Topological generation of algebraic groups

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Finite groups

- **Minimal generation.** $d(G) = \min\{|S| \,:\, G = \langle S \rangle\}$.

- **Invariable generation.** $\{x_1, \dots, x_t\} \subseteq G$ invariably generates if
$$G = \langle x_1^{g_1}, \dots, x_t^{g_t} \rangle$$
for all $g_i \in G$. Let $d_I(G) = \min\{|S| \,:\, S \text{ invariably generates } G\}$.

- **Random generation.** Let
$$\mathbb{P}_t(G) = \frac{|\{(x_1, \dots, x_t) \in G^t \,:\, G = \langle x_1, \dots, x_t \rangle\}|}{|G|^t}$$
be the probability that t randomly chosen elements generate G.

- **Conjugate generation.** If $G = \langle g^G \rangle$ then define
$$\kappa(g) = \min\{|S| \,:\, S \subseteq g^G,\ G = \langle S \rangle\}.$$

Finite simple groups

Theorem. Let G be a (non-abelian) finite simple group.

- $d(G) = 2$ (Steinberg, 1962)
- $d_I(G) = 2$ (Kantor, Lubotzky & Shalev, 2011)
- $\mathbb{P}_2(G) \to 1$ as $|G| \to \infty$ (Liebeck & Shalev, 1995)
- If $G = Cl_n(q)$ is a classical group with $n \geqslant 5$, then
$$\kappa(g) \leqslant n+1 \text{ for all } 1 \neq g \in G$$
(Guralnick & Saxl, 2003)

Problem. Can we establish analogous results for **algebraic groups**?

First observations

Let G be a **simple algebraic group** over an algebraically closed field k of characteristic $p \geqslant 0$, e.g. $\mathrm{SL}_n(k)$, $\mathrm{Sp}_n(k)$, E_8, etc.

- G is **not** finitely generated:

$$\langle x_1, \dots, x_t \rangle \leqslant G(F) < G$$

  for some subfield $F = k_0(\lambda_1, \dots, \lambda_m)$ of k (with k_0 the prime field).

- $S \subseteq G$ is a **topological generating set** if $\langle S \rangle$ is (Zariski-)dense in G.

- If k is algebraic over a finite field, then G is locally finite.

We will always assume that k is not algebraic over a finite field.

Topological 2-generation

Theorem (Guralnick, 1998).
If $p = 0$, then $\Delta := \{(g, h) \in G^2 \,:\, G = \overline{\langle g, h\rangle}\}$ is dense in G^2.

- If V is a finite dimensional kG-module, then

$$\{(g, h) \in G^2 \,:\, \overline{\langle g, h\rangle} \text{ acts irreducibly on } V\} \text{ is open in } G^2$$

- Let V_1 be the adjoint module for G and V_2 an irreducible kG-module such that every finite subgroup of G is reducible. Then

$$\Delta = \{(g, h) \in G^2 \,:\, \overline{\langle g, h\rangle} \text{ is irreducible on } V_1 \text{ and } V_2\} \text{ is open}$$

- If $g \in G$ is non-central and $h \in G$ is a regular semisimple element such that $\overline{\langle h\rangle}$ is a maximal torus, then $G = \overline{\langle g, h^a\rangle}$ for some $a \in G$.

  Therefore, Δ is non-empty and thus dense.

A generalisation

Notation. Let Ω be a (locally closed) irreducible subset of G^t, e.g.

$$G^t,\ \{g\} \times G^{t-1} \text{ or } C_1 \times \cdots \times C_t, \text{ with } C_i = g_i^G$$

For $x = (x_1, \ldots, x_t) \in \Omega$, set $G(x) = \overline{\langle x_1, \ldots, x_t \rangle}$ and define

$$\Delta = \{x \in \Omega \,:\, G(x) = G\}.$$

Theorem (BGG, 2019). If Δ is non-empty, then Δ is dense in Ω.

- As a special case, $\{x \in G^2 \,:\, G(x) = G\}$ is dense in G^2 for all $p \geqslant 0$.
- By considering $\Omega = C_1 \times \cdots \times C_t$, it follows that all topological generating sets for G are "almost invariable".

Comments on the proof

Theorem (BGG, 2019). If Δ is non-empty, then Δ is dense in Ω.

- Assume $\Delta \neq \emptyset$ and write $\Delta = \Delta^+ \cap \Lambda$, where

$$\Delta^+ = \{x \in \Omega \,:\, \dim G(x) > 0\}$$
$$\Lambda = \{x \in \Omega \,:\, G(x) \not\leqslant H \text{ for any } H \in \mathcal{M}\}$$

  and $\mathcal{M}$ is the set of maximal closed pos. diml. subgroups of G.

- By considering a finite collection of irreducible kG-modules, we can construct an open subset Γ of Ω with $\Delta \subseteq \Gamma \subseteq \Lambda$.

- **Key step:** $\Delta^+ \neq \emptyset \implies \Delta^+$ is dense in Ω.

- Then $\Delta = \Delta^+ \cap \Lambda = \Delta^+ \cap \Gamma$ is dense in Ω.

Exceptional algebraic groups

Theorem (BGG, 2019).

Let G be an exceptional group and set $N = 4$ if $G = G_2$, otherwise $N = 5$. Let $\Omega = C_1 \times \cdots \times C_t$, where $t \geqslant N$ and each $C_i = g_i^G$ is non-central. **Then Δ is dense in Ω.**

- The bound $t \geqslant N$ is best possible in all cases.

  e.g. if $G = E_8$ and $C = g^G$ is the class of long root elements, then $\dim C_V(g) = 190$ on the adjoint module V, so $\Delta = \emptyset$ if $\Omega = C^4$.

- Excluding a handful of classes, we can show that Δ is dense if $t \geqslant 3$.

- We expect the same bounds are best possible for the corresponding **finite** exceptional groups.

  Here [GS, 2003] gives $\kappa(g) \leqslant \mathrm{rank}(G) + 4$ for all $1 \neq g \in G(q)$.

Key lemma

For $H \leqslant G$ and $g \in G$, set

$$X = G/H,\ X(g) = \{x \in X \,:\, x^g = x\},\ \alpha(G,H,g) = \frac{\dim X(g)}{\dim X}$$

Lemma. Let G be a simple algebraic group and set $\Omega = C_1 \times \cdots \times C_t$, where $t \geqslant 3$ and each $C_i = g_i^G$ is non-central. Then Δ is dense if

$$\sum_{i=1}^{t} \alpha(G,H,g_i) < t-1 \qquad (\star)$$

for all $H \in \mathcal{M}$.

This relies on the fact that G has only finitely many classes of positive dimensional maximal closed subgroups (Liebeck & Seitz, 2004).

Fixed point spaces for exceptional groups

Lemma. Let G be an exceptional group and set

$$\beta(G) = \max\{\alpha(G, H, g) \,:\, g \in G \text{ non-central}, \, H \in \mathcal{M}\}.$$

- Then $\beta(G) < 1 - \frac{1}{N}$, where $N = 4$ if $G = G_2$, otherwise $N = 5$.
- More precisely:

G	E_8	E_7	E_6	F_4	G_2
$\beta(G)$	15/19	7/9	10/13	3/4	2/3

Corollary. If $\Omega = C_1 \times \cdots \times C_t$ with $t \geqslant N$ and $C_i = g_i^G$, then

$$\sum_{i=1}^{t} \alpha(G, H, g_i) \leqslant t \cdot \beta(G) < t\left(1 - \frac{1}{N}\right) \leqslant t - 1$$

for all $H \in \mathcal{M}$, so $(\star)$ holds and Δ is dense.

Computing dimensions

Lemma (Lawther, Liebeck & Seitz, 2002). If $g \in H$, then

$$\dim X(g) = \dim X - \dim g^G + \dim(g^G \cap H).$$

Example (LLS). Let $G = E_8$, $H = P_8$, $g \in G$ a long root element.

- We may assume $g \in L'$, where $L = T_1E_7$ is a Levi factor. Then

$$\dim(g^G \cap H) = \frac{1}{2}(\dim g^G + \dim g^{L'}) = \frac{1}{2}(58 + 34) = 46$$

- The lemma now gives $\dim X(g) = 57 - 58 + 46 = 45$, so

$$\alpha(G, H, g) = \frac{\dim X(g)}{\dim X} = \frac{45}{57} = \frac{15}{19} = \beta(G)$$

An application to random generation

Let L be a finite group and let r, s be primes dividing $|L|$.

Write $\mathbb{P}_{r,s}(L)$ for the probability that L is generated by a randomly chosen element of order r and a random element of order s.

Theorem. Let r, s be primes with $(r, s) \neq (2, 2)$ and let G_i be a sequence of finite simple exceptional groups such that $|G_i| \to \infty$ and r, s divide $|G_i|$ for all i.

- **Guralnick, Liebeck, Lübeck & Shalev, 2019.**

  If $(r, s) = (2, 3)$, then $\mathbb{P}_{r,s}(G_i) \to 1$ as $i \to \infty$.

- **BGG, 2019.** The same conclusion holds for all r and s.

Another key lemma

Let $G(q) = G_\sigma$ be a finite quasisimple exceptional group of Lie type over $\mathbb{F}_q$, where σ is a suitable Steinberg endomorphism of G.

Let r, s be prime divisors of $|G(q)/Z(G(q))|$ with $(r, s) \neq (2, 2)$ and define

$$\mathcal{C}(G, r, q) = \max\{\dim g^G : g \in G(q) \text{ has order } r \text{ modulo } Z(G)\}.$$

e.g. if $G = E_8$ and $r = 3$, then $\mathcal{C}(G, r, q) = 168$ for all q.

Lemma. Let $g_r \in G$ be any element of order r modulo $Z(G)$ with $\dim g_r^G = \mathcal{C}(G, r, q)$ and define $g_s \in G$ similarly. Then

$$\alpha(G, H, g_r) + \alpha(G, H, g_s) < 1$$

for all positive dimensional maximal closed subgroups H of G.

Some comments on the proof

- Set $\Omega = C_1 \times C_2$, where $C_1 = g_r^G$ and $C_2 = g_s^G$ as before, with $C_i(q) := C_i \cap G(q) \neq \emptyset$ for $i = 1, 2$.

- From the lemma, we deduce that

$$\Delta = \{(g,h) \in \Omega \,:\, G = \overline{\langle g, h \rangle}\} \text{ is dense in } \Omega$$

  and then a general theorem [GLLS, 2019] implies that the proportion of pairs in $C_r(q) \times C_s(q)$ generating $G(q)$ tends to 1 as $q \to \infty$.

- But almost all pairs of elements of order r and s (modulo $Z(G)$) in $G(q)$ are contained in $C_r(q) \times C_s(q)$ for such classes C_r and C_s.

Conjecture (GLLS, 2019). Let r, s be primes with $\{r, s\} \not\subseteq \{2, 3\}$ and let G_i be a sequence of finite simple groups such that $|G_i| \to \infty$ and r, s divide $|G_i|$ for all i. Then $\mathbb{P}_{r,s}(G_i) \to 1$ as $i \to \infty$.