) \leq \dim H + 1$.
- (ii) *If $b^0(G, H) = c$ then there exists a non-empty open subvariety U of Ω^c such that $|\bigcap_i G_{\alpha_i}|$ is finite for all $(\alpha_1, \dots, \alpha_c) \in U$.*
- (iii) $b^0(G, H) \geq \dim G / \dim \Omega$.
- (iv) $b^1(G, H) \leq b^0(G, H) + 1$.
- (v) *If H is finite and nontrivial then $b^0(G, H) = 1$ and $b(G, H) = b^1(G, H) = 2$.*

Proof. First consider (i). We may assume H is positive-dimensional. Let H_j be the intersection of j distinct G -conjugates of H and assume $\dim H_j > 0$. Let K_j denote the connected component of H_j . We claim that there exists an intersection $H_{j+1} \leq H_j$ of $j+1$ conjugates of H such that $\dim H_{j+1} < \dim H_j$. If not, then $K_j = K_j \cap H^g$ for all $g \in G$, which implies that K_j is a positive-dimensional normal subgroup of G contained in H . This is a contradiction since H is core-free. It follows that there is a chain of subgroups

$$H = H_1 > H_2 > \dots > H_m,$$

where each H_j is an intersection of j conjugates of H , H_m is finite and $\dim H_{j+1} < \dim H_j$ for all j . Therefore $b^0(G, H) \leq m$ and the bound in (i) follows since the dimension drops by at least 1 at each stage of the above chain. For the remainder set $b^0(G, H) = c$.

Part (ii) follows immediately from Lemma 2.3(ii), and part (iii) is an easy consequence of the fact that G has an orbit of dimension $\dim G$ on the product variety $\Omega^{b^0(G, H)}$. For part (iv), consider the induced action of G on $X \times Y$, where $X = \Omega^c$ and $Y = \Omega$. By applying part (ii) (with respect to the action of G on X) it follows that the hypotheses of Lemma 2.4 are satisfied, whence $b^1(G, H) \leq c + 1$ as required. Finally, part (v) follows immediately from (iv). \square

Remark 2.6. Let G be a simple algebraic group over an algebraically closed field of characteristic $p \geq 0$ and suppose $\Omega = G/H$ is a primitive G -variety.

- (i) There are examples with $b^0(G, H) > \lceil \dim G / \dim \Omega \rceil$. For instance, if $G = E_6$, $H = A_1 A_5$ and $p \neq 2$ then $\dim G = 78$, $\dim H = 38$ and $b^0(G, H) = 3$ (see Lemma 3.19). Similarly, if $G = \mathrm{Sp}_6$ and H is of type $\mathrm{Sp}_2 \wr S_3$ then $\dim G = 21$, $\dim H = 9$ and $b^0(G, H) = 3$ (see Section 4.2).
- (ii) By inspecting the proof of Theorems 4 – 7, we observe that $b^0(G, H) = b(G, H)$ in almost all cases. Indeed, the only known exceptions are the cases with H finite.

Let G be a simple algebraic group over the algebraic closure of \mathbb{F}_p , where p is a prime, and let $\Omega = G/H$ be a faithful G -variety. Let $\sigma : G \rightarrow G$ be a Frobenius morphism of G , so the set of fixed points G_σ is a finite group of Lie type over \mathbb{F}_q for some p -power q . Assume H is σ -stable. Then the action of G on Ω induces an action of G_σ on G_σ/H_σ , and we write $b(G_\sigma, H_\sigma)$ for the corresponding base size. In addition, let $b^\infty(G_\sigma, H_\sigma)$ be the *asymptotic base size* of G_σ , which is the smallest integer c such that $P(G_\sigma, c)$ tends to 1 as q tends to infinity, where $P(G_\sigma, c)$ is the probability that c randomly chosen elements of G_σ/H_σ form a base for G_σ .

By definition, if $b^\infty(G_\sigma, H_\sigma) = c$ and q is sufficiently large then almost every c -tuple of points in G_σ/H_σ forms a base for G_σ . Notice that the generic base size $b^1(G, H)$ of G captures this asymptotic property at the algebraic group level, in the sense that if $b^1(G, H) = c$ then there exists a dense subset Λ of Ω^c such that every c -tuple in Λ is a base for G . Part (i) of the next result reveals that the generic and asymptotic base sizes do indeed coincide.

Proposition 2.7. *With the notation established, the following hold:*

- (i) $b^\infty(G_\sigma, H_\sigma) = b^1(G, H)$.
- (ii) *If $q > 2$ then $b^0(G, H) \leq b(G_\sigma, H_\sigma)$.*
- (iii) *If q is sufficiently large then $b(G_\sigma, H_\sigma) \leq b^\infty(G_\sigma, H_\sigma)$.*

Proof. First consider (i). Suppose $b^1(G, H) = c$ and set $\Gamma = \Omega^c$ and

$$\Lambda = \{(\alpha_1, \dots, \alpha_c) \in \Gamma \mid \bigcap_i G_{\alpha_i} \neq 1\},$$

so Λ is contained in a proper closed subvariety of Γ . Let $\Lambda(q)$ and $\Gamma(q)$ denote the set of \mathbb{F}_q -rational points in Λ and Γ , respectively. By considering the Lang-Weil estimates [46], we deduce that the ratio

$$1 - P(G_\sigma, c) = \frac{|\Lambda(q)|}{|\Gamma(q)|} \approx q^{\dim \Lambda - \dim \Gamma}$$

tends to zero as q tends to infinity, whence $b^\infty(G_\sigma, H_\sigma) \leq b^1(G, H)$. A similar argument shows that $b^1(G, H) \leq b^\infty(G_\sigma, H_\sigma)$, hence equality holds.

Next consider (ii). Suppose $b(G_\sigma, H_\sigma) = c$ and $b^0(G, H) > c$. Fix distinct points $\alpha_1, \dots, \alpha_c$ in Ω and set $L = \bigcap_i G_{\alpha_i}$. Since $b^0(G, H) > c$, the connected component L^0 is infinite, whence the hypothesis $q > 2$ implies that $(L^0)_\sigma$ is nontrivial (see [34, Proposition 8.1]). In particular, the stabilizer in G_σ of any c points in G_σ/H_σ is nontrivial. This is a contradiction, hence (ii) follows.

Finally, note that if q is sufficiently large then $P(G_\sigma, c) > 0$, where $c = b^\infty(G_\sigma, H_\sigma)$, so G_σ admits a base of size c and (iii) follows. \square

Remark 2.8. There are examples with $b(G_\sigma, H_\sigma) < b^\infty(G_\sigma, H_\sigma)$ for all values of q . For example, if nq is odd, $G_\sigma = \mathrm{PGL}_n(q)$ and H is of type $O_n(q)$ then $b(G_\sigma, H_\sigma) = 2$ and $b^\infty(G_\sigma, H_\sigma) = 3$ (see [18]).

The main goal of this paper is to determine the three base measures $b^0(G)$, $b(G)$ and $b^1(G)$ for every primitive action of a simple algebraic group G . Of course, if $b^0(G) \geq c$ and $b^1(G) \leq c$ for an integer c , then we immediately deduce that

$$b^0(G) = b(G) = b^1(G) = c.$$

Therefore, our initial aim is to obtain accurate lower and upper bounds on $b^0(G)$ and $b^1(G)$, respectively. In Proposition 2.5(iii) we established a useful lower bound on $b^0(G)$, so let us consider the generic base size $b^1(G)$. Our main result is Theorem 2.13 below, which provides an effective upper bound on $b^1(G, H)$ in terms of the dimensions of some specific conjugacy classes in G and H , assuming that H^0 is reductive. In order to prove this key theorem, we require a couple of preliminary results.

In [62], Lusztig proved that a simple algebraic group contains only finitely many conjugacy classes of unipotent elements. We require the following extension to algebraic groups with reductive connected component.

Lemma 2.9. *Let G be an algebraic group with G^0 reductive. Then there are only finitely many unipotent classes in G , and only finitely many conjugacy classes of elements of a given finite order.*

Proof. This is the main theorem of [38]. \square

Proposition 2.10. *Let G be an algebraic group with G^0 reductive, and let Ω be an irreducible G -variety. Let \mathcal{C} be the set of conjugacy classes of G containing elements of prime*

order (or arbitrary nontrivial unipotent elements if $p = 0$) and set $\Lambda = \bigcup_{C \in \mathcal{C}} \Omega(C)$, where

$$\Omega(C) = \bigcup_{x \in C} C_\Omega(x)$$

and $C_\Omega(x) = \{\alpha \in \Omega \mid x\alpha = \alpha\}$ is the fixed point space of x . Then either Λ is contained in a proper closed subvariety of Ω , or $\Omega(C)$ contains a non-empty open subvariety of Ω for some $C \in \mathcal{C}$.

Proof. Suppose Λ is not contained in a proper closed subvariety of Ω , so Λ is dense in Ω . It suffices to show that $\Omega(C)$ is dense in Ω for some $C \in \mathcal{C}$: if $x \in C$ then $C_\Omega(x)$ is closed and the morphism $\phi : G \times C_\Omega(x) \rightarrow \Omega$ defined by $\phi(g, \alpha) = g\alpha$ has image $\Omega(C)$, so $\Omega(C)$ contains a non-empty open subvariety of $\overline{\Omega(C)} = \Omega$. Set $m = \min\{\dim G_\alpha \mid \alpha \in \Omega\}$ and note that $\Lambda = \{\alpha \in \Omega \mid G_\alpha \neq 1\}$.

If $m > 0$ then every $\alpha \in \Omega$ is fixed by a torus or a unipotent subgroup of G , and so either by an element of order $2 + \delta_{2,p}$, or a unipotent element (of order p if $p > 0$). By Lemma 2.9, there are only finitely many G -classes of such elements, say C_1, \dots, C_r , whence $\Omega = \bigcup_{i=1}^r \Omega(C_i)$ and the irreducibility of Ω implies that $\Omega(C_i)$ is dense in Ω for some i .

Finally, suppose $m = 0$. By Lemma 2.3, there exists a non-empty open subvariety U of Ω and a positive integer n such that $|G_u| \leq n$ for all $u \in U$. If $n = 1$ then $U \subseteq \Omega \setminus \Lambda$, which contradicts our initial assumption. Therefore $n > 1$ and thus $U = \bigcup_{i=1}^s \Omega(C_i)$ for some G -classes C_i of elements of prime order dividing n . Since U is irreducible we deduce that $\Omega = \overline{U} = \overline{\Omega(C_i)}$ for some i , as required. \square

Corollary 2.11. *Let G be a simple algebraic group and let $\Omega = G/H$ be a transitive G -variety with H^0 reductive. Let $c \geq 2$ be an integer and let Γ be the irreducible G -variety Ω^{c-1} . If*

$$\dim \Gamma(C) < \dim \Gamma$$

for every H -class C of elements of prime order in H (including all nontrivial unipotent elements if $p = 0$) then $b^1(G, H) \leq c$.

Proof. Consider the action of H on Γ . By Proposition 2.10, the hypothesis $\dim \Gamma(C) < \dim \Gamma$ for all relevant H -classes C implies that $\{\alpha \in \Gamma \mid H_\alpha = 1\}$ contains a non-empty open subvariety of Γ . We conclude that $b^1(G, H) \leq c$. \square

Lemma 2.12. *Let G be an algebraic group, let H be a closed subgroup of G and let $\Omega = G/H$. Then for $x \in H$,*

$$\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H).$$

Proof. This is [51, Proposition 1.14]. \square

Let \mathcal{P} be the set of elements of prime order in H (including all nontrivial unipotent elements in H if $p = 0$) and let $c \geq 2$ be an integer. We define

$$\mathcal{Q}(G, c) = \frac{c}{c-1} \cdot \sup_{x \in \mathcal{P}} \left\{ \frac{\dim(x^G \cap H)}{\dim x^G} \right\}.$$

The next result is a key tool in our later analysis.

Theorem 2.13. *Let G be a simple algebraic group and let $\Omega = G/H$ be a transitive G -variety, where H^0 is reductive. Let $c \geq 2$ be an integer such that $\mathcal{Q}(G, c) < 1$. Then $b^1(G, H) \leq c$.*

Proof. Let $x \in \mathcal{P}$ and set $C = x^H$, $\Gamma = \Omega^{c-1}$ and $\Gamma(C) = \bigcup_{y \in C} C_\Gamma(y)$. By Corollary 2.11, we need to show that $\dim \Gamma(C) < \dim \Gamma$.

First we claim that

$$\dim \Gamma(C) \leq (c-1) \dim C_\Omega(x) + \dim x^H.$$

To see this, let $C_\Omega(x)^{c-1} = C_\Omega(x) \times \cdots \times C_\Omega(x)$ (with $c-1$ factors) and consider the morphism

$$\phi : H \times C_\Omega(x)^{c-1} \rightarrow \overline{\Gamma(C)} \quad (1)$$

sending $(h, \alpha_1, \dots, \alpha_{c-1})$ to $(h\alpha_1, \dots, h\alpha_{c-1})$. Now $\text{im}(\phi) = \Gamma(C)$ and

$$\phi((hy, y^{-1}\alpha_1, \dots, y^{-1}\alpha_{c-1})) = \phi((h, \alpha_1, \dots, \alpha_{c-1}))$$

for all $y \in C_H(x)$, so $\dim \phi^{-1}(\alpha) \geq \dim C_H(x)$ for all $\alpha \in \text{im}(\phi)$. Therefore

$$\dim \Gamma(C) \leq \dim H + (c-1) \dim C_\Omega(x) - \dim C_H(x)$$

as claimed.

Now, Lemma 2.12 gives

$$\dim C_\Omega(x) = \dim \Omega - \dim x^G + \dim(x^G \cap H),$$

and we may assume $x^G \cap H = x^H$ since $x^G \cap H$ is a finite union of H -classes (see [38, Theorem 1.2]). Since $c \dim x^H < (c-1) \dim x^G$ we conclude that

$$\dim \Gamma(C) \leq (c-1) \dim C_\Omega(x) + \dim x^H = \dim \Gamma - (c-1) \dim x^G + c \dim x^H < \dim \Gamma,$$

as required. \square

Corollary 2.14. *Let $c \geq 2$ be an integer such that*

$$\dim x^H < (1 - c^{-1}) \dim x^G$$

for all $x \in \mathcal{P}$. Then $b^1(G, H) \leq c$.

Proposition 2.15. *Suppose $\Omega = G/H$ with H^0 reductive. Let $x \in H$ be a semisimple element of prime order such that*

$$\dim Z(C_G(x)^0) + \text{rank } H > \text{rank } G.$$

Then there exists $y \in H$ of order $2 + \delta_{2,p}$ such that $C_\Omega(x) \subseteq C_\Omega(y)$.

Proof. The bound $\dim Z(C_G(x)^0) + \text{rank } H > \text{rank } G$ implies that $Z(C_G(x)^0) \cap H$ contains a positive-dimensional torus L . Choose $y \in L$ so that x and y are of coprime order and set $z = xy$. We claim that $C_\Omega(z) = C_\Omega(x)$.

Clearly, $C_\Omega(z) \subseteq C_\Omega(x)$ since x is a power of z . Let T be a maximal torus of H containing L . Let $x = x_1, x_2, \dots, x_m$ in T represent the distinct H -classes in $x^G \cap H$, so $x_i = x^{w_i}$ for some w_i in the Weyl group of H . Also set $z_i = y^{w_i}$ for all i , and note that the z_i are also in distinct H -classes. It is an easy exercise to see that $C_\Omega(x)$ is the union of m disjoint sets, each an orbit of $C_G(x_i)/C_H(x_i)$ for some i . Similarly, $C_\Omega(z)$ contains the $C_G(z_i)/C_H(z_i)$ -orbits. Since $C_H(z_i) \leq C_H(x_i)$ and $C_G(x) = C_G(z)$, it follows that $C_\Omega(x) \subseteq C_\Omega(z)$. This justifies the claim.

Let S be the set of elements $xy \in L$ such that the order of $y \in L$ is relatively prime to the order of x . Then S is dense in L , so by the previous claim we have $C_\Omega(x) = C_\Omega(z)$ for all $z \in S$ and thus

$$C_\Omega(x) = C_\Omega(\overline{S}) = C_\Omega(L) = \bigcap_{y \in L} C_\Omega(y).$$

In particular, if $y \in L$ has order $2 + \delta_{2,p}$ then $C_\Omega(x) \subseteq C_\Omega(y)$, as required. \square

Note that if H has maximal rank then Proposition 2.15 applies to any semisimple element $x \in G$ with $\dim Z(C_G(x)^0) > 0$. The next result allows us to slightly weaken the conditions in the statement of Corollary 2.14. In order to state the result, let $\mathcal{P}' \subseteq \mathcal{P}$ be the union of the set of unipotent elements in \mathcal{P} and the set of semisimple elements $x \in \mathcal{P}$ with the property that either x has order $2 + \delta_{2,p}$ or

$$\dim Z(C_G(x)^0) + \text{rank } H \leq \text{rank } G.$$

Corollary 2.16. *Let $c \geq 2$ be an integer such that*

$$\dim x^H < (1 - c^{-1}) \dim x^G$$

for all $x \in \mathcal{P}'$. Then $b^1(G, H) \leq c$.

Proof. By Corollary 2.11 it suffices to show that $\dim \Gamma(C) < \dim \Gamma$, where $\Gamma = \Omega^{c-1}$, $C = x^H$, $x \in H$ is semisimple of prime order $r > 2 + \delta_{2,p}$ and

$$\dim Z(C_G(x)^0) + \text{rank } H > \text{rank } G.$$

By Proposition 2.15, there exists an element $z \in H$ of order $2 + \delta_{2,p}$ such that $C_\Omega(x) \subseteq C_\Omega(z)$, so $\Gamma(C) \subseteq \Gamma(C')$ for $C' = z^H$. Since $z \in \mathcal{P}'$ we have $\dim z^H < (1 - c^{-1}) \dim z^G$, so $\dim \Gamma(C') < \dim \Gamma$ as required. \square

The next proposition is a generalization of a result of Guralnick and Lawther [39].

Proposition 2.17. *Suppose $\Omega = G/H$ with H^0 reductive, and let $c \geq 2$ be an integer such that the following conditions hold:*

- (i) *There exists a prime $r \neq p$ such that $\dim x^H < (1 - c^{-1}) \dim x^G$ for all $x \in H$ of order r ;*
- (ii) *$\dim x^H \leq (1 - c^{-1}) \dim x^G$ for all unipotent elements $x \in \mathcal{P}$.*

Then $b^0(G, H) \leq c$.

Proof. Let $\Gamma = \Omega^{c-1}$. We need to show that there exists a non-empty open subvariety $U \subseteq \Gamma$ such that $H_\alpha := \bigcap_{i=1}^{c-1} H_{\alpha_i}$ is finite for all $\alpha = (\alpha_1, \dots, \alpha_{c-1}) \in U$. Seeking a contradiction, suppose that no such U exists. Then H_α is infinite for all $\alpha \in \Gamma$, so H_α either contains a torus or a 1-dimensional unipotent subgroup.

By (i), the set of $\alpha \in \Gamma$ such that H_α contains a torus is contained in a proper closed subvariety of Γ (namely, the subvariety $\bigcup_{y \in \Lambda} C_\Gamma(y)$, where Λ is the set of elements of order r in H , and $r \neq p$ is the prime in (i)). Therefore, $(H_\alpha)^0$ is unipotent for all α in a non-empty open subvariety of Γ . Since H_α is infinite and there are only finitely many unipotent classes in H (see Lemma 2.9), it follows that there is a nontrivial unipotent element $x \in H$ such that $\dim(x^H \cap H_\alpha) > 0$ for all α in a non-empty open subvariety $W \subseteq \Gamma$. Set $C = x^H$ and $\Gamma(C) = \bigcup_{y \in C} C_\Gamma(y)$. As noted in the proof of Theorem 2.13, $\Gamma(C)$ is the image of the morphism

$$\phi : H \times C_\Omega(x)^{c-1} \rightarrow \Gamma$$

defined in (1), sending $(h, \alpha_1, \dots, \alpha_{c-1})$ to $(h\alpha_1, \dots, h\alpha_{c-1})$. Since $\dim(x^H \cap H_\alpha) > 0$ for all $\alpha \in W$, it follows that $\Gamma(C)$ contains a non-empty open subvariety of Γ and thus $\dim \Gamma = \dim \Gamma(C)$. Clearly, ϕ is still a dominant morphism if we replace x by x^i for any positive integer i , so if $p > 0$ we can assume that x has order p .

Set $V(x) = \{(y, \alpha) \mid y \in x^H, \alpha \in \Gamma, y\alpha = \alpha\}$. Then $V(x)$ surjects onto x^H and $\Gamma(C)$ via the two projection maps. Therefore, by considering the fibers of the first projection we deduce that

$$\dim V(x) = \dim x^H + \dim C_\Gamma(x)$$

and by applying Lemma 2.12 and the condition in (ii) we get

$$\dim V(x) = \dim \Gamma + c \dim x^H - (c-1) \dim x^G \leq \dim \Gamma.$$

Similarly, the second projection shows that

$$\dim V(x) = \dim \Gamma(C) + \dim(x^H \cap H_\alpha)$$

for some $\alpha \in \Gamma$ with $\dim(x^H \cap H_\alpha) > 0$. Therefore,

$$\dim \Gamma \geq \dim V(x) > \dim \Gamma(C),$$

which is a contradiction. \square

Let X be a simple algebraic group with root system Φ and root subgroups

$$U_\alpha = \{x_\alpha(t) \mid t \in K\}, \quad \alpha \in \Phi.$$

Recall that if α is a long root then U_α is a *long root subgroup*, and $x \in X$ is a *long root element* if x is X -conjugate to $x_\alpha(t)$ for some long root α and $t \in K^*$.

Proposition 2.18. *Suppose $\Omega = G/H$ with H^0 simple. Let $C = x^H$, where $x \in H^0$ is a long root element of H^0 , and assume that each long root subgroup of H^0 is a long root subgroup of G . Let $c \geq 2$ be an integer and set $\Gamma = \Omega^{c-1}$. Then*

$$\dim \Gamma(C) \leq \dim H + (c-1) \dim C_\Omega(x) - \dim C_H(x) - 1.$$

Proof. Define the morphism $\phi : H \times C_\Omega(x)^{c-1} \rightarrow \overline{\Gamma(C)}$ as in the proof of Theorem 2.13 (see (1)). It suffices to show that $\dim \phi^{-1}(\alpha) \geq \dim C_H(x) + 1$ for all $\alpha \in \text{im}(\phi)$.

First we claim that x belongs to a unique long root subgroup of G . To see this, let U be a long root subgroup of G containing x , and suppose that $x \in V$ is in another long root subgroup of G . Set $P = N_G(U)$ and note that P is a parabolic subgroup of G . Since U and V are G -conjugate we have $V = U^{awb}$, where $a, b \in P$ and w is in the Weyl group of G . Therefore, $U \cap V = U \cap U^{wb}$ is conjugate to $U \cap U^w$, which is either trivial or equal to U . But $x \in U \cap V$, so $U = V$ as required.

Let W be a long root subgroup of H^0 containing x . By assumption, W is a long root subgroup of G , so the previous claim implies that $U = W \leq H^0$. Therefore, $C_\Omega(x)$ is invariant under a 1-dimensional torus normalizing U (but not centralizing x), so the proof of Theorem 2.13 yields the desired bound $\dim \phi^{-1}(\alpha) \geq \dim C_H(x) + 1$. \square

Corollary 2.19. *Suppose $\Omega = G/H$ with H^0 simple. Assume that each long root subgroup of H^0 is a long root subgroup of G . Let $c \geq 2$ be an integer such that*

$$\dim x^H \leq (1 - c^{-1}) \dim x^G$$

for all $x \in \mathcal{P}$, with equality only if $x \in H^0$ is a long root element. Then $b^1(G, H) \leq c$.

We close this preliminary section with some remarks on the underlying field. In the following sections we will often work over the algebraic closure k of \mathbb{F}_p , but it is important to note that the same results hold if we replace k by any algebraically closed field of characteristic p . Indeed, in almost all cases the arguments do not depend on the choice of field, but there are some cases where we deduce the results for the algebraic group G from results for the corresponding finite group G_σ (the fixed points of a suitable Frobenius morphism σ). We first make some elementary observations.

Fix a prime p and let K be an algebraically closed field of characteristic p . Let G be a simple algebraic group over K and let H be a maximal closed subgroup of G . Note that G and H are both defined over k . Let c be a positive integer and let X_c denote the product variety $(G/H)^c$, with the natural G -action.

The algebraic group $G(K)$ has an orbit on $X_c(K)$ with finite stabilizer if and only if the same is true for the action of $G(k)$ on $X_c(k)$ (since dimension remains constant under base change). This shows that the connected base size $b^0(G, H)$ remains constant under base change. Similarly, the generic stabilizer in $G(K)$ of a point in $X_c(K)$ is finite of size n if and only if the same is true for the $G(k)$ -stabilizer of a generic point in $X_c(k)$ (because this condition holds on an open subvariety). Therefore, the generic base size $b^1(G, H)$ is also constant under base change.

Next we show that the base size $b(G, H)$ cannot increase under base change. Indeed, if $G(k)$ has a regular orbit on $X_c(k)$ then this orbit remains regular for $G(K)$ on $X_c(K)$ (the stabilizer in $G(K)$ is still zero-dimensional, and thus finite, but the orbits of $\text{Aut}(K/k)$ are either trivial or infinite, so this stabilizer must be trivial). In fact, $b(G, H)$ is constant under base change, and all of these assertions follow from more general considerations – see [37, p.81-82].

Finally, by a straightforward ultraproduct construction, we see that for some algebraically closed field of characteristic 0 (and therefore for all by the previous discussion), the quantities $b^0(G, H)$, $b(G, H)$ and $b^1(G, H)$ in characteristic 0 will be the same as for all sufficiently large characteristics.

3. INVOLUTION-TYPE SUBGROUPS

Let G be a simple algebraic group over an algebraically closed field K of characteristic p . Let Ω be a primitive G -variety with point stabilizer H . In this section we consider the special case where $H = C_G(\tau)$ for an involution $\tau \in \text{Aut}(G)$. Our goal here is to prove Theorem 8, as well as some auxiliary results that may be of independent interest. We are mostly interested in the case $p \neq 2$, although some of the results will apply for all p .

Our strategy is as follows. In Lemma 3.6 we show that there exists a unique G -class of involutions which invert a maximal torus of G , say $C = x^G$. By Proposition 3.7, every element of G is a product of two elements in C , so there exists $g \in G$ such that $xx^g = u$ is a regular unipotent element of G , and thus $C_G(u)$ is abelian and contains no involutions for $p \neq 2$. Moreover, x inverts $C_G(u)$ by Proposition 3.9, and using this we deduce that $b^0(G, H) = b(G, H) = 2$ if $\tau \in C$, otherwise $b^0(G, H) \geq 3$. We begin by recording some preliminary lemmas.

3.1. Preliminaries. We begin with an elementary result that will be needed in the proof of Proposition 3.7. Here $M_n(K)$ denotes the algebra of $n \times n$ matrices with entries in the field K .

Lemma 3.1. *Let K be an algebraically closed field and let $A \in M_n(K)$. Then A is similar to a symmetric matrix.*

Proof. Let p denote the characteristic of K . We may assume that A is in Jordan canonical form. Furthermore, we may assume that A is a single Jordan block, and we may take A to be nilpotent. Let $B = (b_{i,j}) \in M_n(K)$, where $b_{i,n+1-i} = 1$ for $1 \leq i \leq n$, and all other entries are 0. Then B is symmetric and $BAB^{-1} = A^\top$, where A^\top denotes the transpose of A .

If $p \neq 2$, or a diagonal entry of B is non-zero (i.e. n is odd), then we can write $B = X^\top X$ for some non-singular X (because any such non-singular symmetric matrix is congruent to the identity matrix I , over an algebraically closed field). If $p = 2$ and $n = 2m$ is even, then replace B by $B(I + A)$. Note that $B(I + A)$ is non-singular, symmetric and the $(m + 1)$ -th diagonal entry of $B(I + A)$ is non-zero, so we can write $B(I + A) = X^\top X$ for some X . Also note that $B(I + A)A(I + A)^{-1}B^{-1} = A^\top$.

It is easy to check that XAX^{-1} is symmetric. □

For the remainder of Section 3.1, unless stated otherwise, G will denote a connected reductive algebraic group of rank r over an algebraically closed field K of characteristic $p \geq 0$.

Lemma 3.2. *Let G be a connected algebraic group over an algebraically closed field of characteristic $p \geq 0$, and let $\tau \in \text{Aut}(G)$ be an involution. Then either G is abelian and τ inverts G , or $\dim C_G(\tau) > 0$.*

Proof. By [72, Theorem 7.2], τ normalizes a Borel subgroup $B = TU$ of G , where T is a maximal torus of G and U is the unipotent radical of B . Assume that $\dim C_G(\tau) = 0$, so $C_G(\tau)$ is finite.

First suppose $p = 2$. If U is nontrivial then $C_U(\tau)$ is infinite, so we may assume $G = T$ is a torus. An involutory automorphism of a torus corresponds to an element of $\text{GL}_r(\mathbb{Z})$ (acting on the character group of the torus), and any involution other than $-I_r$ centralizes a positive-dimensional subtorus. Since $C_G(\tau)$ is finite, we conclude that τ inverts G .

Now assume that $p \neq 2$. First we show that U is abelian and τ inverts U . Let W be a minimal connected characteristic subgroup of U . Then W is a vector space over K and $C_W(\tau)$ is a subspace, hence $C_W(\tau)$ is trivial and thus τ acts as inversion on W . Since 2 is invertible in K , the fixed points of τ on U surject onto the fixed points of τ on U/W . So by induction, U/W is abelian and τ acts as inversion on U/W . Therefore $C_U(\tau) \leq C_W(\tau) = 1$. Note that $[U, U] \leq W$ and so since τ acts as inversion on U/W , τ centralizes $[U, U]$, whence U is abelian. Thus, τ acts as inversion on U .

Let T_0 be the subgroup of T of elements of odd order. If we consider the action of τ on UT_0 then the above argument implies that UT_0 is abelian and τ acts as inversion on UT_0 . Therefore $B = T \times U$ is abelian and τ acts as inversion on B . Since B is abelian, it follows that G is solvable and thus $G = B$. \square

Lemma 3.3. *Let $\tau \in \text{Aut}(G)$ be an involution and let $H = C_G(\tau)$. Let $g \in G$ and set $D = C_G(\tau\tau^g)$.*

- (i) *$H \cap H^g = 1$ if and only if τ acts as inversion on D , D is abelian and D contains no involutions.*
- (ii) *$H \cap H^g$ is finite if and only if τ acts as inversion on D^0 , D^0 is abelian and if $p = 2$, D^0 is a torus.*

Proof. Suppose τ inverts D . Then clearly $H \cap H^g$ consists of the involutions in D , whence the backward implications of both statements follow.

Conversely, assume that $H \cap H^g$ is finite. Then $C_D(\tau)$ is finite and the result follows from Lemma 3.2. \square

Lemma 3.4. *Suppose G is semisimple and let $x \in G$ be a regular unipotent element contained in a Borel subgroup $B = TU$ of G . Then $C_G(x) = A \times Z(G)$ where A is an abelian subgroup of U .*

Proof. This is well known. See [61], for example. \square

Corollary 3.5. *Suppose G is semisimple of adjoint type, $p \neq 2$ and there exist conjugate involutions $\tau, \tau' \in \text{Aut}(G)$ such that $\tau\tau'$ is regular unipotent and τ inverts $C_G(\tau\tau')$. Then $C_G(\tau) \cap C_G(\tau') = 1$.*

Proof. By Lemma 3.4, $C_G(\tau\tau')$ is unipotent and therefore contains no involutions since $p \neq 2$. Now apply Lemma 3.3. \square

Lemma 3.6. *There exists an involution $\tau \in \text{Aut}(G)$ that inverts a maximal torus T of G , and any two such involutions are G -conjugate. Also, $\dim C_G(\tau) = \frac{1}{2}(\dim G - r)$. If G is simple and $p \neq 2$, the type of $C_G(\tau)$ and τ is recorded in Table 5.*

G	Type of $C_G(\tau)$	Type of τ
A_n	SO_{n+1}	inner if $n = 1$, otherwise graph
B_n	$\mathrm{SO}_{n+1} \times \mathrm{SO}_n$	inner
C_n	GL_n	inner
D_n	$\mathrm{SO}_n \times \mathrm{SO}_n$	inner if n even, otherwise graph
E_8	D_8	inner
E_7	A_7	inner
E_6	C_4	graph
F_4	$A_1 C_3$	inner
G_2	$A_1 \tilde{A}_1$	inner

TABLE 5. Involutions inverting maximal tori, $p \neq 2$

Proof. By passing to an isogeneous group, we may assume that $G = S \times G_1$ where S is a torus and G_1 is a direct product of simple and simply connected groups. We induct on $\dim G$. The case of a torus is clear and so a minimal counterexample would be a simple group (and again we may assume that it is simply connected). Existence now follows by inspection. (Note that τ is an involution modulo the center of G .) Moreover, $\dim C_G(\tau) = \frac{1}{2}(\dim G - r)$ since τ permutes the root subgroups of G without fixed points, and $C_T(\tau)$ is finite.

Suppose τ and τ' are involutory automorphisms of G which invert a maximal torus. To see that τ and τ' are G -conjugate we may assume, without loss, that τ and τ' both invert the same maximal torus T , so $\tau\tau' \in C_{\mathrm{Aut}(G)}(T) = T$. Therefore τ and τ' belong to the same coset of T in $N_{\mathrm{Aut}(G)}(T)$, and since they both invert T it follows that τ and τ' are G -conjugate. \square

Proposition 3.7. *Let $\tau \in \mathrm{Aut}(G)$ be an involution that inverts a maximal torus of G , and assume that $p \neq 2$. Then*

- (i) τ inverts an element in each conjugacy class of G ; and
- (ii) if $C = \tau^G$ then $G = C^2$.

Proof. Suppose (ii) holds. Let $g \in G$. Replacing g by a conjugate, we may assume $g = \tau\tau^x$ for some $x \in G$. Then $z^\tau = z^{-1}$, where $z = g^{x^{-1}}$ is conjugate to g . Therefore (ii) implies (i). Conversely, if τ inverts $g \in G$, then $\tau\tau^g = g^2$ and since $p \neq 2$, squaring is a surjective morphism on G , whence (i) implies (ii). Therefore, it suffices to show that (i) holds.

We may assume that $G = Z(G)^0 \times A$ where A is simply connected and semisimple. By induction on $\dim G$, we reduce to the case where G is simple and simply connected (the case when G is a torus is trivial).

Let $X = x^G$ be a conjugacy class of G . If x commutes with a non-central semisimple element $t \in G$, then we may pass to the connected reductive group $C_G(t)$ (recall that we are assuming that G is simply connected). We may assume that τ inverts a maximal torus T containing t . In particular, τ acts on the connected reductive group $C_G(t)$ which contains T , and by induction τ inverts some conjugate of x in $C_G(t)$. Consequently, we may assume that x is a *semiregular* unipotent element that commutes with no non-central semisimple element. Thus, $C_G(x) = Z(G) \times U$ for some unipotent subgroup U .

Now G contains a unique class of regular unipotent elements, so if x is such an element then $x^y = x^{-1}$ for some $y \in G$. We can assume that y has order a power of 2, hence y is an involution (modulo $Z(G)$). We want to show that y is conjugate to τ (or equivalently, that y inverts some maximal torus). Note that the dimension of the image of the multiplication

map $\mu : y^G \times y^G \rightarrow G$ is at least $\dim G - r$, where r is the rank of G . In particular,

$$\dim y^G \geq \frac{1}{2}(\dim G - r). \quad (2)$$

We now inspect the various possibilities for G , beginning with the classical groups.

If $G = A_1$ or G_2 , then τ and y are both inner and since there is a unique class of involutions (modulo the center in the case of A_1), the result follows.

If $G = A_r$ (with $r > 1$) then we take τ to be the inverse-transpose automorphism. Then τ inverts any symmetric matrix and the result follows by Lemma 3.1.

If $G = C_r$ (with $r > 1$) then an element is semiregular if and only if it is regular unipotent. Moreover, τ and y are inner. Let x be such an element and recall that $y \in G$ is an involution inverting x . By conjugating we may assume that y is in the standard maximal torus of G and so it inverts each root subgroup corresponding to a simple root. This forces y to be conjugate to τ . The same proof (without any modification) also applies for groups of type B_r (with $r > 1$).

Next suppose $G = D_r$ and $r > 3$. If x does not commute with a non-central semisimple element, then x has Jordan form $[J_{2e+1}, J_{2f+1}]$ on the natural G -module, for some $e, f \geq 0$. Then $x \in L := B_e \times B_f$ (where B_0 is trivial), and we note that $L = C_G(z)$ for a suitable involutory graph automorphism of G . Let $y \in L$ be an involution inverting x and a maximal torus of L . Then either y or yz is conjugate to τ , and inverts x .

Next let us assume G is an exceptional algebraic group of rank r . Let $\{\alpha_1, \dots, \alpha_r\}$ be a set of simple roots for the root system of G , where we label simple roots in the usual way (see [12]). Let $U_\alpha = \{x_\alpha(t) \mid t \in K\}$ be the root subgroup of G corresponding to the root α , and write $\alpha = a_1\alpha_1 + \dots + a_r\alpha_r$ to denote the root $\alpha = \sum_i a_i\alpha_i$. In addition, we adopt the standard Chevalley notation

$$n_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t), \quad h_\alpha(t) = n_\alpha(t)n_\alpha(1)^{-1}$$

for $t \in K^*$.

If $G = F_4$ then x is regular and $\dim y^G \geq 24$ by (2), whence y and τ are G -conjugate.

Next suppose that $G = E_6$. Again, we may assume that x is a regular unipotent element. Further, we may also assume that $x \in H := F_4 = C_G(\gamma)$, where γ is a graph automorphism of G . Choose $y \in H$ an involution which inverts x , and note that γy is also an involution inverting x . There are precisely two conjugacy classes of graph automorphisms of E_6 , with centralizers F_4 and C_4 . Note that $\dim C_H(\gamma y) = \dim C_H(y) = 24$. Therefore, $\dim C_G(\gamma y) \leq 50$ and so $C_G(\gamma y) = C_4$, whence γy is in the conjugacy class of involutions inverting a maximal torus of G .

Next consider $G = E_7$. There are three semiregular classes of unipotent elements in G , with respective centralizers of dimension 7, 9 and 11. Also, there are three classes of involutions in G , with dimensions 52, 64 and 70. Let $y \in G$ be an involution inverting the semiregular unipotent element x (these classes are all real by [58, Corollary 5], for example). As before, it suffices to show that y and τ are G -conjugate. Equivalently, we need to show that $\dim y^G = 70$. Note that $\dim y^G \neq 52$ by (2).

Suppose $\dim y^G = 64$, so $C_G(y) = D_6A_1$. We may view G as a subgroup of $L = E_8$ (note that it is really the double cover of G that is contained in E_8 , but y lifts to an involution in the double cover). We claim that $C_L(y) = E_7A_1$. To see this, we use an argument provided by Ross Lawther (private communication). Take a representative $y = h_{\alpha_3}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1)$ of the D_6A_1 -class in E_7 and observe that

$$\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$$

is a basis of the root system of $C_G(y)$. The root system of $C_L(y)$ has a basis comprising the above roots, together with

$$\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8.$$

These roots form a simple system of type E_7A_1 . This establishes the claim and we deduce that $\dim y^L = 112$.

By [58] we have $\dim C_L(x) = 16, 20$ or 24 . Now $\dim y^L \geq \frac{1}{2}(248 - \dim C_L(x))$ and therefore we may assume $\dim C_L(x) = 24$ (since $\dim y^L = 112$). Let $\mu : y^L \times y^L \rightarrow L$ be the multiplication map and let W be the image of μ . Since $\dim x^L = \dim(y^L \times y^L)$, it follows that if $x \in W$ then x^L is an open dense subset of W and the generic fiber of μ is finite. In particular, W is contained in the set of unipotent elements of L . Thus, the same is true for μ restricted to $y^G \times y^G$, which we denote by μ_G . Therefore, the dimension of the image of μ_G is at most 126 (the dimension of the unipotent variety of G), and hence the generic fiber of μ has dimension at least 2. This is a contradiction since the generic fiber is finite.

We conclude that $\dim y^G = 70$ is the only possibility, so y and τ are G -conjugate, as required.

Finally, let us assume $G = E_8$. There are two classes of involutions in G , of dimensions 128 and 112. Let $x \in G$ be a semiregular unipotent element. The semiregular unipotent classes have centralizers of dimension 8, 10 and 12, and there is an additional class in characteristic 3 with $\dim C_G(x) = 30$.

Suppose $\dim C_G(x) < 30$ and let $y \in G$ be an involution that inverts x (by [58, Corollary 5], the class x^G is real). Then $\dim y^G \geq \frac{1}{2}(\dim G - \dim C_G(x)) \geq 118$ and thus y and τ are G -conjugate.

To complete the proof, we may assume $p = 3$ and $\dim C_G(x) = 30$. Fix a maximal torus T of G and a corresponding set of roots. We may take

$$\begin{aligned} x = & x_{01121100}(1)x_{00111100}(1)x_{11110000}(1)x_{00001110}(1)x_{01121000}(1) \\ & \times x_{00000111}(1)x_{10111000}(1)x_{01011100}(1)x_{01122100}(1) \end{aligned}$$

(this follows by calculating the Jordan blocks of x on the adjoint module – see the class labelled $A_7^{(3)}$ in [47, Table 9]). Let

$$y = h_{\alpha_2}(-1)h_{\alpha_4}(-1)h_{\alpha_7}(-1)h_{\alpha_8}(-1)n_{\alpha_2}n_{\alpha_3}n_{\alpha_5}n_{\alpha_8}.$$

By inspecting the E_8 structure constants given in the appendix of [57] we see that y is an involution in $N_G(T)$ that inverts x . Indeed, y reverses the order of the root elements concerned and negates each coefficient.

Let $w = gT$ be the corresponding element of the Weyl group. Note that the roots fixed by w are 01121000, 01122221, 22343221, 23465421 and their negatives. We find that for each such root α , the root vector e_α is in fact negated by $\text{Ad}(g)$. As w is the product of four reflections in mutually orthogonal roots, the trace of $\text{Ad}(g)$ on the Lie algebra $\text{Lie}(T)$ is 0; hence its trace on $\text{Lie}(G) = \text{Lie}(T) \oplus \bigoplus_{\alpha} Ke_\alpha$ is -8 .

Let s and t be involutions in T such that $C_G(s) = A_1E_7$ and $C_G(t) = D_8$. Since $\dim C_G(s) = 136$, it follows that the trace of $\text{Ad}(s)$ on $\text{Lie}(G)$ is $136 - 112 = 24$. Therefore y is conjugate to t , which is conjugate to τ . \square

Remark 3.8. Part (ii) of Proposition 3.7 gives a conjugacy class C of automorphisms of G with the property $G = C^2$. This observation is related to a well known open conjecture of J.G. Thompson, which asserts that if G is a finite simple group then $G = C^2$ for some conjugacy class C . This has been verified if G is an alternating or sporadic group, and also if G is a simple group of Lie type over \mathbb{F}_q with $q > 8$. We refer the reader to [29] for further details.

Proposition 3.9. *Assume that $p \neq 2$. Let $\tau \in \text{Aut}(G)$ be an involution that inverts a maximal torus of G , and let u be a regular element inverted by τ . Then τ acts as inversion on $C_G(u)$.*

Proof. We argue by induction on $\dim G$. In the usual manner, we first reduce to the case where G is simple and simply connected. Recall that r denotes the rank of G .

Consider the multiplication map $\mu : \tau^G \times \tau^G \rightarrow G$. By Proposition 3.7, this map is surjective. By Lemma 3.6 we have $\dim \tau^G = \frac{1}{2}(\dim G + r)$, so Lemma 2.1 implies that there is a non-empty open subvariety W of G such that $\dim \mu^{-1}(w) = r$ for all $w \in W$. Therefore

$$\dim \mu^{-1}(u) = r = \dim C_G(u).$$

If $x'y' = u$ with $x', y' \in \tau^G$, then $x' \in \tau C_G(u)$. In particular, there exists $v \in C_G(u)$ such that the coset $\tau v C_G(u)^0$ consists of involutions, whence τ acts as inversion on $v C_G(u)^0$. If $C_G(u)$ is connected, this gives the result.

If u is not unipotent then τ normalizes the subgroup of semisimple elements in $C_G(u)$. Since this subgroup properly contains $Z(G)$, we can pass to $C_G(t)$ with $t \in C_G(u)$ semisimple and non-central (note that $C_G(u) \leq C_G(t)$). The result follows by induction.

Finally, suppose u is unipotent. Then $C_G(u) = Z(G) \times U$, where U consists of unipotent elements, and so we may assume that $Z(G) = 1$. If $p = 0$, or if p is a good prime for G then $C_G(u)$ is connected and the result follows. In bad characteristic, $C_G(u)$ is disconnected, but we have $C_G(u) = \langle C_G(u)^0, u \rangle$. Since τ inverts u and a coset of $C_G(u)^0$, we conclude that τ inverts $C_G(u)$. \square

3.2. Proof of Theorem 8. We are ready to prove the main statement of Theorem 8.

Corollary 3.10. *Let G be a simple algebraic group of rank r over an algebraically closed field of characteristic $p \neq 2$. Let $H = C_G(\tau)$, where $\tau \in \text{Aut}(G)$ is an involution that inverts a maximal torus of G , and let $\Omega = G/H$ be the corresponding homogeneous space. Then the following hold:*

- (i) H has a unique regular orbit on Ω , so $b^0(G, H) = b(G, H) = 2$.
- (ii) The generic 2-point stabilizer has order 2^r , i.e. there is a non-empty open subvariety $U \subseteq \Omega \times \Omega$ such that $|G_\alpha \cap G_\beta| = 2^r$ for all $(\alpha, \beta) \in U$.
- (iii) $b^1(G, H) = 3$.
- (iv) If $G < A \leq \text{Aut}(G)$ then A acts on Ω and $b^0(A, C_A(\tau)) = 2$, $b(A, C_A(\tau)) = b^1(A, C_A(\tau)) = 3$.

Proof. By Proposition 3.7, there exists a conjugate τ^g of τ such that $\tau\tau^g = u$ is a regular unipotent element. Then τ inverts u , so Proposition 3.9 implies that τ inverts $C_G(u)$, whence $H \cap H^g = 1$ by Corollary 3.5. In particular, we have $b^0(G, H) = b(G, H) = 2$.

Conversely, suppose that $C_G(\tau) \cap C_G(\tau') = 1$ for some conjugate τ' of τ . Then τ acts fixed-point-freely on $C_G(\tau\tau')$, so τ inverts $C_G(\tau\tau')$ and thus $C_G(\tau\tau')$ is abelian and contains no involutions. Therefore, $u = \tau\tau'$ is a semiregular unipotent element and thus [50, Theorem 1] implies that u is regular. Since any two involutions in $\langle \tau, C_G(u) \rangle$ are conjugate, it follows that $C_G(\tau)$ acts transitively on the set of regular unipotent elements in G inverted by τ . As we have noted above, the points of Ω which belong to a regular $C_G(\tau)$ -orbit are in bijection with the regular unipotent elements inverted by τ . Therefore, H has a unique regular orbit on Ω .

Generically, the product of two conjugates of τ is a regular semisimple element (because the set of such elements is dense in G), whence there exists a non-empty open subvariety U of $\Omega \times \Omega$ such that $G_\alpha \cap G_\beta$ coincides with the set of involutions in a maximal torus

of G , for all $(\alpha, \beta) \in U$. Therefore (i) and (ii) hold, while (iii) follows from Proposition 2.5(iv).

Finally, let us consider (iv). Since the class of involutions inverting a maximal torus is invariant under A , we have $A = N_A(H)G$ and so A acts on Ω (with point stabilizer $C_A(\tau)$). The only possible regular orbit would be the (unique) regular orbit of G , but clearly A is not regular on this orbit so $b(A, C_A(\tau)) \geq 3$. By (ii), it follows that a generic pair of points has finite (but nontrivial) A -stabilizer, so $b^0(A, C_A(\tau)) = 2$ and $b^1(A, C_A(\tau)) = 3$ as claimed. \square

Remark 3.11. Suppose $p = 2$ and $\tau \in \text{Aut}(G)$ is an involution inverting a maximal torus of G . Since tori have no involutions, it is trivial to see that generically $C_G(\tau) \cap C_G(\tau^g) = 1$. Therefore, with respect to the action of G on τ^G , we see that

$$b^0(G, C_G(\tau)) = b(G, C_G(\tau)) = b^1(G, C_G(\tau)) = 2.$$

Finally, the following result completes the proof of Theorem 8.

Proposition 3.12. *Let G be a connected reductive algebraic group over an algebraically closed field of characteristic $p \geq 0$. Let $\tau \in \text{Aut}(G)$ be an involution that does not invert a maximal torus of G . Set $H = C_G(\tau)$ and $\Omega = G/H$. Then $b^0(G, H) \geq 3$.*

Proof. If $p = 2$, the result follows by Lemma 3.3, so assume that $p \neq 2$. We proceed by induction on $\dim G$. Let $\tau' = \tau^g$ be a conjugate of τ .

The result is clear if $\dim G = 1$, or more generally if G is solvable, as in the proof of Lemma 3.3. By induction, we may assume that the solvable radical of G is trivial and so G is semisimple. Again by induction, we may assume that τ permutes the simple components of G transitively and so G is either simple, or a product of two simple groups. In the latter case, we see that generically the product $\tau\tau'$ is regular semisimple and the common centralizer of τ and τ' is a diagonal torus. In particular, $H \cap H^g$ generically has positive dimension, whence $b^0(G, H) \geq 3$.

So we may assume that G is simple. Set $t = \tau\tau'$ and $L = C_G(t)$. If L^0 is nonabelian then Lemma 3.2 implies that $\dim C_L(\tau) > 0$ and the result follows since $C_L(\tau) = C_G(\tau) \cap C_G(\tau')$.

Now assume L^0 is abelian. By [50, Theorem 2], we deduce that either t is regular, or $p = 3$ and $G = G_2$. In the latter case, $\text{Aut}(G)$ has a unique class of involutions so τ must invert a maximal torus, which is a contradiction. Therefore, we may assume t is regular. As before, if $\dim C_L(\tau) > 0$ then the result follows, so let us assume otherwise.

By Lemma 3.2, τ inverts L^0 and we claim that τ inverts a maximal torus of G . By the usual reduction argument, we may assume that t is unipotent. We may also assume that $Z(G) = 1$. It follows that $\gamma C_G(t) = \tau C_G(t)$, where $\gamma \in \text{Aut}(G)$ is an involution that inverts a maximal torus of G . Therefore γ and τ are conjugate (since γ inverts $C_G(t)$) and thus τ inverts a maximal torus. This final contradiction completes the proof. \square

3.3. Applications. We will now use Theorem 8 to settle some special cases of Theorems 5 and 7. Let G be a simple algebraic group over an algebraically closed field K of characteristic $p \geq 0$ and suppose H is an *involution-type* subgroup of G . This means that H is a maximal subgroup of G with the same structure as a centralizer $C_{\tilde{G}}(\tau)$, where \tilde{G} is a simple algebraic group over an algebraically closed field of characteristic $r \neq 2$, $\tau \in \text{Aut}(\tilde{G})$ is an involution (where $\text{Aut}(\tilde{G})$ denotes the group of algebraic automorphisms of \tilde{G}), and the root systems of G and \tilde{G} are isomorphic. For example, A_1E_7 and D_8 are the involution-type subgroups of E_8 .

Note that certain involution-type subgroups of symplectic and orthogonal groups act reducibly on the natural module; we will deal separately with these subspace actions in

Section 4.1. The non-subspace involution-type subgroups we are interested in here are listed in Table 6.

G	Type of H
SL_n	$\mathrm{GL}_{n/2} \wr S_2, \mathrm{Sp}_n, \mathrm{SO}_n$
Sp_n	$\mathrm{Sp}_{n/2} \wr S_2$ ($n \geq 8$), $\mathrm{GL}_{n/2}$
SO_n	$O_{n/2} \wr S_2, \mathrm{GL}_{n/2}$
E_8	$A_1 E_7, D_8$
E_7	$A_1 D_6, T_1 E_6, A_7$
E_6	$D_5 T_1, C_4$ ($p \neq 2$), $A_1 A_5, F_4$
F_4	B_4, C_4 ($p = 2$), $A_1 C_3$ ($p \neq 2$)
G_2	$A_1 \tilde{A}_1$

TABLE 6. Non-subspace involution-type subgroups

Our main result on involution-type subgroups is the following:

Theorem 3.13. *Let G be a simple algebraic group over an algebraically closed field K of characteristic $p \geq 0$, let H be a non-subspace involution-type subgroup of G and let $\Omega = G/H$ be the corresponding coset variety. Then one of the following holds:*

- (i) $b^0(G, H) = b(G, H) = 2$, $b^1(G, H) = 3$, and either $G = \mathrm{SL}_2$ and H is of type $\mathrm{GL}_1 \wr S_2$, or $p \neq 2$ and
 $(G, H) = (\mathrm{SL}_n, \mathrm{SO}_n), (\mathrm{Sp}_n, \mathrm{GL}_{n/2}), (\mathrm{SO}_n, O_{n/2} \wr S_2), (E_8, D_8), (E_7, A_7.2),$
 $(E_6, C_4), (F_4, A_1 C_3)$ or $(G_2, A_1 \tilde{A}_1)$;
- (ii) $b^0(G, H) = b(G, H) = b^1(G, H) = b$ and (G, H, b) is recorded in Table 7;
- (iii) $2 = b^0(G, H) \leq b(G, H) \leq b^1(G, H) = 3$, $p = 2$, $G = \mathrm{SO}_n$ and H is of type $O_{n/2} \wr S_2$, where $n \equiv 0 \pmod{4}$ and $n \geq 8$;
- (iv) $2 = b^0(G, H) \leq b(G, H) \leq b^1(G, H) \leq 3$, $p = 2$ and $(G, H) = (E_7, A_7.2), (E_6, A_1 A_5)$ or $(G_2, A_1 \tilde{A}_1)$.

G	Type of H	Conditions	b
SL_n	$\mathrm{GL}_{n/2} \wr S_2$	$n \geq 4$	3
	Sp_n	$n \geq 6$	$3 + \delta_{6,n}$
Sp_n	$\mathrm{Sp}_{n/2} \wr S_2$	$n \geq 8$	3
SO_n	$\mathrm{GL}_{n/2}$	$n \geq 10$	3
E_8	$A_1 E_7$		3
	D_8	$p = 2$	2
E_7	$A_1 D_6$		3
	$T_1 E_6$		3
E_6	F_4		4
	$D_5 T_1$		3
	$A_1 A_5$	$p \neq 2$	3
F_4	B_4		4
	C_4	$p = 2$	4

TABLE 7. Values of b in Theorem 3.13(ii)

We prove Theorem 3.13 in a sequence of lemmas. First we record a couple of useful preliminary results. Let V be a finite dimensional vector space over K . We say that $g \in \mathrm{GL}(V)$ is a *quadratic element* if its minimal polynomial over K is quadratic (or equivalently if g is semisimple, then g has precisely two distinct eigenvalues).

Lemma 3.14. *Let $g, h \in \mathrm{GL}(V)$ be quadratic elements and set $G = \langle g, h \rangle$. Then every composition factor of the KG -module V has dimension at most 2.*

Proof. We argue by induction on $n = \dim V$. The result is clear if $n \leq 2$. By induction, it suffices to prove that G acts reducibly on V .

Let $U \subseteq V$ be a g -eigenspace of largest dimension. Since g is quadratic, we have $\dim U \geq n/2$. If h has an eigenvector in U , then G has a 1-dimensional invariant subspace, so let us assume otherwise. Now, if $0 \neq v \in hU \cap U$ then the span of v and hv is G -invariant. Therefore, we may assume hU is a complement to U in V , whence n is even and $\dim U = n/2$. Let W be an h -eigenspace of largest dimension. The same argument shows that $\dim W = n/2$, so we may assume that $V = U \oplus W = W \oplus gW$.

Thus, with respect to an appropriate choice of basis, we have

$$g = \begin{pmatrix} aI_{n/2} & A \\ 0 & bI_{n/2} \end{pmatrix}, \quad h = \begin{pmatrix} cI_{n/2} & 0 \\ B & dI_{n/2} \end{pmatrix},$$

where A and B are invertible. Conjugating by an appropriate block diagonal matrix, we may assume that A is the identity matrix and B is diagonal, whence G clearly has a 2-dimensional invariant subspace. \square

Corollary 3.15. *Suppose $G = \mathrm{SL}_n(K)$, $A \in \mathrm{GL}_n(K)$ is a quadratic element and $H = N_G(K[A])$. Then*

$$\dim(H \cap H^g) \geq \begin{cases} n/2 - 1 & \text{if } n \text{ is even} \\ (n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

for all $g \in G$.

Proof. Let V be the natural KG -module. There exists a non-empty open subset U of G such that $\langle A, A^g \rangle$ contains a regular semisimple element of G for all $g \in U$. Therefore, if $g \in U$ then Lemma 3.14 implies that V is a direct sum of 1- and 2-dimensional non-isomorphic irreducible $K\langle A, A^g \rangle$ -modules, whence $C_G(A) \cap C_G(A^g)$ contains a torus of dimension $n/2 - 1$ if n is even, and one of dimension $(n-1)/2$ if n is odd. The resulting lower bound on $\dim(H \cap H^g)$ holds for all $g \in \overline{U}$, hence all $g \in G$ since U is dense in G . \square

We are now ready to give the proof of Theorem 3.13. For the remainder of this section, let \mathcal{P} be the set of elements of prime order in H (including all nontrivial unipotent elements if $p = 0$).

Lemma 3.16. *Theorem 3.13 holds if G is a classical group.*

Proof. First suppose $G = \mathrm{SL}_n$. If H is of type $\mathrm{GL}_{n/2} \wr S_2$ then $\dim x^H \leq \frac{1}{2} \dim x^G$ for all $x \in \mathcal{P}$ (see the proof of [14, Proposition 2.1]), whence Corollary 2.14 yields $b^1(G, H) \leq 3$. Now $H = N_G(K[A])$ for a suitable quadratic element $A \in \mathrm{GL}_n$, so if $n \geq 3$ then Corollary 3.15 implies that $b^0(G, H) \geq 3$ and thus

$$b^0(G, H) = b(G, H) = b^1(G, H) = 3. \quad (3)$$

Now suppose $n = 2$, so $H = N_G(T)$ is the normalizer of a maximal torus T of G . If $p \neq 2$ then $H = C_G(\tau)$, where τ is an involution inverting T , so in this case Theorem 8 implies that

$$b^0(G, H) = b(G, H) = 2, \quad b^1(G, H) = 3. \quad (4)$$

Now assume $p = 2$ and let X_1, X_2 be distinct tori in G . If X_1 and X_2 are not contained in a common Borel subgroup then it is straightforward to see that the intersection $N_G(X_1) \cap N_G(X_2)$ has order 2. On the other hand, if X_1, X_2 are contained in the same Borel subgroup of G then $N_G(X_1) \cap N_G(X_2)$ is trivial, whence we have the same answer as for $p \neq 2$.

If $H = \mathrm{Sp}_n$ then [34, Theorem 1.1] yields

$$b^0(G, H) = b(G, H) = b^1(G, H) = 3 + \delta_{6,n} + 2\delta_{4,n}$$

(note that the case $n = 4$ is equivalent to a subspace action – see Table 1). Finally, if H is of type SO_n then $p \neq 2$ (since H is a maximal subgroup of G) and $H = C_G(\tau)$ for a suitable involutory graph automorphism τ . By Theorem 8, since $p \neq 2$ and τ inverts a maximal torus of G , we conclude that (4) holds.

Next assume $G = \mathrm{Sp}_n$. If H is of type $\mathrm{Sp}_{n/2} \wr S_2$ then $\dim H > \frac{1}{2} \dim G$ and thus $b^0(G, H) \geq 3$ by Proposition 2.5(iii). Here [14, Proposition 2.1] states that

$$\dim(x^G \cap H) \leq \left(\frac{1}{2} + \frac{1}{n} \right) \dim x^G$$

for all $x \in \mathcal{P}$, so

$$\dim x^H < \frac{2}{3} \dim x^G \tag{5}$$

if $n \geq 8$. In particular, Corollary 2.14 implies that (3) holds when $n \geq 8$. (As noted in Table 1, if $n = 4$ then the action of G is equivalent to a subspace action – see Remark 4.11.) Now suppose H is of type $\mathrm{GL}_{n/2}$. If $p = 2$ then H is contained in a subgroup of type SO_n , so we may assume $p \neq 2$. Here $H = C_G(\tau)$, where $\tau \in G$ is an involution inverting a maximal torus of G , so (4) holds by Theorem 8.

Finally, let us turn to the case $G = \mathrm{SO}_n$. First suppose H is of type $\mathrm{GL}_{n/2}$. Here $\dim H > \frac{1}{2} \dim G$ and thus $b^0(G, H) \geq 3$ by Proposition 2.5(iii). According to the proof of [13, Lemma 4.2], if $n \geq 10$ then

$$\dim x^H \leq \left(\frac{1}{2} + \frac{1}{n-2} \right) \dim x^G$$

for all $x \in \mathcal{P}$, so Corollary 2.14 implies that (3) holds. (Note that if $n = 8$ then the action of G is equivalent to the action of SO_8 on non-degenerate 2-spaces of the natural module – see Table 1.)

Now suppose H is of type $O_{n/2} \wr S_2$, so H is the stabilizer of a pair of complementary non-degenerate spaces. If $p \neq 2$ then $H = C_G(\tau)$ for an involution $\tau \in \mathrm{Aut}(G)$ which inverts a maximal torus of G , whence (4) holds by Theorem 8. Now assume $p = 2$, so $n/2$ is even. By [14, Proposition 2.1] we have $\dim x^H \leq \frac{1}{2} \dim x^G$ for all $x \in \mathcal{P}$ (with equality if and only if x is an involution of type c_s (with $2 \leq s \leq n/2$ and s even) or $a_{n/2}$, in the notation of Aschbacher and Seitz [2]), so Corollary 2.13 yields $b^1(G, H) \leq 3$. In fact, by applying Lemma 4.25 (see Section 4.1.4) we deduce that $b^0(G, H) = 2$ and $b^1(G, H) = 3$ (the fact that $b^0(G, H) = 2$ also follows from Proposition 2.17). However, we have been unable to determine the exact value of $b(G, H)$ in this case. \square

In order to complete the proof of Theorem 3.13, we may assume that G is an exceptional group, and we will consider each possibility for G in turn. Let us say a few words on the notation and terminology we will use in the remainder of this section. Given a semisimple subgroup $X \leq G$ we write $\Phi(X)$ (respectively, $\Phi^+(X)$) for the set of roots (respectively, positive roots) of X with respect to a fixed maximal torus. If W is a KG -module then $W \downarrow X$ denotes the restriction of W to X . For each simple factor Y of X we fix a set of fundamental dominant weights $\{\lambda_1, \lambda_2, \dots\}$ (numbered in the usual way, following [12]),

and we write $L(\lambda)$ for the irreducible KY -module with highest weight λ . If W is a KX -module then W^* denotes its dual. The Lie algebra of X is denoted by $\text{Lie}(X)$, and we write T_i for an i -dimensional torus. In addition, J_i denotes a standard unipotent Jordan block of size i , and we adopt the notation of [47] for labelling the unipotent classes in G .

Let Φ be a root system and let Ψ be a subsystem of Φ . Following [51, Section 5], we say that Ψ is A_2 -dense in Φ if every subsystem of Φ of type A_2 meets Ψ . Note that if Φ_1 is a subsystem of Φ , and Ψ is A_2 -dense in Φ , then $\Psi \cap \Phi_1$ is A_2 -dense in Φ_1 . Such subsystems are called *anti-open* in [48], and the complete list of all proper anti-open subsystems of irreducible root systems is given in [48] (also see [51, Lemma 5.1]).

Lemma 3.17. *Theorem 3.13 holds if $G = E_8$.*

Proof. Here $H = N_G(X)$ with $X = A_1E_7$ or D_8 (see Table 7). Suppose $X = A_1E_7$, so $\dim H = 136$ and $b^0(G, H) \geq 3$ by Proposition 2.5(iii). We claim that (5) holds for all $x \in \mathcal{P}$. If x is unipotent then we can calculate the precise dimensions of x^H and x^G from the information on class fusions recorded in [49, Table 23] (see [58, Chapter 22] for a convenient list of unipotent class dimensions in exceptional algebraic groups), and the claim quickly follows. Now assume x is semisimple. Since $\dim x^H \leq 128$, we may assume that $\dim x^G \leq 192$, in which case the desired result follows from [51, Theorem 2]. For example, if $C_G(x)$ does not have an E_7 or D_8 factor then [51, Table 7.4] indicates that $\dim x^G - \dim x^H \geq 70$ and the result follows. This justifies the claim and we conclude that (3) holds.

Now assume $X = D_8$. If $p \neq 2$ then $H = C_G(\tau)$ and $\tau \in G$ is an involution which inverts a maximal torus, so (4) holds by Theorem 8. Now suppose $p = 2$. If $x \in H$ is an involution then by inspecting [49, Table 22] we quickly deduce that

$$\dim x^H < \frac{1}{2} \dim x^G. \quad (6)$$

For the remainder, let us assume $x \in G$ is a semisimple element of prime order r with $D = C_G(x)$. If $r = 3$ then $D^0 = A_8, A_2E_6, E_7T_1$ or D_7T_1 (see [51, Proposition 1.2]). In the latter case we have $\dim x^G = 156$ and [51, Theorem 2] states that $\dim x^G - \dim x^H \geq 80$, so $\dim x^H \leq 76 < \frac{1}{2} \dim x^G$. Next suppose $D^0 = E_7T_1$, so $\dim x^G = 114$ and [51, Theorem 2] yields $\dim x^H \leq 58$. In fact, we claim that $\dim x^H \leq 56 < \frac{1}{2} \dim x^G$. First observe that $D^0 < L = E_7A_1$. By [51, Lemma 5.1], the root systems $\Phi(L)$ and $\Phi(H)$ are A_2 -dense in $\Phi(G)$, so $\Phi(L \cap H)$ is A_2 -dense in both $\Phi(L)$ and $\Phi(H)$. A further application of [51, Lemma 5.1] implies that $L \cap H = A_7T_1$ or $D_6A_1^2$. Therefore $D \cap H = A_7T_1$ or $D_6A_1T_1$, whence $\dim x^H \leq 56$ as claimed. The other two cases are similar. For example, suppose $D^0 = A_8$ so $\dim x^G = 168$. Since $\Phi(D \cap H)$ is A_2 -dense in $\Phi(D)$, [51, Lemma 5.1] indicates that $\Phi(D \cap H) = A_1A_6, A_2A_5$ or A_3A_4 , so $\dim x^H \leq 80 < \frac{1}{2} \dim x^G$. Similarly, if $D^0 = A_2E_6$ then $\dim x^G = 162$ and once again the A_2 -density of $\Phi(D \cap H)$ in $\Phi(D)$ implies that $\dim x^H \leq 80 < \frac{1}{2} \dim x^G$.

Next suppose $r \geq 5$ and D^0 is semisimple, so D^0 does not contain a positive-dimensional central torus. Then it is easy to see that $r = 5$ and $D^0 = A_4A_4$ is the only possibility, so $\dim x^G = 200$. Now $\Phi(D \cap H)$ is A_2 -dense in $\Phi(D)$, and by applying [51, Lemma 5.1] we deduce that $|\Phi^+(D \cap H)| \geq |\Phi^+(A_2^2A_1^2)| = 8$, whence

$$\dim x^G - \dim x^H = 2(|\Phi^+(G)| - |\Phi^+(H)| - |\Phi^+(D)| + |\Phi^+(D \cap H)|) \geq 104$$

and thus $\dim x^H \leq 96$. We conclude that (6) holds for all $x \in H$ of order 2 or 3, and also for any $x \in H$ of prime order $r \geq 5$ such that $C_G(x)^0$ is semisimple. Therefore Corollary 2.16 implies that $b^1(G, H) = 2$. \square

Lemma 3.18. *Theorem 3.13 holds if $G = E_7$.*

Proof. Here $H = N_G(X)$ with $X = A_1D_6, T_1E_6$ or A_7 . First assume $X = A_1D_6$, so $\dim H = 69$ and thus $b^0(G, H) \geq 3$. We claim that (5) holds for all $x \in \mathcal{P}$. If x is unipotent then the desired bound follows from the fusion information in [49, Table 19], so let us assume x is semisimple with centralizer $D = C_G(x)$. If D contains an E_6, D_6 or A_7 factor then (5) follows from [51, Theorem 2]. For example, if $D^0 = A_7$ then [51, Table 7.4] indicates that $\dim x^G - \dim x^H \geq 32$, whence $\dim x^H \leq 38 < \frac{2}{3} \dim x^G = 140/3$. For the remaining semisimple elements, [51, Theorem 2] yields $\dim x^G - \dim x^H \geq 40$, so we may assume $\dim x^G \geq 120$. However, $\dim x^H \leq 62$ for all $x \in H$ and thus (5) holds in all cases. This justifies the claim and we conclude that (3) holds.

Next suppose $X = T_1E_6$. As in the previous case, we have $b^0(G, H) \geq 3$ since $\dim H > \frac{1}{2} \dim G$, and again we claim that (5) holds for all $x \in \mathcal{P}$, giving (3). If x is unipotent then the conjugacy classes x^H and x^G have the same Bala-Carter label, and using this observation it is straightforward to check that (5) holds. The argument for semisimple elements is entirely similar to the previous case, using [51, Theorem 2].

Finally, let us assume $X = A_7$, so $H = A_{7.2}$. If $p \neq 2$ then $H = C_G(\tau)$ for a suitable involution τ that inverts a maximal torus of G . Therefore (4) holds by Theorem 8. Now assume $p = 2$. As in the statement of Corollary 2.16, let \mathcal{P}' be the set of $x \in H$ of prime order r , where either $r \leq 3$ or $C_G(x)^0$ is semisimple. We claim that $\dim(x^G \cap H) \leq \frac{1}{2} \dim x^G$ for all $x \in \mathcal{P}'$, with equality if and only if $r = 2$ and x belongs to one of the classes labelled $(3A_1)', (3A_1)''$ or $4A_1$. In particular, a combination of Corollary 2.16 and Proposition 2.17 implies that

$$2 = b^0(G, H) \leq b(G, H) \leq b^1(G, H) \leq 3, \quad (7)$$

but we are unable to determine the exact values of $b(G, H)$ and $b^1(G, H)$ in this case.

To justify the claim, first assume $x \in \mathcal{P}'$ has odd order r . Let $D = C_G(x)$. Since $\dim Z(D^0) > 0$ if $r > 3$, we may assume $r = 3$ and thus $D^0 = E_6T_1, D_6T_1, A_6T_1, A_1D_5T_1$ or A_2A_5 (see [51, Proposition 1.2]). If $D^0 \in \{E_6T_1, A_6T_1, A_1D_5T_1\}$ then the bound in (6) follows from [51, Theorem 2]. For example, if $D^0 = A_1D_5T_1$ then $\dim x^G = 84$ and [51, Theorem 2] states that $\dim x^G - \dim x^H \geq 44$, giving the required bound. Next suppose $D^0 = D_6T_1$. Here $\dim x^G = 66$ and [51, Theorem 2] yields $\dim x^H \leq 34$. In order to improve this bound, first note that $D^0 < L = D_6A_1$, and the root systems $\Phi(L)$ and $\Phi(H)$ are A_2 -dense in $\Phi(G)$ (see [51, Lemma 5.1]), so $\Phi(L \cap H)$ is A_2 -dense in both $\Phi(L)$ and $\Phi(H)$. Using [51, Lemma 5.1] to determine the possibilities for $L \cap H$, we deduce that $|\Phi^+(D \cap H)| \geq 12$ and thus $\dim x^H \leq 32$ as required. Finally, suppose that $D = A_2A_5$, so $\dim x^G = 90$. Using the A_2 -density of $\Phi(D \cap H)$ in $\Phi(D)$, we deduce that $|\Phi^+(D \cap H)| \geq 6$ and thus $\dim x^H \leq 44$. This justifies the claim for semisimple elements.

Finally let us assume $r = 2$. The fusion of the H -classes of involutions in the connected component $H^0 = A_7$ is recorded in [49, Table 20], and we quickly deduce that (6) holds for all involutions $x \in H^0$, unless x has Jordan form $[J_2^4]$ on the natural module for A_7 (where J_2 denotes a standard unipotent Jordan block of size 2). Here x is in the G -class labelled $(3A_1)'$, so $\dim x^H = \frac{1}{2} \dim x^G = 32$. Finally, suppose $x \in H \setminus H^0$ is an involution. Let V_{56} be the 56-dimensional irreducible KG -module. Then [55, Proposition 2.3] gives

$$V_{56} \downarrow A_7 = L(\lambda_2) \oplus L(\lambda_6) = L(\lambda_2) \oplus L(\lambda_2)^*$$

and x interchanges the A_7 -modules $L(\lambda_2)$ and $L(\lambda_2)^*$, so x has Jordan form $[J_2^{28}]$ on V_{56} . In particular, [47, Table 7] indicates that x is in one of the G -classes labelled $(3A_1)''$ or $4A_1$. In fact, if $C_{H^0}(x) = C_{C_4}(t)$ (where $t \in C_4$ is a long root element) then the proof of [51, Lemma 4.1] reveals that x is in $4A_1$, so $\dim x^H = \frac{1}{2} \dim x^G = 35$. Now, if $C_{H^0}(x) = C_4$ then we can calculate the Jordan form of x on the Lie algebra $\text{Lie}(G)$ (using the fact that $\text{Lie}(G) \downarrow A_7 = \text{Lie}(A_7) \oplus L(\lambda_4)$, as noted in [55, Proposition 2.1]). We find that x has

Jordan form $[J_2^{53}, J_1^{27}]$, and by inspecting [47, Table 8] we conclude that x is in the class $(3A_1)''$. Therefore $\dim x^H = \frac{1}{2} \dim x^G = 27$. This justifies the claim. \square

Lemma 3.19. *Theorem 3.13 holds if $G = E_6$.*

Proof. We have $H = N_G(X)$ and $X \in \{D_5T_1, C_4(p \neq 2), A_1A_5, F_4\}$. First assume $X = D_5T_1$. Since $\dim H = 46 > \frac{1}{2} \dim G$ we deduce that $b^0(G, H) \geq 3$ by Proposition 2.5(iii). We claim that (5) holds for all $x \in \mathcal{P}$. If x is unipotent then the Bala-Carter labels for the classes x^G and x^H are the same, and we quickly deduce that (5) holds. Now assume $x \in H$ is semisimple and set $D = C_G(x)$. If D has a D_5 or A_5 factor then (5) follows from [51, Theorem 2]; in all other cases, the same result gives $\dim x^G - \dim x^H \geq 20$, so we may assume $\dim x^G \geq 60$. In fact, since $\dim x^H \leq 40$ for all $x \in H$, we reduce to the case $\dim x^G = 60$, so $D^0 = A_2^2T_2$ or A_3T_3 . Here the proof of [19, Lemma 4.17] yields $\dim x^G - \dim x^H \geq 24$. This justifies the claim, and we conclude that $b^0(G, H) = b(G, H) = b^1(G, H) = 3$.

Next consider the case $X = C_4$. Here $p \neq 2$ and $H = C_G(\tau)$, where $\tau \in \text{Aut}(G)$ is an involutory graph automorphism that inverts a maximal torus of G . Therefore Theorem 8 implies that (4) holds.

Next suppose $X = F_4$. Here $\dim H = 52 = \frac{2}{3} \dim G$, so $b^0(G, H) \geq 3$. We claim that $b^0(G, H) = b(G, H) = b^1(G, H) = 4$. To see this, let $\{\omega_1, \dots, \omega_6\}$ be a set of fundamental dominant weights for G and let $V = L(\omega_1)$ be the irreducible 27-dimensional KG -module with highest weight ω_1 . Then H is the G -stabilizer of a generic 1-dimensional subspace of V , so we can identify Ω with a non-empty open subvariety in the projective space $\mathbb{P}(V)$. We may also identify V with the coset variety E_7/P_7 , where P_7 is a maximal parabolic subgroup of E_7 with Levi subgroup $L = E_6T_1$. Now F_4 is the L -stabilizer of a generic point in E_7/P_7 , so L/F_4 is open in V (with T_1 acting as scalars). In particular, the generic 3-point (respectively 4-point) stabilizer for the original action of G on Ω is the same as the generic 5-point (respectively 6-point) stabilizer in the action of E_7 on E_7/P_7 . In Proposition 5.5 we will show that

$$b^0(E_7, P_7) = b(E_7, P_7) = b^1(E_7, P_7) = 6,$$

whence $b^0(G, H) = b(G, H) = b^1(G, H) = 4$ as claimed.

Finally, let us assume $X = A_1A_5$. If $p \neq 2$ then $H = C_G(z)$ for a suitable involution $z \in G$, and Theorem 8 implies that $b^0(G, H) \geq 3$ since z does not invert a maximal torus of G . We claim that (5) holds for all $x \in \mathcal{P}$, whence $b^0(G, H) = b(G, H) = b^1(G, H) = 3$ via Corollary 2.14. If $x \in H$ is unipotent then the desired bound quickly follows from the information in [49, Table 17], so assume x is semisimple and let $D = C_G(x)$. If D has a D_5 or A_5 factor then [51, Theorem 2] is sufficient. In all other cases we have $\dim x^G - \dim x^H \geq 24$ by [51, Theorem 2], so we may assume $\dim x^G = 72$. However, $\dim x^H \leq 32$ for all $x \in H$, and the claim follows.

Now suppose $p = 2$. As in Corollary 2.16, let \mathcal{P}' be the set of $x \in H$ of prime order r , where either $r \leq 3$ or $C_G(x)^0$ is semisimple. We claim that $\dim(x^G \cap H) \leq \frac{1}{2} \dim x^G$ for all $x \in \mathcal{P}'$, with equality if and only if $r = 2$ and x belongs to one of the G -classes labelled $2A_1$ or $3A_1$. In particular, a combination of Corollary 2.16 and Proposition 2.17 implies that (7) holds, but we have not determined the exact values of $b(G, H)$ and $b^1(G, H)$ in this case.

If $x \in H$ is an involution then the claim quickly follows from the fusion information presented in [49, Table 17]. For example, if $x = ([J_2], [J_2^3]) \in H$ then x is in the G -class labelled $3A_1$ and thus $\dim x^H = 2 + 18 = 20$ and $\dim x^G = 40$. Now assume $x \in \mathcal{P}'$ has odd prime order r . Set $D = C_G(x)$. If $r > 3$ then $\dim Z(D^0) > 0$, so we may assume $r = 3$, in which case $D^0 = A_5T_1, D_4T_2$ or A_2^3 (see [51, Proposition 1.2]). According to [51,

Lemma 5.1], the root system $\Phi(H)$ is A_2 -dense in $\Phi(G)$, whence $\Phi(D \cap H)$ is A_2 -dense in $\Phi(D)$. First assume $D = A_5T_1$, so $\dim x^G = 42$. Here the A_2 -density of $\Phi(D \cap H)$ in $\Phi(D)$ implies that $\Phi(D \cap H) = A_3A_1$ or A_2^2 . In particular, $|\Phi^+(D \cap H)| \geq 6$ so $\dim x^H \leq 20$ as required. Next suppose $D^0 = D_4T_2$. Here $\dim x^G = 48$ and the usual argument implies that $\Phi(D \cap H) = A_3$, whence $|\Phi^+(D \cap H)| \geq 6$ and thus $\dim x^H \leq 20$. Finally, if $D^0 = A_2^3$ then $\dim x^G = 54$ and the result follows since $\dim y^H \leq 26$ for all $y \in H$ of order 3. This justifies the claim. \square

Lemma 3.20. *Theorem 3.13 holds if $G = F_4$.*

Proof. Here $H = N_G(X)$ with $X \in \{B_4, C_4(p=2), A_1C_3(p \neq 2)\}$. First assume $X = B_4$, so $\dim H = 36 > \frac{2}{3} \dim G$ and thus $b^0(G, H) \geq 4$. A combination of [49, Table 13] and [51, Theorem 2] implies that $\dim x^H \leq \frac{3}{4} \dim x^G$ for all $x \in \mathcal{P}$, with equality if and only if x is a long root element. Therefore $b^1(G, H) \leq 4$ by Corollary 2.19 (note that every long root subgroup of $H = H^0$ is a long root subgroup of G), hence $b^0(G, H) = b(G, H) = b^1(G, H) = 4$. The case $X = C_4$ (with $p = 2$) now follows immediately since subgroups of type C_4 and B_4 are conjugate in $\text{Aut}(G)$; they are interchanged by an involutory graph automorphism.

Finally, suppose $X = A_1C_3$ and $p \neq 2$ (note that $A_1C_3 < F_4$ is non-maximal when $p = 2$; see [57, Table 10.3]). Here $H = C_G(\tau)$, where $\tau \in G$ is an involution which inverts a maximal torus of G . Therefore $b^0(G, H) = b(G, H) = 2$ and $b^1(G, H) = 3$ by Theorem 8. \square

Lemma 3.21. *Theorem 3.13 holds if $G = G_2$.*

Proof. Here $H = A_1\tilde{A}_1$. If $p \neq 2$ then $H = C_G(\tau)$, where $\tau \in G$ is an involution that inverts a maximal torus of G , so (4) holds by Theorem 8.

Now assume $p = 2$. We claim that $\dim(x^G \cap H) \leq \frac{1}{2} \dim x^G$ for all $x \in \mathcal{P}$, with equality if and only if x is an involution in the G -class labelled \tilde{A}_1 . To justify the claim, first assume $x \in H$ is semisimple. If $C_H(x)^0 = A_2$ then $\dim x^G = 6$ and [51, Theorem 2] indicates that $\dim x^H = 2$; in every other case we have $\dim x^G \geq 10$ and the desired bound follows since $\dim x^H \leq 4$ for all $x \in H$. Finally, if $x \in H$ is an involution then the desired result is easily deduced from [49, Table 10]. Therefore Proposition 2.17 implies that (7) holds when $p = 2$, but we are unable to determine the exact values of $b(G, H)$ and $b^1(G, H)$. \square

4. CLASSICAL GROUPS

Let G be a simple classical algebraic group over an algebraically closed field K of characteristic $p \geq 0$ with natural module V . In this section we complete the proofs of Theorems 4 and 5.

The main theorem on the subgroup structure of G is a result of Liebeck and Seitz [56], which provides a natural algebraic group analogue of Aschbacher's celebrated structure theorem [1] for finite classical groups. Following [56, Section 1], we introduce six natural, or *geometric*, collections of closed subgroups of G , labelled \mathcal{C}_i for $1 \leq i \leq 6$, and we set $\mathcal{C} = \bigcup_i \mathcal{C}_i$. A rough outline of the subgroups in each \mathcal{C}_i collection is given in Table 8.

The main theorem of [56] provides the following description of the maximal closed subgroups of G .

Theorem 4.1. *Let H be a closed subgroup of G . Then one of the following holds:*

- (i) H is contained in a member of \mathcal{C} ;

	Rough description
\mathcal{C}_1	Stabilizers of subspaces of V
\mathcal{C}_2	Stabilizers of orthogonal decompositions $V = \bigoplus_i V_i$
\mathcal{C}_3	Stabilizers of totally singular decompositions $V = V_1 \oplus V_2$
\mathcal{C}_4	Stabilizers of tensor product decompositions $V = \bigotimes_i V_i$
\mathcal{C}_5	Normalizers of symplectic-type r -groups, $r \neq p$ prime
\mathcal{C}_6	Classical subgroups

TABLE 8. The \mathcal{C}_i collections

- (ii) *modulo scalars, H is almost simple and $E(H)$ (the unique quasisimple normal subgroup of H) is irreducible on V . Further, if $G = \mathrm{SL}(V)$ then $E(H)$ does not fix a non-degenerate form on V . In addition, if H is infinite then $E(H) = H^0$ is tensor-indecomposable on V .*

Proof. This is [56, Theorem 1]. □

We will write \mathcal{S} to denote the collection of maximal closed subgroups of G that arise in part (ii) of Theorem 4.1. Note that the subgroups in \mathcal{C}_5 are finite so they can be discarded. Also notice that the members of \mathcal{C}_3 and \mathcal{C}_6 are involution-type subgroups, which we considered in the previous section.

In studying bases for a classical group G , it is natural to make a distinction between the primitive actions of G in which the point stabilizer H acts reducibly on V (the so-called *subspace actions* of G), and those in which H is irreducible. Indeed, for subspace actions it is easy to see that the various base measures can be arbitrarily large, whereas Theorem 2 states that $b^1(G, H) \leq 4$ when H is irreducible. We begin by considering subspace actions.

4.1. Subspace actions. Let G be a classical algebraic group in a primitive *subspace action* on a G -variety Ω . As defined in the Introduction, this means that either

- (i) Ω is an orbit of subspaces of the natural G -module V ; or
- (ii) The action of G on Ω is equivalent to the action of an isomorphic classical group L on an orbit of subspaces of the natural L -module (see Table 1).

The purpose of this section is to prove Theorem 4, and we begin our analysis by dealing with the linear groups.

4.1.1. *Linear groups.*

Proposition 4.2. *Let $G = \mathrm{SL}_n$ and let Ω be the set of d -dimensional subspaces of V , with $d \leq n/2$. Set $k = \lceil n/d \rceil$.*

- (i) *If d divides n then $b^0(G) = b(G) = b^1(G) = k + \epsilon$, where*

$$\epsilon = \begin{cases} 3 & \text{if } 1 < d = n/2 \\ 2 & \text{if } 1 < d < n/2 \\ 1 & \text{if } d = 1. \end{cases}$$

- (ii) *If d does not divide n then*

$$k + 1 \leq b^0(G) = b(G) = b^1(G) \leq k + 2 + \delta_{3,k}.$$

Proof. First note that the stabilizer in $\mathrm{GL}(V)$ of any given collection of subspaces of V coincides with the unit group of a suitable K -algebra and is therefore connected. Thus, the same is true in $G = \mathrm{SL}(V)$. It follows that $b^0(G) = b(G) = b^1(G)$ in all cases.

Consider (i). The case $d = 1$ is trivial. Next suppose $d = n/2$ and $n \geq 4$. Choose four generic d -dimensional subspaces of V , say V_1, V_2, V_3 and V_4 . By generic we mean that $V_i \cap V_j = 0$ for all $i \neq j$ (note that this is an open condition), so $V = V_1 \oplus V_2$ in particular. Let L be the stabilizer in G of V_1 and V_2 , so L is of type $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. Conjugating by L , we may assume that $V_3 = \{(v, f(v)) \mid v \in V_1\}$ and $V_4 = \{(v, g(v)) \mid v \in V_1\}$ for generic isomorphisms $f, g : V_1 \rightarrow V_2$. By fixing a suitable basis we have $f, g \in \mathrm{M}_{n/2}(K)$ (the algebra of $n/2 \times n/2$ matrices over K) and we are free to assume that f is the identity matrix. Since V_4 has been chosen generically, it follows that g is a regular semisimple matrix.

Suppose $x \in G$ fixes each of the subspaces V_1, V_2 and V_3 . Then x is a block-diagonal matrix of the form $\mathrm{diag}[y, y]$ with $y \in \mathrm{GL}_{n/2}(K)$, so x fixes V_4 if and only if y commutes with g . It follows that the common stabilizer of V_1, \dots, V_4 is a torus of dimension $n/2 - 1$, whence $b^0(G) > 4$. Now take $V_5 = \{(v, h(v)) \mid v \in V_1\}$ so that g and h generate the full matrix algebra $\mathrm{M}_{n/2}(K)$ (note that this is another generic condition). If $x = \mathrm{diag}[y, y] \in G$ stabilizes each V_i ($1 \leq i \leq 5$) then y commutes with both g and h , whence y (and thus x) is a scalar and we conclude that $b^1(G) = 5$, as claimed.

Now suppose $1 < d < n/2$ and d divides n , so $k = n/d$. Let V_1, \dots, V_k be generic d -dimensional subspaces of V , so $V = V_1 \oplus \dots \oplus V_k$. Choose another d -dimensional subspace V_{k+1} so that its intersection with any sum of the subspaces V_1, \dots, V_k (except $V = \sum_{i \leq k} V_i$ of course) is trivial. The G -stabilizer of V_1, \dots, V_k and V_{k+1} is isomorphic to $\mathrm{GL}(V_1)$ (embedded diagonally), whence $b^0(G) > k + 1$. Now take

$$V_{k+2} = \{(v, f_2(v), \dots, f_k(v)) \mid v \in V_1\},$$

where the matrices $f_i \in \mathrm{M}_d(K)$ generate the full matrix algebra. As before we deduce that the stabilizer in G of these $k + 2$ subspaces consists of scalars, hence $b^1(G) = k + 2$ as claimed.

Finally, let us turn to (ii), so d does not divide n . Since $k = \lceil n/d \rceil$ we have

$$(k - 1)d + 1 \leq n \leq kd - 1.$$

First we claim that $b^0(G) \geq k + 1$. Let V_1, \dots, V_k be generic d -dimensional subspaces. We may assume that $W = \sum_{i < k} V_i$ is a direct sum, and we may write $V_k = U \oplus (V_k \cap W)$ for some nontrivial subspace U . Clearly, any $x \in G$ that preserves U and acts as a scalar on W preserves each V_i . The claim follows.

To complete the proof, it suffices to produce $k + 2 + \delta_{3,k}$ subspaces of dimension d whose common stabilizer in G consists of scalars (for then the generic stabilizer of $k + 2 + \delta_{3,k}$ d -dimensional subspaces is finite, and therefore trivial by the remark at the beginning of the proof).

First assume that $k \geq 5$. Let V_1, \dots, V_k be generic d -dimensional spaces and set $W_1 = \sum_{i < k} V_i$ and $W_2 = \sum_{i > 1} V_i$. We may assume that W_1 and W_2 are direct sums. Let U_1 be the diagonal d -dimensional subspace of W_1 , and let $U_2 = \{(v, f_3(v), \dots, f_k(v)) \mid v \in V_2\}$ where each f_i is a generic isomorphism from V_2 to V_i . Arguing as above, if $x \in G$ preserves U_1, U_2 and each V_i then x must be a scalar on $V_2 \oplus \dots \oplus V_{k-1}$. Furthermore, since x preserves U_1 and U_2 it follows that x induces the same scalar on V_1 and V_k , whence $b^1(G) \leq k + 2$ as required.

For $k = 4$ we need to work a bit harder. Again, let V_1, \dots, V_4 be generic d -dimensional spaces. We may assume that $V = \sum_{i=1}^4 V_i$, $\dim(V_1 \cap V_4) = 1$ and that $V_1 + V_2 + V_3$ and $V_2 + V_3 + V_4$ are direct sums. Define U_1 and U_2 as in the previous paragraph. Suppose $x \in G$ preserves U_1, U_2 and each V_i . Let $x_i = x|_{V_i}$ denote the restriction of x to V_i . Then $x_1 = x_2 = x_3$ and x_1 commutes with f_3 . Moreover, x_4 is uniquely determined by x_2 and f_4 . In particular, since x_4 must preserve $V_1 \cap V_4$, generically this forces x_4 to be a scalar, so x itself is a scalar.

Finally, let us assume $k = 3$. Let V_1 and V_2 be generic d -dimensional subspaces. Let U_1 be the standard diagonal d -dimensional subspace of $V_1 \oplus V_2$ and let U_2 be an additional generic d -dimensional subspace of $V_1 \oplus V_2$. Also, let V_3 be another d -dimensional space such that $V = \sum_{i=1}^3 V_i$ and $\dim(V_1 \cap V_3) = 1$. Let U_3 be a generic d -dimensional subspace of $V_2 \oplus V_3$. Arguing as above, we deduce that any $x \in G$ preserving each of the subspaces V_i, U_i (for $1 \leq i \leq 3$) is a scalar, so $b^1(G) \leq 6$. \square

4.1.2. *Symplectic groups.* Now assume $G = \mathrm{Sp}_n$, where $n \geq 4$ is even. There are three cases to consider:

- (i) $H = G_U$ is the stabilizer of a non-degenerate d -dimensional subspace U of V , where $2 \leq d < n/2$ is even;
- (ii) $H = G_U$ is the stabilizer of a totally singular d -dimensional subspace U of V , where $1 \leq d \leq n/2$;
- (iii) $H = O_n$ and $p = 2$ (see Table 1).

First we deal with non-degenerate subspaces. Our main result is the following:

Proposition 4.3. *Let $G = \mathrm{Sp}_n$ and let Ω be the set of d -dimensional non-degenerate subspaces of V , with $d < n/2$. Set $k = \lceil n/d \rceil$. Then either*

$$b^0(G) = b(G) = b^1(G) = k,$$

or $n = 6$, $d = 2$ and $b^0(G) = b(G) = b^1(G) = 4$.

We require the following lemma concerning the (imprimitive) action of G on the set of $\frac{n}{2}$ -dimensional non-degenerate subspaces of V .

Lemma 4.4. *Let $G = \mathrm{Sp}_n$, where $n \equiv 0 \pmod{4}$, and let Ω be the set of $\frac{n}{2}$ -dimensional non-degenerate subspaces of V . Then*

$$b^0(G) = b(G) = b^1(G) = 3 + \delta_{4,n}.$$

Proof. By dimension, $b^0(G) \geq 3$. First assume $n \geq 8$. Let V_1, V_2, V_3 be generic subspaces in Ω . Let W be the orthogonal complement V_1^\perp , so $V = V_1 \perp W$. Without loss of generality we may assume that $V_i = \{(v, f_i(v)) \mid v \in V_1\}$ for $i = 2, 3$, where each f_i is an isomorphism $f_i : V_1 \rightarrow W$. Suppose $x \in G$ stabilizes each V_i . Since x stabilizes V_1 we can write $x = (x_1, x_2) \in \mathrm{Sp}(V_1) \times \mathrm{Sp}(W)$. Now x stabilizes V_2 and V_3 if and only if

$$(x_1(v), x_2 f_i(v)) = (x_1(v), f_i x_1(v))$$

for all $v \in V_1$ and $i = 2, 3$, or equivalently, $x_2 f_i = f_i x_1$ for $i = 2, 3$. Therefore, $t := f_2^{-1} f_3$ must commute with x_1 . Now generically, t is a regular semisimple element of $\mathrm{GL}_{n/2}$ and so its centralizer is a maximal torus T of $\mathrm{GL}_{n/2}$ (and an open subvariety of maximal tori in $\mathrm{GL}_{n/2}$ are of this form). If $n \geq 8$ and $T < \mathrm{GL}_{n/2}$ is a generic maximal torus then the linear span of any even number of eigenspaces will be non-degenerate, whence $T \cap \mathrm{Sp}_{n/2}$ is central and the result holds in this case.

Finally, suppose $n = 4$. Here $T \cap \mathrm{Sp}_2$ is a 1-dimensional torus and so the stabilizer of any three non-degenerate 2-dimensional spaces is positive-dimensional. Using the notation as above, let V_4 be another 2-dimensional space in Ω . Then x_1 must centralize $f_2^{-1} f_3$ and $f_3^{-1} f_4$, and generically they will have a trivial common centralizer. Therefore $b^0(G) = b(G) = b^1(G) = 4$ in this case. \square

Proof of Proposition 4.3. First observe that $k \geq 3$ and $(k-1)d + 2 \leq n \leq kd$, whence $b^0(G) \geq k$. Indeed, if V_1, \dots, V_{k-1} are generic elements of Ω then $W = V_1 \oplus \dots \oplus V_{k-1}$ is non-degenerate of dimension $(k-1)d$, so $\mathrm{Sp}(W^\perp)$ stabilizes each V_i and has positive dimension.

To begin with, let us assume $d \geq 4$. First consider the case $k = 3$. Let V_1, V_2, V_3 be generic elements of Ω , so $W = V_1 \oplus V_2$ is non-degenerate of dimension $2d$. Further, we may assume that there is a non-degenerate d -dimensional subspace W_1 of W such that $V_3 = \{(u, f(u)) \mid u \in W_1\}$ (in terms of the decomposition $V = W \perp W^\perp$) for some surjective linear map $f : W_1 \rightarrow W^\perp$. Suppose $x \in G$ preserves each V_i , and consider the restriction of x to W , which we denote by $x_1 \in \text{Sp}(W)$. Then x_1 preserves V_1, V_2 and W_1 , so Lemma 4.4 implies that x_1 is a scalar. Without loss of generality we may assume that $x_1 = 1$. Let x_2 denote the restriction of x to W^\perp . Then $x(u, f(u)) = (u, x_2 f(u)) = (u, f(u))$ and thus $x_2 = 1$ on the image of f . The result follows.

Next suppose $d, k \geq 4$. Let V_1, \dots, V_k be generic elements of Ω and assume $x \in G$ preserves each V_i . By the analysis of the case $k = 3$ in the previous paragraph, x acts as a scalar on each $V_i \oplus V_j \oplus V_\ell$ with $1 \leq i < j < \ell \leq k$, and the desired result follows.

Finally, let us consider the case $d = 2$. First assume $k = 3$ (so $n = 6$). Let V_1, V_2 and V_3 be generic 2-spaces in Ω . Set $W_1 = V_1$, $W_2 = V_1^\perp \cap V_{12}$ and $W_3 = V_{12}^\perp$, where $V_{12} = V_1 \oplus V_2$, so $V = W_1 \perp W_2 \perp W_3$. Note that if $x \in G$ stabilizes each V_i then it also stabilizes each W_i . We may assume that

$$V_2 = \{(v, f(v), 0) \mid v \in V_1\}, \quad V_3 = \{(v, f_2(v), f_3(v)) \mid v \in V_1\},$$

where $f, f_2 : V_1 \rightarrow W_2$ and $f_3 : V_1 \rightarrow W_3$ are isomorphisms. In particular, if $x \in G$ stabilizes each V_i then we may write $x = (x_1, x_2, x_3) \in \text{Sp}_2 \times \text{Sp}_2 \times \text{Sp}_2$, where $x_2 f = f x_1$, $x_2 f_2 = f_2 x_1$ and $x_3 f_3 = f_3 x_1$. It is straightforward to see that generically x_1 belongs to a torus of Sp_2 , whence $b^0(G) > 3$. Arguing as above shows that $b^1(G) = 4$, as required.

Now assume that $d = 2$ and $k \geq 4$. Let V_1, \dots, V_k be generic 2-spaces in Ω . Set W_i to be the (direct) sum of all $V_j, j \neq i$. Then each W_i is non-degenerate of codimension 2 in V . Assume that we have handled the case $k = 4$, then by induction any x preserving V_j ($j \neq i$) is a scalar on W_i whence on V . So consider the case $k = 4$ and assume that x preserves each V_i . Write $V = W_4 \oplus W_4^\perp$. Let V_4' be the projection of V_4 into W_4 with respect to this orthogonal decomposition of V . Then generically V_4' is a non-degenerate 2-space. If x preserves each V_i , then x also preserves V_4' and so by the case $k = 3$, x is a scalar on W_4 (and so similarly on W_i for each i), whence x is a scalar. We conclude that $b^1(G) = k$. \square

Next, let us turn our attention to stabilizers of totally singular subspaces.

Proposition 4.5. *Let $G = \text{Sp}_n$ and let Ω be the set of d -dimensional totally singular subspaces of V , with $d \leq n/2$. Set $k = \lceil n/d \rceil$. Then either*

$$b^0(G) = b(G) = b^1(G) = k,$$

or one of the following holds:

- (i) $n = 6, d = 2$ and $b^0(G) = b(G) = b^1(G) = 4$;
- (ii) $d = n/2$ and $b^0(G) = b(G) = 4, b^1(G) = 5 - \delta_{2,p}$.

We prove this result in a sequence of lemmas. First observe that $b^0(G) \geq k$. Indeed, if V_1, \dots, V_{k-1} are elements of Ω then there is a positive-dimensional unipotent subgroup of G that acts trivially on a hyperplane containing $V_1 + \dots + V_{k-1}$.

Lemma 4.6. *If $d = 1$ then $b^0(G) = b(G) = b^1(G) = n$.*

Proof. As above, $b^0(G) \geq n$. First assume $n = 4$. Let $\{e_1, e_2, e_3, e_4\}$ be a generic basis for V and let $\{f_1, f_2\}$ be a basis for $\langle e_1, e_2 \rangle^\perp$, so $e_3 = a_1 e_1 + a_2 e_2 + b_1 f_1 + b_2 f_2$ with all coefficients non-zero. Suppose $x \in G$ stabilizes each $\langle e_i \rangle$. Then $x e_3 = c e_3$ for some scalar $c \in K$, so $x e_1 = c e_1$ and $x e_2 = c e_2$ since x preserves $\langle f_1, f_2 \rangle = \langle e_1, e_2 \rangle^\perp$. Since $\langle e_1, e_2 \rangle$ is

non-degenerate, it follows that $c = \pm 1$. Therefore $xe_i = \pm e_i$ for all i , so x is a scalar and thus $b^1(G) = 4$ as required.

Now assume $n \geq 6$. Let V_1, \dots, V_n be generic elements of Ω . In particular, we may assume that any four distinct V_i generate a non-degenerate 4-dimensional subspace. Suppose that $x \in G$ preserves each V_i . By the previous paragraph, x is a scalar on the sum of any given four of the V_i . But the V_i generate V , so this implies that x is a scalar on V . \square

Lemma 4.7. *If $d = n/2$ then $b^0(G) = b(G) = 4$ and $b^1(G) = 5 - \delta_{2,p}$.*

Proof. Let H be the stabilizer of an element of Ω and let Q denote the unipotent radical of H . A generic 2-point stabilizer is a Levi subgroup $L = \mathrm{GL}_{n/2}$ of H (see Lemma 5.3 in Section 5.1). Moreover, since Q has a dense regular orbit on Ω , it suffices to compute the base size for the action of L on Q by conjugation.

As an L -module, Q is isomorphic to the symmetric square of the natural L -module, so a stabilizer in the conjugation action of L on Q corresponds to the stabilizer of a non-degenerate symmetric bilinear form. Now, if $p \neq 2$ then such a stabilizer is an orthogonal group $O_{n/2}$, and Theorem 3.13(i) implies that the intersection of two generic conjugates of $O_{n/2}$ is finite but not trivial. We conclude that $b^0(G) = b(G) = 4$ and $b^1(G) = 5$.

Now assume $p = 2$ and consider the L -stabilizer of a pair of generic non-degenerate symmetric bilinear forms. By conjugating we may assume that the first form is represented by the identity matrix $I = I_{n/2}$ and the second is represented by an invertible symmetric matrix S . The stabilizer of this pair consists of all $x \in L$ with $xx^\top = I$ and $xSx^\top = xSx^{-1} = S$. Generically, S is a regular semisimple matrix, so x is a polynomial in S and therefore x is symmetric. Thus, $x^\top = x = x^{-1}$ and so $x^2 = 1$. However, no involution commutes with a regular semisimple element, so the pairwise stabilizer is trivial and thus $b^0(G) = b(G) = b^1(G) = 4$. \square

Lemma 4.8. *Suppose $d \geq 2$ and $k = 3$. Then*

$$b^0(G) = b(G) = b^1(G) = 3 + \delta_{2,d}.$$

Proof. By definition of k we have $2d + 2 \leq n \leq 3d$. Let V_1, V_2, V_3 be generic elements of Ω . We may assume that $W_1 = V_1 \oplus V_2$ is non-degenerate of dimension $2d$. Set $W_3 = W_1^\perp$, so $V = V_1 \oplus V_2 \oplus W_3$ and note that we may assume that $V_3 = \{(v, f_2(v), f_3(v)) \mid v \in V_1\}$, where $f_2 : V_1 \rightarrow V_2$ is an isomorphism and $f_3 : V_1 \rightarrow W_3$ is a linear surjection. Suppose $x \in G$ stabilizes each V_i (and therefore also W_3). Let x_1, x_2 denote the restriction of x to V_1, V_2 , respectively, and let $x_3 \in \mathrm{Sp}_{n-2d}$ be the restriction of x to W_3 . Note that x preserves each V_i if and only if

$$x_2 = x_1^{-\top}, \quad x_1^\top f_2 x_1 = f_2 \quad \text{and} \quad x_3 f_3 = f_3 x_1.$$

It is not difficult to see that for a generic f_2 , the subgroup $\{y \in \mathrm{GL}_d \mid y^\top f_2 y = f_2\}$ is a torus T of dimension $\lfloor d/2 \rfloor$ (and the only scalar in T is in the center of Sp_d). Note that $x_1 \in T$, so x_1 preserves the form defined by f_2 . If $n < 3d$ then $d \geq 3$ and f_3 has a nontrivial kernel L , whence L must be x_1 -invariant (since $x_3 f_3 = f_3 x_1$). Given a generic subspace L , no nontrivial element of T preserves L , whence x_1 is trivial. Since $x_2 = x_1^{-\top}$ and $x_3 f_3 = f_3 x_1$, we deduce that x_2 and x_3 are also trivial, so x is trivial and thus $b^0(G) = b(G) = b^1(G) = 3$.

Finally, let us assume $n = 3d$, so $d \geq 2$ is even. Here f_3 is an isomorphism and thus $x_3 = f_3 x_1 f_3^{-1}$. If $d \geq 4$ then $f_3 T f_3^{-1} \cap \mathrm{Sp}_d$ coincides with the center of Sp_d , so $x_1 = \pm 1$ and x is a scalar. Again, we conclude that $b^0(G) = b(G) = b^1(G) = 3$. However, if $d = 2$ then the same argument shows that the stabilizer of three generic subspaces is a 1-dimensional torus, whence $b^1(G) \geq 4$. An easy argument now yields $b^0(G) = b(G) = b^1(G) = 4$. \square

Lemma 4.9. *Suppose $d \geq 2$ and $k \geq 4$. Then $b^0(G) = b(G) = b^1(G) = k$.*

Proof. Let V_1, \dots, V_k be generic subspaces in Ω and suppose $x \in G$ fixes each V_i . Set $W = V_1 + V_2 + V_3$ and note that we may assume this is a direct sum. Further, if d is even then we may assume W is non-degenerate. If $d \geq 4$ is even then Lemma 4.8 implies that the restriction of x to W is a scalar, and the result quickly follows. Now, if $d \geq 3$ is odd then we may assume that W has a 1-dimensional radical R and that each V_i ($1 \leq i \leq 3$) intersects R trivially. By Lemma 4.8, x is a scalar on W/R and is therefore a scalar on each V_i (and necessarily the same scalar). Again the result follows.

Finally, suppose $d = 2$. Arguing as above, we see that it suffices to prove the result for $k = 4$ (so $n = 8$). A minor variation of the previous argument gives the result; we leave the reader to check the details. \square

This completes the proof of Proposition 4.5. Finally, we deal with the one extra case that arises when $p = 2$.

Proposition 4.10. *Suppose $G = \mathrm{Sp}_n$, $p = 2$ and $\Omega = G/H$, where $H = O_n$. Then $b^0(G) = b(G) = n$ and $b^1(G) = n + 1$.*

Proof. We may view G as acting indecomposably on the orthogonal module M of dimension $n + 1$, so we can identify Ω with the set of non-degenerate hyperplanes in M . Now, if V_1, \dots, V_{n-1} are generic hyperplanes in Ω then their intersection is a 2-dimensional non-degenerate subspace X of M . Therefore, there is a positive-dimensional subgroup of G acting trivially on M/X , whence $b^0(G) \geq n$.

Let $V_0 \in \Omega$ denote the non-degenerate hyperplane fixed by H . Let V_1, \dots, V_{n-1} be generic elements of Ω , so $U_i = V_i \cap V_0$ is a hyperplane in V_0 for all $i \geq 1$. Generically, the radical of each U_i (with respect to the H -invariant alternating form on V_0) will be a 1-dimensional non-degenerate subspace. Let Λ denote the set of 1-dimensional non-degenerate subspaces of V_0 . By Lemma 4.23 (see Section 4.1.4), the stabilizer in H^0 of $n - 1$ generic elements in Λ is trivial, but the corresponding stabilizer in H has order 2 (indeed, there is a transvection $x \in H \setminus H^0$ fixing all $n - 1$ hyperplanes; see Remark 4.24). Therefore $b^0(G) = n$ and $b^1(G) = n + 1$.

To complete the proof, note that we can choose V_1, \dots, V_{n-1} so that the intersection of the V_i , $0 \leq i \leq n - 1$, is a 1-dimensional totally singular subspace of M . In this situation, there is no nontrivial element of H fixing the hyperplanes V_1, \dots, V_{n-1} , so $b(G) = n$ as required. \square

Remark 4.11. Proposition 4.10 implies that $b^0(G) = b(G) = n$ and $b^1(G) = n + 1$ for the equivalent action of $G = \mathrm{SO}_{n+1}$ (with $p = 2$) on the set of 1-dimensional non-singular subspaces of the natural module for G (see Table 1). In particular, we deduce that $b^0(G) = b(G) = 4$ and $b^1(G) = 5$ if $G = \mathrm{Sp}_4$, $p = 2$ and H is a \mathcal{C}_2 -subgroup $\mathrm{Sp}_2 \wr S_2$ (by Lemma 4.14, the same conclusion holds if $p \neq 2$).

4.1.3. *Orthogonal groups, $p \neq 2$.* In this section we deal with the subspace actions of orthogonal groups SO_n , where $p \neq 2$. We start by considering the stabilizers of non-degenerate subspaces.

Lemma 4.12. *Let $G = \mathrm{SO}_n$, where $p \neq 2$ and $n \geq 4$ is even. Let Ω be the set of $\frac{n}{2}$ -dimensional non-degenerate subspaces of V . Then $b^0(G) = b(G) = 2$ and $b^1(G) = 3$.*

Proof. Let H be the stabilizer of a subspace in Ω , so H is of type $O_{n/2} \times O_{n/2}$. If $n \geq 6$ then the result follows from Theorem 3.13(i), so let us assume $n = 4$. Here $G = A_1 A_1$ and H is contained in the normalizer of a maximal torus, so the same conclusion holds in this case too. \square

Proposition 4.13. *Let $G = \mathrm{SO}_n$ with $n \geq 7$ and let Ω be the set of d -dimensional non-degenerate subspaces of V , with $1 \leq d < n/2$. Set $k = \lceil n/d \rceil$ and assume $p \neq 2$. Then either*

$$b^0(G) = b(G) = b^1(G) = k,$$

or $n = (k-1)d + 1$, $b^0(G) = b(G) = k-1$ and $b^1(G) = k - \epsilon$, where $\epsilon = 1$ if n is even, otherwise $\epsilon = 0$.

The proof of Proposition 4.13 is given in the next two lemmas.

Lemma 4.14. *If $d = 1$ then $b^0(G) = b(G) = n-1$ and $b^1(G) = n - \epsilon$, where $\epsilon = 1$ if n is even, otherwise $\epsilon = 0$.*

Proof. It is convenient to prove this result for all $n \geq 3$. First observe that $b^0(G) \geq n-1$. Indeed, the sum of $n-1$ generic non-degenerate 1-spaces is a non-degenerate hyperplane, so the sum of $n-2$ generic elements of Ω is non-degenerate and thus their common stabilizer is positive-dimensional.

If $n = 3$ then the stabilizer of a non-degenerate 1-space is the normalizer of a maximal torus, whence the result is clear in this case (see the proof of Lemma 3.16, for example). Now assume $n \geq 4$. By induction, any $x \in G$ stabilizing $n-1$ generic elements of Ω must act as ± 1 on the corresponding non-degenerate hyperplane (the sum of $n-1$ spaces). If n is even, this forces x to be a scalar and the result follows. Now assume n is odd. Here, either x is a scalar or $-x$ is a reflection, so in this situation we have $b^1(G) = n$. Now we can also choose $n-1$ elements of Ω so that their sum is a hyperplane with a 1-dimensional radical; this forces x to be a scalar, so $b^0(G) = b(G) = n-1$. \square

Lemma 4.15. *If $d \geq 2$ then either $b^0(G) = b(G) = b^1(G) = k$, or $n = (k-1)d + 1$, $b^0(G) = b(G) = k-1$ and $b^1(G) = k - \epsilon$, where $\epsilon = 1$ if n is even, otherwise $\epsilon = 0$.*

Proof. By definition of k we have $(k-1)d + 1 \leq n \leq kd$ and $k \geq 3$. First assume $n > (k-1)d + 1$. As before, we have $b^0(G) \geq k$. Suppose $k = 3$ and let V_1, V_2, V_3 be generic elements of Ω . Without loss of generality we may assume that $W = V_1 \oplus V_2$ is a non-degenerate $2d$ -space. By Lemma 4.12, the stabilizer of V_1 and V_2 in $\mathrm{SO}(W)$ is finite. The common G -stabilizer of V_1, V_2 and V_3 preserves the orthogonal projection of V_3 into W , so this stabilizer acts as a scalar on $V = V_1 + V_2 + V_3$ and thus $b^0(G) = b(G) = b^1(G) = 3$. More generally, if $k \geq 4$ and V_1, V_2, V_3 are generic elements of Ω then any $x \in G$ that preserves each V_i acts as a scalar on $V_1 \oplus V_2 \oplus V_3$. In particular, if $x \in G$ stabilizes k generic elements of Ω then x is a scalar and the result follows.

Finally, let us assume $n = (k-1)d + 1$. Here $b^0(G) \geq k-1$. Let V_1, \dots, V_{k-1} be generic elements of Ω . Then $W = V_1 \oplus \dots \oplus V_{k-1}$ is a non-degenerate hyperplane. By the previous paragraph, any $x \in G$ preserving each V_i acts as ± 1 on W . If n is even then we immediately deduce that $b^1(G) = k-1$. If n is odd, let x be the reflection with fixed space W . Then $-x \in G$ fixes each of the V_i , whence $b^1(G) = k$ in this case.

We can also choose the V_i so that W is a hyperplane with a 1-dimensional radical R . It follows by induction that any $x \in G$ preserving each V_i must be a scalar on W/R . Since we may assume that R is not contained in any of the V_i , this implies that x is a scalar on W , and thus a scalar on the whole space V . We conclude that $b(G) = k-1$. \square

This completes the proof of Proposition 4.13. Next we turn our attention to totally singular subspaces, and we continue to assume that $p \neq 2$. Let H be the stabilizer of a totally singular d -dimensional subspace of V and set $\Omega = G/H$. Note that if $d < n/2$ then Ω is the set of all totally singular d -dimensional subspaces of V , whereas if $d = n/2$ then there are two distinct G -orbits on such subspaces, which are interchanged by a graph automorphism of $G = \mathrm{SO}_n$. In particular, if $d = n/2$ then the two G -actions are

permutation isomorphic. Our main result is the following, which we prove in Lemmas 4.17 – 4.21 below.

Proposition 4.16. *Let $G = \mathrm{SO}_n$ with $n \geq 7$, let H be the stabilizer of a totally singular d -dimensional subspace of V with $1 \leq d \leq n/2$ and set $\Omega = G/H$. Assume $p \neq 2$ and set $k = \lceil n/d \rceil$. Then either*

$$b^0(G) = b(G) = b^1(G) = k,$$

or one of the following holds:

- (i) $d = n/2$, $n \neq 10$ and $b^0(G) = b(G) = b^1(G) = c(n)$, where $c(8) = 7$, $c(12) = 6$ and $c(n) = 5$ for all $n \geq 14$;
- (ii) $n = 10$, $d = 5$ and $5 \leq b^0(G) \leq b^1(G) \leq 6$;
- (iii) $k = 3$ and $b^0(G) = b(G) = b^1(G) = 4 - \delta_{n,3d}$;
- (iv) $k \geq 4$, $n = (k-1)d + 1$, $b^0(G) = b(G) = k - 1$ and $b^1(G) = k - \epsilon$, where $\epsilon = 1$ if n is even, otherwise $\epsilon = 0$.

Lemma 4.17. *If $n \geq 5$ and $d = 1$ then $b^0(G) = b(G) = n - 1$ and $b^1(G) = n - \epsilon$, where $\epsilon = 1$ if n is even, otherwise $\epsilon = 0$.*

Proof. Clearly, $b^0(G) \geq n - 1$. We can choose $n - 1$ subspaces in Ω so that their sum is a hyperplane with a 1-dimensional radical (and the radical does not coincide with any of the $n - 1$ spaces). It follows that any element stabilizing this hyperplane is a scalar, whence $b^0(G) = b(G) = n - 1$. Generically, the hyperplane is non-degenerate and we now complete the argument by proceeding as in the proof of Lemma 4.14. \square

Lemma 4.18. *If $d = n/2$ then either $n = 10$ and $5 \leq b^0(G) \leq b^1(G) \leq 6$, or $b^0(G) = b(G) = b^1(G) = c(n)$ where $c(8) = 7$, $c(12) = 6$ and $c(n) = 5$ for all $n \geq 14$.*

Proof. First observe that $\dim \Omega = n^2/8 - n/4$ so Proposition 2.5(iii) yields $b^0(G) \geq 5$. To begin with, let us assume d is even. The intersection of two generic conjugates of H is a Levi subgroup $L \cong \mathrm{GL}_d$ of H (see Lemma 5.3). Let Q be the unipotent radical of H . Now Q has a dense orbit on Ω , so the intersection of three generic conjugates of H coincides with the stabilizer in L of a non-degenerate alternating form on the natural d -dimensional L -module U . Since d is even, this stabilizer is a symplectic group Sp_d . Consequently, a generic 4-point stabilizer in G is the intersection of the L -stabilizers of two non-degenerate alternating forms on U . The desired result now follows from [34, Theorem 1.1].

Next suppose $d \geq 7$ is odd. Let V_1, \dots, V_5 be generic subspaces in Ω . Then $W = V_1 + V_2$ is a hyperplane with a 1-dimensional radical R (note that any two complementary d -dimensional totally singular subspaces of V are in different G -orbits, since d is odd). In particular, if $i > 2$ then $V_i \cap W$ is a $(d - 1)$ -dimensional totally singular subspace of W (which intersects R trivially). We may assume that the $V_i \cap W$ are all in the same $\mathrm{SO}(W/R)$ -orbit, so by the previous paragraph it follows that the common G -stabilizer of the V_i induces a scalar on W/R . In fact, since $W = \sum_i (V_i \cap W)$, it follows that the common G -stabilizer acts as a scalar on W , and thus a scalar on V . We conclude that $b^0(G) = b(G) = b^1(G) = 5$ as required.

Finally, the same argument shows that $b^1(G) \leq 6$ if $d = 5$. \square

To complete the proof of Proposition 4.16, we may assume that $k \geq 3$ and $d \geq 2$.

Lemma 4.19. *Suppose $k \geq 5$ and $d \geq 2$. Then either $b^0(G) = b(G) = b^1(G) = k$, or $n = (k - 1)d + 1$, $b^0(G) = b(G) = k - 1$ and $b^1(G) = k - \epsilon$, where $\epsilon = 1$ if n is even, otherwise $\epsilon = 0$.*

Proof. By definition of k we have $(k-1)d+1 \leq n \leq kd$. For now let us assume $n > (k-1)d+1$, in which case $b^0(G) \geq k$ (since the sum of any $k-1$ subspaces in Ω has codimension at least 2). Let V_1, \dots, V_k be generic elements of Ω and let $W = V_1 \oplus \dots \oplus V_k$, so $\dim W = kd$ and W is non-degenerate. We may also assume that $V_1 \oplus V_2$ and $V_3 \oplus V_4$ are non-degenerate. By (the proof of) Lemma 4.12, the common stabilizer of $V_1 \oplus V_2$ and $V_3 \oplus V_4$ in $\mathrm{SO}(W)$ is an elementary abelian 2-group. The same is true for any such combination of V_1, \dots, V_k into complementary non-degenerate $2d$ -dimensional spaces. It quickly follows that the common stabilizer of V_1, V_2, V_3 and V_4 in G induces only scalars on W . The same is true for the sum of any four of the V_i , so the G -stabilizer of V_1, \dots, V_k acts as scalars on V and the result follows.

Now assume $n = (k-1)d+1$, so $b^0(G) \geq k-1$. Let V_1, \dots, V_{k-1} be generic elements of Ω , so $W = \sum_i V_i$ is a non-degenerate hyperplane. The common G -stabilizer of the V_i acts as a scalar ± 1 on W , so $b^0(G) = b(G) = k-1$ and in the usual way we deduce that $b^1(G) = k - \epsilon$, where $\epsilon = 1$ if n is even, otherwise $\epsilon = 0$. \square

Lemma 4.20. *Suppose $k = 4$ and $d \geq 2$. Then either $b^0(G) = b(G) = b^1(G) = 4$, or $n = 3d+1$, $b^0(G) = b(G) = 3$ and $b^1(G) = 4 - \epsilon$, where $\epsilon = 1$ if n is even, otherwise $\epsilon = 0$.*

Proof. Note that $3d+1 \leq n \leq 4d$ and

$$\frac{\dim G}{\dim \Omega} = \frac{n(n-1)}{d(2n-3d-1)} \geq 3,$$

with equality if and only if $n = 3d+1$.

If $n = 3d+1$ then $b^0(G) \geq 3$ (by Proposition 2.5(iii)) and the result follows by repeating the argument in the final paragraph of the proof of Lemma 4.19. Now assume $n \geq 3d+2$, so $b^0(G) \geq 4$. Let V_1, \dots, V_4 be generic subspaces in Ω . Let $W = V_1 + V_2$, so W is non-degenerate and $2d$ -dimensional. Generically, the orthogonal projections of V_3, V_4 into W and W^\perp are injective with non-degenerate images (these are open conditions, so one only has to see that it is possible). If $x \in G$ preserves each V_i then x preserves two non-degenerate d -dimensional subspaces of W , namely the projections of V_3 and V_4 . By (the proof of) Lemma 4.12, the stabilizer of these d -spaces in $\mathrm{SO}(W)$ is a finite 2-group. However, x also preserves V_1 and V_2 , so x must induce a scalar on W . By symmetry, the same is true for each combination $V_i + V_j$, whence x is a scalar on V and the result follows. \square

Finally, let us assume $k = 3$. Note that $d \geq 3$ since $n \geq 7$.

Lemma 4.21. *Suppose $k = 3$ and $d \geq 3$. Then $b^0(G) = b(G) = b^1(G) = 4 - \delta_{n,3d}$.*

Proof. Since $k = 3$ we have $2d+1 \leq n \leq 3d$. As in the proof of the previous lemma we have $\dim G/\dim \Omega \geq 3$, with equality if and only if $n = 3d$. Thus, $b^0(G) \geq 4$ unless possibly $n = 3d$, in which case $b^0(G) \geq 3$.

Let H be the stabilizer of a subspace in Ω and note that the generic intersection of two conjugates of H is a Levi subgroup $L \cong \mathrm{GL}_d \times \mathrm{SO}_{n-2d}$ (see Lemma 5.3). Now $H = QL$, where Q is the unipotent radical of H , and since Q has a regular dense orbit on Ω it suffices to compute the base size for the action of L on Q by conjugation. Let X and Y be the natural modules for GL_d and SO_{n-2d} , respectively.

Now Q has a normal subgroup Q_1 with $Q_1 \cong \Lambda^2(X)$ as a GL_d -module (with SO_{n-2d} acting trivially) and $Q/Q_1 \cong Q_2 \cong X \otimes Y$ as an L -module. Since $Q \cong Q_1 \times Q_2$ as varieties, it is sufficient to consider the action of L on $Q_1 \times Q_2$. Here the point stabilizer in L of a generic point (q_1, q_2) is isomorphic to the intersection of the stabilizer $L_1 < \mathrm{GL}_d$ of a non-degenerate alternating form on X and a subgroup L_2 fixing an $(n-2d)$ -dimensional subspace X' of X (more precisely, L_2 acts as $\mathrm{SO}(X')$ on X').

First assume $n = 3d$. Then $L_1 \cong \mathrm{Sp}_d$ and $L_2 \cong \mathrm{SO}_d$, embedded diagonally in L . In particular, we see that $L_1 \cap L_2 = 1$, whence $b^1(G) = 3$ and the result follows. Now assume $n < 3d$. It is straightforward to show that the intersection of two generic conjugates of such a subgroup $L_1 \cap L_2$ of L is trivial. For example, if $n = 2d + 1$ then L_1 is the stabilizer of an alternating form and L_2 is the stabilizer of a vector $x \in X$. The result follows. \square

This completes the proof of Proposition 4.16.

4.1.4. *Orthogonal groups, $p = 2$.* To complete the analysis of subspace actions we may assume that $G = \mathrm{SO}_n$, where $n \geq 7$ and $p = 2$. The arguments are similar (and often easier) to the case $p \neq 2$. The main difference here is that we may assume n is even. In addition, in the analysis of non-degenerate d -dimensional subspaces we may assume that $d = 1$ or d is even. Indeed, any odd dimensional space has a radical when considered as an alternating space, so the action is imprimitive if the dimension is greater than 1. (For convenience, we will refer to 1-dimensional *non-degenerate* subspaces, although strictly speaking we should use the term *non-singular*.)

Proposition 4.22. *Let $G = \mathrm{SO}_n$, where $p = 2$ and $n \geq 8$ is even. Let Ω be the set of d -dimensional non-degenerate subspaces of V , where $d < n/2$ and either $d = 1$ or d is even. Set $k = \lceil n/d \rceil$. Then $b^0(G) = b(G) = b^1(G) = k - \epsilon$, where $\epsilon = 1$ if $n = (k - 1)d + 1$, otherwise $\epsilon = 0$.*

Lemma 4.23. *Assume that $n \geq 4$ is even and let Ω be the set of 1-dimensional non-degenerate or totally singular subspaces of V . Then $b^0(G) = b(G) = b^1(G) = n - 1$.*

Proof. We induct on n . First assume that $n = 4$, so $G = \mathrm{SL}_2 \times \mathrm{SL}_2$. The stabilizer of a singular 1-space is a Borel subgroup and thus $b^0(G) = b(G) = b^1(G) = 3$. The stabilizer of a non-degenerate 1-space is a diagonal copy of SL_2 (the centralizer of an outer involution), and the result is an easy computation. (In the latter case we could start the induction at $n = 2$, where the stabilizer of a non-degenerate 1-space is trivial.)

Clearly we have $b^0(G) \geq n - 1$. Let V_1, \dots, V_{n-1} be generic subspaces in Ω . We may assume that the sum $W = \sum_i V_i$ is a hyperplane with a 1-dimensional radical R (with respect to the underlying symmetric form on V). Moreover, we may assume that the defining quadratic form on V does not vanish on R . By induction, it follows that the common G -stabilizer of each V_i is trivial on W/R , and therefore trivial on each V_i . In particular, the common stabilizer is trivial on W and so also on V . The result follows. \square

Remark 4.24. In the previous lemma, if $G = O_n$ is the full orthogonal group and Ω is the set of 1-dimensional non-degenerate subspaces of V , then $b^0(G) = b(G) = n - 1$ and $b^1(G) = n$. More precisely, the G -stabilizer of $n - 1$ generic elements of Ω contains a transvection and has order 2.

The next result shows that the conclusion to Lemma 4.12 also holds when $p = 2$.

Lemma 4.25. *Suppose $n \equiv 0 \pmod{4}$ and let Ω be the set of $\frac{n}{2}$ -dimensional non-degenerate subspaces of V . Then $b^0(G) = b(G) = 2$ and $b^1(G) = 3$.*

Proof. Let H be the stabilizer of a subspace in Ω , so H is of type $O_{n/2} \times O_{n/2}$. It is straightforward to see that the G -stabilizer of two generic subspaces in Ω coincides with the intersection in $\mathrm{GL}_{n/2}$ of two generic conjugates of $O_{n/2}$.

Now, the generic intersection in $\mathrm{GL}_{n/2}$ of two conjugates of $\mathrm{Sp}_{n/2}$ is isomorphic to the direct product of $n/4$ copies of SL_2 . Therefore, the intersection of generic conjugates of $O_{n/2}$ and $\mathrm{Sp}_{n/2}$ is the normalizer of a torus in the direct product $(\mathrm{SL}_2)^{n/4}$. Consequently,

we deduce that the intersection in $\mathrm{GL}_{n/2}$ of two generic conjugates of $O_{n/2}$ is elementary abelian of order $2^{n/4}$, and it can be trivial. The result follows. \square

The remainder of the proof of Proposition 4.22 is entirely similar to the argument given in the case $p \neq 2$, the only difference being that certain cases do not arise when $p = 2$. We leave the details to the reader.

Finally, let us consider the stabilizers of totally singular subspaces.

Proposition 4.26. *Let $G = \mathrm{SO}_n$, where $p = 2$ and $n \geq 8$ is even. Let H be the stabilizer of a totally singular d -dimensional subspace of V with $1 < d \leq n/2$ and set $\Omega = G/H$ and $k = \lceil n/d \rceil$. Then either*

$$b^0(G) = b(G) = b^1(G) = k,$$

or one of the following holds:

- (i) $d = n/2$, $n \neq 10$ and $b^0(G) = b(G) = b^1(G) = c(n)$, where $c(8) = 7$, $c(12) = 6$ and $c(n) = 5$ for all $n \geq 14$;
- (ii) $n = 10$, $d = 5$ and $5 = b^0(G) \leq b(G) \leq b^1(G) \leq 6$;
- (iii) $k = 3$ and $b^0(G) = b(G) = b^1(G) = 4 - \delta_{n,3d}$;
- (iv) $k \geq 4$, $n = (k-1)d + 1$ and $b^0(G) = b(G) = b^1(G) = k - 1$.

Once again, the proof of this proposition is very similar to the case $p \neq 2$ (see Proposition 4.16). We leave the reader to make the necessary minor modifications. Note that $b^0(G) = 5$ in case (ii): the usual argument yields $b^0(G) \geq 5$, and a straightforward MAGMA calculation gives $b(G_\sigma) \leq 5$ for the corresponding action of $G_\sigma = \Omega_{10}^+(4)$, so $b^0(G) \leq 5$ by Proposition 2.7(ii).

This completes the proof of Theorem 4.

4.2. Non-subspace actions. Here we complete the proof of Theorem 5. Let G be a simple classical algebraic group over an algebraically closed field of characteristic $p \geq 0$ and let Ω be a primitive non-subspace G -variety with point stabilizer H . By the main theorem of [56] (see Theorem 4.1), we may assume that H is a positive-dimensional subgroup in one of the collections $\mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_6$ or \mathcal{S} . In fact, in view of Theorem 3.13, we may assume that $H \in \mathcal{C}_2 \cup \mathcal{C}_4 \cup \mathcal{S}$. Our first result deals with the tensor product subgroups in \mathcal{C}_4 and the irreducible almost simple subgroups in \mathcal{S} .

Proposition 4.27. *If $H \in \mathcal{C}_4 \cup \mathcal{S}$ is positive-dimensional then one of the following holds:*

- (i) $b^0(G, H) = b(G, H) = b^1(G, H) = 2$; or
- (ii) $(G, H) = (\mathrm{SO}_7, G_2)$ (with $p \neq 2$) or (Sp_6, G_2) (with $p = 2$), and

$$b^0(G, H) = b(G, H) = b^1(G, H) = 4.$$

Proof. Assume $p > 0$ and let K be the algebraic closure of the prime field \mathbb{F}_p . Let σ be a Frobenius morphism of G such that G_σ is an almost simple classical group over \mathbb{F}_q , where q is a p -power. We may assume H is σ -stable. If $(G, H) \neq (\mathrm{SO}_7, G_2)$ or (Sp_6, G_2) then the proof of the main theorem of [18] implies that $b(G_\sigma, H_\sigma) = b^\infty(G_\sigma, H_\sigma) = 2$ with respect to the action of G_σ on G_σ/H_σ . Therefore Proposition 2.7(i) yields $b^1(G, H) = 2$, hence (i) holds. (Note that we could verify this independently of [18], by applying Theorem 2.13, but it is convenient to use our results for the corresponding finite group actions. In this way we see that the same conclusion holds if K is *any* algebraically closed field of characteristic $p \geq 0$ (this is discussed in more detail at the end of Section 2).)

Now assume $(G, H) = (\mathrm{SO}_7, G_2)$ or (Sp_6, G_2) . By considering the corresponding action of G_σ on the set of cosets of $G_2(q)$, and by inspecting the proof of [54, Proposition 2], we

deduce that the generic 2-point stabilizer in the action of G on Ω has connected component A_2 . Now $\dim G_2 + \dim A_2 > \dim G$, so every 3-point stabilizer is positive-dimensional and thus $b^0(G, H) \geq 4$. According to the proof of [13, Lemma 7.7] we have $\dim x^H \leq \frac{3}{4} \dim x^G$ for all $x \in H$ of prime order (including all nontrivial unipotent elements if $p = 0$), with equality if and only if x is a long root element. Therefore Corollary 2.19 implies that $b^1(G, H) \leq 4$, as required (note that each long root subgroup of $H = H^0$ is a long root subgroup of G). \square

Proof of Theorem 5.

We may assume H is a \mathcal{C}_2 -subgroup that stabilizes a direct sum decomposition

$$V = V_1 \oplus \cdots \oplus V_t$$

with $t \geq 3$ (if $t = 2$ then H is one of the involution-type subgroups considered in Lemma 3.16). If $G = \mathrm{SL}_n$ or SO_n then [14, Proposition 2.1] implies that $\dim x^H \leq \frac{1}{t} \dim x^G$ for all $x \in H$ of prime order, whence Corollary 2.14 yields

$$b^0(G, H) = b(G, H) = b^1(G, H) = 2. \quad (8)$$

Now assume $G = \mathrm{Sp}_n$ and H is of type $\mathrm{Sp}_{n/t} \wr S_t$ with $t \geq 3$. Here [14, Proposition 2.1] yields

$$\dim x^H \leq \left(\frac{1}{t} + \frac{2}{n+2} \right) \dim x^G$$

for all $x \in H$ of prime order (and all nontrivial unipotent elements if $p = 0$), so Corollary 2.14 implies that (8) holds unless $(n, t) = (6, 3)$. Here $b^1(G, H) \leq 3$ and we claim that

$$b^0(G, H) = b(G, H) = b^1(G, H) = 3.$$

By [34, Lemma 4.1], there is a *self-adjoint* element $g \in \mathrm{GL}_6$ such that $C_{\mathrm{GL}_6}(g) = H$, so according to [34, Lemma 2.2] there exists $x \in \mathrm{GL}_6$ with $G \cap G^x = H$. In particular, if $y \in G$ then

$$H \cap H^y = G \cap G^x \cap G^{xy},$$

so [34, Lemma 5.7] implies that $\dim(H \cap H^y) > 0$. Therefore $b^0(G, H) \geq 3$ and the claim follows.

This completes the proof of Theorem 5.

5. EXCEPTIONAL GROUPS

In this section we complete the proof of Theorems 6 and 7. Let G be a simple exceptional algebraic group over an algebraically closed field K of characteristic $p \geq 0$. Let us recall the main theorem on the subgroup structure of G , which is due to Liebeck and Seitz [57].

Theorem 5.1. *Let H be a positive-dimensional maximal closed subgroup of G . Then one of the following holds:*

- (i) H is a parabolic subgroup;
- (ii) $G = E_7$, $p \neq 2$ and $H = (2^2 \times D_4).S_3$;
- (iii) $G = E_8$, $p \neq 2, 3, 5$ and $H = A_1 \times S_5$;
- (iv) $H = N_G(X)$, with X given in Table 9.

Proof. This is [57, Corollary 2]. Note that in Table 9, $D_4 < F_4$ is the subgroup generated by all long root subgroups, and if $p = 2$ we write $\tilde{D}_4 < F_4$ to denote the subgroup generated by all short root subgroups. Similarly, we define $A_2 < G_2$ and $\tilde{A}_2 < G_2$ (if $p = 3$). \square

G	X	$N_G(X)/X$
E_8	$A_1, B_2, A_1A_2, A_1G_2^2 (p \neq 2), G_2F_4$	$1, 1, Z_2, Z_2, 1$
	$D_8, A_1E_7, A_8, A_2E_6, A_4^2, D_4^2$	$1, 1, Z_2, Z_2, Z_4, Z_2 \times S_3$
	A_2^4, A_1^8, T_8	$\mathrm{GL}_2(3), \mathrm{AGL}_2(3), 2.O_8^+(2)$
E_7	$A_1, A_2, A_1^2, A_1G_2, A_1F_4, G_2C_3$	$1, Z_2, 1, 1, 1, 1$
	$T_1E_6, A_1D_6, A_7, A_2A_5$	$Z_2, 1, Z_2, Z_2$
	$A_1^3D_4, A_1^7, T_7$	$S_3, \mathrm{GL}_3(2), Z_2 \times \mathrm{Sp}_6(2)$
E_6	$A_2, G_2, C_4 (p \neq 2), F_4, A_2G_2$	$Z_2, 1, 1, 1, Z_2$
	$T_1D_5, T_2D_4, A_1A_5, A_2^3, T_6$	$1, S_3, 1, S_3, O_6^-(2)$
F_4	A_1, G_2, A_1G_2, A_1C_3	$1, 1, 1, 1$
	$B_4, C_4 (p = 2), D_4, \tilde{D}_4 (p = 2), A_2\tilde{A}_2$	$1, 1, S_3, S_3, Z_2$
G_2	$A_1, A_1\tilde{A}_1, A_2, \tilde{A}_2 (p = 3)$	$1, 1, Z_2, Z_2$

TABLE 9. Some maximal non-parabolic subgroups of exceptional groups

5.1. Parabolic actions. First let us consider Theorem 6, so $\Omega = G/H$ and H is a maximal parabolic subgroup of G . Recall that H is conjugate to a standard parabolic subgroup P_i for some $1 \leq i \leq r$, where r denotes the rank of G . Further, this notation indicates that if L_i is a Levi subgroup of P_i then the root system of the semisimple group L_i' corresponds to the Dynkin diagram of G with the i -th node deleted. We continue to follow Bourbaki [12] in the labelling of Dynkin diagrams.

Proposition 5.2. *Let G be a simple exceptional algebraic group and let $\Omega = G/H$, where $H = P_i$ is a maximal parabolic subgroup of G . Then*

$$c - \epsilon \leq b^0(G, H) \leq b(G, H) \leq b^1(G, H) \leq c,$$

where c is defined in Table 11. Here an asterisk indicates that $\epsilon = 1$, otherwise $\epsilon = 0$ and thus $b^0(G, H) = b(G, H) = b^1(G, H) = c$.

Proof. Let $P_i = Q_iL_i$ be a Levi decomposition of P_i and observe that $\dim \Omega = \dim Q_i = |\Phi^+(G)| - |\Phi^+(L_i')|$; for the reader's convenience we record this dimension in Table 10. By Proposition 2.5(iii) we have $b^0(G, H) \geq \dim G / \dim \Omega$, while an upper bound for $b^1(G, H)$ is obtained by combining Proposition 2.7(i) and [19, Theorem 3]. The result follows. \square

	$H = P_1$	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$G = E_8$	78	92	98	106	104	97	83	57
E_7	33	42	47	53	50	42	27	
E_6	16	21	25	29	25	16		
F_4	15	20	20	15				
G_2	5	5						

 TABLE 10. G exceptional, $\dim G/P_i$

In order to complete the proof of Theorem 6 we may assume that (G, H) is one of the following cases:

$$(E_8, P_7), (E_7, P_3), (E_7, P_7), (E_6, P_1), (E_6, P_2), (E_6, P_6).$$

The next lemma is a key result in our analysis, and it holds for any semisimple algebraic group G over an algebraically closed field.

	$H = P_1$	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$G = E_8$	4	3	3	3	3	3	4*	5
E_7	5	4	4*	3	3	4	6*	
E_6	6*	5*	4	4*	4	6*		
F_4	5*	4*	4*	5*				
G_2	4*	4*						

TABLE 11. G exceptional, H parabolic

Lemma 5.3. *Let P be a maximal parabolic subgroup of G and set $\Omega = G/P$. Let $P = QL$ be a Levi decomposition of P , and assume P is G -conjugate to its opposite parabolic $P^- = UL$. Then the generic 2-point stabilizer in the action of G on Ω is conjugate to L .*

Proof. First observe that $Q \cap P^- = 1$, so Q has a regular orbit on Ω . Since $\dim \Omega = \dim G/P = \dim Q$, this orbit is open and dense in Ω . Moreover, this orbit is also L -invariant. Therefore, if $P = G_\alpha$ then the 2-point stabilizer $G_{\alpha,\beta}$ is a conjugate of L for any point β in the open Q -orbit. The result follows. \square

Note that if G is an exceptional group, the previous lemma applies unless $G = E_6$ and $P = P_1, P_3, P_5$ or P_6 .

Proposition 5.4. *Suppose $G = E_6$ and $H = P_1$ or P_6 . Then $b^0(G, H) = b(G, H) = b^1(G, H) = 6$.*

Proof. Since P_1 and P_6 are interchanged by an involutory graph automorphism of G , we may assume $H = P_1$. Here $\dim H = 62$, $\dim \Omega = 16$ and

$$5 \leq b^0(G, H) \leq b^1(G, H) \leq 6$$

(see Proposition 5.2), so it remains to show that the generic 5-point stabilizer is positive-dimensional. To do this, we may assume that $p > 0$.

Let q be a p -power. In the terminology of Cohen and Cooperstein [24], the corresponding action of $E_6(q)$ is equivalent to the action on the subset of *white points* in the standard 27-dimensional $E_6(q)$ -module. This transitive action has permutation rank 3, and from the description of the suborbits (see [24, (P.1), p.470]) we deduce that the generic 2-point stabilizer for the original parabolic action of G is of the form UD_4T_2 , where U is a 16-dimensional unipotent subgroup. Moreover, U is a vector space and $U \downarrow D_4 = U_1 \oplus U_2$, where U_1 and U_2 are distinct irreducible 8-dimensional modules for D_4 . It follows that U has a 16-dimensional regular orbit \mathcal{O} on $\Omega = G/H$, whence \mathcal{O} is open (and thus dense) in Ω . In particular, we may identify \mathcal{O} with U and thus the generic 5-point stabilizer of G on Ω is the same as the generic 2-point stabilizer of D_4T_2 on U .

Consider two generic points in $U = U_1 \oplus U_2$, say $u_1 + u_2$ and $v_1 + v_2$, where $u_i, v_i \in U_i$ and each $\langle u_i, v_i \rangle$ is a non-degenerate 2-space. The D_4 -stabilizer of these two generic points is the subgroup fixing each vector u_1, v_1, u_2, v_2 , which is of the form $D_3 \cap D_3^g$ for some $g \in D_4$. Now $\dim D_4 = 28$ and $\dim D_3 = 15$, so $\dim(D_3 \cap D_3^g) \geq 2$ and thus the generic 5-point stabilizer of G on Ω is at least 2-dimensional. Therefore $b^0(G, H) \geq 6$ and the result follows. \square

Proposition 5.5. *Suppose $(G, H) = (E_7, P_7)$. Then $b^0(G, H) = b(G, H) = b^1(G, H) = 6$. Moreover, the generic 5-point stabilizer is 8-dimensional.*

Proof. Here $\dim H = 106$, $\dim \Omega = 27$ and $5 \leq b^0(G, H) \leq b^1(G, H) \leq 6$ (see Proposition 5.2), so as in the proof of the previous proposition we need to show that $b^0(G, H) > 5$. Let $H = QL$ be a Levi decomposition, so $L = E_6T_1$ and Q is abelian. By Lemma 5.3, we

may assume that L is the generic 2-point stabilizer. Moreover, Q has a regular open orbit \mathcal{O} on Ω , on which L acts by conjugation, so it suffices to show that the generic 3-point stabilizer in the conjugation action of L on Q is positive-dimensional. We may assume $p > 0$.

Let q be a p -power. At the finite level, we may identify Q with the standard 27-dimensional $E_6(q)$ -module. In the terminology of Cohen and Cooperstein [24], the generic $E_6(q)$ -orbit on this module coincides with the subset of *black points*. This orbit has point stabilizer $F_4(q)$, so we see that F_4 is the generic 3-point stabilizer in the original action of G . Further, by considering [24, Table 2] we deduce that the generic 4-point stabilizer has connected component D_4 . Now

$$Q \downarrow D_4 = V_1 \oplus V_2 \oplus V_3 \oplus 0 \oplus 0 \oplus 0,$$

where V_1, V_2, V_3 are the distinct irreducible 8-dimensional D_4 -modules, and 0 is the 1-dimensional trivial D_4 -module (see [55, Proposition 2.3]). A generic vector in Q is of the form $v = v_1 + v_2 + v_3$, where each $v_i \in V_i$ spans a non-degenerate subspace, so the generic 5-point stabilizer in G is the intersection in D_4 of three conjugates of a subgroup $B_3 < D_4$. Since $\dim D_4 = 28$ and $\dim B_3 = 21$, it follows that the generic 5-point stabilizer is at least 7-dimensional, whence $b^0(G, H) > 5$ as required.

Finally, let us show that the generic 5-point stabilizer is 8-dimensional. First observe that the intersection of two generic conjugates of $B_3 < D_4$ is a subgroup $G_2 < D_4$ (one way to see this is to consider the corresponding situation at the level of finite groups; see the proof of [54, Proposition 3]). Moreover, the intersection of G_2 with an additional generic conjugate of B_3 is isomorphic to A_2 (again, this follows from the proof of [54, Proposition 3]). The claim follows. Indeed, this shows that the generic 5-point stabilizer is precisely A_2 . \square

Remark 5.6. Recall that Theorem 13 states that there are infinitely many non-standard finite almost simple primitive permutation groups with base size 6. This quickly follows from Proposition 5.4 above. Indeed, assume $p > 0$, let $G = E_6$ and let H be a σ -stable P_1 parabolic subgroup of G , where G is defined over the algebraic closure $\overline{\mathbb{F}}_p$ and σ is a Frobenius morphism of G so that G_σ has socle $E_6(q)$ for some p -power q . Now $b^0(G, H) = 6$ by Proposition 5.4, so Proposition 2.7(ii) implies that $b(G_\sigma, H_\sigma) \geq 6$ for all $q > 2$, while the main theorem of [19] yields $b(G_\sigma, H_\sigma) \leq 6$. Therefore $b(G_\sigma, H_\sigma) = 6$ for all $q > 2$, and this establishes Theorem 13. In fact, by using a suitable permutation representation of $E_6(2)$, it is straightforward to show that $b(G_\sigma, H_\sigma) = 6$ when $q = 2$ (see [19, Remark 1]). Similarly, if $G = E_7$ and $H = P_7$ then Proposition 5.5 implies that $b(G_\sigma, H_\sigma) = 6$ for all q . (Since the generic 5-point stabilizer in G is 8-dimensional, it is not a split torus and thus [34, Proposition 8.1] implies that every 5-point stabilizer in G_σ is nontrivial when $q = 2$.)

In order to complete the proof of Theorem 6 we may assume that $(G, H) = (E_8, P_7)$, (E_7, P_3) or (E_6, P_2) . In particular, note that Lemma 5.3 applies in each of these cases.

We need to introduce some new notation and terminology that we will use for the remainder of this section. Fix a maximal torus T of G , let Φ denote the root system of G , $\Delta = \{\alpha_1, \dots, \alpha_r\}$ a set of simple roots (with the usual labelling), Φ^+ the corresponding set of positive roots, and let $\{U_\alpha \mid \alpha \in \Phi\}$ be the root subgroups of G . Suppose $H = P_i$. Let Φ_J be the root system spanned by the simple roots $J = \Delta \setminus \{\alpha_i\}$ and set $\Phi_J^+ = \Phi_J \cap \Phi^+$. By replacing H by a suitable conjugate, we may assume that $H = QL$ is a Levi decomposition of H , with Levi factor $L = \langle T, U_{\pm\alpha} \mid \alpha \in J \rangle$ and unipotent radical $Q = \prod U_{-\beta}$, the product taken over all $\beta \in \Phi^+ \setminus \Phi_J^+$.

Let $\beta \in \Phi^+ \setminus \Phi_J^+$, say $\beta = d_i\alpha_i + \sum_{j \neq i} c_j\alpha_j$. Following [3], we define the *level* and *height* of β by

$$\text{level}(\beta) = d_i, \quad \text{height}(\beta) = d_i + \sum_{j \neq i} c_j.$$

For each positive integer j we define $Q_j = \prod U_{-\beta}$, where the product is over the roots $\beta \in \Phi^+ \setminus \Phi_J^+$ of level j . Finally, again following [3], we say that G is *special* if $(G, p) = (F_4, 2)$, $(G_2, 3)$ or $(G_2, 2)$.

The next result is a special case of [3, Theorem 2].

Theorem 5.7. *Let G be a simple exceptional algebraic group and assume that G is not special. Let $H = QL$ be a maximal parabolic subgroup of G , let $j \geq 1$ be an integer and define $Q_j \leq Q$ as above. Let $T_{L'}$ be a maximal torus of L' contained in T . The following hold:*

- (i) Q_j is invariant under conjugation by L .
- (ii) Q_j is an irreducible KL' -module with highest weight $-\beta|_{T_{L'}}$, where $\beta \in \Phi^+$ is the unique root of minimal height with $\text{level}(\beta) = j$.
- (iii) L has an open dense orbit on Q_j .

Let (G, H) be one of the remaining cases that we have to consider and let $H = QL$ be a Levi decomposition. By Lemma 5.3, we may assume that L is the generic 2-point stabilizer in the action of G on the coset variety $\Omega = G/H$. Moreover, Q has a regular dense orbit on Ω so we can reduce the problem to computing the base size for the action of L on Q . As an L -variety,

$$Q \cong Q_1 \times Q_2 \times \cdots \times Q_m$$

where $m \geq 1$ is the maximal level of a root $\beta \in \Phi^+ \setminus \Phi_J^+$. In particular, the stabilizer in L of a generic point in Q is the intersection of the generic stabilizers of L on each Q_j .

The derived subgroup L' is a product of simple groups L_1, \dots, L_k for some $k \geq 1$. By Theorem 5.7(ii), Q_j is an irreducible KL' -module, so we can write $Q_j \cong L(\mu_1) \otimes \cdots \otimes L(\mu_k)$ as KL' -modules, where $L(\mu_i)$ denotes the irreducible $KL_{i'}$ -module with highest weight μ_i . For each factor L_i we express μ_i in terms of a set of fundamental dominant weights $\{\lambda_1, \lambda_2, \dots\}$ (with respect to the usual ordering), unless $L_i = A_1$ when we will write $L(m)$ rather than $L(m\lambda_1)$. We write 0 for the trivial 1-dimensional $KL_{i'}$ -module. Finally, let $\{\omega_1, \dots, \omega_r\}$ be a set of fundamental dominant weights for G .

Proposition 5.8. *Suppose $(G, H) = (E_8, P_7)$ or (E_7, P_3) . Then $b^0(G, H) = b(G, H) = b^1(G, H) = 4$.*

Proof. According to Proposition 5.2, in both of these cases we have $b^1(G, H) \leq 4$, so it suffices to show that the intersection of three generic conjugates of H is positive-dimensional. First consider the case $(G, H) = (E_8, P_7)$. By Lemma 5.3, the generic 2-point stabilizer is a Levi subgroup $L = E_6A_1T_1$ of H and so by the above discussion it suffices to show that $\dim C_L(q) > 0$ for a generic element $q \in Q$, where Q is the unipotent radical of H . Let j be a positive integer and define Q_j as above. By applying Theorem 5.7 we deduce that each Q_j is an irreducible KL' -module with

$$Q_1 \cong L(\lambda_6) \otimes L(1), \quad Q_2 \cong L(\lambda_1) \otimes 0, \quad Q_3 \cong 0 \otimes L(1)$$

as KL' -modules. For example, $\beta = \alpha_7$ is clearly the unique root of minimal height at level 1, so Q_1 is an irreducible KL' -module with highest weight $-\alpha_7 = \omega_6 - 2\omega_7 + \omega_8$ (restricted to a suitable maximal torus of $L' = E_6A_1$), whence $Q_1 \cong L(\lambda_6) \otimes L(1)$ as claimed.

A generic point in $Q_1 \times Q_2 \times Q_3$ has the form $q = (a_1 \otimes b_1 + a_2 \otimes b_2, c, d)$, where $a_1, a_2 \in L(\lambda_6)$, $b_1, b_2, d \in L(1)$ and $c \in L(\lambda_1)$. As in the proof of Proposition 5.5, we see

that F_4 is the generic stabilizer in the action of E_6 on the 27-dimensional modules $L(\lambda_1)$ and $L(\lambda_6)$, so $C_{E_6}(q)$ is the intersection of three conjugates of F_4 in E_6 . By Theorem 7(ii) (see the proof of Lemma 3.19), the intersection of any three conjugates of F_4 in E_6 is positive-dimensional, so $\dim C_L(q) \geq \dim C_{E_6}(q) > 0$ and thus $b^0(G, H) = b(G, H) = b^1(G, H) = 4$ as required.

The case $(G, H) = (E_7, P_3)$ is similar. Here the generic 2-point stabilizer is $L = A_1 A_5 T_1$ and once again it suffices to show that $\dim C_L(q) > 0$ for a generic $q \in Q$. In this case, using Theorem 5.7, we calculate that

$$Q_1 \cong L(1) \otimes L(\lambda_2), \quad Q_2 \cong 0 \otimes L(\lambda_4), \quad Q_3 \cong L(1) \otimes 0$$

as KL' -modules, and a generic point $q \in Q_1 \times Q_2 \times Q_3$ has the form $q = (a_1 \otimes b_1 + a_2 \otimes b_2, c, d)$, where $a_1, a_2, d \in L(1)$, $b_1, b_2 \in L(\lambda_2)$ and $c \in L(\lambda_4)$. The generic stabilizer in A_5 with respect to the 15-dimensional modules $L(\lambda_2)$ and $L(\lambda_4) \cong L(\lambda_2)^*$ is C_3 (note that $L(\lambda_2) = \Lambda^2(W)$, where W is the natural A_5 -module, so $L(\lambda_2)$ and $L(\lambda_4)$ can be identified with the space of alternating forms on W). Therefore $C_{A_5}(q)$ is the intersection of three conjugates of C_3 in A_5 , which is positive-dimensional by Theorem 5(ii) (see [34, Theorem 1.1]). The desired conclusion follows as before. \square

Proposition 5.9. *If $(G, H) = (E_6, P_2)$ then $b^0(G, H) = b(G, H) = b^1(G, H) = 5$.*

Proof. By Proposition 5.2 we have $b^1(G, H) \leq 5$, so it suffices to show that the intersection of four generic conjugates of H in G is positive-dimensional. By Lemma 5.3, the generic 2-point stabilizer is $L = A_5 T_1$. By applying Theorem 5.7 we deduce that $Q \cong Q_1 \times Q_2$ (as an L -variety), where $Q_1 \cong L(\lambda_3)$ and $Q_2 \cong 0$ as KL' -modules (note that $L(\lambda_3) = \Lambda^3(W)$, where W is the natural module for A_5). If $v \in Q_1$ is generic then $C_{A_5}(v)$ is a C_2 -subgroup of type $\mathrm{GL}_3 \wr S_2$. By Theorem 5 (see the proof of Lemma 3.16), the intersection of any two such centralizers in A_5 is positive-dimensional, so the generic 2-point stabilizer of L on Q is also positive-dimensional, whence $b^0(G, H) > 4$ as required. \square

This completes the proof of Theorem 6.

Remark 5.10. A similar approach can also be used to investigate the remaining cases where $G = F_4, G_2$ or $(G, H) = (E_6, P_4)$. However, the analysis here is more complicated and we do not get better results than the bounds provided in Proposition 5.2. If G is *special*, that is, if $(G, p) = (F_4, 2), (G_2, 3)$ or $(G_2, 2)$, then in these cases we can calculate $b^0(G, H)$ via Proposition 2.7(ii) and a suitable computation with $F_4(4), G_2(3)$ and $G_2(4)$, using MAGMA [11]. We find that $b^0(G, H) = 4$ if $(G, p) = (F_4, 2)$ and $H = P_1$ or P_4 , otherwise $b^0(G, H) = 3$.

5.2. Non-parabolic actions. In this section we complete the proof of Theorem 7 on non-parabolic actions of exceptional groups. By Theorem 5.1, one of the following holds:

- (i) $G = E_7, H = (2^2 \times D_4).S_3$ and $p \neq 2$;
- (ii) $G = E_8, H = A_1 \times S_5$ and $p \neq 2, 3, 5$;
- (iii) $H = N_G(X)$, with X given in Table 9.

We adopt the notation introduced earlier (see the discussion preceding the statement of Lemma 3.17). In particular, $\mathrm{Lie}(G)$ is the Lie algebra of G and $C_{\mathrm{Lie}(G)}(x)$ denotes the fixed point space of $x \in G$ on $\mathrm{Lie}(G)$, with respect to the adjoint representation. Note that

$$\dim C_{\mathrm{Lie}(G)}(x) \leq \dim C_G(x)$$

for all $x \in G$, with equality if x is semisimple (see [43, Section 1.10], for example). Given a simple algebraic group X , we will write $W(\lambda)$ for the Weyl module for X with highest weight λ , and we will express λ in terms of a set of fundamental dominant weights $\{\lambda_1, \lambda_2, \dots\}$ for X (unless $X = A_1$, when we write $W(m)$ rather than $W(m\lambda_1)$). We denote the trivial 1-dimensional KL_i -module by 0 and we will write \mathcal{P} for the set of elements in H of prime order (including all unipotent elements if $p = 0$). We will use the Aschbacher-Seitz [2] notation for involutions in classical groups when $p = 2$.

Proposition 5.11. *Theorem 7 holds for $G = E_8$.*

Proof. If $x \in G$ is nontrivial then $\dim x^G \geq 58$ (minimal if x is a long root element), hence Corollary 2.14 immediately implies that

$$b^0(G, H) = b(G, H) = b^1(G, H) = 2 \quad (9)$$

if $\dim H < 29$. For the remainder, let us assume $\dim H \geq 29$.

By Theorem 5.1, we have $H = N_G(X)$ with X given in Table 9. First assume H is not a maximal rank subgroup of G , so $H^0 = A_1G_2^2$ ($p \neq 2$) or G_2F_4 since $\dim H \geq 29$. In both cases we claim that

$$\dim x^H < \frac{1}{2} \dim x^G \quad (10)$$

for all $x \in \mathcal{P}$, so (9) follows from Corollary 2.14. This is clear if $H^0 = A_1G_2^2$ since $\dim x^H \leq 26$ for all $x \in H$ (note that if $x \in H \setminus H^0$ has prime order then x is a semisimple involution and thus $\dim x^G \geq 112$; see [35, Table 4.3.1], for example).

Next assume $H = H^0 = G_2F_4$. Here $\dim x^H \leq 60$ for all $x \in H$, so we may assume that $\dim x^G \leq 120$. Suppose x is unipotent, so the bound on $\dim x^G$ implies that x belongs to one of the G -classes labelled $A_1, 2A_1, 3A_1$ or A_2 (see [51, Table 2]). The fusion of unipotent classes in H is described in [49, Table 38] and we quickly deduce that (9) holds.

Now assume x is semisimple and $\dim x^G \leq 120$, so $C_G(x) = A_1E_7$ or E_7T_1 . If $C_G(x) = A_1E_7$ then $p \neq 2$ and x is an involution, so $\dim x^H \leq 36$ since there is a unique class of involutions in G_2 (of dimension 8), and exactly two such classes in F_4 (dimensions 28 and 16). Finally, assume $C_G(x) = E_7T_1$, so $\dim x^G = 114$ and $\dim C_{\text{Lie}(G)}(x) = 134$. Now

$$\text{Lie}(G) \downarrow G_2F_4 = \text{Lie}(G_2F_4) \oplus (W(\lambda_1) \otimes W(\lambda_4))$$

(see [55, Proposition 2.4]). If $\dim x^H \geq 58$ then $C_H(x) = T_6$ or A_1T_5 , and from the above description of $\text{Lie}(G) \downarrow G_2F_4$ it is straightforward to see that $\dim C_{\text{Lie}(G)}(x) < 134$, which is a contradiction. For example, suppose $x = x_1x_2$ and $C_H(x) = T_6$. Up to conjugacy, x_1 acts on $W(\lambda_1)$ as a diagonal matrix $[I_3, \lambda I_2, \lambda^{-1} I_2]$ for some $\lambda \in K^*$ with $\lambda \neq \pm 1$, so [60, Lemma 3.7] implies that $\dim C_{W(\lambda_1) \otimes W(\lambda_4)}(x) \leq 78$ and thus $\dim C_{\text{Lie}(G)}(x) \leq 84$. This establishes (10) and we conclude that (9) holds (see Corollary 2.14).

For the remainder we may assume $H = N_G(X)$ is a maximal rank subgroup with

$$X \in \{A_2^4, D_4^2, A_4^2, A_2E_6, A_8, A_1E_7, D_8\}$$

(see Table 9). The cases $H^0 = D_8$ and A_1E_7 were handled in Lemma 3.17. In each of the remaining cases we claim that (10) holds for all $x \in \mathcal{P}$, in which case Corollary 2.14 implies that (9) holds.

First assume $H^0 = A_2^4$, so $H/H^0 = \text{GL}_2(3)$ and $\dim H = 32$. If $x \in G$ is a long root element then $x \in H^0$ (see [51, Proposition 1.13(iii)]), so $\dim x^H \leq 24$, $\dim x^G = 58$ and the required bound follows. On the other hand, if x is not a long root element then $\dim x^G \geq 92$ and again the claim holds.

Next suppose $H^0 = D_4^2$. Here $H/H^0 = Z_2 \times S_3$, where Z_2 swaps the two factors, and S_3 induces graph automorphisms (simultaneously on the two D_4 factors). Now $\dim x^H \leq 48$

for all $x \in H$, so we may assume x is a unipotent element in one of the G -classes labelled A_1 or $2A_1$ (with respective dimensions 58 and 92). By [55, Proposition 2.1] we have

$$\begin{aligned} \text{Lie}(G) \downarrow D_4 D_4 &= \text{Lie}(D_4 D_4) \oplus (W(\lambda_1) \otimes W(\lambda_1)) \oplus (W(\lambda_3) \otimes W(\lambda_3)) \\ &\quad \oplus (W(\lambda_4) \otimes W(\lambda_4)), \end{aligned}$$

where $W(\lambda_1)$ is the natural D_4 -module, and $W(\lambda_3), W(\lambda_4)$ are the two distinct irreducible spin modules for D_4 .

First assume $p \neq 2$. We claim that $\dim(x^G \cap H) = 10$ if $x \in A_1$, and $\dim(x^G \cap H) = 20$ if $x \in 2A_1$. To see this, let $u, v \in D_4$ be elements with respective Jordan forms $[J_2^2, J_1^4]$ and $[J_3, J_1^5]$ on the natural module $W(\lambda_1)$. We calculate that $[J_3, J_2^8, J_1^9]$ and $[J_3^6, J_1^{10}]$ are the respective Jordan forms of u and v on $\text{Lie}(D_4)$. In addition, we note that u has Jordan form $[J_2^2, J_1^4]$ on both $W(\lambda_3)$ and $W(\lambda_4)$, and v has Jordan form $[J_2^4]$ on these modules. Using the above decomposition for $\text{Lie}(G) \downarrow D_4 D_4$ we can calculate the Jordan form of $(u, 1), (v, 1), (u, u) \in H^0$ on $\text{Lie}(G)$, and then use [47, Table 9] to determine the G -class of these elements. In this way, we deduce that $(u, 1) \in A_1$ and $(v, 1), (u, u) \in 2A_1$. Moreover, one can check that these elements represent the only H -classes that are in A_1 and $2A_1$. (For example, we find that $(u, v) \in 3A_1$ and $(v, v) \in A_2$. Also, if $p = 3$ and $x \in H \setminus H^0$ induces a triality automorphism on each D_4 factor then x cyclically permutes the modules $W(\lambda_1), W(\lambda_3)$ and $W(\lambda_4)$, so the Jordan form of x on $\text{Lie}(G)$ has at least 64 Jordan blocks of size 3 and thus x is not in A_1 nor $2A_1$.) This justifies the claim.

Similarly, if $p = 2$ then careful calculation reveals that $x^G \cap H$ is a union of two H -classes when $x \in A_1$, with representatives $(a_2, 1), (b_1, b_1) \in H^0$ (in the notation of [2]), whence $\dim(x^G \cap H) = 14$. Similarly, if $x \in 2A_1$ then $x^G \cap H$ comprises two H -classes, with representatives (a_2, a_2) and $(c_2, 1)$, so $\dim(x^G \cap H) = 20$ as before. (Note that if $x \in H \setminus H^0$ interchanges the two D_4 factors then the Jordan form of x on $\text{Lie}(G)$ has at least 96 Jordan blocks of size 2, so x is not in A_1 nor $2A_1$.) We conclude that (10) holds if $H^0 = D_4^2$.

Next consider the case $H^0 = A_4^2$. Here $H/H^0 = Z_4$ and $\dim x^H \leq 40$ for all $x \in H$, so we may assume $x \in G$ is a long root element. In particular, $x \in H^0$ (see [51, Proposition 1.13(iii)]) and by inspecting [49, Table 26] we deduce that $\dim x^H \leq 8$.

It remains to deal with the cases $H^0 = A_2 E_6$ and $H^0 = A_8$. First suppose $H^0 = A_2 E_6$. Here $H/H^0 = Z_2$ and $\dim x^H \leq 78$ for all $x \in H$, so we may assume $\dim x^G \leq 156$. In particular, if x is semisimple then $C_G(x) = E_7 A_1, E_7 T_1, D_8$ or $D_7 T_1$ (see [31]), and by applying [51, Theorem 2] we deduce that (10) holds. Now assume x is unipotent. If $x \in H^0$ then the fusion information in [49, Table 24] is sufficient, so let us assume $p = 2$ and $x \in H \setminus H^0$. There are two H -classes of involutions in $H \setminus H^0$, represented by x_1 and x_2 say, where $C_{H^0}(x_1) = A_1 F_4$ and $C_{H^0}(x_2) = A_1 C_{F_4}(t)$, where $t \in F_4$ is a long root element (each x_i acts as a graph automorphism on the A_2 and E_6 factors). By [55, Proposition 2.1] we have

$$\text{Lie}(G) \downarrow A_2 E_6 = \text{Lie}(A_2 E_6) \oplus (W(\lambda_1) \otimes W(\lambda_6)) \oplus (W(\lambda_2) \otimes W(\lambda_1))$$

and using this we calculate that the Jordan form of x_1 and x_2 on $\text{Lie}(G)$ is $[J_2^{110}, J_1^{28}]$ and $[J_2^{120}, J_1^8]$, respectively. Therefore, by inspecting [47, Table 9], we see that $x_1 \in 3A_1$ and $x_2 \in 4A_1$, so $\dim x_1^H = 31$, $\dim x_1^G = 112$ and $\dim x_2^H = 47$, $\dim x_2^G = 128$.

Finally suppose $H^0 = A_8$, so $H/H^0 = Z_2$ and we may assume $\dim x^G \leq 144$ since $\dim x^H \leq 72$ for all $x \in H$. If x is semisimple then $C_G(x) = E_7 A_1, E_7 T_1$ or D_8 , and it is easy to check that [51, Theorem 2] is sufficient. Similarly, if $x \in H^0$ is unipotent then the desired bound follows from the information in [49, Table 25]. Finally, suppose $p = 2$ and $x \in H \setminus H^0$ is an involution (so x is a graph automorphism of A_8). Now

$$\text{Lie}(G) \downarrow A_8 = \text{Lie}(A_8) \oplus W(\lambda_3) \oplus W(\lambda_6)$$

(see [55, Proposition 2.1]) and we deduce that x has Jordan form $[J_2^{120}, J_1^8]$ on $\text{Lie}(G)$. Therefore [47, Table 9] indicates that x is in the G -class labelled $4A_1$, so $\dim x^H = 44$ and $\dim x^G = 128$. \square

Proposition 5.12. *Theorem 7 holds for $G = E_7$.*

Proof. If $x \in G$ is nontrivial then $\dim x^G \geq 34$ (minimal if x is a long root element), so we may as well assume $\dim H \geq 17$. According to Theorem 5.1, one of the following holds:

- (i) $p \neq 2$ and $H = (2^2 \times D_4).S_3$; or
- (ii) $H = N_G(X)$ with $X \in \{A_1G_2, A_1F_4, G_2C_3, T_1E_6, A_1D_6, A_7, A_2A_5, A_1^3D_4, A_1^7\}$.

If $H^0 = A_7, A_1D_6$ or T_1E_6 then H is an involution-type subgroup and we refer the reader to Lemma 3.18. In each of the remaining cases we claim that (10) holds for all $x \in \mathcal{P}$, so

$$b^0(G, H) = b(G, H) = b^1(G, H) = 2.$$

Let V_{56} be the 56-dimensional irreducible KG -module.

First suppose $H = (2^2 \times D_4).S_3$. Here $p \neq 2$ and $\dim x^H \leq 24$ for all $x \in \mathcal{P}$. In particular, if x is not a long root element then $\dim x^G \geq 52$ and thus $\dim x^H < \frac{1}{2} \dim x^G$, so it suffices to show that there are no long root elements in H . To see this, first observe that $H^0 = D_4$ belongs to an A_7 subgroup of G (embedded via the natural 8-dimensional module for D_4), and the root subgroups of this A_7 are also root subgroups of G . Since SO_8 does not contain any transvections when $p \neq 2$, it follows that there are no long root elements of G in H^0 . By [51, Proposition 1.13(iii)], there are also no long root elements in $H \setminus H^0$. Therefore, (10) holds for all $x \in \mathcal{P}$.

If $H = H^0 = A_1G_2$ then $\dim x^H \leq 14$ for all $x \in \mathcal{P}$, and the claim follows since $\dim x^G \geq 34$. Next suppose $H^0 = A_1^7$, so $H/H^0 = \text{GL}_3(2)$. If $x \in H^0$ then $\dim x^H \leq 14$ and the result follows. On the other hand, if $x \in H \setminus H^0$ then x is not a long root element (see [51, Proposition 1.13(iii)]), so $\dim x^G \geq 52 > 2 \dim H$.

Now consider the case $H = H^0 = A_1F_4$. Here we may assume $\dim x^G \leq 100$ since $\dim x^H \leq 50$ for all $x \in H$. By [55, Proposition 2.5(i)],

$$V_{56} \downarrow A_1F_4 = (W(1) \otimes W(\lambda_4)) \oplus (W(3) \otimes 0).$$

If $x \in H$ is unipotent then it is straightforward to calculate the Jordan form of x on V_{56} , and by inspecting [47, Table 7] we can determine the G -class of x . In this way we quickly deduce that (10) holds for all unipotent elements $x \in \mathcal{P}$.

Now assume $x \in \mathcal{P}$ is semisimple. The possibilities for $D = C_G(x)$ with $\dim x^G \leq 100$ are listed in Table 12 (see [32], for example).

D^0	$\dim x^G$				
$A_3^2A_1$	100	$A_5A_1T_1$	94	A_7	70
$D_4A_1T_2$	100	A_5A_2	90	D_6T_1	66
$A_4A_2T_1$	100	D_5T_2	86	D_6A_1	64
$D_4A_1^2T_1$	98	$D_5A_1T_1$	84	T_1E_6	54
A_5T_2	96	A_6T_1	84		

TABLE 12. $D = C_G(x)$, x semisimple, $\dim x^G \leq 100$

First assume $p \neq 2$ and x is an involution, so $D^0 = D_6A_1, A_7$ or T_1E_6 (see [35, Table 4.3.1]). The largest class of involutions in F_4 has dimension 28, so $\dim x^H \leq 30$ and thus we may assume $D^0 = T_1E_6$, whence $\dim x^G = 54$. Write $x = x_1x_2$, where $x_1 \in A_1$

and $x_2 \in F_4$. If $C_{F_4}(x_2) \neq A_1C_3$ then $\dim x^H \leq 18$, so let us assume $C_{F_4}(x_2) = A_1C_3$. According to [55, Proposition 2.4] we have

$$\mathrm{Lie}(G) \downarrow A_1F_4 = \mathrm{Lie}(A_1F_4) \oplus (W(2\lambda_1) \otimes W(\lambda_4)), \quad (11)$$

and we note that $\dim C_{\mathrm{Lie}(G)}(x) = \dim C_G(x) = 79$. However, $\dim C_{W(\lambda_4)}(x_2) = 14$ and we deduce that $\dim C_{\mathrm{Lie}(G)}(x) = 69$ if $x_1 = 1$, and $\dim C_{\mathrm{Lie}(G)}(x) = 63$ if $x_1 \neq 1$. This is a contradiction and thus (10) holds for all involutions.

For the remainder, we may assume that $x \in H$ has odd prime order. Suppose $\dim x^G = 98$ or 100 , so $\dim C_{\mathrm{Lie}(G)}(x) = 33$ or 35 . Write $x = x_1x_2$ as before. We may assume that $\dim x^H = 50$, so x_1 and x_2 are both regular. In particular, since x_1 is regular, [60, Lemma 3.7] implies that $\dim C_{W(2\lambda_1) \otimes W(\lambda_4)}(x) \leq 26$, whence $\dim C_{\mathrm{Lie}(G)}(x) \leq 5 + 26$, a contradiction.

Now assume $D^0 = A_5T_2$, so $\dim x^G = 96$, $\dim C_{\mathrm{Lie}(G)}(x) = 37$ and we reduce to the case $\dim x_2^{F_4} = 46$ or 48 . If $x_1 \neq 1$ then $\dim C_{W(2\lambda_1) \otimes W(\lambda_4)}(x) \leq 26$ as above, so $\dim C_{\mathrm{Lie}(G)}(x) \leq 7 + 26$ and we reach a contradiction. Now suppose $x_1 = 1$, so x_2 is regular and $\dim C_{\mathrm{Lie}(G)}(x) = 7 + 3\alpha$, where $\alpha = \dim C_{W(\lambda_4)}(x_2)$. We claim that $\alpha \leq 8$.

First observe that we may assume $x_2 \in D_4 < D_5 < E_6$. Let $\{\omega_1, \dots, \omega_6\}$ and $\{\xi_1, \dots, \xi_5\}$ be fundamental dominant weights for E_6 and D_5 , respectively, and let V_{27} be the 27-dimensional irreducible module for E_6 with highest weight ω_1 . By [55, Table 8.7] we have

$$V_{27} \downarrow D_5 = W(\xi_1) \oplus W(\xi_4) \oplus 0, \quad (12)$$

where $W(\xi_1)$ is the natural module for D_5 , and $W(\xi_4)$ is one of the irreducible spin modules. The 26-dimensional F_4 -module $W(\lambda_4)$ is a section of V_{27} . Since $x_2 \in D_4$ is regular it follows that $\dim C_{W(\xi_1)}(x_2) \leq 4$. Now the restriction of the D_5 spin module $W(\xi_4)$ to D_4 is a sum of two non-isomorphic spin modules for D_4 . Therefore the regularity of x_2 implies that $\dim C_{W(\xi_4)}(x_2) \leq 2 + 2$ and thus $\alpha \leq 4 + 2 + 2 = 8$ as claimed. In particular, $\dim C_{\mathrm{Lie}(G)}(x) \leq 7 + 24 = 31$, which is a contradiction. The case $D^0 = A_5A_1T_1$ is entirely similar. If $D^0 = A_5A_2$ then x has order 3 and we deduce that $\dim x^H \leq 2 + 36 = 38 < \frac{1}{2} \dim x^G$ (see [35, Table 4.7.1]).

Next consider the case $D^0 = D_5T_2$, so $\dim x^G = 86$ and we may assume $\dim x^H \geq 44$. As before, if $x_1 \neq 1$ then (11) implies that $\dim C_{\mathrm{Lie}(G)}(x) \leq 11 + 26$, which is a contradiction. Now suppose $x_1 = 1$, so $\dim C_{F_4}(x_2) = 4, 6$ or 8 , and

$$\dim C_{\mathrm{Lie}(G)}(x) = 3 + \dim C_{F_4}(x_2) + 3\alpha \quad (13)$$

with $\alpha = \dim C_{W(\lambda_4)}(x_2)$. Since $\dim C_{\mathrm{Lie}(G)}(x) = 47$, we reduce to the case $\dim C_{F_4}(x_2) = 8$, so $C_{F_4}(x_2)^0 = T_2A_1^2$ is the only possibility. Here $x_2 \in D_4 < D_5$ and using (12) we calculate that $\alpha \leq 10$. For example, if $C_{D_4}(x_2) = \mathrm{GL}_2 \times \mathrm{GL}_2$ then $\dim C_{W(\mu_1)}(x_2) = 2$ and the proof of [13, Lemma 7.4] yields $\dim C_{W(\mu_4)}(x_2) \leq 8$. Therefore $\dim C_{\mathrm{Lie}(G)}(x) \leq 41 < 47$. We conclude that (10) holds when $D^0 = D_5T_2$. A similar argument applies when $D^0 = D_5A_1T_1$ or A_6T_1 .

To complete the analysis of the case $H = A_1F_4$ we may assume $D^0 = D_6T_1$ or E_6T_1 (in the latter case we may also assume x has odd order). Suppose $D^0 = D_6T_1$, so $\dim x^G = 66$ and we may assume $\dim x^H \geq 34$. If $x = x_1x_2$ and x_1 is nontrivial then the usual argument implies that $\dim C_{\mathrm{Lie}(G)}(x) \leq 21 + 26$, which is a contradiction. Now assume $x_1 = 1$ and note that (13) holds. By arguing as in the proof of [13, Lemma 7.4] we calculate that $\alpha \leq 14$. However, $\dim C_{F_4}(x_2) \leq 18$ since $x_1 = 1$ and $\dim x^H \geq 34$, so $3 + \dim C_{F_4}(x_2) + 3\alpha \leq 63$, which contradicts (13) since $\dim C_{\mathrm{Lie}(G)}(x) = 67$. An entirely similar argument applies if $D^0 = E_6T_1$. We conclude that (9) holds when $H = A_1F_4$.

Now suppose $H = H^0 = G_2C_3$. First note that we may assume $\dim x^G \leq 60$. If x is unipotent then the relevant classes are labelled A_1 , $2A_1$ and $(3A_1)''$, with respective dimensions 34, 52 and 54. As explained in [49, Section 5.12], complete information on the fusion of unipotent classes can be deduced from [49, Table 38], and it is straightforward to check that (10) holds.

Now assume $x \in H$ is semisimple. Here the hypothesis $\dim x^G \leq 60$ implies that $C_G(x) = E_6T_1$, hence $\dim x^G = 54$ and we may assume $\dim x^H \geq 28$, so $C_H(x) = T_5$ or A_1T_4 , and thus x has odd prime order (if $x \in H$ is an involution then $\dim x^H \leq 20$). By [55, Proposition 2.4] we have

$$\mathrm{Lie}(G) \downarrow G_2C_3 = \mathrm{Lie}(G_2C_3) \oplus (W(\lambda_1) \otimes W(\lambda_2))$$

and we note that $\dim C_{\mathrm{Lie}(G)}(x) = 79$. Write $x = x_1x_2$, where $x_1 \in G_2$ and $x_2 \in C_3$. Let s denote the codimension of the largest eigenspace of x_1 on $W(\lambda_1)$. Since $C_H(x) = T_5$ or A_1T_4 , we calculate that $s \geq 2$ and thus [60, Lemma 3.7] implies that the codimension of the largest eigenspace of x on $W(\lambda_1) \otimes W(\lambda_2)$ is at least 28. Therefore, $\dim C_{\mathrm{Lie}(G)}(x) \leq 7+70$, a contradiction.

Next suppose $H^0 = A_1^3D_4$. Here $H/H^0 = S_3$ and $\dim x^H \leq 30$ for all $x \in H$, so we may assume $\dim x^G \leq 60$. If x is semisimple then $C_G(x) = E_6T_1$ and [51, Theorem 2] implies that $\dim x^H \leq 18 < \frac{1}{2} \dim x^G$. Now assume x is unipotent. The relevant G -classes are labelled A_1 , $2A_1$ and $(3A_1)''$, and we calculate that

$$\begin{aligned} V_{56} \downarrow A_1^3D_4 &= (W(1) \otimes 0 \otimes 0 \otimes W(\lambda_1)) \oplus (0 \otimes W(1) \otimes 0 \otimes W(\lambda_3)) \\ &\oplus (0 \otimes 0 \otimes W(1) \otimes W(\lambda_4)) \oplus (W(1) \otimes W(1) \otimes W(1) \otimes 0). \end{aligned} \quad (14)$$

If $p = 2$ then x is an involution, so $\dim x^H \leq 22$ and therefore we may assume x is a long root element. In particular, $x \in H^0$ (see [51, Proposition 1.13(iii)]). According to [47, Table 7], a long root element has Jordan form $[J_2^{12}, J_1^{32}]$ on V_{56} , and by considering the above decomposition (14) we deduce that x is H -conjugate to $x_1x_2x_3x_4 \in A_1^3D_4$, where

- (i) $x_1 = J_2$ and $x_i = 1$ for all $i \geq 2$; or
- (ii) $x_4 = a_2$ and $x_i = 1$ for all $i \leq 3$.

Therefore, $\dim x^H \leq 10$ and the result follows. Finally, suppose $p \neq 2$ and x is unipotent. If $p = 3$ and $x \in H \setminus H^0$ then we calculate that x has Jordan form $[J_3^{18}, J_1^2]$ on V_{56} , so x belongs to one of the classes labelled $2A_2$ or $2A_2 + A_1$ (see [47, Table 7]). In particular, $\dim x^H < \frac{1}{2} \dim x^G$ as required. Similarly, if $x \in H^0$ then we can determine the Jordan form of x on V_{56} ; in this way, the reader can check that if x is in one of the relevant classes A_1 , $2A_1$ or $(3A_1)''$ then $\dim x^H \leq 14 < \frac{1}{2} \dim x^G$.

Finally, let us consider the case $H^0 = A_2A_5$. Here $H/H^0 = Z_2$ and $\dim x^H \leq 36$ for all $x \in H$, so we may assume $\dim x^G \leq 72$. In particular, if x is semisimple then $C_G(x)^0 = D_6A_1$, A_7 , E_6T_1 or D_6T_1 , and the bound provided by [51, Theorem 2] is sufficient. For example, if $C_G(x)^0$ has a D_6 factor then [51, Theorem 2] yields $\dim x^G - \dim x^H \geq 39$, so $\dim x^H \leq 27 < \frac{1}{2} \dim x^G$. Now assume x is unipotent. Here the relevant classes are labelled A_1 , $2A_1$, $(3A_1)''$, $(3A_1)'$, A_2 and $4A_1$. If $x \in H \setminus H^0$ then $p = 2$ and x acts as a graph automorphism on the A_2 and A_5 factors of H^0 , so $\dim x^H \leq 5+20 = 25$. In addition, [51, Proposition 1.13(iii)] implies that x is not a long root element, so $\dim x^G \geq 52$ and the desired bound follows. Finally, if $x \in H^0$ is unipotent then the G -class of x is given in [49, Table 21] and the result quickly follows. \square

Proposition 5.13. *Theorem 7 holds for $G = E_6$.*

Proof. We may assume that $\dim H \geq 11$ since $\dim x^G \geq 22$ for all nontrivial $x \in G$. According to Theorem 5.1, we have $H = N_G(X)$ with

$$X \in \{G_2, A_2G_2, T_2D_4, A_2^3, F_4, A_1A_5, T_1D_5, C_4 (p \neq 2)\}.$$

If $H^0 = F_4, A_1A_5, T_1D_5$ or C_4 (with $p \neq 2$) then H is an involution-type subgroup, and these cases have already been dealt with in Lemma 3.19. In each of the remaining cases we claim that (10) holds for all $x \in \mathcal{P}$, so (9) follows. Let $\{\omega_1, \dots, \omega_6\}$ be a set of fundamental dominant weights for G , and let V_{27} be the 27-dimensional irreducible KG -module with highest weight ω_1 .

If $H = H^0 = G_2$ then [49, Table 31] indicates that there are no long root elements in H , so $\dim x^G \geq 32$ for all $x \in \mathcal{P}$ and the claim follows since $\dim x^H \leq 12$.

Next assume $H^0 = A_2G_2$. Here $H/H^0 = Z_2$ and $\dim x^H \leq 18$ for all $x \in \mathcal{P}$, so we may assume $\dim x^G \leq 36$. Suppose $x \in H$ is unipotent, so the relevant classes are labelled A_1 and $2A_1$ (with respective dimensions 22 and 32). If $x \in H \setminus H^0$ then $p = 2$ and [51, Proposition 1.13(iii)] implies that x is not a long root element, so $\dim x^G \geq 32$. Moreover, $\dim x^H \leq 5+8 = 13$ since x is an involution, and the result follows. For unipotent elements $x \in H^0$, Ross Lawther has determined the G -class of x (using the method described in [49]), and we quickly deduce that (10) holds in all cases. For completeness, we record this information in Table 13. For instance, if $x = x_1x_2 \in A_2G_2$ is unipotent, where $x_1 \in A_2$ is regular and $x_2 \in G_2$ belongs to the class labelled $\tilde{A}_1^{(3)}$ (in which case $p = 3$), then x is in the G -class labelled $2A_2 + A_1$ and thus $\dim x^H = 6 + 8 = 14$, $\dim x^G = 54$.

	1	A_1	\tilde{A}_1	$\tilde{A}_1^{(3)}$	$G_2(a_1)$	G_2
1	1	A_1	$3A_1$	$3A_1$	A_2	D_4
A_1	$2A_1$	$3A_1$	$A_2 + A_1$ $3A_1 (p = 2)$	$A_2 + A_1$	$A_2 + 2A_1$	$D_5(a_1)$ $D_4 (p = 2)$
A_2	$2A_2$	$2A_2 + A_1$	$A_3 + A_1$ $2A_2 (p = 3)$	$2A_2 + A_1$	$D_4(a_1)$ $2A_2 + A_1 (p = 3)$	$E_6(a_3)$ $D_5(a_1) (p = 2)$

TABLE 13. The fusion of unipotent classes, $A_2G_2 < E_6$

Now suppose $x \in H$ is semisimple, so $C_G(x)^0 = D_5T_1$ since $\dim x^G \leq 36$. Here $\dim x^G = 32$, so we may as well assume $\dim x^H \geq 16$, whence $C_H(x)^0 = T_4$ or A_1T_3 . Now $\dim C_{\text{Lie}(G)}(x) = 46$ and [55, Proposition 2.4] gives

$$\text{Lie}(G) \downarrow A_2G_2 = \text{Lie}(A_2G_2) \oplus (\text{Lie}(A_2) \otimes W(\lambda_1)).$$

First assume $C_H(x)^0 = T_4$. Write $x = x_1x_2 \in A_2G_2$. Since x_1 and x_2 are both regular semisimple elements, with respect to suitable bases we calculate that x_1 acts on $\text{Lie}(A_2)$ as the diagonal matrix $[I_2, \lambda I_2, \lambda^{-1}I_2, \lambda^2, \lambda^{-2}]$, and x_2 acts on $W(\lambda_1)$ as $[I_3, \mu I_2, \mu^{-1}I_2]$, for some $\lambda, \mu \in K^*$. In particular, $\dim C_{\text{Lie}(A_2) \otimes W(\lambda_1)}(x) \leq 18$ and thus $\dim C_{\text{Lie}(G)}(x) \leq 22 < 46$, which is a contradiction. Similar reasoning eliminates the case $C_H(x)^0 = A_1T_3$, and we conclude that (10) holds for all $x \in \mathcal{P}$.

Next suppose $H^0 = T_2D_4$, so $H/H^0 = S_3$. Since $\dim x^H \leq 24$ for all $x \in H$, we may assume $\dim x^G \leq 48$. In particular, if x is semisimple then $C_G(x)^0 = T_1D_5, A_5A_1, A_5T_1$ or T_2D_4 , and the bound supplied by [51, Theorem 2] is sufficient. Now assume $x \in H$ is unipotent. By [55, Proposition 2.3] we have

$$V_{27} \downarrow D_4 = W(\lambda_1) \oplus W(\lambda_3) \oplus W(\lambda_4) \oplus 0^3$$

and in the usual way we can compute the Jordan form of x on V_{27} (and subsequently determine the G -class of x via [47, Table 5]). In particular, if $p = 2$ and $x \in H \setminus H^0$ then x

induces a b_1 or b_3 involution on the D_4 factor (in the notation of [2]); in the former case, x has Jordan form $[J_2^{10}, J_1^7]$, otherwise it is $[J_2^{12}, J_1^3]$; it follows that the respective G -classes are $2A_1$ and $3A_1$, and the result follows. Similarly, if $p = 3$ and $x \in H \setminus H^0$ has order 3 then x induces a triality graph automorphism on the D_4 factor and we calculate that x has Jordan form $[J_3^9]$ on V_{27} (there are two classes of triality graph automorphisms; they have the same Jordan form on V_{27}). Therefore, x is in one of the classes $2A_2$ or $2A_2 + A_1$, so $\dim x^H \leq 20 < \frac{1}{2} \dim x^G$ as required.

Finally suppose $H^0 = A_2^3$, in which case $H/H^0 = S_3$. Here $\dim x^H \leq 18$ for all $x \in \mathcal{P}$, so we may assume $\dim x^G \leq 36$. As before, if $x \in H$ is semisimple then [51, Theorem 2] is sufficient (note that $C_G(x)^0 = D_5T_1$ is the only possibility with $\dim x^G \leq 36$), so let us assume x is unipotent. Here the relevant G -classes are labelled A_1 and $2A_1$, with respective dimensions 22 and 32. If $x \in H^0$ then the desired bound quickly follows from the information in [49, Table 18]. Now assume $x \in H \setminus H^0$, so $p = 2$ or 3 . By [55, Proposition 2.3] we have

$$V_{27} \downarrow A_2^3 = (W(\lambda_1) \otimes W(\lambda_2) \otimes 0) \oplus (W(\lambda_2) \otimes 0 \otimes W(\lambda_1)) \oplus (0 \otimes W(\lambda_1) \otimes W(\lambda_2)).$$

If $p = 3$ then x cyclically permutes the A_2 factors of H^0 , so from the above decomposition we deduce that x has Jordan form $[J_3^9]$ on V_{27} and thus [47, Table 5] indicates that x is in one of the classes labelled $2A_2$ or $2A_2 + A_1$, a contradiction. Finally, let us assume $p = 2$ and $x \in H \setminus H^0$ is an involution. By [51, Proposition 1.13(iii)], x is not a long root element, so we may assume $x \in 2A_1$ and thus $\dim x^G = 32$. Now x acts as a transposition on the A_2 factors, and it either centralizes or induces an involutory automorphism on the fixed factor. Therefore $\dim x^H \leq 8 + 5 = 13 < \frac{1}{2} \dim x^G$ as required. \square

Proposition 5.14. *Theorem 7 holds for $G = F_4$.*

Proof. In view of Theorem 5.1 and Lemma 3.20, we may assume $H = N_G(X)$ with

$$X \in \{D_4, \tilde{D}_4 (p = 2), A_1, G_2, A_1G_2, A_2\tilde{A}_2\}.$$

Note that if $p = 2$ then the subgroups D_4 and \tilde{D}_4 are interchanged by a graph automorphism of G , so we only need to consider D_4 , which is generated by the long root subgroups. Let $\{\omega_1, \dots, \omega_4\}$ be a set of fundamental dominant weights for G , and let $V_{26} = W(\omega_4)$ be the 26-dimensional Weyl module for G with highest weight ω_4 .

First consider the case $H^0 = D_4$. Here $H/H^0 = S_3$ and $\dim H > \frac{1}{2} \dim G$, so $b^0(G, H) \geq 3$ by Proposition 2.5(iii). We claim that $\dim x^H < \frac{2}{3} \dim x^G$ for all $x \in \mathcal{P}$, so

$$b^0(G, H) = b(G, H) = b^1(G, H) = 3$$

(see Corollary 2.14). Note that

$$V_{26} \downarrow D_4 = W(\lambda_1) \oplus W(\lambda_3) \oplus W(\lambda_4) \oplus 0^2. \quad (15)$$

If x is semisimple then the claim follows from [51, Theorem 2], so let us assume x is unipotent. If $x \in H^0$ then we can use the above decomposition (15) to determine the G -class of x . For example, suppose $p \neq 2$ and $x \in H^0$ has Jordan form $[J_3, J_1^5]$ on the natural D_4 -module $W(\lambda_1)$. Then x has Jordan form $[J_2^4]$ on $W(\lambda_3)$ and $W(\lambda_4)$ (the two spin modules for D_4), so $[J_3, J_2^8, J_1^7]$ is the Jordan form of x on V_{26} and thus [47, Table 3] indicates that x belongs to the G -class labelled \tilde{A}_1 . In this way, it is straightforward to verify the claim for all unipotent elements $x \in H^0$. (Note that if $p = 2$ then $x \in D_4 < B_4$ and the fusion of unipotent B_4 -classes is stated explicitly in the proof of [51, Lemma 4.6]; in particular, involutions of type c_2 or a_4 in H^0 belong to the G -class labelled $\tilde{A}_1^{(2)}$.)

To complete the analysis of the case $H^0 = D_4$ we may assume $x \in H \setminus H^0$ is unipotent. Suppose $p = 2$ and x is an involution; there are two such H -classes in $H \setminus H^0$. As before, using the decomposition (15), it is easy to calculate the Jordan form of x on V_{26} ; if x is

a b_1 -involution we get $[J_2^{10}, J_1^6]$, and $[J_2^{12}, J_1^2]$ is the Jordan form of a b_3 -involution. The result now follows by inspecting [47, Table 3]. Finally, suppose $x \in H \setminus H^0$ and $p = 3$, so x acts on D_4 as a triality graph automorphism. There are two such H -classes in $H \setminus H^0$, and we calculate that x has Jordan form $[J_3^8, J_2]$ on V_{26} , so x is in one of the classes labelled \tilde{A}_2 or $\tilde{A}_2 + A_1$ (with respective dimensions 30 and 36). In particular, if $C_{D_4}(x) = G_2$ then $\dim x^H = 14 < \frac{2}{3} \dim x^G$. On the other hand, if $C_{D_4}(x) \neq G_2$ then $\dim x^H = 20$ and we need to show that x belongs to the G -class labelled $\tilde{A}_2 + A_1$. To see this, first observe that

$$\mathrm{Lie}(G) \downarrow D_4 = \mathrm{Lie}(D_4) \oplus W(\lambda_1) \oplus W(\lambda_3) \oplus W(\lambda_4).$$

(see [55, Table 8.4]). Now x has Jordan form $[J_3^8, J_2^2]$ on $\mathrm{Lie}(D_4)$, so x has Jordan form $[J_3^{16}, J_2^2]$ on $\mathrm{Lie}(G)$. In particular, [47, Table 4] indicates that x is in one of the G -classes labelled $A_2 + \tilde{A}_1$ or $\tilde{A}_2 + A_1$, so in view of our earlier analysis of $V_{26} \downarrow x$ we deduce that x is in the class $\tilde{A}_2 + A_1$, as required.

In each of the remaining cases we claim that (10) holds for all $x \in \mathcal{P}$ (and thus (9) follows). Since $\dim x^G \geq 16$, the case $H^0 = A_1$ is clear.

Next suppose $H^0 = A_2 \tilde{A}_2$. Here $H/H^0 = Z_2$ and $\dim x^H \leq 12$ for all $x \in \mathcal{P}$, so we may assume $\dim x^G \leq 24$. The claim quickly follows from [51, Theorem 2] if x is semisimple, so let us assume x is unipotent. If $x \in H^0$ then the fusion information in [49, Table 16] is sufficient, so we may assume $p = 2$ and $x \in H \setminus H^0$ is an involution. Here x acts as a graph automorphism on each A_2 factor, so $\dim x^H = 10$. By [51, Proposition 1.13(iii)], x is not a root element, so $\dim x^G \geq 22$ and the result follows.

Now assume $H = H^0 = G_2$. Here $p = 7$ (see [56, Corollary 2]) and we may as well assume $\dim x^G \leq 24$. In particular, if x is unipotent then x belongs to one of the G -classes labelled A_1 or \tilde{A}_1 , and the information in [49, Table 28] is sufficient. Now assume x is semisimple. If x has odd order then $\dim x^G \geq 30$, so we may assume x is an involution. There is a unique class of involutions $x \in G_2$ (this class has dimension 8) and it suffices to show that $C_G(x) = A_1 C_3$ (rather than B_4). To see this, first note that

$$\mathrm{Lie}(G) \downarrow G_2 = \mathrm{Lie}(G_2) \oplus W(\lambda_1 + \lambda_2)$$

(see [55, Proposition 2.4]). Now x acts on $\mathrm{Lie}(G_2)$ as $[-I_8, I_6]$, so it remains to show that x acts on $W(\lambda_1 + \lambda_2)$ as $[-I_{20}, I_{18}]$, rather than $[-I_8, I_{30}]$. In terms of fundamental dominant weights $\{\mu_1, \mu_2\}$, the restriction of the G_2 -module $W(\lambda_1 + \lambda_2)$ to $A_2 < G_2$ is given by

$$W(\lambda_1 + \lambda_2) \downarrow A_2 = \mathrm{Lie}(A_2) \oplus W(2\mu_1 + \mu_2) \oplus W(\mu_1 + 2\mu_2).$$

Now $x = [-I_2, I_1] \in A_2$ acts on $\mathrm{Lie}(A_2)$ as $[-I_4, I_4]$ and [40, Theorem 8.3] rules out the possibility that x acts on $W(2\mu_1 + \mu_2)$ (and also $W(\mu_1 + 2\mu_2) = W(2\mu_1 + \mu_2)^*$) as $[-I_2, I_{13}]$. Therefore, x must act on $W(\lambda_1 + \lambda_2)$ as $[-I_{20}, I_{18}]$, so $C_G(x) = A_1 C_3$ as claimed.

Finally, let us assume $H = H^0 = A_1 G_2$, so $p \neq 2$ (see [56, Corollary 2]). Since $\dim x^H \leq 14$ for all $x \in H$, we may assume that $\dim x^G \leq 28$. Consequently, if x is unipotent then x belongs to one of the G -classes labelled A_1 , \tilde{A}_1 or $A_1 \tilde{A}_1$, and in each case the required bound follows from the fusion information in [49, Table 29]. If x is semisimple and $\dim x^G \leq 28$ then $C_G(x) = B_4$ or $A_1 C_3$. In particular, x is an involution. Since $\dim x^H \leq 10$ for all involutions $x \in H$, we reduce to the case $C_G(x) = B_4$. We claim that $\dim x^H = 2$. By [55, Proposition 2.4] we have

$$\mathrm{Lie}(G) \downarrow A_1 G_2 = \mathrm{Lie}(A_1 G_2) \oplus (W(4) \otimes W(\lambda_1)).$$

Write $x = x_1 x_2$, where $x_1 \in A_1$ and $x_2 \in G_2$. Suppose x_1 and x_2 are both nontrivial. There is a unique class of involutions in both A_1 and G_2 , and it is easy to see that x_1 acts on $W(4)$ as $[-I_2, I_3]$, and x_2 acts on $W(\lambda_1)$ as $[-I_4, I_3]$. It follows that x acts on $\mathrm{Lie}(G)$ as $[-I_{28}, I_{24}]$, so $C_G(x) = A_1 C_3$. In this way, we deduce that $C_G(x) = B_4$ if and only if $x_1 \neq 1$ and $x_2 = 1$, whence $\dim x^H = 2$ as claimed. \square

Proposition 5.15. *Theorem 7 holds for $G = G_2$.*

Proof. According to Theorem 5.1 we have $H = N_G(X)$ with

$$X \in \{A_1\tilde{A}_1, A_1, A_2, \tilde{A}_2 (p = 3)\}.$$

Note that if $p = 3$ then the subgroups A_2 and \tilde{A}_2 are interchanged by a graph automorphism of G , so we only need to consider A_2 , which is generated by the long root subgroups. The case $H^0 = A_1\tilde{A}_1$ corresponds to an involution-type subgroup and this has already been dealt with in Lemma 3.21. If $H = A_1$ then $\dim x^H \leq 2$ for all $x \in \mathcal{P}$, so $\dim x^H < \frac{1}{2} \dim x^G$ (since $\dim x^G \geq 6$) and thus Corollary 2.14 implies that (9) holds.

Finally, suppose $H^0 = A_2$. Here $H/H^0 = Z_2$ and $\dim H = 8 > \frac{1}{2} \dim G$ so $b^0(G, H) \geq 3$. We claim that $\dim x^H \leq \frac{2}{3} \dim x^G$ for all $x \in \mathcal{P}$, with equality if and only if $x \in H^0$ is a long root element. In particular, Corollary 2.19 implies that $b^1(G, H) \leq 3$, so

$$b^0(G, H) = b(G, H) = b^1(G, H) = 3.$$

First assume $x \in H$ is a unipotent element. If $x \in H^0$ then the claim follows from the information in [49, Table 11], so let us assume $p = 2$ and $x \in H \setminus H^0$ is an involution. Here x acts as a graph automorphism on H^0 , so $\dim x^H = 5$ and we calculate that x has Jordan form $[J_2^3, J_1]$ on the 7-dimensional Weyl module V_7 (since $V_7 \downarrow A_2 = W(\lambda_1) \oplus W(\lambda_2) \oplus 0$). Therefore [47, Table 1] indicates that x is in the G -class \tilde{A}_1 , and thus $\dim x^G = 8$. Finally, suppose x is semisimple. Since $\dim x^H \leq 6$ we may as well assume $\dim x^G \leq 8$, so $C_G(x)^0 = A_1\tilde{A}_1$ or A_2 . In the latter case, x centralizes H so assume $C_G(x)^0 = A_1\tilde{A}_1$. Here $p \neq 2$, x is an involution and $\dim x^H = 4$. The result follows. \square

This completes the proof of Theorem 7.

6. NORMALIZERS OF TORI

In this final section we prove Theorem 9. Let G be a simple algebraic group of rank r over an algebraically closed field of characteristic $p \geq 0$, let T be a maximal torus of G and consider the action of G on $\Omega = G/H$, where $H = N_G(T)$. Recall that Theorem 9 states that either $b^1(G, H) = 2$, or $G = A_1$ and the generic 2-point stabilizer has order 2.

Suppose $G = A_r$, so H is a \mathcal{C}_2 -subgroup of G of type $\mathrm{GL}_1 \wr S_{r+1}$. If $r = 1$ and $p \neq 2$ then the desired result follows from Theorem 8, and the case $p = 2$ is handled in the proof of Lemma 3.16. If $r > 1$ then Theorem 5 yields $b^1(G, H) = 2$ (see Section 4.2). Similarly, if $G = D_r$ (with $r \geq 4$) then H is a \mathcal{C}_2 -subgroup of type $O_2 \wr S_r$ and once again the result follows from Theorem 5. In each of the remaining cases, we claim that

$$\dim x^H < \frac{1}{2} \dim x^G \tag{16}$$

for all $x \in \mathcal{P}$, where \mathcal{P} is the set of elements of prime order in H (including all nontrivial unipotent elements if $p = 0$). In particular, by applying Corollary 2.14, we deduce that $b^1(G, H) = 2$.

If G is an exceptional algebraic group then $\dim x^G > 2r$ for all $x \in \mathcal{P}$, whence (16) holds. Indeed, $\dim x^G \geq \alpha$ where α is defined as follows:

G	E_8	E_7	E_6	F_4	G_2
α	58	34	22	16	6

Now, if $G = C_r$ (with $r \geq 2$) then either x is a long root element and $\dim x^G = 2r$, or $\dim x^G \geq 4r - 4$ (see [13, Proposition 2.9]). We immediately deduce that (16) holds, unless x is a long root element, or if $r = 2$, $p \neq 2$ and $x = [-I_2, I_2]$ is an involution. In the latter case we calculate that $\dim(x^G \cap H) = 1$ and $\dim x^G = 4$. Similarly, if $x \in H$ is a

long root element then $p = 2$ and $\dim(x^G \cap H) = 1$. Finally, let us assume $G = B_r$ (with $r \geq 3$ and $p \neq 2$). Here $\dim x^G \geq 2r$, with equality if and only if x is a long root element or $x = [-I_{2r}, I_1]$. But H does not contain any long root elements, and if $x = [-I_{2r}, I_1]$ we calculate that $\dim(x^G \cap H) = 1$. The result follows.

This completes the proof of Theorem 9.

Finally, we sketch a proof of Corollary 10. Define G , H and r as above, and assume that G is of adjoint type and the center of $\text{Lie}(G)$ is trivial. Let X be the product variety $G/H \times G/H$.

We first show that the action of G on X is generically free. By Theorem 9, the generic orbits of G on X are free, so we need to show that there exists a non-empty open subvariety U of X such that for all $x \in U$, the map ϕ_x sending $\text{Lie}(G)$ to the tangent space of x in X is injective. Of course, this is equivalent to the fact that H acts generically freely on G/H .

By [25, Proposition XI.5.9], H is a smooth group scheme. This implies that the kernel of ϕ_x is just the intersection of two maximal Cartan subalgebras of $\text{Lie}(G)$. It is easy to see that generically this intersection is just the center of the Lie algebra, which is trivial by assumption. Indeed, a Cartan subalgebra is the centralizer of a generic regular semisimple element, and two such elements generically generate the Lie algebra, so the common centralizer is the center.

By [26, Example 7.3(b)], the action of H on G/H is versal. Recall that one of the equivalent definitions of $\text{ed}(H)$ is the minimal value of $\dim X - \dim H$, where X ranges over all generically free versal H -varieties (see [67, Remark 2.6]). Thus,

$$\text{ed}(H) \leq \dim G - 2r$$

where $r = \dim H$ is the rank of G . As we have noted, the inequality $\text{ed}(G) \leq \text{ed}(H)$ was already proved by Springer (see [66, Proposition 4.3]).

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