Meanders, hyperelliptic pillowcase covers and the Johnson filtration

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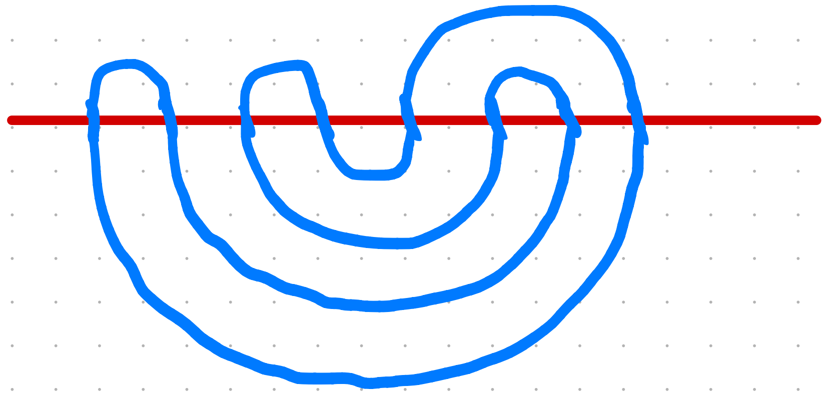
Weihnachtsworkshop 2022

Meanders:



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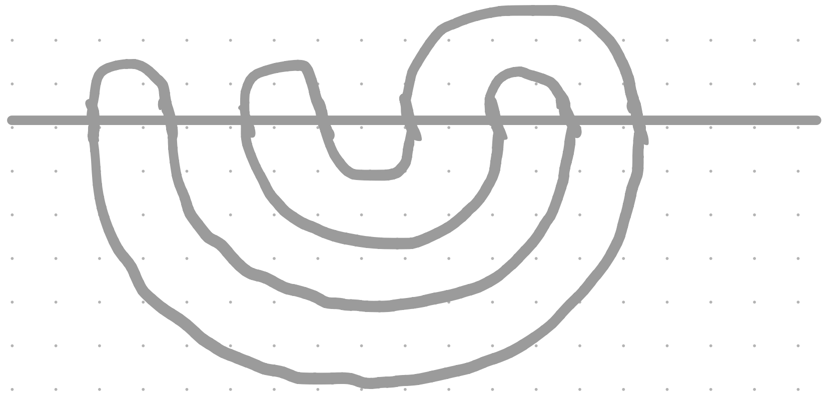
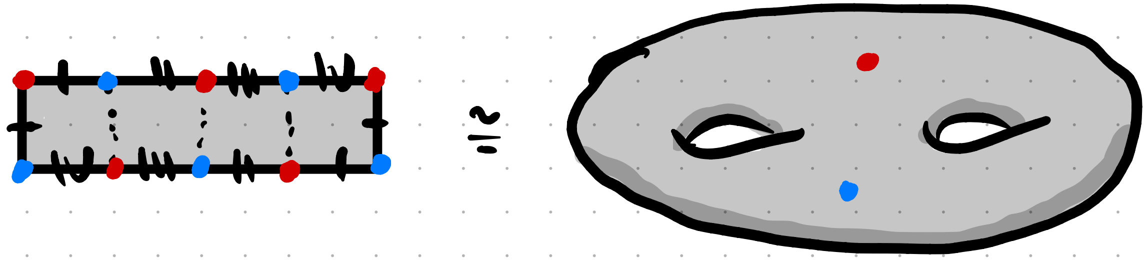


Pillowcase covers:



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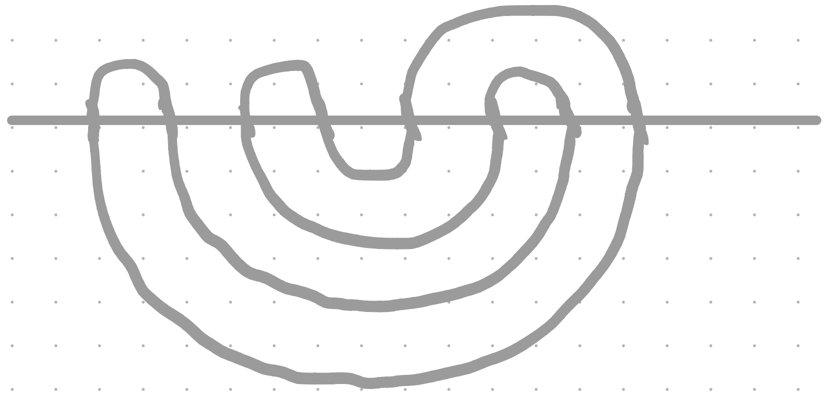

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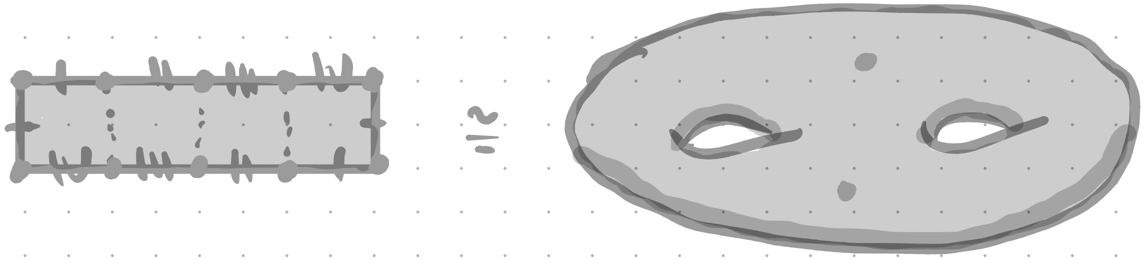


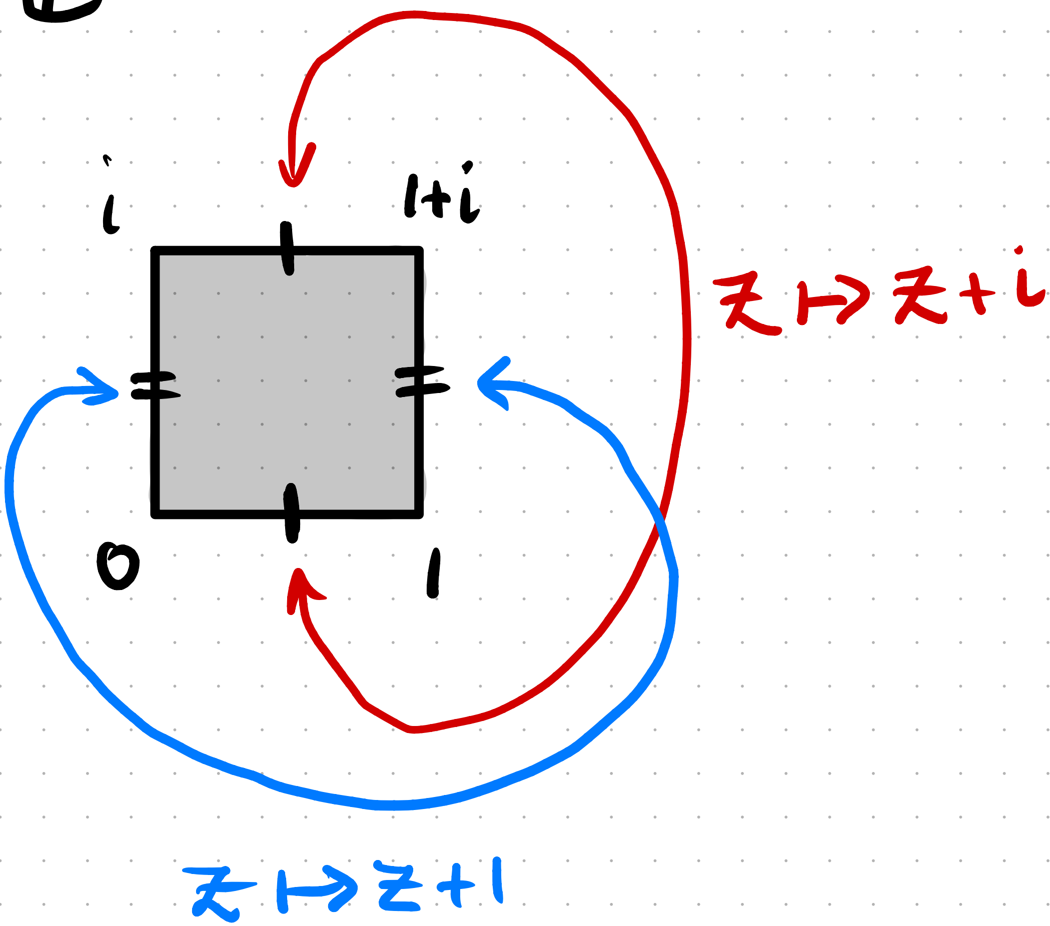
The Johnson filtration:

$$MCG(S) := {}^{\mathrm{Homeo}^+(S)}\big/_{\mathrm{isotopy}} = I^0(S) \geq I^1(S) \geq I^2(S) \geq \dots$$

Pillowcase Covers

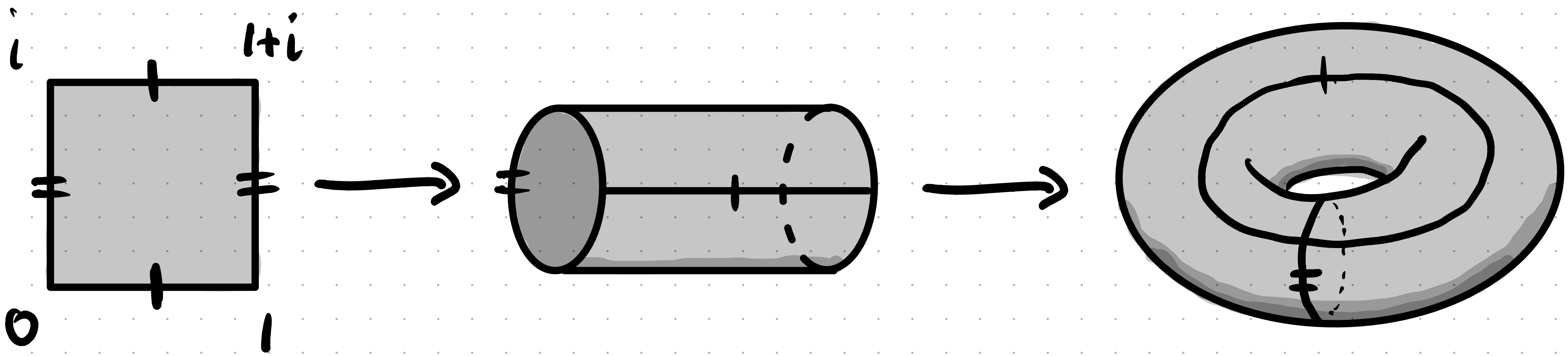
Motivation 1 : The square torus

$\mathbb{C}$



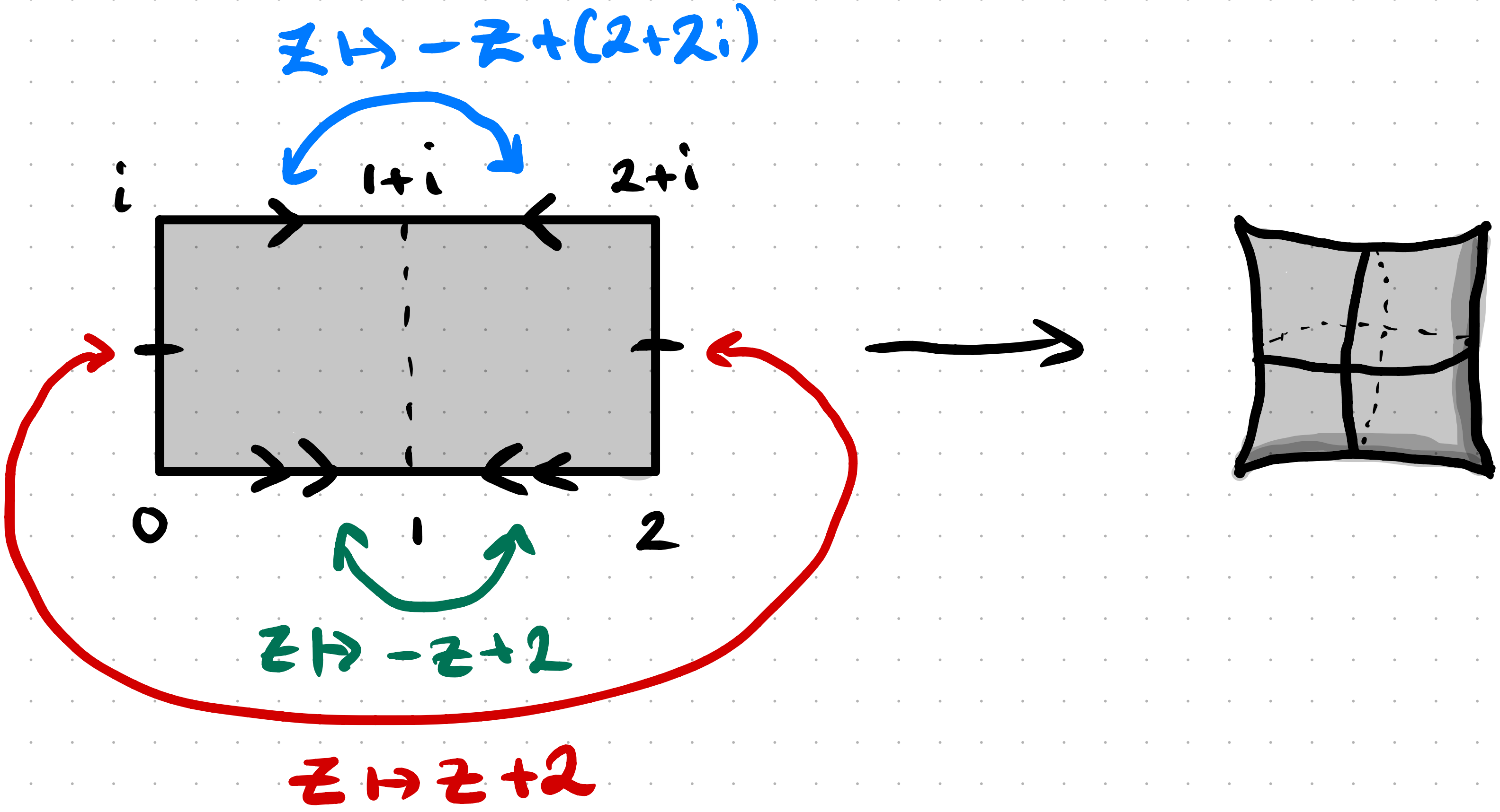
Motivation 1 : The square torus

$\mathbb{C}$



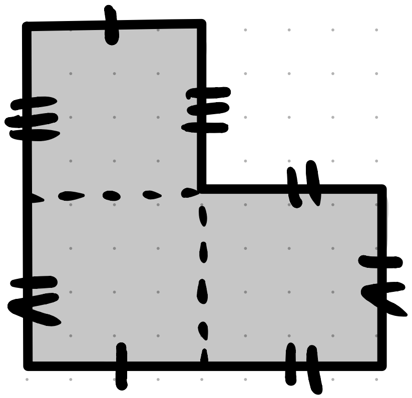
Motivation 2: The pillowcase

$\mathbb{C}$

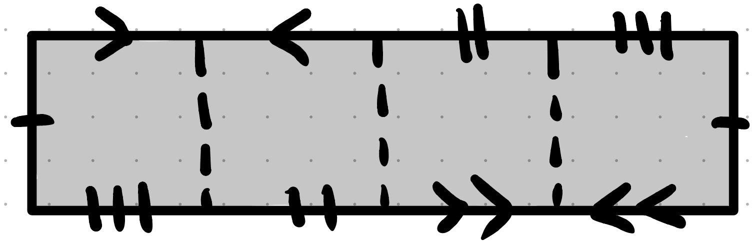


Pillowcase cover: A pillowcase cover is a closed, connected, orientable surface obtained by identifying the sides of a collection of unit squares by translations ($z \mapsto z+c$) or half-translations ($z \mapsto -z+c$).

Examples:



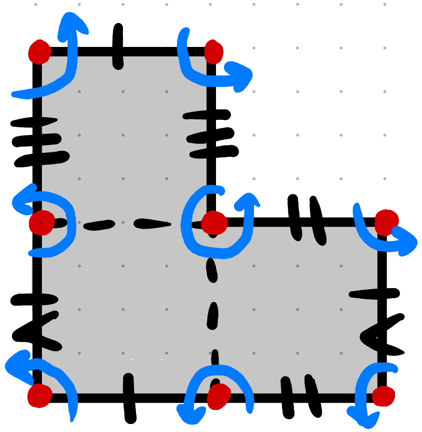
genus 2

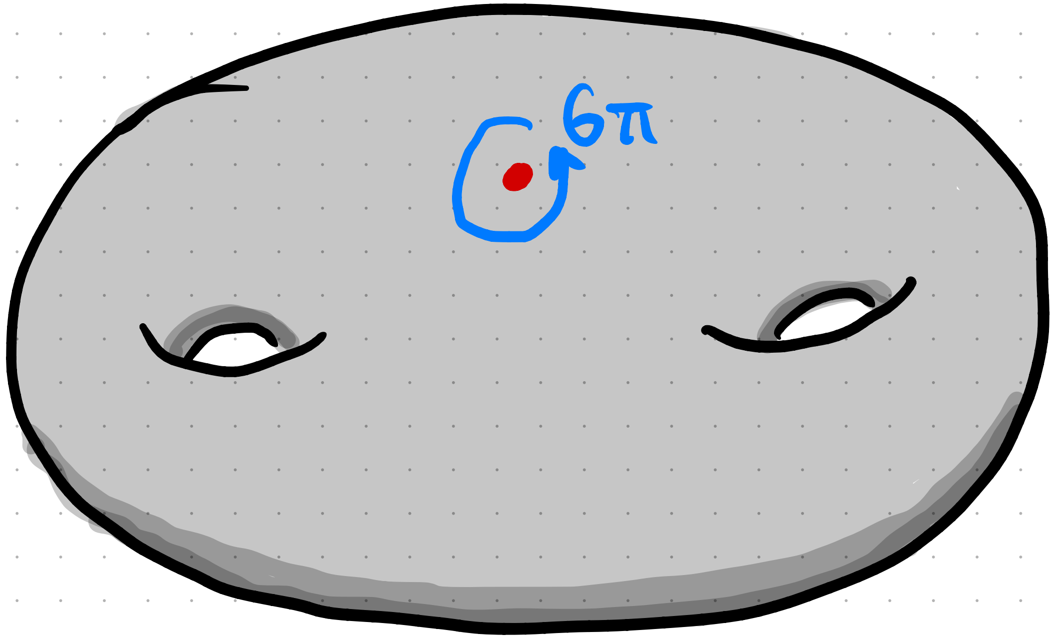


genus 1

Singularity data:

$$6\pi = 2\pi + 2 \cdot 2\pi$$



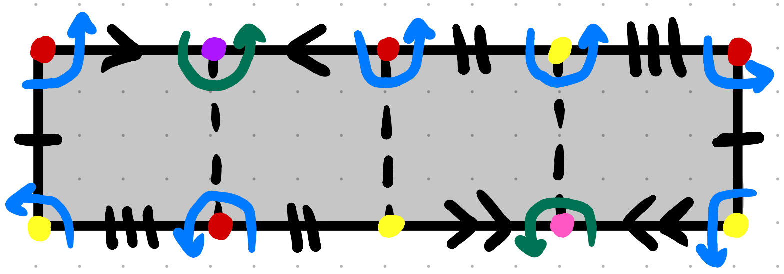


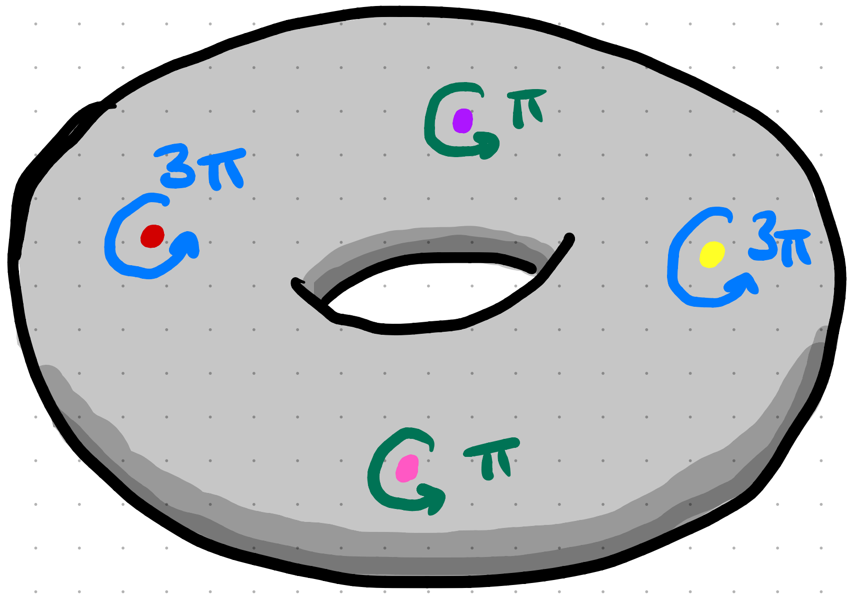
Since we only used translations, we record the excess angle in multiples of 2π.

We record the singularity data (2).

Singularity data:

$$3\pi = 2\pi + 1\cdot\pi$$
$$\pi = 2\pi + (-1)\cdot\pi$$





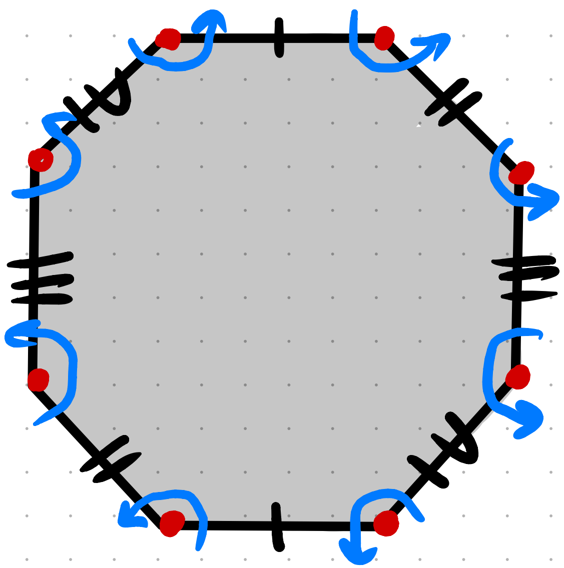
Since we used a half-translation we record the excess angle in multiples of π.

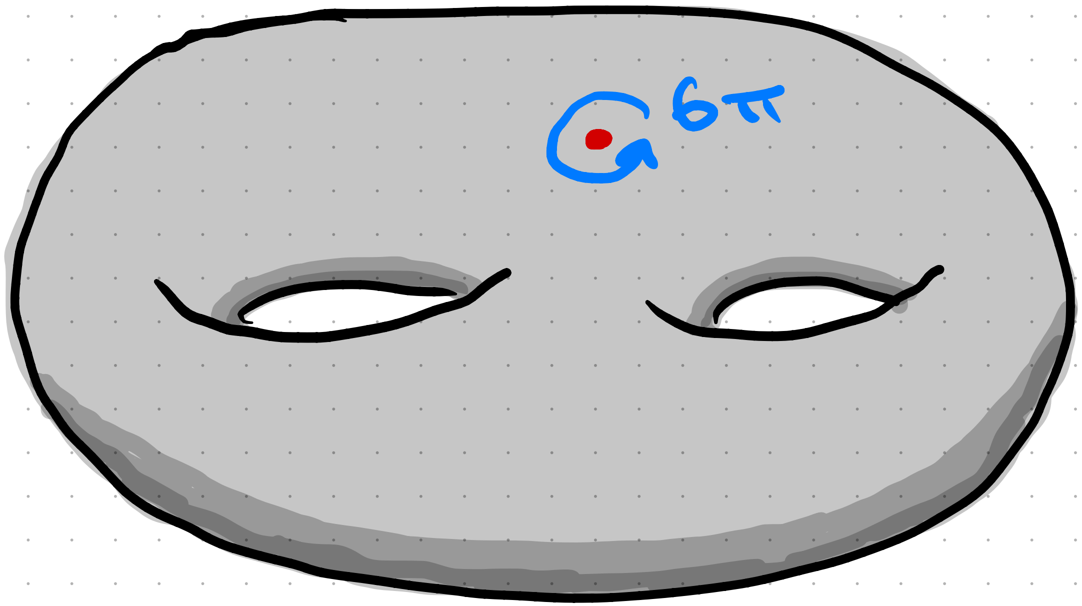
We record the singularity data $(1,1,-1,-1) =: (1^2,-1^2)$.

Why study pillowcase covers?

The moduli space of flat surfaces:

A flat surface is the generalisation of a pillowcase cover allowing arbitrary polygons.





Flat surfaces are useful for studying many low-dimensional dynamical systems:

- billiards in polygons
- interval exchange transformations

It is often necessary to study the 'full space' of flat surfaces.

$$\{\text{flat surfaces of genus } g\} \big/ _{MCG(S)} = \mathcal{H}_g \sqcup \mathcal{Q}_g$$

Only translations

At least one half-translation

Stratified by singularity data:

$$\mathcal{H}_g = \bigsqcup \underbrace{\mathcal{H}_g(k_1, \dots, k_n)}_{} \qquad \mathcal{Q}_g = \bigsqcup \underbrace{\mathcal{Q}_g(k_1, \dots, k_n)}_{}$$

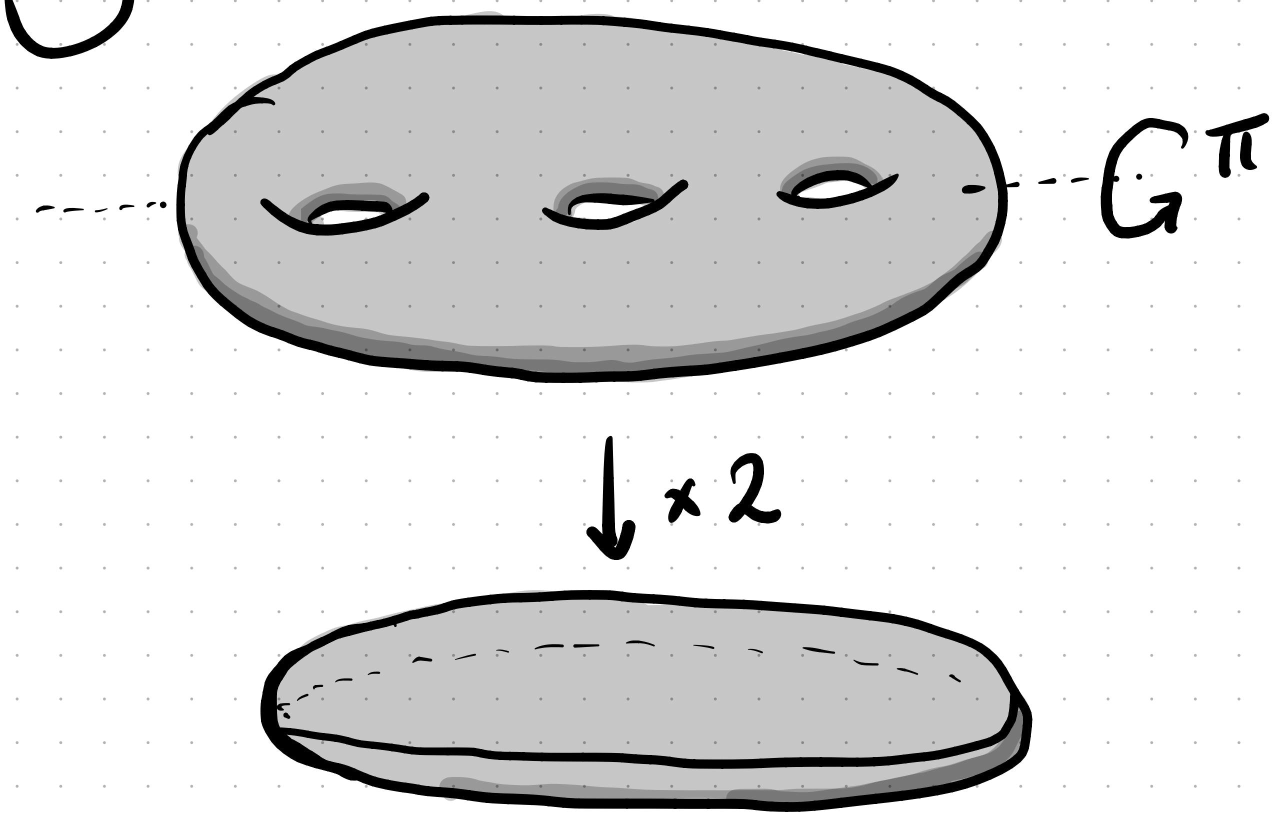
Can be disconnected

Kontsevich-Zorich '03

Lanneau '08

Chen-Möller '14

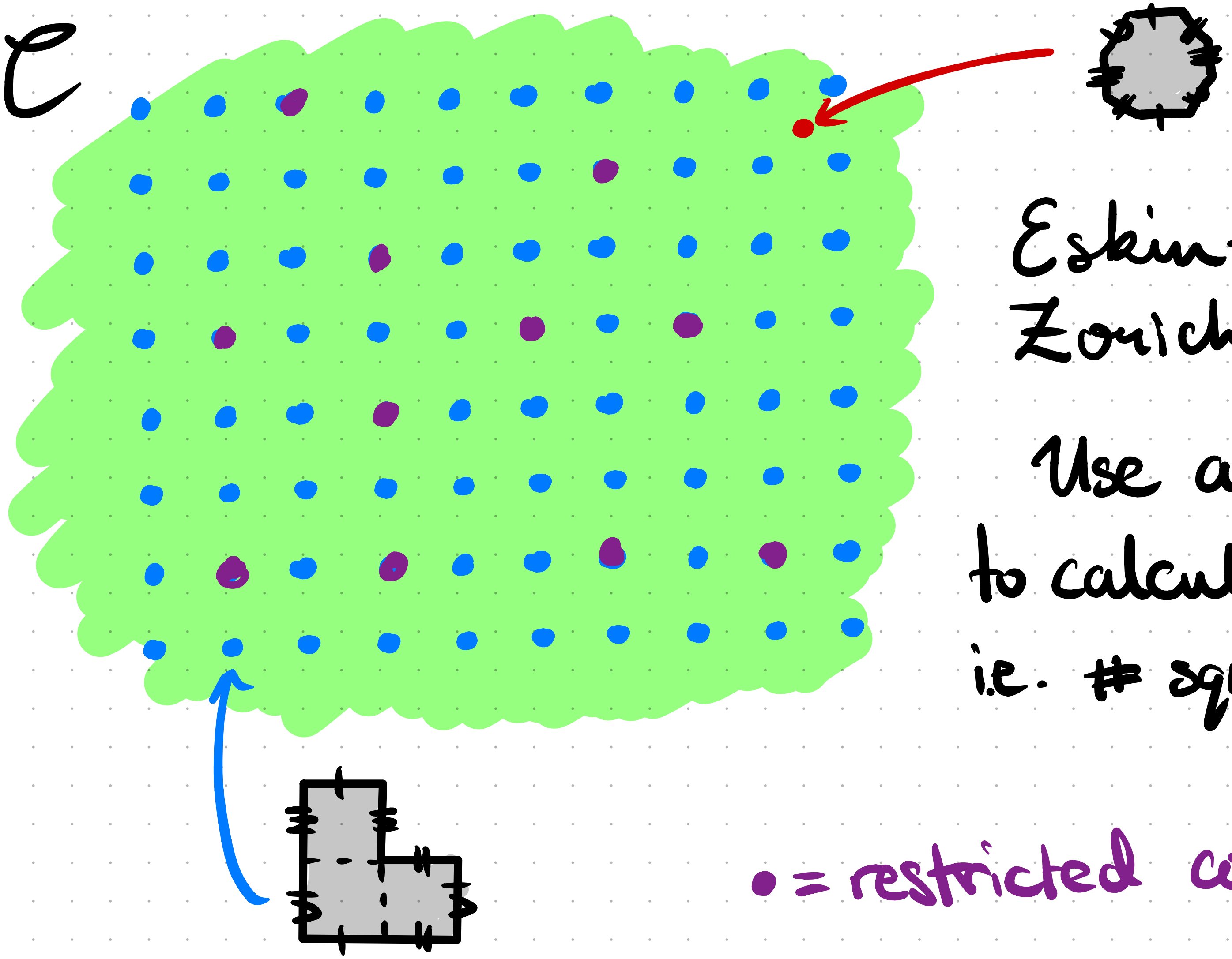
A flat surface is said to be hyperelliptic if it possesses an isometric involution inducing a double cover of the sphere.



We will say that a connected component $\mathcal{C}$ of a stratum $\mathcal{H}(k_1, \ldots, k_n)$ or $\mathcal{Q}(k_1, \ldots, k_n)$ is hyperelliptic if it consists entirely of hyperelliptic flat surfaces.

Otherwise we will call it non-hyperelliptic.

Let $\mathcal{C}$ be a connected component.



Eskin-Okounkov '01
Zorich '02

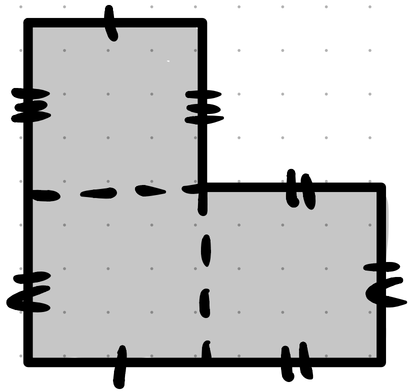
Use asymptotic counts to calculate volumes i.e. # squares $\longrightarrow \infty$.

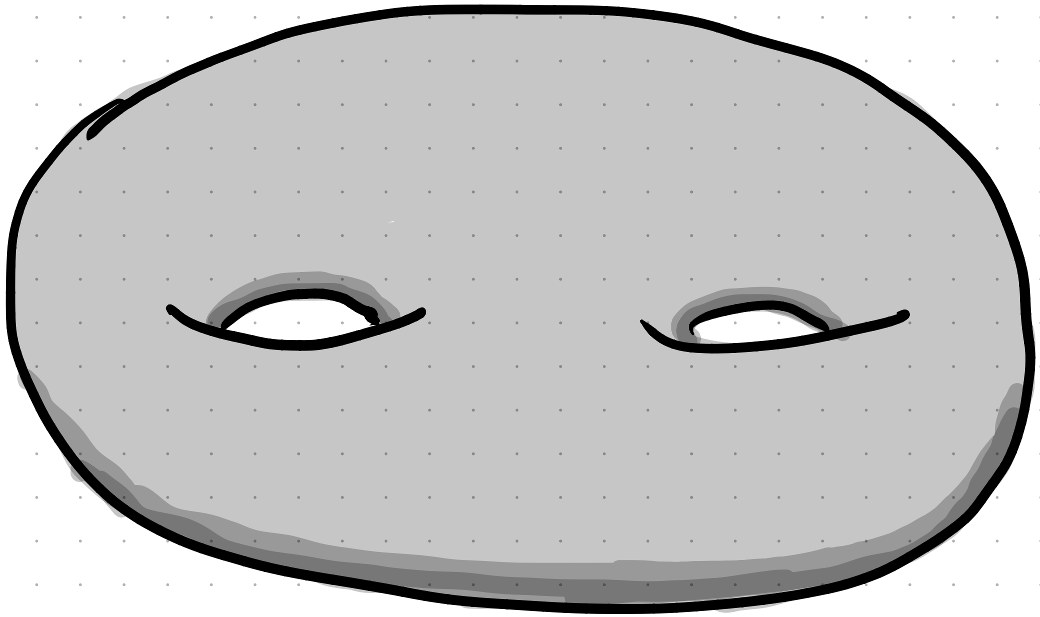
• = restricted combinatorics

$[1,1]$-pillowcase covers

[1,1]-pillowcase covers:

Every pillowcase cover is naturally associated to a collection of curves.





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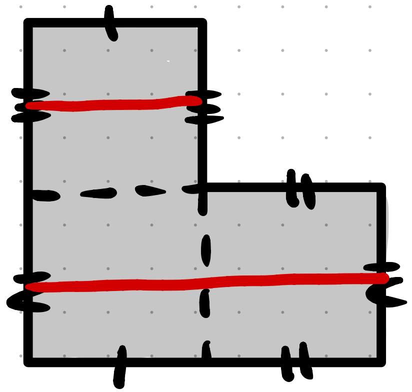

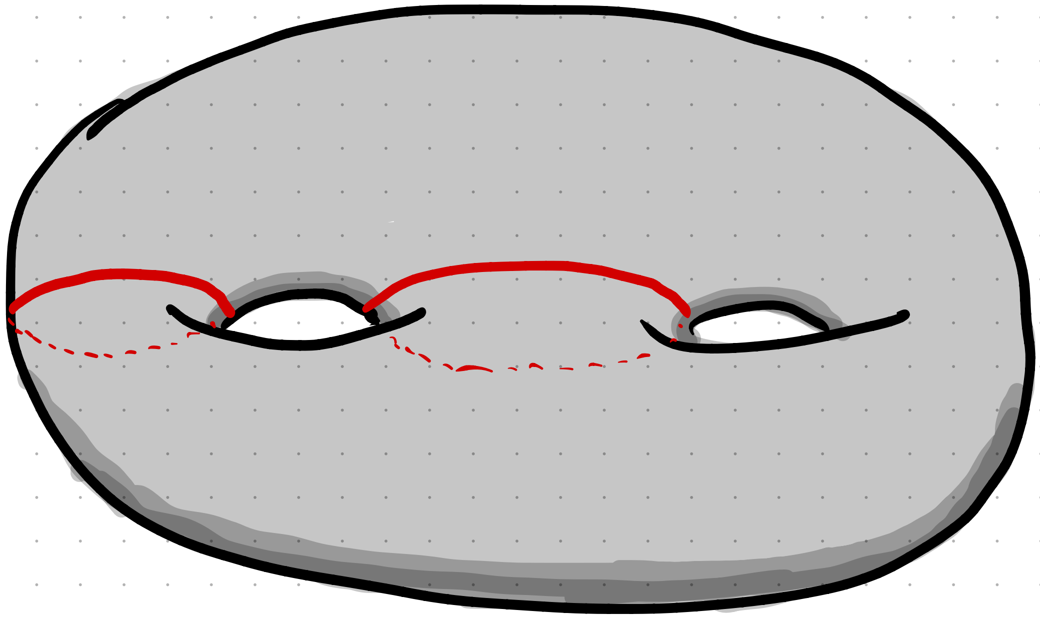

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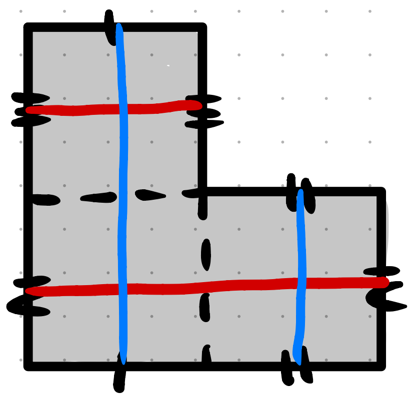
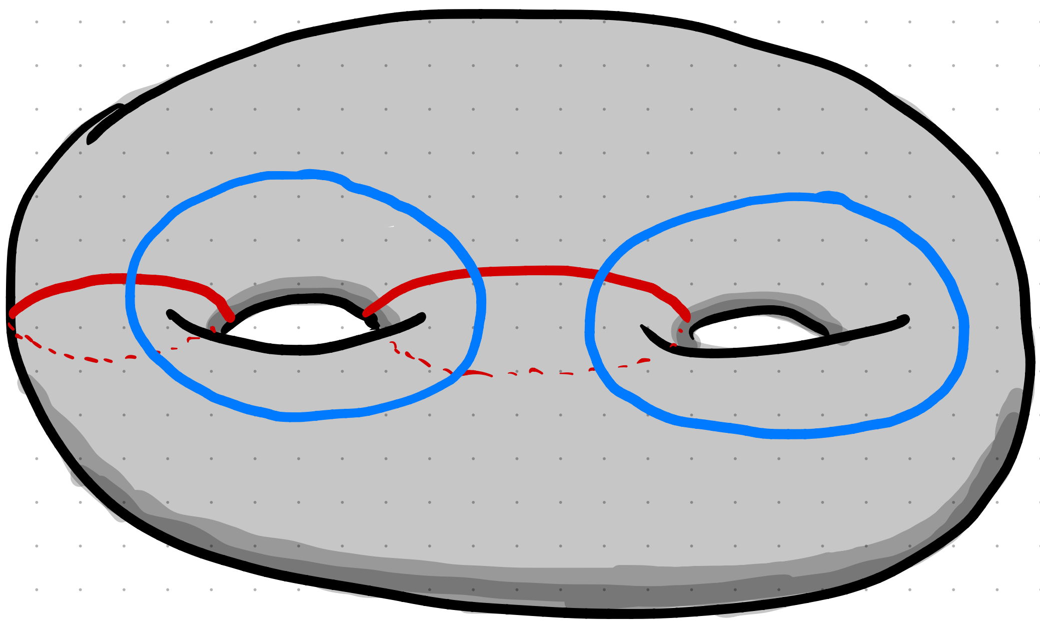
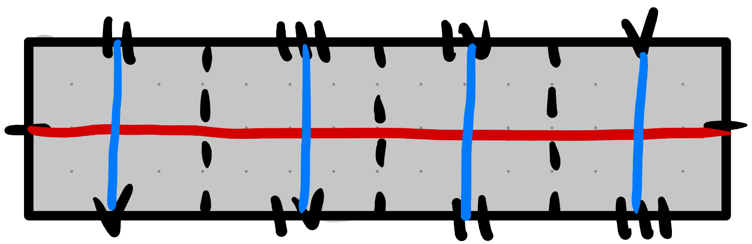
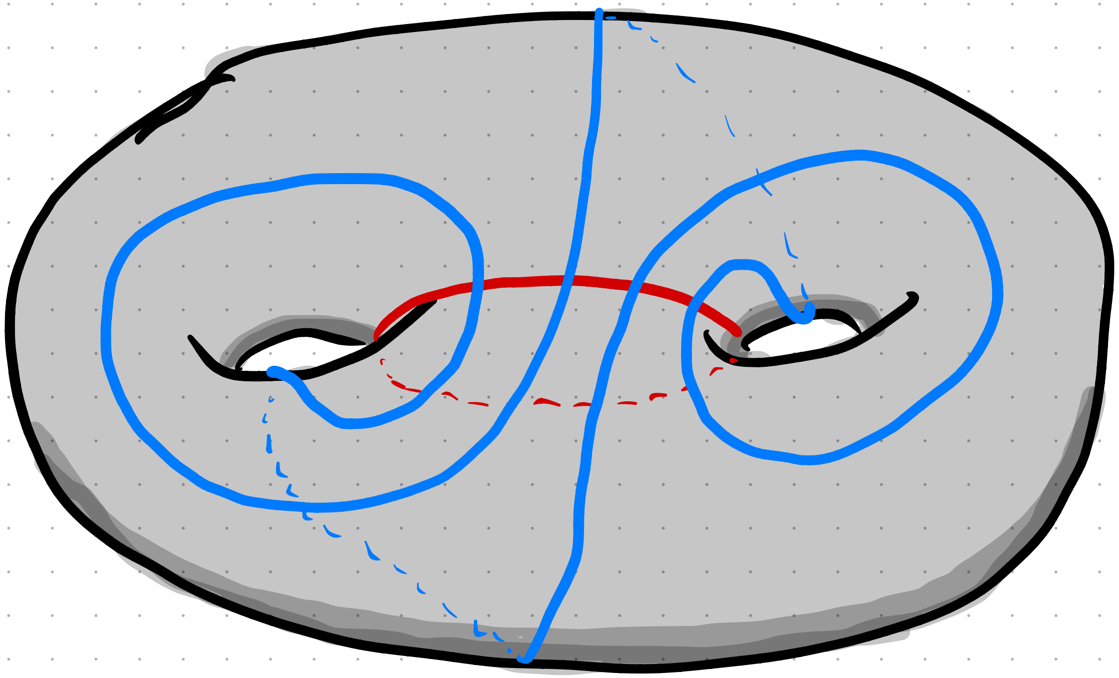




[1,1]-pillowcase covers:

[1,1]-pillowcase covers are those associated to a pair of curves.





The curves are said to be a filling pair.

Recent results:

Delecroix - Goujard - Zograf - Zorich '21-'22

Asymptotic counts to determine volume contributions. (Again # squares $\to \infty$).

What can be said for a small number of squares?

Any pillowcase cover of genus g with n singularities requires at least

$$N_{min} := 2g + n - 2$$

many squares.

Question 1: Given a connected component $\mathcal{C}$, can you construct a $[1,1]$-pillowcase cover in $\mathcal{C}$ using only N_{min} squares?

Theorem 1 (J, '21+'22) Fix $g \geq 2$.

If $\mathcal{C}$ is non-hyperelliptic, then the answer is yes.

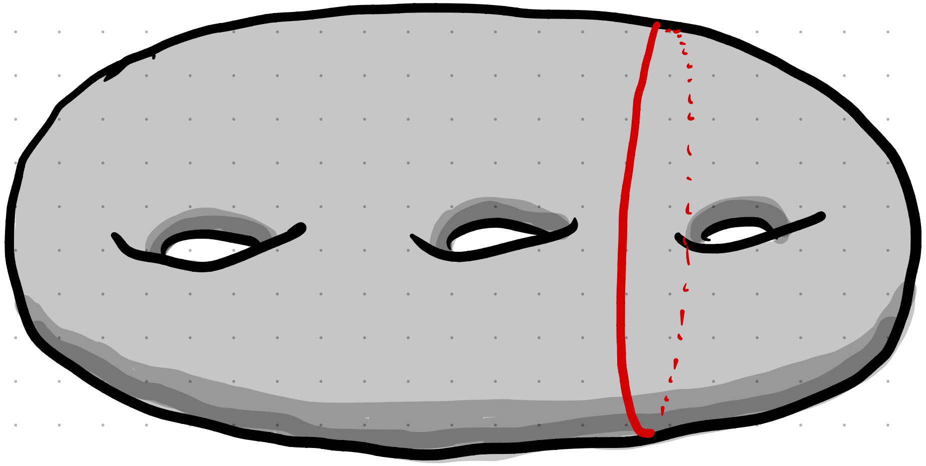
If $\mathcal{C}$ is hyperelliptic, the answer is no. Here $[1,1]$-pillowcase covers require at least $4g+2n-8 > N_{min}$.

A refinement: Can we get more control over the topological properties of the filling pair?

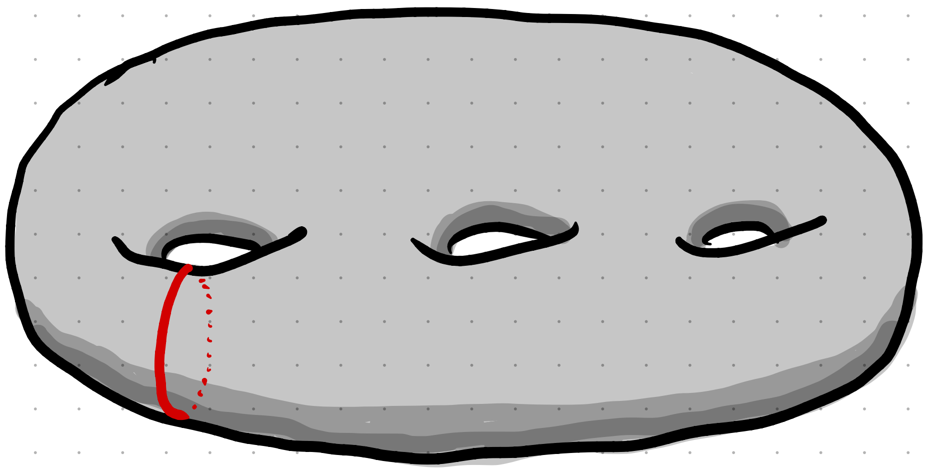
Separating vs non-separating:

A key property of a curve is whether or not it separates the surface.

Separating:



Non-separating:



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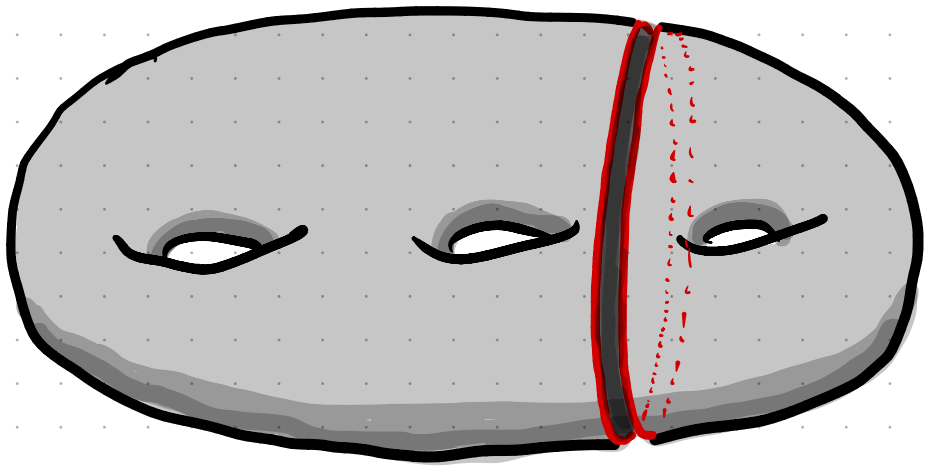
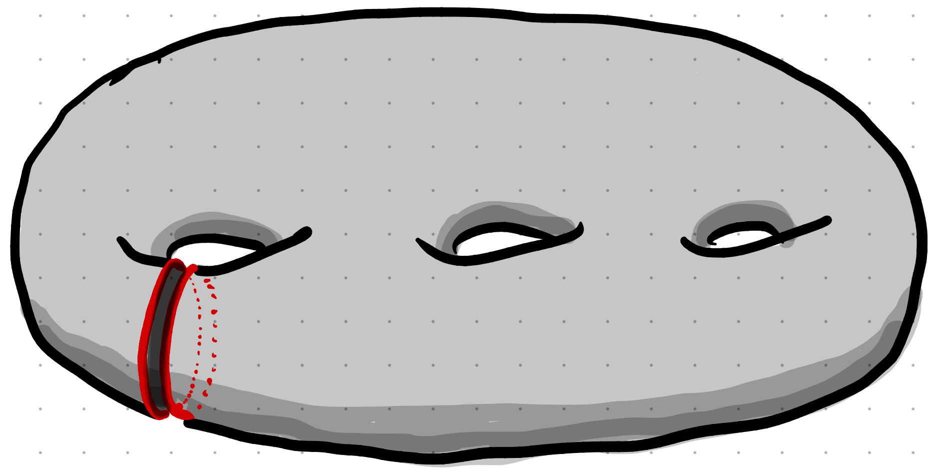
Separating:



Non-separating:



All the curves obtained from the $[1,1]$-pillowcase covers in Theorem 1 were non-separating curves.

Question 2:

Given a connected component $\mathcal{C}$ of a stratum $Q(k_1, \ldots, k_n)$, how many squares are required to construct a $[1,1]$-pillowcase cover in $\mathcal{C}$ whose filling pair has at least one separating curve?

Main result:

Theorem 2 (J, '22)

Let $\mathcal{C}$ be any hyperelliptic component. Then we have the following:

Connected component	Minimum number of squares required to produce a [1,1]-pillowcase cover whose cylinders are both non-sep.	one sep. one non-sep.	both sep.
$\mathcal{H}^{hyp}(2g-2), g \geq 2$	$4g-4$	n/a	n/a
$\mathcal{H}^{hyp}(g-1,g-1), g \geq 2$	$4g-2$	n/a	n/a
$\mathcal{Q}^{hyp}(4j+2,4k+2), k \geq j \geq 0$	$4j+4k+4$	$\max\{8j+6, 8k+4\}$	$16k-8j+8$
$\mathcal{Q}^{hyp}(4j+2,2k-1,2k-1), j \geq 1, k \geq 0, j \geq k$	$4j+4k+2$	$8j+4$	$16j-8k+12$
$\mathcal{Q}^{hyp}(4j+2,2k-1,2k-1), k > j \geq 0$	$4j+4k+2$	$8k$	$16k-8j$
$\mathcal{Q}^{hyp}(2j-1,2j-1,2k-1,2k-1), k \geq 1, j \geq 0, k \geq j$	$4j+4k$	$\max\{8j+2, 8k\}$	$16k-8j+4$
$\mathcal{Q}^{hyp}(2,-1,-1)$	3	4	12

Main result:

Theorem 2 (J. '22)

Let $\mathcal{C}$ be any hyperelliptic component. Then we have the following:

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$\mathcal{Q}^{hyp}(4j+2, 2k-1, 2k-1), k > j \geq 0$	$4j+4k+2$	$8k$	$16k-8j$
$\mathcal{Q}^{hyp}(2j-1, 2j-1, 2k-1, 2k-1), k \geq 1, j \geq 0, k \geq j$	$4j+4k$	$\max\{8j+2, 8k\}$	$16k-8j+4$
$\mathcal{Q}^{hyp}(2, -1, -1)$	3	4	12

sum max difference

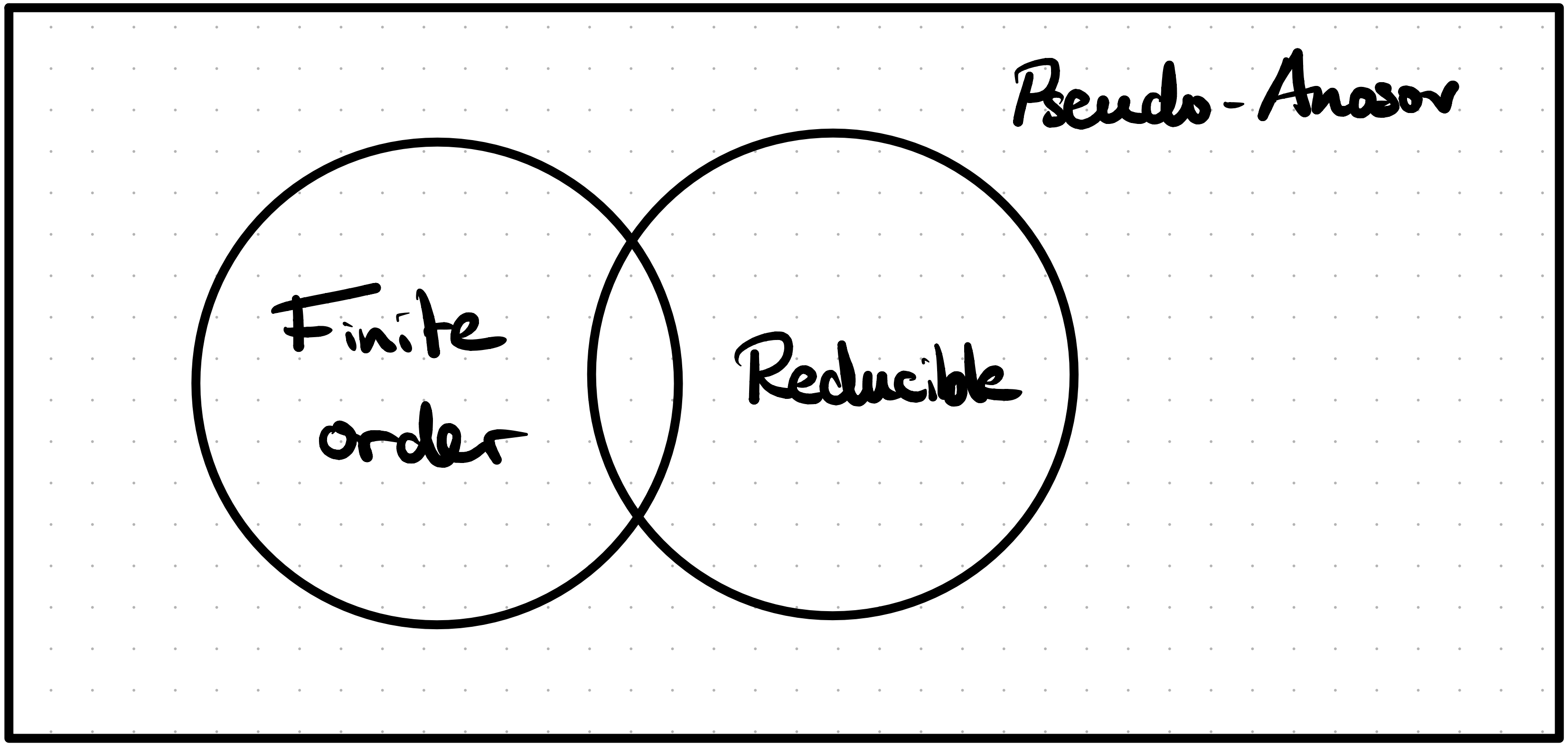
Applications

Aim:

We want to study **pseudo-Anosov homeomorphisms** that satisfy a geometric condition related to **Teichmüller space** and the **curve graph**.

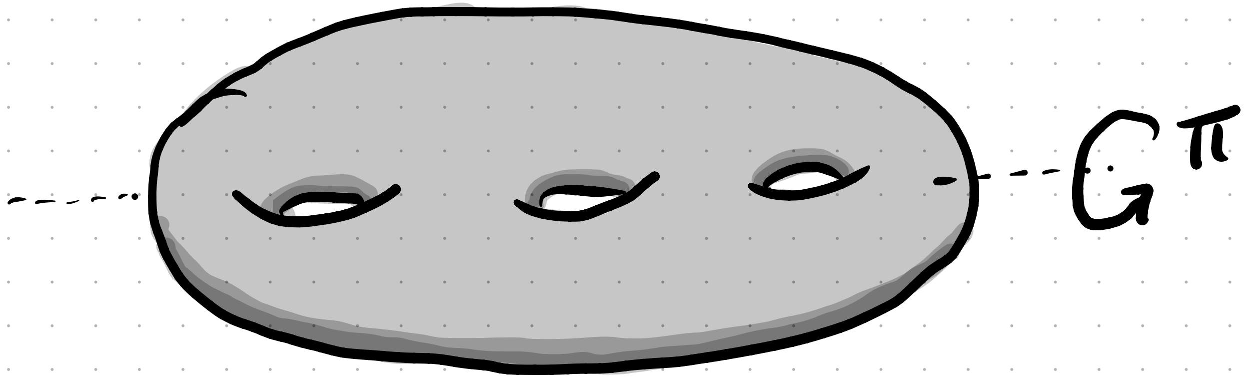
Nielsen-Thurston Classification

$$MCG(S) := \text{Homeo}^+(S) / \text{isotopy}$$

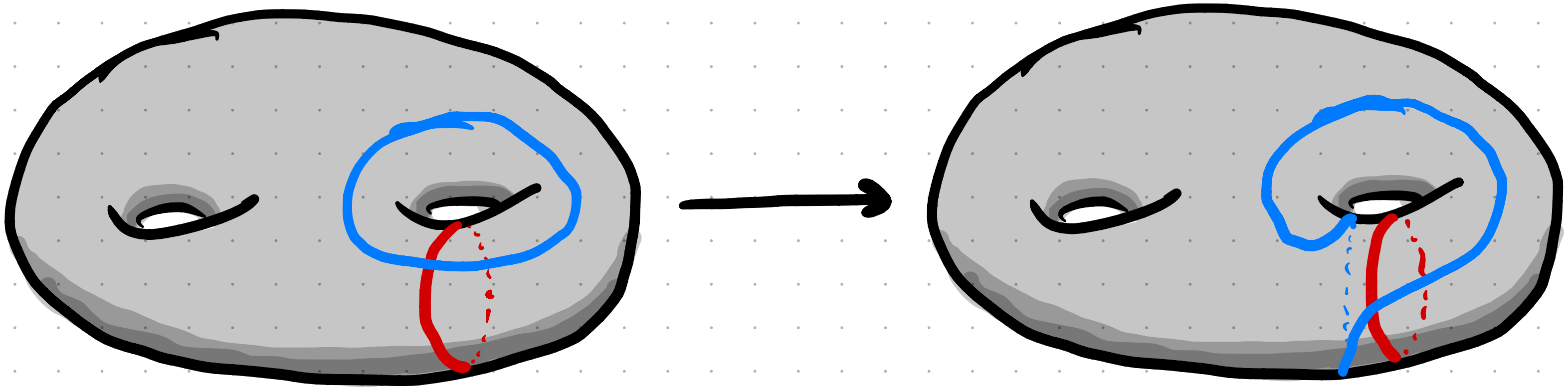


Nielsen-Thurston Classification

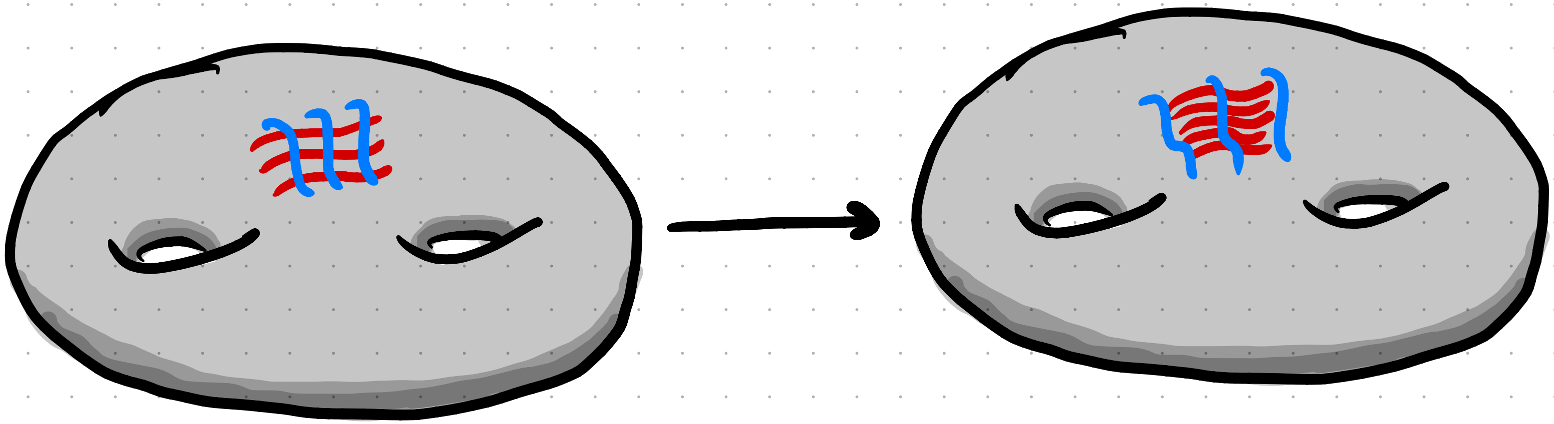
Finite order:



Reducible:



Pseudo-Anosov:



Teichmüller space

$$T(S) := \{\text{marked hyperbolic metrics on } S\} / \text{isotopy}$$

$MCG(S)$ acts on $T(S)$ by isometries.

The translation length of a pseudo-Anosov f is

$$\ell_T(f) = \log \lambda_f$$

Teichmüller disks :

The Teichmüller disk of a flat surface X is the image of an embedding

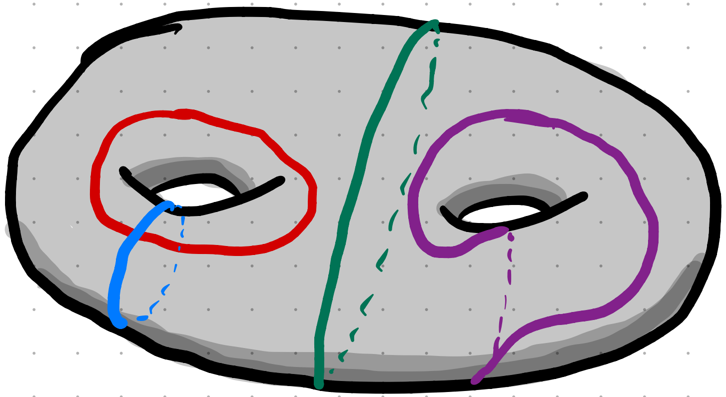
$$SO(2,\mathbb{R})\backslash SL(2,\mathbb{R})\cdot X \longrightarrow T(S)$$

$$\begin{array}{c} \cong \\ \mathbb{H} \end{array}$$

The curve graph

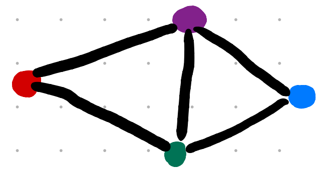
Vertices – isotopy classes of curves

Edges $\longleftrightarrow$ disjoint realisations



MCG(S) acts by isometry.

The translation length of a pseudo-Anosov



is $\ell_C(f) := \lim_{n\to\infty} \dfrac{d_C(\alpha, f^n\alpha)}{n}$, for any $\alpha \in C(S)$.

Ratio-optimising pseudo-Anosovs

A pseudo-Anosov $f \in MCG(S)$ is said to be ratio-optimising if

$$\frac{\ell_c(f)}{\ell_\tau(f)} \gtrsim \frac{1}{\log(g)}.$$

Crucial in the study of the 'systole map'.

Connection to pillowcase covers:

Theorem (Aougab-Taylor, '17)

Given a filling pair with intersection number N, $\exists D>0$ for which there are infinitely many pseudo-Anosovs f satisfying $\frac{\ell_C(f)}{\ell_T(f)} \geq \frac{1}{\log(D\cdot N)}$ and stabilising the Teichmüller disk of the $[1,1]$-pillowcase cover associated to the filling pair.

For $D\cdot N \simeq g$, these pseudo-Anosovs are ratio-optimising.

Question: In which connected components can these [1,1]-pillowcase covers lie?

Answer: By Theorem 1, the answer is any connected component.

In other words, ratio-optimising pseudo-Anosovs exist stabilising the Teichmüller disk of a flat surface in any connected component.

A refinement: Can we control the algebraic properties of the ratio-optimising pseudo-Anosovs?

Theorem (Aougab-Taylor, '17)

If both curves in the filling pair are separating, then the ratio-optimising pseudo-Anosovs can be produced arbitrarily deep in the **Johnson filtration**.

The Johnson filtration:

Let $\Gamma_1 = \pi_1(S)$, and define $\Gamma_{i+1} := [\Gamma_1, \Gamma_i]$.

$MCG(S)$ acts on Γ_1 / Γ_{i+1}.

Define $\mathcal{I}^i(S)$ to be the kernel of this action.

We get:

$$\mathcal{I}^0(S) = MCG(S) \geq \mathcal{I}^1(S) \geq \mathcal{I}^2(S) \geq \dots$$

The Torelli group acts trivially on $H_1(S, \mathbb{Z})$

The Johnson kernel is useful in 3-manifold theory.

Corollary of Theorem 2:

Let $\mathcal{C}$ be any hyperelliptic connected component of Q_g, then there exist ratio-optimising pseudo-Anosovs lying arbitrarily deep in the Johnson filtration stabilising the Teichmüller disk of a $[1,1]$-pillowcase cover in $\mathcal{C}$.

Proof of Theorem 2

A history of filling pairs on hyperelliptic surfaces

A history of filling pairs on hyperelliptic surfaces

Genus 2:

The minimum intersection number for a filling pair on a surface of genus two is 4 rather than the theoretical $3 = N_{min} = 2g+n-2$, $n=1$.

Proof by contradiction of Margalit:

1. The filling pair descends to an arc system on the quotient sphere

2. Checking the possible configurations you can find an arc that lifts to a disjoint curve on the surface.

A history of filling pairs on hyperelliptic surfaces

Proof for hyperelliptic components of Theorem 1

1. Associated filling pair descends to an arc system on quotient sphere

2. An Euler characteristic argument gives a lower bound on the number of intersections for a filling arc system.

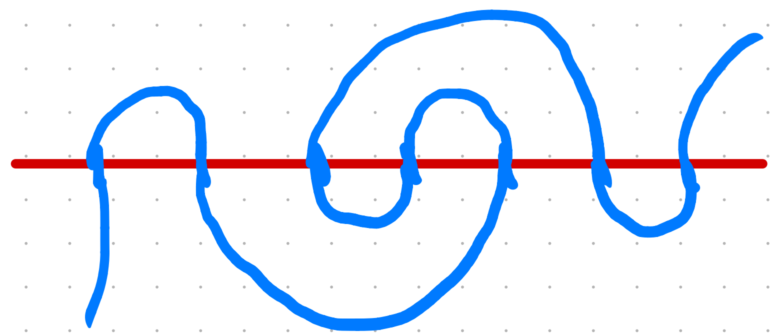
A history of filling pairs on hyperelliptic surfaces

Observation: If the filling pair contains 1 or 2 separating curves, then the image on the sphere will be an arc and a curve, or a pair of curves.

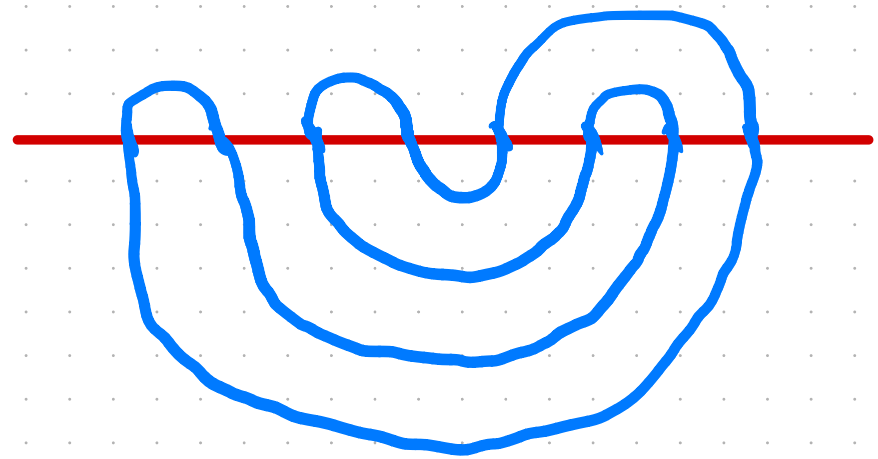
Realisation: These curve and arc systems are combinatorial objects called **meanders**.

Meanders

Open meanders:



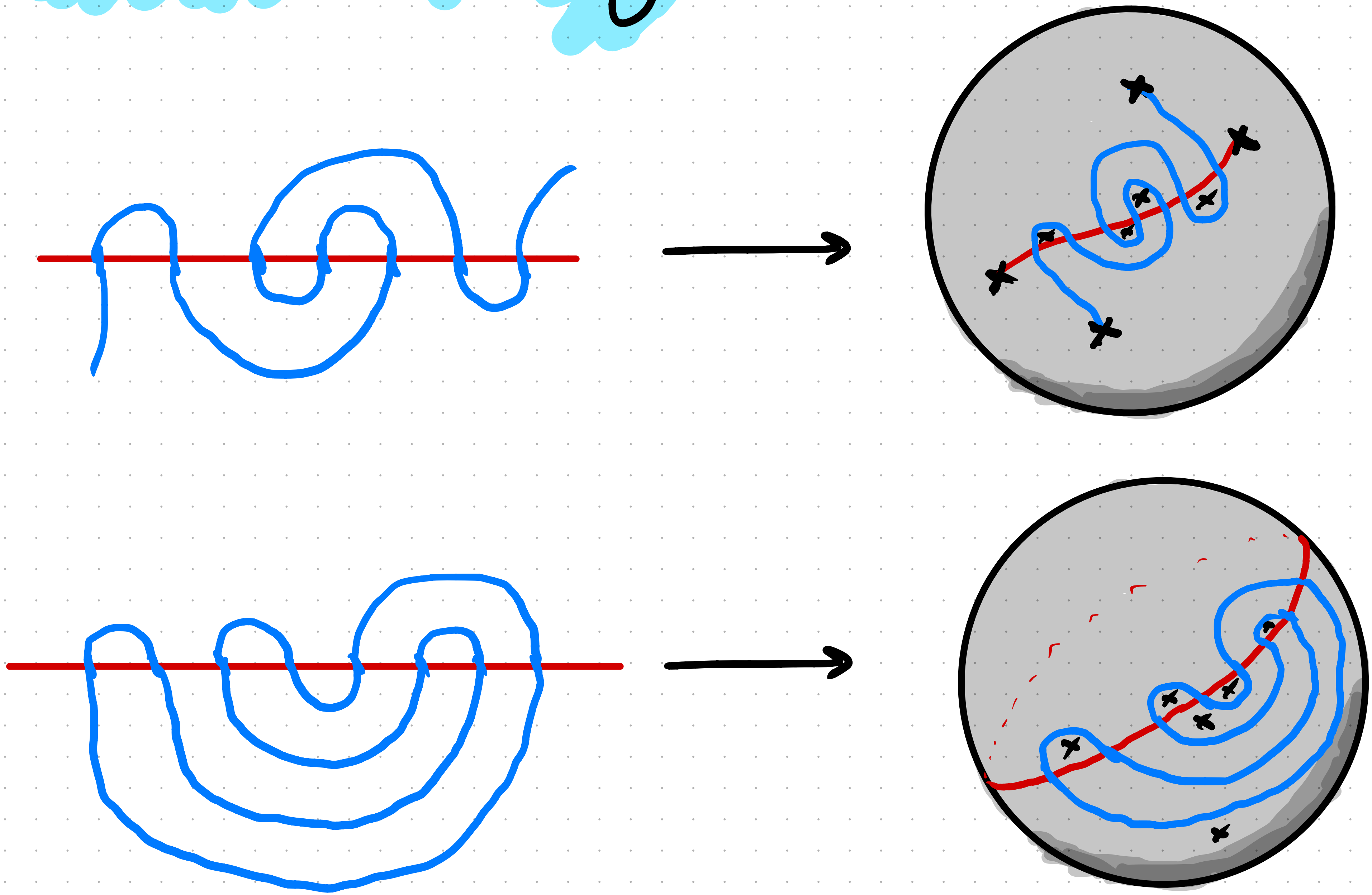
Closed meanders:



Applications of meanders:

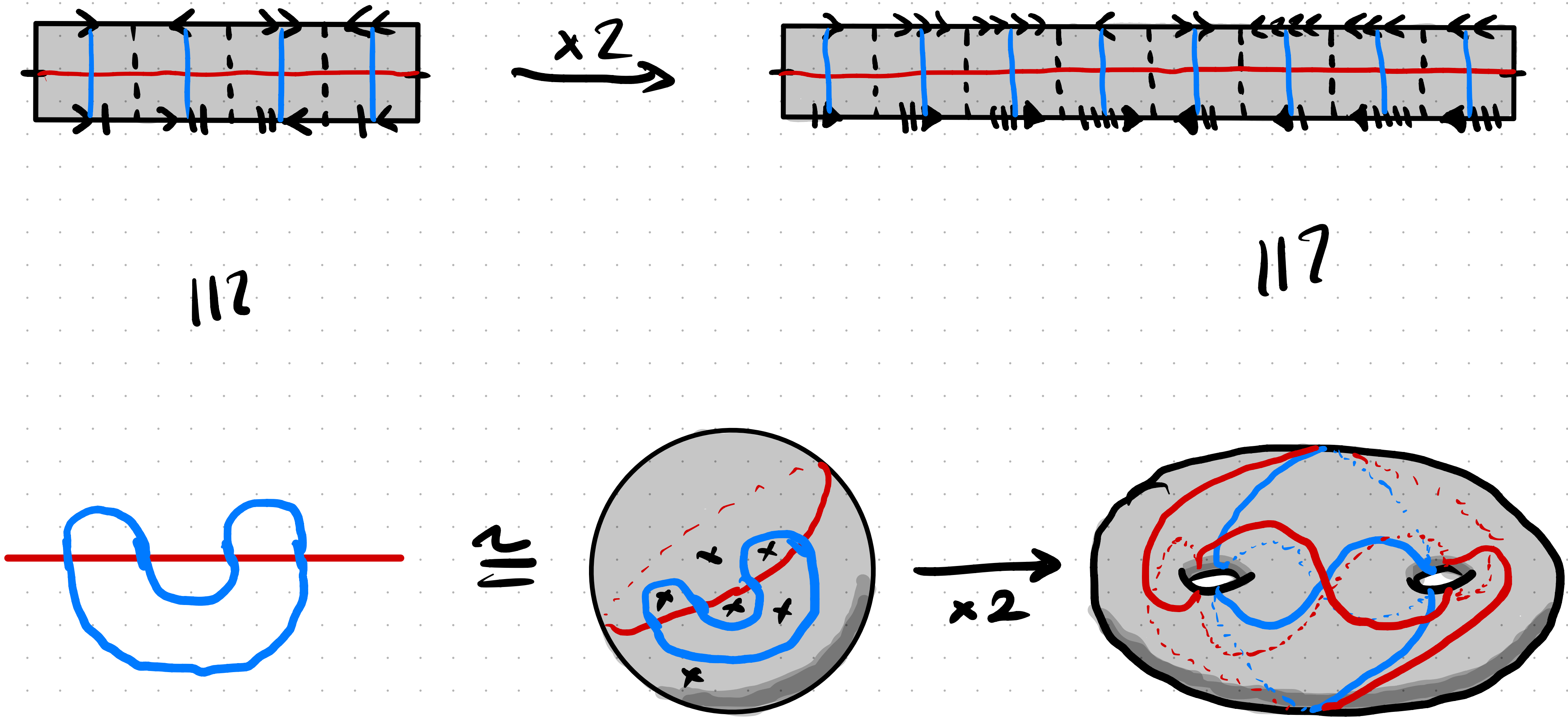
- Compact polymer folding
- Temperley-Lieb algebra
- (2+1)-dimensional gravity
- 3-manifold invariants

As arc and curve systems:



The lifting construction:

$$\mathcal{Q}(-1^4) \xrightarrow{\times 2} \mathcal{Q}^{hyp}(2^2)$$



Remaining questions:

- What can be said about separating filling pairs in non-hyperelliptic components?

- Are there applications to the study of limit sets of Teichmüller disks?