Single-cylinder square-tiled surfaces: Constructions and applications

Luke Jeffreys

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School of Mathematics and Statistics
College of Science and Engineering
University of Glasgow

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For Emily and my family

Abstract

This thesis investigates the combinatorial properties of square-tiled surfaces and studies the connections of these surfaces to the constructions of pseudo-Anosov homeomorphisms, and filling curves on punctured surfaces.

We begin by constructing, in every connected component of every stratum of the moduli space of Abelian differentials, square-tiled surfaces having a single vertical and single horizontal cylinder. We show that, for all but the hyperelliptic components, this can be achieved in the minimum number of squares required for a square-tiled surface in the ambient stratum. Moreover, for the hyperelliptic components, we show that the number of squares required is strictly greater and construct surfaces realising these bounds.

Using these surfaces, we demonstrate that pseudo-Anosov homeomorphisms optimising the ratio of Teichmüller to curve graph translation length are, in a reasonable sense, ubiquitous in the connected components of strata of Abelian differentials.

We consider the construction of filling pairs on punctured surfaces. We begin by determining the minimal intersection number of a filling pair on a genus two surface with an odd number, at least three, of punctures completing the work of Aougab-Huang and Aougab-Taylor. We then present a further application of the single-cylinder square-tiled surfaces constructed above by constructing filling pairs on punctured surfaces whose algebraic and geometric intersection numbers are equal.

Finally, we extend the constructions of single-cylinder square-tiled surfaces to certain strata of the moduli space of quadratic differentials.

In Chapter 1, we give the necessary background to describe the main results of this thesis. In Chapter 2, we prove the lemmas that are necessary for the construction of single-cylinder square-tiled surfaces in Chapter 3. Chapter 4 contains the construction of ratio-optimising pseudo-Anosov homeomorphisms, and the constructions of filling pairs on punctured surfaces are given in Chapter 5. In Chapter 6, we extend the constructions of Chapter 3 to certain strata of quadratic differentials. Finally, in Chapter 7, we present some remaining open questions and possible directions for future research. Appendix A gives an alternative proof of Proposition 3.2, and Appendix B contains the python code that realises the construction of Chapter 3.

Contents

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Declaration

All work in this thesis was carried out by the author.

List of Symbols

S	a surface
S_g	a surface of genus g
$S_{g,p}$	a surface of genus g with p punctures
α	an essential simple closed curve on the surface S
$i(\alpha, \beta)$	the geometric intersection number of the curves α and β
$\widehat{i}(\alpha, \beta)$	the algebraic intersection number of the curves α and β
$\mathrm{Mod}(S)$	the mapping class group of the surface S
$\mathcal{C}(S)$	the curve graph of the surface S
$\mathcal{T}(S)$	the Teichmüller space of the surface S
$\mathcal{M}(S)$	the moduli space of conformal structures on the surface S
$\mathcal{H}$	the moduli space of Abelian differentials
$\mathcal{H}(k_1, \ldots, k_n)$	a stratum of the moduli space of Abelian differentials
$\mathcal{H}^{hyp}(k_1, \ldots, k_n)$	the hyperelliptic component of a stratum
$\mathcal{H}^{nonhyp}(k_1, \ldots, k_n)$	the nonhyperelliptic component of a stratum
$\mathcal{H}^{even}(k_1, \ldots, k_n)$	the even component of a stratum
$\mathcal{H}^{odd}(k_1, \ldots, k_n)$	the odd component of a stratum
$\mathcal{Q}$	the moduli space of quadratic differentials
$\mathcal{Q}(k_1, \ldots, k_n)$	a stratum of the moduli space of quadratic differentials
$\mathcal{Q}^{reg}(k_1, \ldots, k_n)$	the regular component of a stratum
$\mathcal{Q}^{irr}(k_1, \ldots, k_n)$	the irregular component of a stratum

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Chapter 1

Introduction

The Teichmüller space $\mathcal{T}(S)$ of a surface S is a space that, in a sense, parameterises the hyperbolic metrics that S can carry. More specifically, it is the space of marked hyperbolic metrics on the surface S up to isometries isotopic to the identity.

Teichmüller space is an important object in many areas of modern research. Indeed, the quotient of Teichmüller space by the mapping class group $\mathrm{Mod}(S)$, the group of symmetries of the surface S, is the moduli space of Riemann surfaces. As such, the study of Teichmüller space has important applications in algebraic geometry. Moreover, the action of $\mathrm{Mod}(S)$ on Teichmüller space has played a crucial role in the study of the mapping class group in geometric group theory. Teichmüller space is also key to the study of certain dynamical systems. Specifically, questions about billiards in rational polygons can be answered by studying the properties of geodesics in Teichmüller space. The dynamical and geometric properties of Teichmüller and moduli space are an extremely active area of modern research and important results in this field can be found, for example, in the works of Mirzakhani and McMullen, among others. Furthermore, being intimately related to the geometry of surfaces, Teichmüller space has been an important tool in low-dimensional topology where, for example, it features heavily in Thurston's classification of surface homeomorphisms and his theorems relating to hyperbolic 3-manifolds.

Recent research has focussed on certain spaces related to Teichmüller space - namely, the moduli spaces of Abelian and quadratic differentials. The volumes of the strata of these spaces are important pieces of information and key objects in the determination of these volumes are surfaces called square-tiled surfaces. The main focus of this thesis is the construction of square-tiled surfaces of a particular combinatorial type in every connected component of every stratum of the moduli space of Abelian differentials. This construction is carried out in Chapter 3.

Using the connection between filling pairs of curves on surfaces and square-tiled surfaces of the type we construct in Chapter 3, we then present applications of this construction to the study of the coarse-geometry of Teichmüller space, and the study of sets of filling curves on surfaces. This work is carried out in Chapter 4 and Chapter 5, respectively.

In Chapter 6, we extend the constructions of single-cylinder square-tiled surfaces given in Chapter 3 to many strata of the moduli space of quadratic differentials.

Finally, in Chapter 7, we present some open questions relating to the work carried out in this thesis as well as some new directions for future research.

This work contains (in Chapters 2-4 and Section 5.2) results from a preprint [29] and (in Section 5.1) paper [28] of the author.

1.1 Preliminaries

We begin by presenting the necessary background on Teichmüller space, and the moduli spaces of Abelian and quadratic differentials. We include the basic definitions and results necessary to state our main theorems in the following sections. For more details on the mapping class group and Teichmüller space we refer the reader to the textbook of Farb-Margalit [21], while the surveys of Forni-Matheus [22], Yoccoz [53] and Zorich [56] contain more details on Abelian differentials and related topics.

1.1.1 Surfaces, curves and mapping class groups

In the sequel, we will let S denote a closed, connected, oriented surface. That is, S is a compact, connected topological manifold with no boundary that locally looks like $\mathbb{R}^2$ with a fixed choice of orientation. Examples of such objects include spheres and tori. We will let S_g denote a *surface of genus g*. As such, a surface S_0 is a sphere and a surface S_1 is a torus. A *surface of genus g with p punctures*; that is, a surface homeomorphic to the complement of p points in the surface S_g, will be denoted by $S_{g,p}$. If the genus and number of punctures are clear from the context then we will omit the subscripts.

Curves on surfaces

An *essential simple closed curve* α on the surface S is an embedding $\alpha : \mathbb{S}^1 \hookrightarrow S$ of the circle into S whose image is not isotopic to a point or to a puncture. We use the notation $\boldsymbol{\alpha} := [\alpha]$ to denote the isotopy class of the curve α. We will abuse notation by identifying α with its image in S and, from here on, by *curve* we will mean an essential simple closed curve. A curve α is *non-separating* if $S \setminus \alpha$ is connected, and is *separating* otherwise.

Given two curves α and β on the surface S, we define their *geometric intersection number* to be

$$i(\alpha, \beta) := \min_{\gamma \in \boldsymbol{\beta}} |\alpha \cap \gamma|.$$

We remark that the definition is symmetric. That is,

$$\min_{\gamma \in \boldsymbol{\beta}} |\alpha \cap \gamma| = \min_{\gamma \in \boldsymbol{\alpha}} |\gamma \cap \beta|.$$

We also define the geometric intersection number of two isotopy classes, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, to be $i(\alpha, \beta)$ for any choice of representatives $\alpha \in \boldsymbol{\alpha}$ and $\beta \in \boldsymbol{\beta}$. We will denote it by $i(\boldsymbol{\alpha}, \boldsymbol{\beta})$. We will say that two curves α and β are in *minimal position* if $|\alpha \cap \beta| = i(\alpha, \beta)$.

A pair of curves $\{\alpha, \beta\}$ in minimal position on the surface S are said to be a *filling pair* if any other curve γ on the surface has to intersect at least one of α or β. Equivalently, $\{\alpha, \beta\}$ are a filling pair if their complement $S \setminus (\alpha \cup \beta)$ is a disjoint union of disks or once-punctured disks.

We also define the *algebraic intersection number* of a pair of oriented curves α and β on the surface S. Denoted by $\widehat{i}(\alpha, \beta)$, this is defined to be the signed count of the intersections between the curves α and β where an intersection has sign $+1$ if its orientation agrees with the orientation of S, and has sign -1 otherwise. This does not depend on the isotopy classes of α and β and so we can speak about $\widehat{i}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Indeed, the algebraic intersection number extends to a well-defined symplectic form on the first homology of S. Contrary to the geometric intersection number, the algebraic intersection number of two curves is skew-symmetric; that is, $\widehat{i}(\alpha, \beta) = -\widehat{i}(\beta, \alpha)$.

Mapping class groups

Let $S = S_g$ be a surface of genus g and consider the group $\mathrm{Homeo}^+(S)$ of orientation-preserving self-homeomorphisms of the surface S. If two elements $f_1, f_2 \in \mathrm{Homeo}^+(S)$ are isotopic then we will write $f_1 \sim f_2$. Isotopy of homeomorphisms is an equivalence relation and so we can define the *mapping class group* $\mathrm{Mod}(S)$ of the surface S to be the quotient group

$$\mathrm{Mod}(S) := \mathrm{Homeo}^+(S)/\sim.$$

The isotopy class $[f]$ of a homeomorphism f is called its *mapping class*.

Let $[f] \in \mathrm{Mod}(S)$. We will say that $[f]$ is:

i. a *periodic* mapping class if there exists an integer n such that $[f]^n = [\mathrm{id}]$, where id is the identity homeomorphism;

ii. a *reducible* mapping class if it fixes up to isotopy a disjoint union of curves in S; or

iii. a *pseudo-Anosov* mapping class if it is neither periodic nor reducible.

We will extend these adjectives to a homeomorphism f, depending on the characterisation of $[f]$. In particular, it will make sense to speak about a pseudo-Anosov homeomorphism. This classification of surface homeomorphisms is now known as the Nielsen-Thurston classification.

A periodic homeomorphism of particular interest is a hyperelliptic involution of a surface S. For example, if we consider the surface S with all of the genus aligned along an axis, then a *hyperelliptic involution* of the surface can be realised by the homeomorphism given by a rotation by π about this axis. The quotient of the surface by action of this involution is a sphere with $2g+2$ punctures; that is, the surface $S_{0,2g+2}$.

An important example of a reducible homeomorphism is that of a Dehn twist. Consider the annulus $A = \mathbb{S}^1 \times [0,1]$ with a fixed choice of orientation, and define the homeomorphism

$$\begin{aligned} T_A : A &\to A \\ (\theta, t) &\mapsto (\theta + 2\pi t, t). \end{aligned}$$

Given a curve α in S, we can take an annular neighbourhood A_α of α and consider a homeomorphism $\phi : A \to A_\alpha$. We then define the *Dehn twist about* α to be the homeomorphism $T_\alpha : S \to S$ given by

$$T_\alpha(x) = \begin{cases} \phi \circ T_A \circ \phi^{-1}(x), \text{ if } x \in A_\alpha \\ x, \text{ otherwise.} \end{cases}$$

The mapping class group is finitely generated by Dehn twists around curves. Indeed, Dehn proved that $\mathrm{Mod}(S_g)$ is generated by $2g(g-1)$ many Dehn twists [11]. The number of Dehn twists required has since been improved by Lickorish [38] and Humphries [27].

We now describe a special case of a construction of pseudo-Anosov homeomorphisms due to Thurston. Let α and β be two curves that together fill the surface S and let T_α and T_β be the Dehn twists around α and β, respectively. If $i(\alpha, \beta) \geq 2$, then the group $\langle T_\alpha, T_\beta \rangle$ is isomorphic to a free group of rank two. Furthermore, one can show that $\langle T_\alpha, T_\beta \rangle$ contains no periodic elements and the only reducible elements are conjugate to a power of one of T_α, T_β, or $T_\alpha T_\beta$. The latter case only occurs when $i(\alpha, \beta) = 2$. This construction therefore gives rise to infinitely many pseudo-Anosov homeomorphisms. We call such a construction a *Thurston construction*.

The curve graph

We now define the *curve graph* $\mathcal{C}(S)$ of a surface $S = S_g$, for $g \geq 2$. The vertices of $\mathcal{C}(S)$ are isotopy classes of curves on S, and two vertices are joined by an edge if and only if their respective isotopy classes have representatives that can be realised disjointly from one another on S. In fact, the curve graph can be extended to a $(3g-3)$-dimensional simplicial flag complex in which the curve graph $\mathcal{C}(S)$ is the 1-skeleton. This complex was first introduced by the Harvey [24]. The curve graph, as well as the full flag complex, carries a natural action of the mapping class group and, as such, has played a crucial role in the study of the mapping class group.

We will equip $\mathcal{C}(S)$ with the path metric $d_{\mathcal{C}}$. That is, $d_{\mathcal{C}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the length of the shortest path between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, where every edge is given length one. Masur-Minsky showed that the metric space $(\mathcal{C}(S), d_{\mathcal{C}})$ is δ-hyperbolic [40] - a seminal work in this field. Given a pseudo-Anosov homeomorphism $f : S \to S$, we define the *asymptotic translation length of f on $\mathcal{C}(S)$* to be

$$\ell_{\mathcal{C}}(f) := \liminf_{n \to \infty} \frac{d_{\mathcal{C}}(\boldsymbol{\alpha}, f^n(\boldsymbol{\alpha}))}{n},$$

for any $\boldsymbol{\alpha} \in \mathcal{C}^0(S)$. It can be shown that this is finite and does not depend on the choice of $\boldsymbol{\alpha}$. Moreover, from the work of Bowditch [9] and Masur-Minsky [40] it follows that this is a positive limit.

1.1.2 Teichmüller space and moduli space

A *complex atlas* on the surface S is a set $\{\psi_\alpha : U_\alpha \to V_\alpha\}_\alpha$ of homeomorphisms between open sets $U_\alpha \subset S$ and $V_\alpha \subset \mathbb{C}$, such that $\bigcup_\alpha U_\alpha = S$, and every transition map $\psi_\beta \circ \psi_\alpha^{-1} : \psi_\alpha(U_\alpha \cap U_\beta) \to \psi_\beta(U_\alpha \cap U_\beta)$ is a biholomorphism. Two complex atlases are equivalent if their union is also a complex atlas. A *complex structure* on the surface S is an equivalence class of complex atlases. Each complex structure contains a unique maximal atlas. A *Riemann surface* X of genus g is a topological surface S equipped with a maximal complex atlas. Observe that a Riemann surface is a complex manifold of dimension one. In other words, it is a complex curve. A map $f : X \to Y$ between two Riemann surfaces X and Y is said to be biholomorphic if for all charts $\psi_\alpha : U_\alpha \to V_\alpha$ on X and $\phi_i : U_i' \to V_i'$ on Y, with $U_\alpha \cap f^{-1}(U_i')$, the map $\phi_i \circ f \circ \psi_\alpha^{-1}$ is biholomorphic in the usual complex analytical sense. Two Riemann surfaces are said to be *isomorphic* if there exists a biholomorphism between them.

Teichmüller space

We now fix a topological surface S of genus g. A *marked Riemann surface* is a pair (X, φ), where X is a Riemann surface of genus g and $\varphi : S \to X$ is a homeomorphism. Two marked Riemann surfaces (X_1, φ_1) and (X_2, φ_2) are equivalent if $\varphi_2 \circ \varphi_1^{-1} : X_1 \to X_2$ is isotopic to an isomorphism of Riemann surfaces. The *Teichmüller space* $\mathcal{T}(S)$ of the surface S is then defined to be the space of equivalence classes of marked Riemann surfaces.

For $g \geq 2$, every Riemann surface X can be realised as the quotient of the hyperbolic plane $\mathbb{H}$ by a subgroup Γ of $\mathrm{Isom}^+(\mathbb{H}) \cong \mathrm{PSL}(2, \mathbb{R})$. As such, the natural hyperbolic metric on $\mathbb{H}$ descends to a hyperbolic metric on X. We can equivalently define Teichmüller space to be the space of equivalence classes of marked hyperbolic surfaces where two such marked hyperbolic surfaces (X_1, φ_1) and (X_2, φ_2) are equivalent if $\varphi_2 \circ \varphi_1^{-1}$ is isotopic to an isometry.

Moduli space

We can define an action $\mathrm{Mod}(S) \curvearrowright \mathcal{T}(S)$ of the mapping class group on Teichmüller space by setting $[f] \cdot [(X, \varphi)] = [(X, \varphi \circ f^{-1})]$. One can check that this is well-defined in the sense that the point $[(X, \varphi \circ f^{-1})]$ does not depend on the choice of representatives of $[f]$ or $[(X, \varphi)]$. As such, we can define the quotient of Teichmüller space by this action. Indeed, we define the *moduli space of Riemann surfaces* to be

$$\mathcal{M}(S) := \mathcal{T}(S) / \mathrm{Mod}(S).$$

For $g \geq 2$, the moduli space of Riemann surfaces is a complex orbifold of dimension $3g - 3$.

Teichmüller metric

Viewed as a map between tangent spaces, the derivative of a conformal map between Riemann surfaces sends circles to circles. More generally, the derivative of a map between Riemann surfaces sends circles to ellipses. For a map $f : X \to Y$ between Riemann surfaces that is smooth outside of a finite set of points, we define the *dilatation of f at a smooth point $p \in X$* to be

$$K_f(p) = \frac{|f_z(p)| + |f_{\bar{z}}(p)|}{|f_z(p)| - |f_{\bar{z}}(p)|},$$

which can be interpreted as the ratio of the length of the major axis to the length of the minor axis of the ellipse that is the image under df of the unit circle in the tangent space to X at p. We then define the *dilatation of f* to be

$$K_f = \sup K_f(p),$$

where the supremum is taken over all smooth points $p \in X$. If $K_f < \infty$, then we say that f is *quasiconformal* or, more specifically, *K_f-quasiconformal*. If $K_f = 1$ then f is conformal.

With this in mind, we define a function $d_{\mathcal{T}} : \mathcal{T}(S) \times \mathcal{T}(S) \to \mathbb{R}$ as follows. Given

$x = [(X, f)]$ and $y = [(Y, g)]$ in $\mathcal{T}(S)$, we define

$$d_{\mathcal{T}}(x, y) = \inf_{h \sim g \circ f^{-1}} \frac{1}{2} \log K_h,$$

where the infimum is taken over all quasiconformal maps h isotopic to the change of marking map $g \circ f^{-1} : X \to Y$. This function is well defined and defines a Finsler metric on $\mathcal{T}(S)$ called the *Teichmüller metric*.

Given a pseudo-Anosov homeomorphism $f : S \to S$, we define the *translation length of f on $\mathcal{T}(S)$* to be

$$\ell_{\mathcal{T}}(f) := \inf_{x \in \mathcal{T}(S)} d_{\mathcal{T}}(x, f(x)) > 0.$$

When viewed as a map between Riemann surfaces, it follows from Bers' proof of the classification of surface homeomorphisms that the translation length of f is equal to $\frac{1}{2}\log(K_f)$ [7]. Furthermore, there is a geodesic axis γ in $\mathcal{T}(S)$ fixed by f and along which f acts by translations of distance $\ell_{\mathcal{T}}(f) = \frac{1}{2}\log(K_f)$. We remark that one can show that the asymptotic translation length on Teichmüller space, defined analogously to the asymptotic translation length on the curve graph, can be shown to be equal to $\ell_{\mathcal{T}}$.

1.1.3 Abelian differentials

For $g \geq 1$, consider the set of *Abelian differentials*; that is, the set of pairs (X, ω), where X is a compact Riemann surface of genus g and ω is a non-zero holomorphic 1-form on X, also called an Abelian differential. We then define the *moduli space of Abelian differentials* $\mathcal{H}_g$ to be the quotient of this set by the action of the mapping class group $\mathrm{Mod}(S)$, where $\mathrm{Mod}(S)$ acts on Riemann surfaces by precomposition with charts and on 1-forms by pullback. We will however abuse notation and denote an equivalence class $[(X, \omega)]$ by its representative (X, ω). We also drop the subscript g from $\mathcal{H}_g$ if the genus is clear from the context.

Recall that for genus greater than or equal to two the moduli space of Riemann surfaces has complex dimension $3g - 3$, and that the space of holomorphic 1-forms on a Riemann surface has complex dimension g. It can be shown that $\mathcal{H}$ is a complex orbifold of complex dimension $4g - 3$.

Translation surfaces

Let S be a topological surface of genus g. By a *translation atlas on S*, we will mean an atlas of charts $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_\alpha$ with $U_\alpha \subset S$, $V_\alpha \subset \mathbb{C}$ open sets, $\bigcup_\alpha U_\alpha = S \setminus \Sigma$, for a finite set of points $\Sigma \subset S$, such that all transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are given by translations

$z \mapsto z + c$. Two translation atlases are equivalent if their union is also a translation atlas, and an equivalence class of translation atlases is called a *translation structure*. A surface S equipped with a maximal translation atlas will be called a *translation surface*. We claim that a translation surface is equivalent to a Riemann surface equipped with an Abelian differential.

Given an Abelian differential ω on a Riemann surface X, let Σ denote the set of zeros of ω. This is the finite set of points $p \in X$ such that $\omega(p) = 0$. Now let $p \in X \setminus \Sigma$ and let U_p be a path connected open neighbourhood of p that does not contain any zeros of ω. In other words, $U_p \cap \Sigma = \emptyset$.

We claim that ω is a closed 1-form. Indeed, for any C^1 1-form $\sigma = u\,dz + v\,d\bar{z}$, we have $d\sigma = (u_{\bar{z}} - v_z)\,d\bar{z} \wedge dz$. Since ω is holomorphic, it locally satisfies $\omega = f\,dz$, where f is a holomorphic function. Hence, we have that $d\omega = (f_{\bar{z}} - 0)\,d\bar{z} \wedge dz = 0$, since $f_{\bar{z}} = 0$ for a holomorphic function f. As such, we can define a coordinate function $\phi_p : U_p \to \mathbb{C}$ by defining

$$\phi_p(x) = \int_p^x \omega.$$

This is well-defined since ω is closed. Moreover, up to a modification of U_p, ϕ_p is a biholomorphism. Note that, for $x \in U_p \cap U_q$, given $p, q \in X \setminus \Sigma$, we have

$$\phi_q(x) = \int_q^x \omega = \int_p^x \omega + \int_q^p \omega = \phi_p(x) + \int_q^p \omega.$$

Hence, we see that the transition maps satisfy $\phi_p \circ \phi_q^{-1}(z) = z + c$, for the constant $c = \int_p^q \omega \in \mathbb{C}$. This construction then gives us an atlas of charts on $X \setminus \Sigma$ with transition maps being translations. The Riemann removable singularity theorem allows us to extend this atlas to an atlas of charts on X compatible with the original Riemann surface structure. A maximal atlas of this nature then gives rise to a translation surface structure on the underlying topological surface.

Conversely, let S be a translation surface. This structure endows S with a Riemann surface structure and the local pullback of the 1-form dz on $\mathbb{C}$ gives rise to an Abelian differential ω on this Riemann surface. This is well-defined since $d(z + c) = dz$, and c is constant for each transition map. With this correspondence in mind, $\mathcal{H}$ may also be called the *moduli space of translation surfaces*, and we may call the pair (X, ω) a translation surface.

Given a translation surface (X, ω), the pullback of the flat metric on $\mathbb{C}$ gives rise to a flat metric on $X \setminus \Sigma$ with trivial holonomy. For genus greater than or equal to two, by the Gauss-Bonnet theorem, the metric completion on the whole of X must have negative curvature. Indeed, we have that each zero of ω of order k gives rise to a cone-type singularity with cone-angle equal to $(k + 1)2\pi$. That is, in a sense, the negative curvature is concentrated in the finitely many points of Σ. Furthermore, away from the zeros of ω, the pullbacks of the vector fields $\partial/\partial x$ and $\partial/\partial y$ give rise to a canonical choice of horizontal and vertical directions.

Translation surfaces and their generalisations half-translation surfaces (introduced in Section 6.1) arise naturally in the study of various dynamical systems such as billiards in rational polygons, electron transport in Fermi surfaces, and certain self-maps of the interval. More on these motivations can be found in the survey of Zorich [56].

Stratification of $\mathcal{H}$

By the Riemann-Roch theorem, the sum of the orders of the zeros of an Abelian differential on a Riemann surface of genus g is equal to $2g-2$ and this data can be used to stratify $\mathcal{H}$. The stratum $\mathcal{H}(k_1,\ldots,k_n) \subset \mathcal{H}$, with $k_i \geq 1$ and $\sum k_i = 2g-2$, is the subset of $\mathcal{H}$ consisting of Abelian differentials with n distinct zeros of orders $k_1,\ldots,k_n$. Each stratum is an orbifold of complex dimension $2g+n-1$. When it is convenient to do so, we may use power notation for the k_i. For example, we can denote $\mathcal{H}(1,1,1,1)$ by $\mathcal{H}(1^4)$.

The individual strata of $\mathcal{H}$ may have a number of connected components and the work of Kontsevich-Zorich completely classified these components [34, Theorems 1 and 2].

Square-tiled surfaces

A useful way to construct a translation surface is by taking a finite collection of polygons in $\mathbb{C}$ with pairs of parallel sides of equal length identified by translations such that the quotient of the polygons by these identifications is a closed connected oriented surface. The standard way of realising the square torus as the quotient of the unit square by identifying opposite sides by translations realises the torus as a translation surface. More generally, if a translation surface is realised by identifying the sides (left sides to right sides and top sides to bottom sides) of a collection of unit squares in $\mathbb{C}$, then we call such a translation surface a *square-tiled surface.*

A square-tiled surface is therefore a branched cover of the square torus branched over one point, and one can also think of square-tiled surfaces as being the integral points of the period coordinates on a stratum (a coordinate system defined by taking the ω-periods of a relative homology basis). The period coordinates give rise to a measure on strata, called the Masur-Veech measure, which when restricted to the area one locus of the stratum gives a finite measure. This fact was proved independetly by Masur [39] and Veech [50,51]. Since square-tiled surfaces can be thought of as lattice points, understanding square-tiled surfaces has played a crucial role in calculating the volumes of strata of Abelian differentials which are important pieces of information for dynamical calculations. See, for example, the works of Zorich [55] and Eskin-Okounkov [20]. Such volume calculations depend on asymptotic counts of square-tiled surfaces and these counts can be simplified by counting square-tiled surfaces of different combinatorial type separately.

It is natural to ask what is the minimum number of squares required to build a square-tiled surface in a stratum $\mathcal{H}_g(k_1, \ldots, k_n)$. An Euler-characteristic argument can be used to show that a square-tiled surface in this stratum must be constructed from at least $2g + n - 2$ squares. We will make reference to this lower bound throughout the thesis.

One important piece of combinatorial data for a square-tiled surface is the number of maximal, flat horizontal or vertical cylinders. A *cylinder* is a maximal embedded annulus in the surface, not containing any singularities in its interior. For example, the horizontal curves intersecting the sides labelled by 0s in Figure 1.1 are the core curves of the horizontal cylinders of the surfaces. One can also see that the surface on the left also has two vertical cylinders while the one on the right has a single vertical cylinder. If a square-tiled surface has a single vertical cylinder and a single horizontal cylinder then we shall call it a *1,1-square-tiled surface*. A square-tiled surface in $\mathcal{H}_g(k_1, \ldots, k_n)$ can have between 1 and $g + n - 1$ cylinders in its horizontal or vertical directions. Therefore, 1,1-square-tiled surfaces have the simplest possible decomposition into horizontal and vertical cylinders.

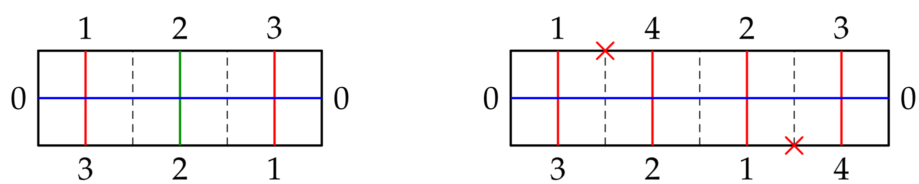

Figure 1.1: Two square-tiled surfaces in $\mathcal{H}(2)$ with a single horizontal cylinder. The surface on the right also has a single vertical cylinder while the one on the left has two vertical cylinders.

The process of splitting a pair of identified sides into two and identifying them as before adds a marked point, a zero of order zero, to the translation surface. That is, pick a pair of sides that are identified by translation and add a vertex to the centre of each side splitting each side into two - a left-side and a right-side. We then identify the right-sides with each other and the left-sides with each other. The resulting surface has zeros of the same orders but the additional vertex is now a zero of order zero - that is, it has cone-angle 2π. If one performs this operation on a square-tiled surface then, after setting the newly formed edges to have length 1, we obtain another square-tiled surface with an additional square. Observe that the surface on the right of Figure 1.1 is obtained from the surface on the left by performing such an operation on the sides labelled 1. The added vertex is shown in red. This operation does not change the connected component of the surface and we will make use of this technique when adding squares to hyperelliptic square-tiled surfaces in Section 3.1.

There is a natural correspondence between 1,1-square-tiled surfaces and filling pairs of curves on surfaces. The core curves of the cylinders of a 1,1-square-tiled surface form a filling pair on the underlying surface with, up to an appropriate choice of orientation, geometric intersection number equal algebraic intersection number. Indeed, the complement of the core curves is a disjoint union of disks, one for each zero. Moreover, since the vertical curve after leaving the top of a square can only enter the bottom of another square, its intersections with the horizontal curve all occur with the same orientation. Conversely, if we have a filling pair on a surface with geometric intersection number equal to algebraic intersection number, then the dual complex of these curves is a square-complex realising the surface as a square-tiled surface. The intersection number condition guarantees that the sides of the squares are identified in the correct manner.

Hyperellipticity

We say that a translation surface (X, ω) is *hyperelliptic* if there exists an isometric involution $\tau : X \to X$, known as a hyperelliptic involution, that induces a ramified double cover $\pi : X \to S_{0,2g+2}$ from X to the $(2g+2)$-times punctured sphere. Note that we must have $\tau^*\omega = -\omega$. The $2g+2$ fixed points of this involution, which map to the punctures on the sphere, are called Weierstrass points. Kontsevich-Zorich showed that the strata $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1, g-1)$ contain connected components, denoted by $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$ respectively, consisting entirely of hyperelliptic translation surfaces. These connected components will be called the *hyperelliptic components*.

We note that, since $\tau^*\omega = -\omega$, the zero of an Abelian differential in $\mathcal{H}^{hyp}(2g-2)$ is fixed by the hyperelliptic involution. Similarly, the two zeros of a hyperelliptic Abelian differential in $\mathcal{H}(g-1, g-1)$ are either fixed or mapped to one another under the hyperelliptic involution. It follows from the discussion below that the zeros of an Abelian differential in $\mathcal{H}^{hyp}(g-1, g-1)$ are symmetric; that is, mapped to one another, under the hyperelliptic involution. We make use of these facts in the proof of Proposition 3.2.

To motivate the definition of the components $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$ we will briefly introduce the notion of a quadratic differential on a surface. We direct the reader to Section 6.1 for a more detailed discussion. A *quadratic differential* on a Riemann surface X is a global section of the symmetric square of the canonical line bundle on X. In other words, a quadratic differential q on X is locally given by $f(z)dz^2$. Observe that the global square of an Abelian differential gives rise to a quadratic differential on X. We denote by $\mathcal{Q}$ the moduli space of non-zero meromorphic quadratic differentials on a surface of genus g that are not global squares of Abelian differentials. We will also assume that all such differentials have at most simple poles. As such, $\mathcal{Q}$ is stratified by subsets $\mathcal{Q}(k_1, \ldots, k_n)$, $k_i \geq 1$ or $k_i = -1$, and $\sum_{i=1}^{n} k_i = 4g - 4$, consisting of quadratic differentials with zeros of the prescribed orders.

There exists a natural double covering construction that takes a pair $(X, q) \in \mathcal{Q}$ to a pair (X', q') where q' is now a global square of an Abelian differential ω'. This gives a series of maps

$$\mathcal{Q}(l_1, \ldots, l_m) \to \mathcal{H}(k_1, \ldots, k_n),$$

and it can be shown that these maps are immersions. In the following two cases

$$\mathcal{Q}(2g-3, -1^{2g+1}) \to \mathcal{H}(2g-2)$$

$$\mathcal{Q}(2g-2, -1^{2g+2}) \to \mathcal{H}(g-1, g-1)$$

these maps give double covers of spheres. Moreover, it can be shown that in these particular cases the dimensions of the domain stratum and the range stratum agree and, since genus zero quadratic strata are connected and the Teichmüller geodesic flow acts ergodically on connected components and equivariantly with respect to this map, the images of these maps must be connected components in the image strata.

Spin structures and parity

The second invariant used to classify the connected components of a stratum is the notion of the parity of a spin structure.

Recall that the *canonical class* K_X of a Riemann surface X is the linear equivalence class of divisors of 1-forms on X. A *spin structure* on a Riemann surface X is a choice of half of the canonical class. That is, a spin structure is a choice of divisor class D in the Picard group of divisor classes $Pic(X)$, such that $2D = K_X$. Let $\Gamma(X, L)$ denote the space of holomorphic sections of the line bundle L corresponding to the divisor class D. Then the *parity of the spin structure D* is defined to be

$$\dim \Gamma(X, L) \bmod 2.$$

Given an Abelian differential $\omega \in \mathcal{H}(2k_1, \ldots, 2k_n)$ the divisor

$$Z_\omega = 2k_1 P_1 + \cdots + 2k_n P_n$$

represents the canonical class K_X. As such, we have a canonical choice of spin structure on X given by the divisor class

$$D_\omega = [k_1 P_1 + \cdots + k_n P_n].$$

Atiyah [6] and Mumford [45] demonstrated that the parity of a spin structure is invariant under continuous deformation. As such, the parity of the canonical spin structure given by an Abelian differential is constant on each connected component of the stratum. We will say that a connected component has *even* or *odd spin structure* depending on whether or not the parity of D_ω is 0 or 1.

Recall, that an Abelian differential ω on X determines a flat metric on X with cone-type singularities. Moreover, this metric has trivial holonomy, and away from the zeros of ω there is a well-defined horizontal direction. We can therefore define the index, $\text{ind}(\gamma)$, of a simple closed curve γ on X, avoiding the singularities, to be the degree of the Gauss map of γ. That is, $\text{ind}(\gamma)$ is the integer such that the total change of angle between the vector tangent to γ and the vector tangent to the horizontal direction determined by ω is $2\pi \cdot \text{ind}(\gamma)$. One can think about the index of a curve as the winding number of the curve with respect to the horizontal unit vector field of ω.

Given $\omega \in \mathcal{H}(2k_1, \ldots, 2k_n)$, we define a function $\Omega_\omega : H_1(X, \mathbb{Z}_2) \to \mathbb{Z}_2$ by

$$\Omega_\omega([\gamma]) = \text{ind}(\gamma) + 1 \mod 2,$$

where γ is a simple closed curve, and extend to a general homology class by linearity. We claim that this function is well-defined. Firstly, if we homotope a simple closed curve γ across a zero of order k then $\text{ind}(\gamma)$ will change by $\pm k$ but since all of our zeros have even order this will fix $\text{ind}(\gamma)$ modulo 2. One can check that for the boundary δ of a small disk not containing a zero, we have that $\text{ind}(\delta) + 1 \equiv 1 + 1 \equiv 0 \mod 2$. For the boundary δ of a small disk containing a zero of order k we have $\text{ind}(\delta) \equiv (k+1) + 1 \equiv 0 \mod 2$ since all of our zeros are of even order. Moreover, it follows from the Poincaré-Hopf Theorem that $\text{ind}(\delta) + 1 \equiv 0 \mod 2$ for any null-homologous simple closed curve δ. Therefore, $\Omega_\omega(\mathbf{0}) \equiv 0 \mod 2$, and so Ω_ω is indeed well-defined.

The function Ω_ω can be shown to be a quadratic form on $H_1(X, \mathbb{Z}_2)$, by which we mean

$$\Omega_\omega(a + b) = \Omega_\omega(a) + \Omega_\omega(b) + a \cdot b,$$

where $a \cdot b$ denotes the standard symplectic intersection form on $H_1(X, \mathbb{Z}_2)$; that is, the intersection form that extends the algebraic intersection number. Now given a choice of representatives $\{\alpha_i, \beta_i\}_{i=1}^g$ of a symplectic basis for $H_1(X, \mathbb{Z}_2)$, we define the *Arf in-*

variant of Ω_ω to be

$$\sum_{i=1}^{g} \Omega_\omega([\alpha_i]) \cdot \Omega_\omega([\beta_i]) \bmod 2 = \sum_{i=1}^{g} (\mathrm{ind}(\alpha_i) + 1)(\mathrm{ind}(\beta_i) + 1) \bmod 2.$$

Arf [5] proved that this number is independent of the choice of symplectic basis and Johnson [31] showed that quadratic forms on $H_1(X, \mathbb{Z}_2)$ are in one-to-one correspondence with spin structures on X. Moreover, Johnson proved that the value of the Arf invariant of Ω_ω coincides with the parity of the canonical spin structure determined by ω. We will make use of this formula when we calculate the parity of spin structures later in the thesis.

Classification of connected components

We are now ready to state the classification result of Kontsevich-Zorich. There are a few low genus cases that need to be dealt with separately, but the classification stabilises for genera greater than or equal to 4.

Theorem 1.1 ([34], Theorem 1)**.** *All connected components of strata of Abelian differentials on Riemann surfaces of genus $g \geq 4$ are described by the following list:*

The stratum $\mathcal{H}(2g-2)$ has three connected components: the hyperelliptic component, $\mathcal{H}^{hyp}(2g-2)$, and two other components: $\mathcal{H}^{even}(2g-2)$ and $\mathcal{H}^{odd}(2g-2)$ corresponding to even and odd spin structures.

The stratum $\mathcal{H}(2l, 2l)$, $l \geq 2$, has three connected components: the hyperelliptic component, $\mathcal{H}^{hyp}(2l, 2l)$, and two other components: $\mathcal{H}^{even}(2l, 2l)$ and $\mathcal{H}^{odd}(2l, 2l)$ corresponding to even and odd spin structures.

All other strata of the form $\mathcal{H}(2l_1, \ldots, 2l_n)$, $l_i \geq 1$, have two connected components: $\mathcal{H}^{even}(2l_1, \ldots, 2l_n)$ and $\mathcal{H}^{odd}(2l_1, \ldots, 2l_n)$ corresponding to even and odd spin structures.

The strata $\mathcal{H}(2l-1, 2l-1)$, $l \geq 2$, has two components: one of them $\mathcal{H}^{hyp}(2l-1, 2l-1)$ is hyperelliptic; the other $\mathcal{H}^{nonhyp}(2l-1, 2l-1)$ is not.

All other strata of Abelian differentials on Riemann surfaces of genus $g \geq 4$ are nonempty and connected.

For lower genera, we have the following classification.

Theorem 1.2 ([34], Theorem 2)**.** *The moduli space of Abelian differentials on a Riemann surface of genus $g = 2$ contains two strata: $\mathcal{H}(1, 1)$ and $\mathcal{H}(2)$. Each of them is connected and coincides with its hyperelliptic component.*

Each of the strata $\mathcal{H}(2, 2)$ and $\mathcal{H}(4)$ of the moduli space of Abelian differentials on a Riemann surface of genus $g = 3$ has two connected components: the hyperelliptic one, and one having odd spin structure. The other strata are connected for genus $g = 3$.

In Chapter 3, we will see that the fact that all Abelian differentials of genus two are hyperelliptic causes us some difficulty in our construction of 1,1-square-tiled surfaces.

The action of SL $(2, \mathbb{R})$

The group $\mathrm{SL}(2, \mathbb{R})$ has a natural action on the moduli space of Abelian differentials. Indeed, recall that (X, ω) can be realised as a collection of polygons $\{P_i\}$ in $\mathbb{C}$ with sides identified by translation. We then define $A \cdot (X, \omega)$ to be the translation surface obtained from the collection of polygons $\{A(P_i)\}$, where $A \in \mathrm{SL}(2, \mathbb{R})$ acts on the polygons by matrix multiplication on the vectors determining the sides. Observe that this action is well defined since equal length parallel vectors are sent to equal length parallel vectors under the action of $\mathrm{SL}(2, \mathbb{R})$.

Since a translation surface determines a complex structure on the underlying topological surface S, and given a choice of homeomorphism $f : S \to X$, we obtain a point in Teichmüller space. The action of $\mathrm{SO}(2, \mathbb{R})$ on an Abelian differential does not change the point in Teichmüller space that it determines. As such, the orbit of an Abelian differential under the action of $\mathrm{SO}(2, \mathbb{R}) \backslash \mathrm{SL}(2, \mathbb{R})$ gives an embedding of $\mathrm{SO}(2, \mathbb{R}) \backslash \mathrm{SL}(2, \mathbb{R}) \cong \mathbb{H}$ into $\mathcal{T}(S)$. We call the image of this embedding the *Teichmüller disk* determined by the Abelian differential. This idea can be generalised to the orbit of a quadratic differential and so we can also talk about the Teichmüller disk determined by a quadratic differential. We refer the reader to the work of Herrlich-Schmithüsen [25] for a more detailed discussion.

The Teichmüller geodesic flow mentioned above is given by the action of the diagonal subgroup

$$\left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}.$$

1.2 Single-cylinder square-tiled surfaces

It is a consequence of recent work of Delecroix-Goujard-Zograf-Zorich [14] that 1,1-square-tiled surfaces make a non-zero contribution to the volumes of strata of Abelian differentials, and moreover equidistribute as the number of squares tends to infinity. Indeed, they showed that this equidistribution is true more generally for square-tiled surfaces of fixed combinatorial type in any $\mathrm{GL}(2, \mathbb{R})$-invariant suborbifold containing a single square-tiled surface [14, Theorem 1.4]. By this we mean, in any finite volume open subset U, a point chosen at random from an ϵ-grid in U is a square-tiled surface having the desired combinatorics with probability that, as ϵ tends to zero, does not depend on U.

Recall that a square-tiled surface in a stratum $\mathcal{H}_g(k_1, \ldots, k_n)$ must have at least

$2g + n - 2$ squares. Square-tiled surfaces realising this number exist in every connected component. For example, one can take the square-tiled surfaces associated to the Jenkins-Strebel permutation representatives given by Zorich [57]. It is not obvious whether forcing a square-tiled surface to have a single horizontal and a single vertical cylinder would require it to have more than $2g + n - 2$ squares. That is, it is natural to ask the following question.

Question 1.3. *Given a connected component $\mathscr{C}$ of a stratum $\mathcal{H}(k_1, \ldots, k_n)$, can a 1,1-square-tiled surface in $\mathscr{C}$ be constructed using only $2g + n - 2$ squares?*

We prove that the answer to this question is yes, unless the connected component $\mathscr{C}$ is a hyperelliptic component. Indeed, our main result is the following.

Theorem 1.4. *Let $\mathscr{C} \neq \mathcal{H}^{hyp}(2g-2), \mathcal{H}^{hyp}(g-1, g-1)$ be a connected component of a stratum $\mathcal{H}_g(k_1, \ldots, k_n)$. Then there exists a 1,1,-square-tiled surface in $\mathscr{C}$ formed of $2g+n-2$ squares; that is, consisting of the minimum number of squares required for a square-tiled surface in the ambient stratum. To construct 1,1-square-tiled surfaces in the components $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$ one instead requires at least $4g-4$ and $4g-2$ squares, respectively. Moreover, there exist 1,1-square-tiled surfaces in these components realising these bounds.*

The theorem demonstrates that, for all but the hyperelliptic components, 1,1-square-tiled surfaces are exhibited in the minimum number of squares possible. For the hyperelliptic components, the cylinders of a 1,1-square-tiled surface are sent to themselves under the action of the hyperelliptic involution and it is this extra symmetry that essentially forces the need for additional squares.

A permutation representative, introduced in Chapter 2, is a useful way of constructing translation surfaces. Zorich [57] constructed permutation representatives for every connected component of every stratum that can be used to build square-tiled surfaces with a single horizontal cylinder. One might expect that a construction of 1,1-square-tiled surfaces could be achieved by applying a sequence of combinatorial moves, called Rauzy moves, to the permutation representatives of Zorich in order to obtain the desired combinatorics. However, this method is not adequate because the complexity of Rauzy diagrams - directed graphs containing all permutation representatives that can be reached by Rauzy moves - grow in such a way as to make this extremely computationally difficult. Moreover, the hope that one would be able to easily find such a sequence of Rauzy moves for each connected component is naive. Indeed, the complexity of such a method is demonstrated, for example, in the case of the strata $\mathcal{H}(2g-5, 1, 1, 1)$. Here, with the permutation representatives given by Zorich, a different sequence of Rauzy moves is required depending on the residue of $2g-5$ modulo 4; see the differing permutation representatives in Proposition 3.10. As such, it seems unreasonable to expect to find a general proof of this nature.

Only in the extremely rigid case of the hyperelliptic components is a proof similar to this achieved. In fact, 1,1-square-tiled surfaces in these components are constructed by hand by adding squares to the combinatorics given by Rauzy. A method of Margalit relating to minimally intersecting filling pairs on the surface of genus two, referenced in a paper of Aougab-Huang [1, Remark 2.18], is then formalised and generalised in order to show that the number of squares achieved for these components is actually the minimum required.

Zorich constructed his permutation representatives by finding permutation representatives for higher dimensional strata and then collapsing zeros together to produce representatives for lower dimensional strata. Although the single horizontal cylinder can be preserved under this collapsing method, the effect on the vertical cylinders is less obvious and harder to control. As such, we will not adopt a collapsing method here.

We instead adopt an inductive method in order to build 1,1-square-tiled surfaces in nonhyperelliptic connected components. More specifically, we show in Chapter 2 that 1,1-square-tiled surfaces in a general connected component can be constructed from 1,1-square-tiled surfaces of lower complexity in such a way that the resulting number of squares and the parity of any resulting spin structure can be easily controlled. We then construct, in Chapter 3, the families of 1,1-square-tiled surfaces required to allow this procedure to be completed. A small number of low complexity exceptional cases were found computationally using the `surface_dynamics` package [13] of SageMath [49]. In Appendix B, we give the Python code that can be used with SageMath and that has been included in the `surface_dynamics` package to produce the 1,1-square-tiled surfaces given by Theorem 1.4.

1.3 Ratio-optimising pseudo-Anosovs

Consider the Teichmüller space $\mathcal{T}(S)$ of marked hyperbolic metrics on the surface S equipped with the Teichmüller metric $d_{\mathcal{T}}$, and the curve graph $\mathcal{C}(S)$ of the surface S equipped with the path metric $d_{\mathcal{C}}$.

The systole map

$$\text{sys}\colon \mathcal{T}(S) \to \mathcal{C}(S)$$

is a coarsely-defined map that sends a marked hyperbolic metric to the isotopy class of the essential simple closed curve of shortest hyperbolic length. Masur-Minsky [40, Consequence of Lemma 2.4] showed that there exists a constant $K > 0$, depending only on g, and a $C \geq 0$ such that $d_{\mathcal{C}}(\text{sys}(x), \text{sys}(y)) \leq K \cdot d_{\mathcal{T}}(x, y) + C$, for all $x, y \in \mathcal{T}(S)$. In other words, the systole map is *coarsely K-Lipschitz*. This result was a key step in their proof that $\mathcal{C}(S)$ is δ-hyperbolic.

It is natural to ask what is the optimum Lipschitz constant, κ_g, defined by

$$\kappa_g := \inf\{K > 0 \mid \exists\, C \geq 0 \,\text{such that sys is coarsely } K\text{-Lipschitz}\}.$$

Gadre-Hironaka-Kent-Leininger determined that the ratio of κ_g to $1/\log(g)$ is bounded from above and below by two positive constants [23, Theorem 1.1]. In such a case, we use the notation $\kappa_g \asymp 1/\log(g)$, and say that κ_g is *comparable* to $1/\log(g)$. To find an upper bound for κ_g, Gadre-Hironaka-Kent-Leininger gave a careful version of the proof of Masur-Minsky that sys is coarsely Lipschitz. They then constructed pseudo-Anosov homeomorphisms - a specific type of surface homeomorphism - for which the ratio $\ell_{\mathcal{C}}(f)/\ell_{\mathcal{T}}(f) \asymp 1/\log(g)$, where $\ell_{\mathcal{C}}(f)$ and $\ell_{\mathcal{T}}(f)$ are the asymptotic translation lengths of f in $\mathcal{C}(S)$ and $\mathcal{T}(S)$, respectively. A lower bound for κ_g then followed by noting that, for any pseudo-Anosov homeomorphism f, we have

$$\kappa_g \geq \frac{\ell_{\mathcal{C}}(f)}{\ell_{\mathcal{T}}(f)}.$$

Using a Thurston construction on filling pairs, Aougab-Taylor constructed an infinite family of pseudo-Anosov homeomorphisms for which $\tau(f) := \ell_{\mathcal{T}}(f)/\ell_{\mathcal{C}}(f)$ was bounded above by a function comparable to $\log(g)$ [4, Theorem 1.1]; such homeomorphisms are said to be *ratio-optimising*. More specifically, given a filling pair $\{\alpha, \beta\}$ on the surface S with geometric intersection number $i(\alpha, \beta) \asymp g$, they used a Thurston construction on $\{\alpha, \beta\}$ to construct pseudo-Anosov homeomorphisms for which $\tau(f) \leq \log(D \cdot i(\alpha, \beta))$, where D is a constant independent of g. Furthermore, they showed that infinitely many conjugacy classes of primitive ratio-optimising pseudo-Anosov homeomorphisms, produced as above, have their invariant axis contained in the Teichmüller disk $\mathcal{D}(\alpha, \beta)$ of the flat structure determined by the filling pair $\{\alpha, \beta\}$. The Teichmüller disk $\mathcal{D}(\alpha, \beta) \subset \mathcal{T}(S)$ is the Teichmüller disk of the Abelian differential given by the filling pair $\{\alpha, \beta\}$.

We can then ask the following.

Question 1.5. *Can the Teichmüller disk $\mathcal{D}(\alpha, \beta)$ be taken to be the Teichmüller disk of an Abelian differential from any connected component of any stratum of the moduli space of Abelian differentials?*

Recall from above that the core curves of the cylinders of a 1,1-square-tiled surface, with a minimal number of squares, form a filling pair with geometric intersection number equal to the number of squares. Hence, as a consequence of Theorem 1.4, we have the following result.

Theorem 1.6. *Given any connected component of any stratum of Abelian differentials, there exist infinitely many conjugacy classes of primitive ratio-optimising pseudo-Anosov homeomorphisms whose invariant axis is contained in the Teichmüller disk of an Abelian differential in that connected component.*

That is, in a reasonable sense, ratio-optimising pseudo-Anosov homeomorphisms are ubiquitous in the connected components of strata of Abelian differentials. We prove this result in Chapter 4.

1.4 Filling pairs on punctured surfaces

Let $S_{g,p}$ denote the surface of genus $g \geq 0$ with $p \geq 0$ punctures. We define $i_{g,p}$ to be the minimal geometric intersection number for a filling pair on $S_{g,p}$. The values of $i_{g,p}$ were determined in almost all cases in the works of Aougab-Huang [1] and Aougab-Taylor [3]. However, in the case of $S_{2,p}$, $p \geq 3$ odd, they showed only the bounds:

$$2g + p - 2 \leq i_{g,p} \leq 2g + p - 1.$$

In Chapter 5, we resolve this final case by proving the following result.

Theorem 1.7. *Let $g = 2$ and $p \geq 3$ be odd, then $i_{g,p} = 2g + p - 2$.*

To prove the existence of such filling pairs, we generalise the construction of filling permutations given by Nieland [46, Theorem 2.1], which are themselves generalisations of the filling permutations introduced by Aougab-Huang [1, Lemma 2.2]. We use these to produce a minimally intersecting filling pair on $S_{2,3}$, and then apply the double-bigon inductive method used by Aougab-Taylor [3, Proof of Lemma 3.1] to extend to all odd $p \geq 3$.

For $g \geq 1$, one can ask whether $i_{g,p}$ can be realised as the algebraic intersection number, $\widehat{i}(\alpha, \beta)$, of a filling pair $\{\alpha, \beta\}$. Aougab-Menasco-Neiland [2] answered this question for the case of $i_{g,0}$; that is, for minimally intersecting filling pairs on closed surfaces. Moreover, they were interested in counting the number of mapping class group orbits of such filling pairs. Their method involves algebraically constructing 1,1-square-tiled surfaces with the minimum number of squares in the stratum $\mathcal{H}(2g-2)$, which they call square-tiled surfaces with connected leaves. The core curves of the cylinders of such surfaces give rise to filling pairs with algebraic intersection number equal to $i_{g,0}$.

Let $n \geq i_{g,p}$, by a *compatible decomposition* of the surface $S_{g,p}$ into $n + 2 - 2g$ many $4k$-gons, we mean a decomposition of the surface into $4k$-gons $P_1, \ldots, P_{n+2-2g}$ such that, if P_i is a $4k_i$-gon, then $\sum(k_i - 1) = 2g - 2$.

The filling pairs obtained from the cylinders of 1,1-square-tiled surfaces in any stratum of Abelian differentials have algebraic intersection number equal to geometric intersection number and also give rise to a decomposition of the surface into a number of $4k$-gons, with the number of polygons and the number of sides of each polygon depending on the stratum of the square-tiled surface. The resulting set of $4k$-gons form a compatible decomposition of the surface. Indeed, each $4k_i$-gon corresponds to a zero of the Abelian differential of order $(k_i - 1)$.

Using a simple modification of the constructions used in the proof of Theorem 1.4, we obtain the following result.

Theorem 1.8. *Let $n \geq i_{g,p}$ and choose a compatible decomposition of $S_{g,p}$ into $n + 2 - 2g$ many $4k$-gons, then there exists a filling pair $\{\alpha, \beta\}$ on the surface $S_{g,p}$ with*

$$\widehat{i}(\alpha, \beta) = i(\alpha, \beta) = n,$$

that gives rise to the specified polygonal decomposition of $S_{g,p}$.

This generalises the existence part of the work of Aougab-Menasco-Neiland to the case of any intersection number on any surface $S_{g,p}$. The proof of this result is contained in Chapter 5.

1.5 Extension to quadratic strata

Introduced in Section 6.1, *pillowcase covers* are a natural generalisation of square-tiled surfaces to the setting of quadratic differentials. In this setting, the notion of cylinders still holds and so one can discuss 1,1-pillowcase covers as the natural analogue of 1,1-square-tiled surfaces. A pillowcase cover in the stratum $\mathcal{Q}_g(k_1, \ldots, k_n)$ requires at least $2g + n - 2$ squares. It is then natural to ask the following question.

Question 1.9. *Given a connected component $\mathcal{C}$ of a non-empty stratum $\mathcal{Q}(k_1, \ldots, k_n)$, what is the minimum number of squares required for a 1,1-pillowcase cover?*

In Chapter 6, we prove the following theorem in answer to this question.

Theorem 1.10. *For $g \geq 2$, let $\mathcal{C} \neq \mathcal{Q}_g^{hyp}(4j + 2, 4k + 2)$, $\mathcal{Q}_g^{hyp}(4j + 2, 2k - 1, 2k - 1)$, or $\mathcal{Q}_g^{hyp}(2j - 1, 2j - 1, 2k - 1, 2k - 1)$ be a connected component of a non-empty stratum $\mathcal{Q}_g(k_1, \ldots, k_n)$ of quadratic differentials with no poles. Then there exists a 1,1-pillowcase cover in $\mathcal{C}$ consisting of $2g + n - 2$ squares. To construct 1,1-pillowcase covers with both core curves being non-separating in the components $\mathcal{Q}_g^{hyp}(4j + 2, 4k + 2)$, $\mathcal{Q}_g^{hyp}(4j + 2, 2k - 1, 2k - 1)$ and $\mathcal{Q}_g^{hyp}(2j - 1, 2j - 1, 2k - 1, 2k - 1)$, one requires at least $4g - 4$, $4g - 2$, and $4g$ squares, respectively. Moreover, there exist 1,1-pillowcase covers realising these bounds.*

For $g = 1$, let $\mathcal{Q}_g(k_1, \ldots, k_n)$ be a non-empty stratum of quadratic differentials with an even number of poles, then there exists a 1,1-pillowcase cover in this stratum consisting of $2g + n - 2$ squares. If instead $\mathcal{Q}_g(k_1, \ldots, k_n)$ is a non-empty stratum of quadratic differentials with an odd number of poles then there exists a 1,1-pillowcase cover in this stratum with at most $2g + n + 1$ squares.

For $g = 0$, let $\mathcal{Q}(k_1, \ldots, k_n, -1^{\kappa+4})$, for $\kappa = \sum k_i$, be a stratum of quadratic differentials. Then there exists a 1,1-pillowcase cover in this stratum with $4\kappa + 2n + 2$ squares.

The case of genus at least two is handled in Sections 6.3 to 6.6. For the hyperelliptic components, we conjecture that the bounds in this theorem are minimal for all 1,1-pillowcase covers in these components. That is, we expect that making one or both of the core curves separating increases the number of squares required.

In genus one, we remark that there are certain strata in the case of an odd number of poles for which the number of squares given by this theorem is not minimal. This is also true for certain genus zero strata. The proof of the above theorem and the possible improvements are given in Section 6.7 for genus one and Section 6.8 for genus zero.

For genus at least one, we are able to apply an inductive method similar to that used to prove Theorem 1.4. However, the combinatorics in the quadratic case are more complicated requiring a greater number of base cases with more delicate structure. Furthermore, this inductive method requires positive genus and so a new method is required for the construction of 1,1-pillowcase covers in genus zero.

Chapter 2

Combination lemmas

The main objects of study in this thesis are the square-tiled surfaces we introduced above. In this chapter, we discuss some associated structures that we will use throughout this thesis. Moreover, we will prove a pair of lemmas that will be essential to the construction of 1,1-square-tiled surfaces in Chapter 3.

2.1 Permutation representatives

An *interval exchange transformation* is a self map of the interval that divides the interval into subintervals and then permutes them. Now consider the translation surface given in Figure 2.1. The first return map to the horizontal transversal T under the upwards vertical flow on the surface induces an interval exchange transformation on T whose permutation is

$$\Pi = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 & 0 \end{pmatrix}. \tag{2.1}$$

That is, under the upwards vertical flow, the interval on T lying below the side labelled 0 returns in position 4 (counting from the left and starting at 0), and so on. In general, the interval below side i returns in position $\Pi^{-1}(i)$. For more details on the connections between translation surfaces and interval exchange transformations we direct the reader to the survey of Yoccoz [53].

The *extended Rauzy class* of this permutation is a class of permutations related under an induction method for interval exchange transformations introduced by Rauzy [47]. Another choice of transversal will give an interval exchange transformation whose permutation lies in the extended Rauzy class of permutation (2.1). Conversely, any translation surface obtained as a suspension of an interval exchange transformation whose permutation lies in the same extended Rauzy class as permutation (2.1) will lie in the same connected component of a stratum as the translation surface in Figure 2.1. Indeed, Veech showed that extended Rauzy classes are in one-to-one correspondence with the connected components of strata [50]. As such, any choice of permutation in an extended Rauzy class will be called a *permutation representative* of the stratum component containing the associated translation surface.

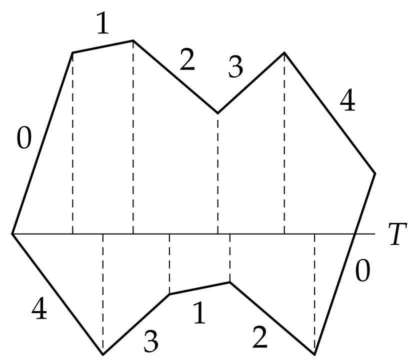

Figure 2.1: The first return map to the horizontal transversal T under the vertical flow induces an interval exchange transformation.

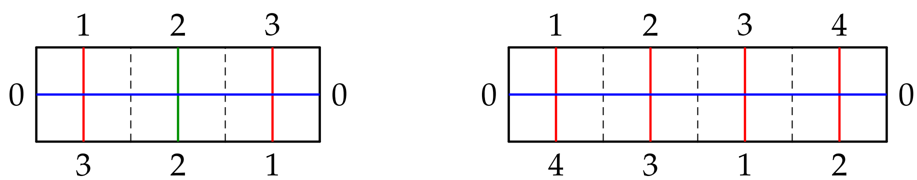

Figure 2.2: Two square-tiled surfaces in $\mathcal{H}(2)$ with a single horizontal cylinder. The surface on the right also has a single vertical cylinder while the one on the left has two vertical cylinders.

If the first symbol of the top row of a permutation representative is equal to the last symbol of the bottom row, then it is possible to construct a translation surface in the associated stratum component having a single horizontal cylinder. Zorich constructed permutation representatives of this form for every connected component of every stratum of Abelian and quadratic differentials [57].

We will be interested in the construction of a square-tiled surface from such a permutation representative. To do this, one takes a line of squares of length one less than the number of symbols in the permutation representative and labels the left and right sides of this line of squares with the first symbol in the top row and last symbol of the bottom row, respectively. From our assumption, these symbols are the same and so the resulting square-tiled surface will have a single horizontal cylinder. We then label the top sides (resp. bottom sides) with the remaining symbols from the top row (resp. bottom row). For example, the square-tiled surface on the right of Figure 2.2 is the one obtained by performing this construction using permutation (2.1) and we see that it does have a single horizontal cylinder, as claimed. The surface on the left of Figure 2.2 can be obtained from the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix}. \tag{2.2}$$

From here on, we will call a square-tiled surface constructed in this manner the square-tiled surface represented by the associated permutation representative.

As briefly mentioned above, we remark that attempting to use Rauzy moves to search the extended Rauzy classes of these permutations for permutations representing 1,1-square-tiled surfaces is not a feasible method for solving our problem. Indeed, Delecroix showed that the cardinality of extended Rauzy classes increases in such a way that this task would be incredibly computationally intensive [12]. Moreover, it is unlikely that examples for different strata could be found using similar sequences of Rauzy moves, and so a general proof of this nature would be difficult to find.

If we have a square-tiled surface with a single horizontal cylinder constructed as above, then, assuming the 0s to be the first symbol of the top row and last symbol of the bottom row, information concerning the vertical cylinders is contained in the permutation obtained by removing the 0s from each row of the permutation representative. Indeed, if the vertical cylinders have width one, then the number of vertical cylinders of the surface is equal to the number of cycles of this permutation. For example, under this modification, permutation (2.2) becomes the permutation $(1,3)(2)$ and indeed the surface on the left of Figure 2.2 has two vertical cylinders. Since we are interested in 1,1-square-tiled surfaces with a minimal number of squares, we will want the vertical cylinder to have width one, and so want this modified permutation to be a cyclic permutation. Indeed, for permutation (2.1) we obtain $(1,4,2,3)$ and it can be checked that the surface on the right of Figure 2.2 does indeed have a single vertical cylinder.

Note that adding a marked point to a side represented by label x, as described above, corresponds to adding a label to the right of x in both rows of the permutation representative.

2.2 Filling pair diagrams

Recall that a pair of essential simple closed curves $\{\alpha, \beta\}$ which are in minimal position on the surface S – that is, $i(\alpha, \beta) := \min_{\gamma \in [\alpha]} |\gamma \cap \beta| = |\alpha \cap \beta|$ – are said to be a filling pair if their complement is a disjoint union of disks. The core curves of the vertical and horizontal cylinders of a 1,1-square-tiled surface form a filling pair on that surface. Since we have an Abelian differential, all intersections occur with the same orientation.

Moreover, each complementary region is a $4k$-gon and corresponds to a zero of order $k-1$ of the associated Abelian differential.

Conversely, given a filling pair whose intersections all occur with the same orientation, the dual complex of the filling pair is a square complex and we can realise the surface as a collection of squares in the plane with sides identified by translations, in other words, as a square-tiled surface. As above, each complementary region with $4k$ sides will give rise to a zero of the Abelian differential of order $k-1$.

With this correspondence in mind, we can form a ribbon graph from the vertical and horizontal cylinders of a 1,1-square-tiled surface. We shall call the underlying oriented combinatorial graph a *filling pair diagram* for the associated square-tiled surface.

We will explain this construction by means of an example. Indeed, consider the 1,1-square-tiled surface in $\mathcal{H}(4)$ with permutation representative

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 & 0 \end{pmatrix}. \tag{2.3}$$

We first draw a horizontal line corresponding to the horizontal cylinder. Note that we will think of the ends being identified even though we do not join them in the diagram. We then add one vertex to the line for every square in the surface, equivalently for every non-zero symbol in the permutation representative. We then add an edge joining the bottom of the vertex corresponding to label x to the top of the vertex corresponding to the label y if and only if y lies below x in the permutation obtained by removing the 0s. The concatenation of these edges represents the vertical cylinder on the surface. The filling pair diagram associated to permutation (2.3) is shown in Figure 2.3. Note that, given a filling pair diagram, the reverse of this process allows us to easily construct a permutation representative.

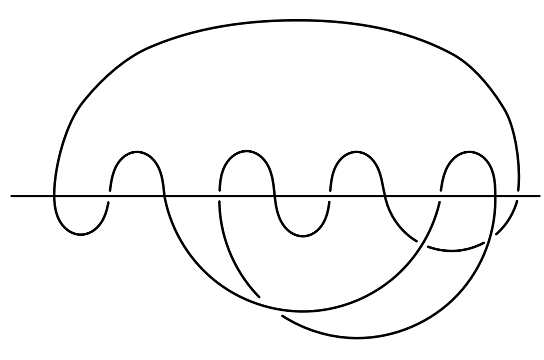

Figure 2.3: A filling pair diagram corresponding to a square-tiled surface represented by permutation (2.3).

Taking a regular neighbourhood of the filling pair diagram gives a ribbon graph with one boundary component for every complementary region of the filling pair. We can construct the square-tiled surface by gluing a saddle with cone-angle $2k\pi$ onto every boundary component with $4k$ sides. Indeed, the ribbon graph obtained from the filling pair diagram in Figure 2.3 has a single boundary component with 20 sides and so we glue in a saddle with cone-angle $10\pi = (4+1)2\pi$. As such, the square-tiled surface will have a single zero of order 4. This agrees with the fact that the permutation representative corresponded to a surface in $\mathcal{H}(4)$.

2.3 Combination lemmas

We now provide the combination lemmas that will be crucial to the construction of 1,1-square-tiled surfaces in the next chapter. The first lemma describes how to combine two 1,1-square-tiled surfaces to produce a single 1,1-square-tiled surface of higher complexity. The second lemma describes how the parity of the spin structure of a surface built in this way depends on the parities of the spin structures of the constituent surfaces.

Before proving the first lemma we will demonstrate the construction through an example. First, consider the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 & 0 \end{pmatrix}$$

representing a 1,1-square-tiled surface in $\mathcal{H}(4)$ with the minimum number of squares. We will describe the process of combining this surface with itself, as in Lemma 2.1 below, to produce a 1,1-square-tiled surface in $\mathcal{H}(4,4)$. The key property we want for this construction is that, on the square-tiled surface represented by this permutation, the bottom of the first square is identified with the top of the second. This can be seen in the permutation if the top row starts 0,1,2, and the second row starts with a 2.

We first realise two copies of this surface as in the top of Figure 2.4. We then relabel the top and bottom sides on the second surface from 6 to 10 and identify the right side of the first surface with the left side of the second and vice versa. This has the effect of concatenating the horizontal cylinders of the two surfaces as can be seen on the surface in the middle of Figure 2.4. We finally change the bottom side with label 1 to have label 6 and the bottom side with label 6 to have label 1. This has the effect of concatenating the vertical cylinders of the two surfaces. Indeed, we obtain the surface at the bottom of Figure 2.4 which can be seen to be a 1,1-square-tiled surface in $\mathcal{H}(4,4)$. The permutation representative for this surface is

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 5 & 4 & 6 & 3 & 7 & 10 & 9 & 1 & 8 & 0 \end{pmatrix}.$$

The construction can easily be performed directly on the permutation representatives and it is also easy to see what the process involves for filling pair diagrams. In the following lemmas, we will call this process of combining surfaces *cylinder concatenation*.

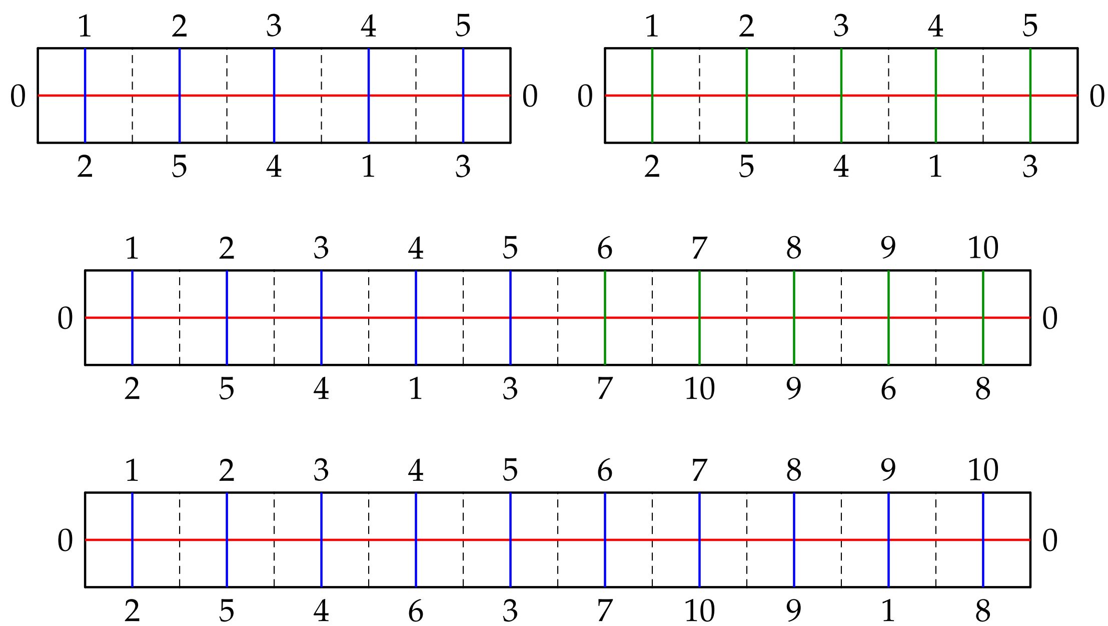

Figure 2.4: Cylinder concatenation construction as in Lemma 2.1.

We observe that the zeros of the constituent surfaces were preserved and that, by using 1,1-square-tiled surfaces with the minimal number of squares required for their respective strata, we obtained a 1,1-square-tiled surface with the minimum number of squares for its stratum. That this is true in general is the content of the following lemma.

Lemma 2.1. *Suppose the permutations*

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & \cdots & N \\ 2 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & \cdots & M \\ 2 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

represent 1,1-square-tiled surfaces S_1 and S_2 in the strata $\mathcal{H}_{g_1}(k_1, \ldots, k_n)$ and $\mathcal{H}_{g_2}(l_1, \ldots, l_m)$, *respectively. We assume only that they have first rows beginning 0,1,2 and second rows beginning with 2. Then the permutation*

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & \cdots & N+M \\ 2 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix} \tag{2.4}$$

obtained from these permutations by the cylinder concatenation method represents a 1,1-square-tiled surface S in $\mathcal{H}_{g_1+g_2-1}(k_1,\dots,k_n,l_1,\dots,l_m)$. In particular, note that if $N = 2g_1 + n - 2$ and $M = 2g_2 + m - 2$, then $N + M = 2(g_1 + g_2 - 1) + (n + m) - 2$.

Proof. Note that the surfaces S_1 and S_2 can be realised as in Figure 2.5. Since the sides labelled by 2 are diagonally opposite, we can construct the blue curves of slope 1 shown in each surface. It can then be seen that the process of cylinder concatenation for these surfaces is equivalent to cutting each surface open along the blue curves and gluing the right side of the slit in each surface to the left side of the slit in the other surface. This action concatenates the cylinders as expected and so we do indeed produce a 1,1-square-tiled surface S. Moreover, it can be seen that the that the zeros of S are exactly the union of the zeros of S_1 and S_2 and, by an Euler characteristic argument, that the genus of S is $g_1 + g_2 - 1$. That is, S lies in $\mathcal{H}_{g_1+g_2-1}(k_1,\dots,k_n,l_1,\dots,l_m)$, as claimed. □

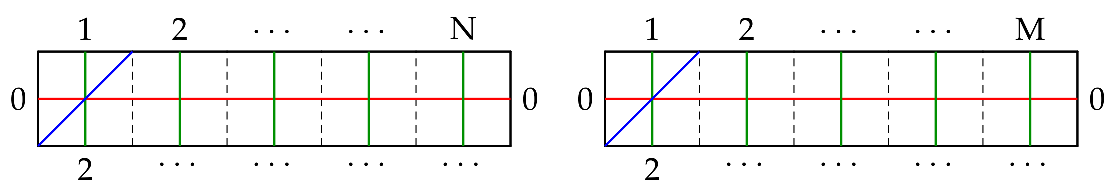

Figure 2.5: Realisation of the surfaces S_1 and S_2.

Note that the surface produced by this method has the necessary form to be a constituent surface; that is, the bottom of the first square is again identified with the top of the second. As such, this process can be iterated.

We now consider how the spin structures of 1,1-square-tiled surfaces behave under cylinder concatenation. Indeed, this is the content of the following lemma.

Lemma 2.2. *Let $S_1 \in \mathcal{H}_{g_1}(2k_1,\dots,2k_n)$ and $S_2 \in \mathcal{H}_{g_2}(2l_1,\dots,2l_m)$ be 1,1-square-tiled surfaces with the form necessary to apply Lemma 2.1. Further assume that S_1 has spin parity ϵ, and S_2 has spin parity η. Let $S \in \mathcal{H}_{g_1+g_2-1}(2k_1,\dots,2k_n,2l_1,\dots,2l_m)$ be the 1,1-square-tiled surface obtained from S_1 and S_2 by applying Lemma 2.1, then S has spin parity*

$$\epsilon + \eta + 1 \mod 2.$$

.

Proof. Consider S_1 and S_2 as in Figure 2.5. In each surface, we can choose the core curves of the horizontal cylinders and the blue curves of slope 1 to form symplectic pairs $\{\alpha_1, \beta_1\}$ and $\{\gamma_1, \delta_1\}$, respectively. Both curves in each pair have constant angle with respect to the horizontal direction and so both have index 0. As such, we have

$$(\text{ind}(\alpha_1)+1)(\text{ind}(\beta_1)+1) = 1 = (\text{ind}(\gamma_1)+1)(\text{ind}(\delta_1)+1).$$

If the sets of curves $\{\alpha_2, \beta_2, \ldots, \alpha_{g_1}, \beta_{g_1}\}$ and $\{\gamma_2, \delta_2, \ldots, \gamma_{g_2}, \delta_{g_2}\}$ complete a symplectic basis on each surface, then we must have

$$\sum_{i=2}^{g_1}(\text{ind}(\alpha_i)+1)(\text{ind}(\beta_i)+1) \equiv \epsilon - 1 \mod 2,$$

and

$$\sum_{i=2}^{g_2}(\text{ind}(\gamma_i)+1)(\text{ind}(\delta_i)+1) \equiv \eta - 1 \mod 2.$$

In S, we can again choose the horizontal core curve and the blue curve of slope 1 to form a symplectic pair $\{\mu, \nu\}$ satisfying $(\text{ind}(\mu)+1)(\text{ind}(\nu)+1) = 1$. A symplectic basis can then be completed by further taking the union of $\{\alpha_2, \beta_2, \ldots, \alpha_{g_1}, \beta_{g_1}\}$ and $\{\gamma_2, \delta_2, \ldots, \gamma_{g_2}, \delta_{g_2}\}$. Each curve will have the same index on S as it did on S_1 or S_2, respectively. Hence, we see that the spin parity of S is

$$1 + (\epsilon - 1) + (\eta - 1) \equiv \epsilon + \eta + 1 \mod 2,$$

as claimed. □

Returning to the example we gave above, one can check that the permutation we combined represented a surface in $\mathcal{H}^{odd}(4)$ and that the resulting permutation represents a surface in $\mathcal{H}^{odd}(4,4)$, as we would expect from Lemma 2.2. Constructions like this will allow us to use a number of constituent surfaces to build 1,1-square-tiled surfaces in the desired connected components of general strata.

Chapter 3

Construction of 1,1-square-tiled surfaces

In this chapter we construct 1,1-square-tiled surfaces in every connected component of every stratum of the moduli space of Abelian differentials using the minimum number of squares possible and hence prove Theorem 1.4. Though all results are stated in terms of permutation representatives, the proofs will make use of the filling pair diagrams introduced in Section 2.2 of Chapter 2. The construction relies heavily on Lemmas 2.1 and 2.2 of the previous chapter.

Outline of proof

We begin by constructing by hand 1,1-square-tiled surfaces in the hyperelliptic components of strata. For a non-hyperelliptic component in an arbitrary stratum $\mathcal{H}(k_1, \ldots, k_n)$ we will employ Lemmas 2.1 and 2.2. That is, for every even k_i we will construct a 1,1-square-tiled surfaces in $\mathcal{H}(k_i)$, and for every pair of odd $\{k_i, k_j\}$ we will construct a 1,1-square-tiled surface in $\mathcal{H}(k_i, k_j)$. Then a 1,1-square-tiled surface in $\mathcal{H}(k_1, \ldots, k_n)$ can be constructed by inductively applying Lemma 2.1 to these surfaces. Moreover, if we can construct 1,1-square-tiled surfaces in the odd and even components of $\mathcal{H}(2k)$, then by inductively applying Lemma 2.2, we can build a 1,1-square-tiled surface in the odd and even components of $\mathcal{H}(2k_1, \ldots, 2k_n)$. However, this method is complicated by the strata $\mathcal{H}(2)$, and $\mathcal{H}(1,1)$ for which there do not exist 1,1-square-tiled surfaces built from the theoretical minimum number of squares. Moreover, there is no $\mathcal{H}^{even}(4)$ component that can be used in our construction. As such, we are required to modify the ideal method described above in such situations.

3.1 Hyperelliptic components

We begin by constructing 1,1-square-tiled surfaces in the hyperelliptic components that realise the number of squares claimed in Theorem 1.4. Indeed, this is the content of the following proposition. We will then prove that these are the minimum number of squares necessary for 1,1-square-tiled surfaces in the hyperelliptic components. The fact that these numbers are strictly greater than the minimum required for square-tiled surfaces in the ambient stratum, particularly for genus two, will cause us difficulty in the sections that follow. Indeed, since the strata $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$ are connected and coincide with their hyperelliptic components, we will not have 1,1-square-tiled surfaces in these strata that can be used to build minimal 1,1-square-tiled surfaces in higher genus strata.

Proposition 3.1. *For $g \geq 2$, the permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & 2g-5 & 2g-4 & 2g-3 \\ 4g-4 & 4g-6 & 4g-5 & 4g-8 & 4g-7 & \cdots & 2g & 2g+1 & 2g-1 \\ & & 2g-2 & 2g-1 & \cdots & 4g-8 & 4g-7 & 4g-6 & 4g-5 & 4g-4 \\ & & 2g-3 & 2g-2 & \cdots & 3 & 4 & 1 & 2 & 0 \end{pmatrix} \tag{3.1}$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & 2g-3 & 2g-2 & 2g-1 \\ 4g-2 & 4g-4 & 4g-3 & 4g-6 & 4g-5 & \cdots & 2g & 2g+1 & 2g-1 \\ & & 2g & 2g+1 & \cdots & 4g-6 & 4g-5 & 4g-4 & 4g-3 & 4g-2 \\ & & 2g-3 & 2g-2 & \cdots & 3 & 4 & 1 & 2 & 0 \end{pmatrix} \tag{3.2}$$

represent 1,1-square-tiled surfaces in $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1,g-1)$, respectively.

Proof. We first note that permutations (3.1) and (3.2) are produced by adding $2g-3$ and $2g-2$ marked points to the standard permutations for $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1,g-1)$, respectively. Essentially originally due to Veech [52] and stated explicitly by Zorich [57, Proposition 6], up to a relabelling, these are

$$\begin{pmatrix} 0 & 1 & \cdots & 2g-2 & 2g-1 \\ 2g-1 & 2g-2 & \cdots & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & \cdots & 2g-1 & 2g \\ 2g & 2g-1 & \cdots & 1 & 0 \end{pmatrix}.$$

Indeed, in the case of $\mathcal{H}^{hyp}(2g-2)$, one can check that we have added marked points by splitting the sides labelled by i for $1 \leq i \leq g-1$, $g+1 \leq i \leq 2g-2$. Similarly, for $\mathcal{H}^{hyp}(g-1,g-1)$ we have split the sides labelled by i for $1 \leq i \leq g-1$, $g+1 \leq i \leq 2g-1$. As such, permutations (3.1) and (3.2) do indeed represent $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1,g-1)$, respectively. Therefore, we need only prove that the permutations have one vertical and one horizontal cylinder when representing a square-tiled surface.

Note that, since the first rows of the permutations begin with 0 and the second rows end with 0, we have one horizontal cylinder in both. For permutation (3.1), the permutation with the 0s removed is a cycle as follows:

$$1 \to 4g-4 \to 2 \to 4g-6 \to 4 \to \cdots \to 2g \to 2g-2 \to 2g-1 \to$$

$$2g-3 \to 2g+1 \to 2g-5 \to 2g+3 \to \cdots \to 4g-7 \to 3 \to 4g-5 \to 1.$$

Since every non-zero symbol in the permutation is contained in this cycle, we see that the square-tiled surface has one vertical cylinder. Hence we do indeed have a 1,1-square-tiled surface. Similarly, for permutation (3.2), the permutation with the 0s removed is a cycle as follows:

$$1 \to 4g-2 \to 2 \to 4g-4 \to 4 \to \cdots \to 2g-2 \to 2g \to 2g-1 \to$$

$$2g+1 \to 2g-3 \to 2g+3 \to 2g-5 \to \cdots \to 4g-5 \to 3 \to 4g-3 \to 1.$$

Again, since every non-zero symbol in the permutation is contained in this cycle, we see that the square-tiled surface has one vertical cylinder and so we have a 1,1-square-tiled surface. Hence, the proposition is proved. □

Observing that the number of squares in a 1,1-square-tiled surface is equal to the number of distinct non-zero symbols in the permutation, we observe that these surfaces exhibit the number of squares claimed in the statement of Theorem 1.4. That is, $4g-4$ squares for $\mathcal{H}^{hyp}(2g-2)$ and $4g-2$ squares for $\mathcal{H}^{hyp}(g-1,g-1)$. To finish the proof of Theorem 1.4 for the hyperelliptic cases we must show that these are in fact the minimum number of squares required for these components.

Proposition 3.2. *A 1,1-square-tiled surface in the stratum $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$ requires at least $4g-4$ or $4g-2$ squares, respectively.*

Proof. We formalise and generalise a method for genus two surfaces attributed to Margalit in a remark in a paper of Aougab-Huang in which they determine the minimal geometric intersection numbers for filling pairs on closed surfaces [1, Remark 2.18]. The idea is to investigate the combinatorics of the images of the filling pair under the quotient by the hyperelliptic involution. If there is an arc between two punctures on the quotient sphere that is disjoint from the images of the filling pair, then this arc lifts to a curve disjoint from the filling pair on the original surface which contradicts the fact that the curves were filling.

Suppose that we have a 1,1-square-tiled surface (X,ω) in $\mathcal{H}^{hyp}(2g-2)$ made from n squares and assume n to be minimal. The core curves, α and β, of the vertical and horizontal cylinders form a filling pair of curves on the surface with geometric intersection number equal to n. Every intersection occurs with the same orientation, and so α and β are nonseparating. Since X is hyperelliptic, there exists an isometric involution

$\tau : X \to X$ and a branched double cover $\pi : X \to S_{0,2g+2}$ of the sphere with $2g+2$ punctures. Since $\tau^*\omega = -\omega$, the vertical and horizontal cylinders are sent to vertical and horizontal cylinders, respectively. Moreover, since τ acts by isometry, the number of such cylinders is fixed. Hence, α and β are nonseparating curves fixed by the hyperelliptic involution and so we have that $\pi(\alpha)$ and $\pi(\beta)$ are simple arcs on $S_{0,2g+2}$.

If n is odd then, since any interior intersection of the arcs $\pi(\alpha)$ and $\pi(\beta)$ will lift to two intersections of α and β on S, $\pi(\alpha)$ and $\pi(\beta)$ must share a single endpoint at a puncture on the sphere and have $(n-1)/2$ interior intersections. The arcs form a graph on the sphere with $3+(n-1)/2$ vertices. Apart from the endpoints of the arcs which have valency 1 or 2, each vertex has valency 4, and so we have $n+1$ edges. It follows from an Euler characteristic argument that the resulting graph has $(n+1)/2$ complimentary regions. As mentioned above, we must have a maximum of one puncture in each complementary region. Three of the punctures lie at the endpoints of the arcs and so we must have

$$\frac{n+1}{2} \geq 2g-1 \Rightarrow n \geq 4g-3.$$

If n is even then, by a similar argument to that given for n odd above, $\pi(\alpha)$ and $\pi(\beta)$ either share both of their endpoints or have disjoint endpoints. In the former case, we have $(n-2)/2$ interior intersections. The arcs form a graph with $2+(n-2)/2$ vertices and n edges. Hence we have $(n+2)/2$ complimentary regions. Two of the punctures lie at the endpoints and so we must have

$$\frac{n+2}{2} \geq 2g \Rightarrow n \geq 4g-2.$$

In the latter case, we have $n/2$ interior intersections. The arcs form a graph with $4+n/2$ vertices and $n+2$ edges. Hence we have $n/2$ complimentary regions. Four punctures lie at endpoints and so we must have

$$\frac{n}{2} \geq 2g-2 \Rightarrow n \geq 4g-4.$$

Hence we see that a 1,1-square-tiled surface in $\mathcal{H}^{hyp}(2g-2)$ requires at least $4g-4$ squares.

Suppose now that X is a 1,1-square-tiled surface in $\mathcal{H}^{hyp}(g-1,g-1)$ with n squares, with n again assumed to be minimal. As above, the core curves of the cylinders, α and β, are nonseparating curves with geometric intersection number equal to n and are fixed by the hyperelliptic involution. Hence, we have that $\pi(\alpha)$ and $\pi(\beta)$ are simple arcs on $S_{0,2g+2}$. Moreover, we must again have a maximum of one puncture in each complimentary region of the arcs. However, since the zeros of ω are by definition symmetric to one another by the hyperelliptic involution, they will correspond to a complementary region between $\pi(\alpha)$ and $\pi(\beta)$ that does not contain a puncture. So, in this case, we require one more complementary region between the arcs than we needed for $\mathcal{H}^{hyp}(2g-2)$. As such, we require an additional interior intersection of the arcs, which corresponds to two additional intersections of the filling pair, and so we must have $n \geq 4g-2$. This completes the proof of the proposition. □

We direct the reader to Appendix A for an alternative proof of this proposition given in the language of saddle connections and Weierstrass points. The details of this alternative proof are relevant to questions about the $\mathrm{SL}(2,\mathbb{Z})$-orbits of 1,1-square-tiled surfaces in $\mathcal{H}^{hyp}(2g-2)$ that we discuss in Section 7.3.

3.2 Even order zeros

In this section we will construct 1,1-square-tiled surfaces in the odd and even components of strata with even order zeros. That, is we construct 1,1-square-tiled surfaces with a minimal number of squares in the odd and even components of strata of the form $\mathcal{H}(2k_1,\ldots,2k_n)$, $k_i \geq 1$ and $\sum k_i = 2g-2$. We do this by building base cases in the odd and even components of the strata $\mathcal{H}(2k)$, $k \geq 2$. These surfaces can then be combined using the cylinder concatenation methods of Lemmas 2.1 and 2.2. We must also deal with the fact that 1,1-square-tiled surfaces in $\mathcal{H}(2)$ cannot be used to construct 1,1-square-tiled surfaces in higher genus strata. Moreover, there are no components $\mathcal{H}^{even}(4)$ and $\mathcal{H}^{even}(2,2)$ and so we have more work to do in order to be able to construct 1,1-square-tiled surfaces in $\mathcal{H}^{even}(4,4,\ldots,4)$ and $\mathcal{H}^{even}(2,2,\ldots,2)$.

Strata of the form $\mathcal{H}(2k)$

We begin by constructing 1,1-square-tiled surfaces in $\mathcal{H}^{odd}(2k)$, for $k \geq 2$.

Proposition 3.3. *The permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 & 0 \end{pmatrix}, \tag{3.3}$$

and, for $g \geq 4$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots & 2g-4 & 2g-3 & 2g-2 & 2g-1 \\ 2 & 5 & 4 & 7 & 3 & 9 & 6 & 11 & 8 & 13 & \cdots & 2g-4 & 1 & 2g-2 & 0 \end{pmatrix} \tag{3.4}$$

represent 1,1-square-tiled surfaces in $\mathcal{H}^{odd}(4)$ and $\mathcal{H}^{odd}(2g-2)$, respectively. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

Proof. We will first prove that the permutations represent the claimed strata. Observe that the filling pair in Figure 3.1 represents permutation (3.3). We observe that its ribbon graph has one boundary component with 20 sides and so corresponds to a single zero of order 4. That is, the permutation represents a 1,1-square-tiled surface in $\mathcal{H}(4)$.

Now consider the filling pair diagram in Figure 3.2 with $2g-1$ vertices. We will modify this diagram to produce a filling pair diagram representing permutation (3.4).

We will perform a series of vertex transpositions on the filling pair diagram. These will not change the fact that we have one vertical and one horizontal cylinder but will change the number of boundary components and the number of sides of the boundary components of the associated ribbon graph. We currently have $2g-1$ boundary components with four sides. Our goal is to produce a filling pair diagram with a single boundary component with $8g-4$ sides.

We first perform two transpositions on the third, fourth and fifth vertices to give the permutation $(3,5,4)$ on the vertices. Note that this gives the first 5 vertices the combinatorics given by the filling pair diagram for $\mathcal{H}(4)$ in Figure 3.1. Moreover, we now have one boundary component with 20 sides and $2g-6$ boundary components with 4 sides. See Figure 3.3.

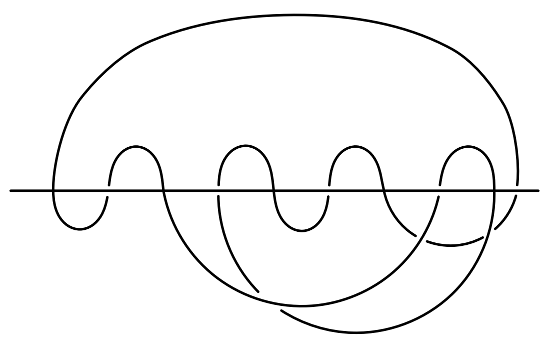

Figure 3.1: A filling pair diagram representing permutation (3.3).

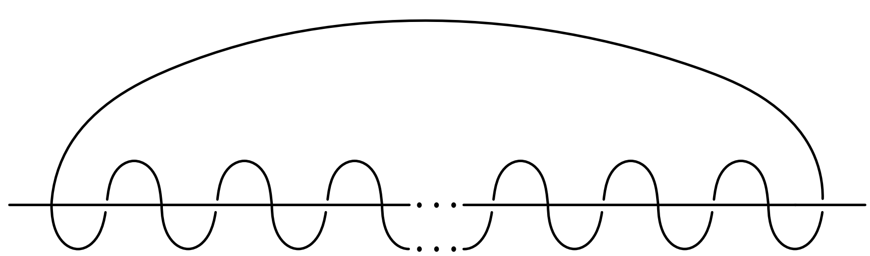

Figure 3.2: A filling pair diagram with $2g-1$ vertices. The associated ribbon graph has $2g-1$ boundary components each with 4 sides.

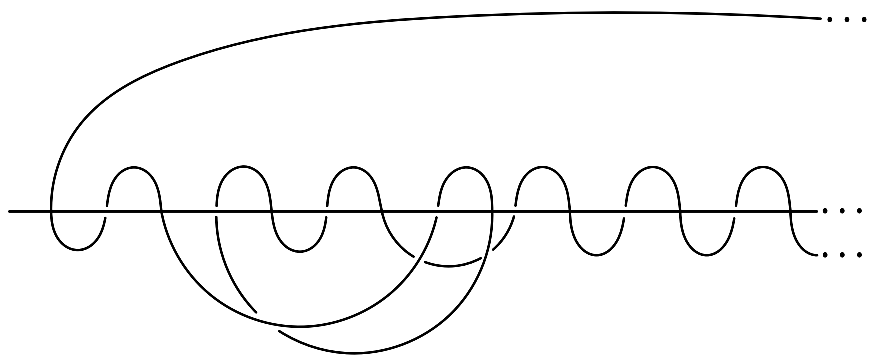

Figure 3.3: The filling pair diagram after applying permutation $(3,5,4)$ on the vertices.

Observe that after this permutation the boundary components around vertices 6-9 have the combinatorics shown on the left of Figure 3.4, where different letters correspond to different boundary components. We then perform a vertex transposition on vertices 6 and 7, as shown on the right of Figure 3.4. We now have a boundary component with 28 sides and $2g-8$ boundary components with 4 sides.

We now observe that, after the vertex transposition, the combinatorics that we had around vertices 6-9 are repeated again around vertices 8-11. As such, we can perform this transposition again to produce a boundary component with 36 sides and $2g-10$ boundary components with 4 sides. Moreover, these combinatorics persist and so we can continue to repeat this transposition for the remaining $g-5$ pairs of vertices ending up with a single boundary component. Since each vertex has valency 4, there are $4g-2$ edges in the filling pair diagram. Each edge will give two sides to the boundary component and so the boundary component will have $8g-4$ sides corresponding to a zero of order $2g-2$, as required.

It is easy to check that this filling pair diagram represents permutation (3.4), and so we have shown that this permutation does indeed represent a 1,1-square-tiled surface in $\mathcal{H}(2g-2)$.

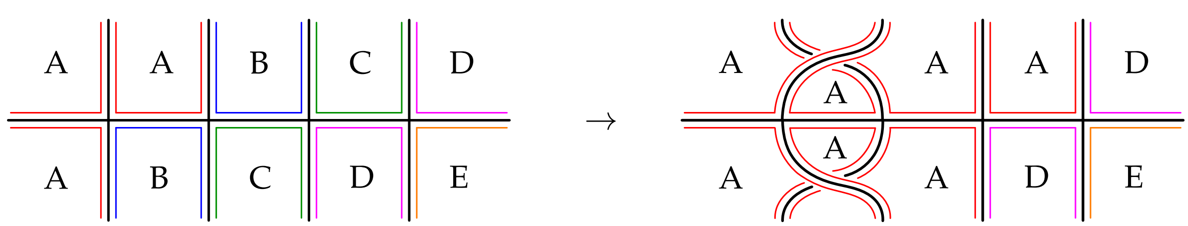

Figure 3.4: The effect of a vertex transposition on the boundary components of the ribbon graph of the filling pair diagram in Figure 3.3.

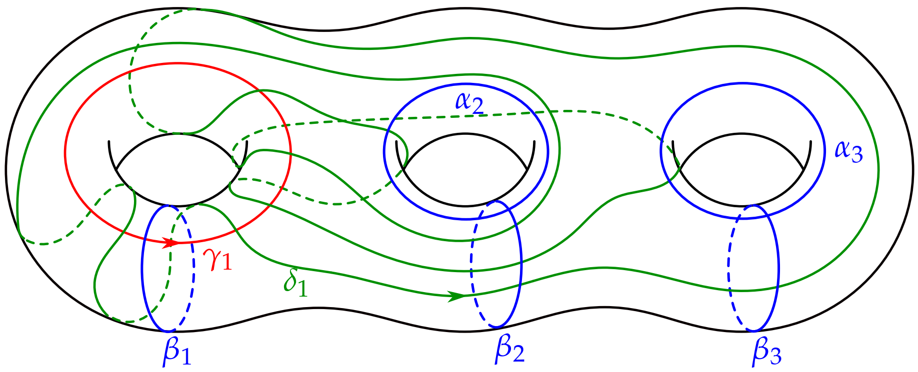

Figure 3.5: Realisation of the filling pair diagram representing permutation (3.3).

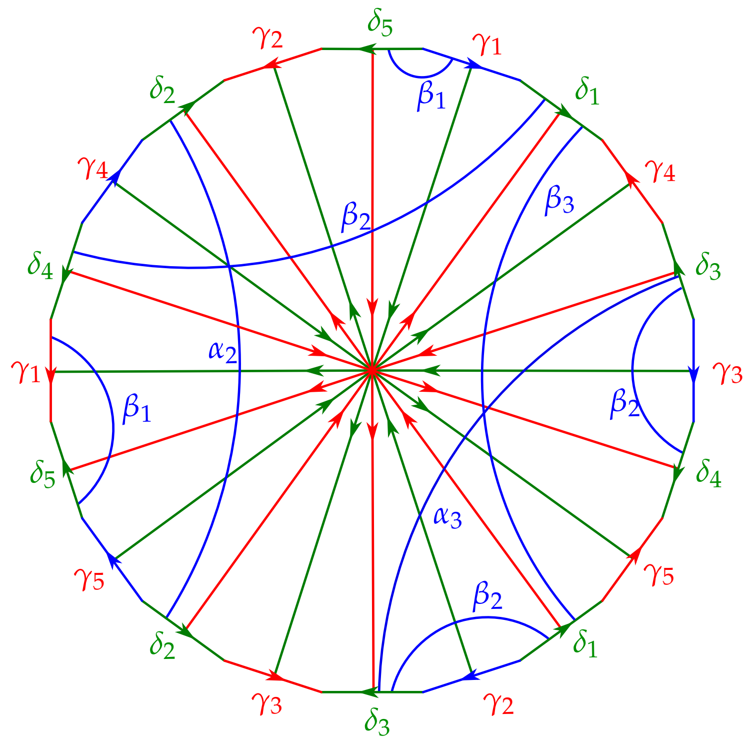

Figure 3.6: Polygonal decomposition of the surface given by the filling pair (γ, δ).

We must now show that these 1,1-square-tiled surfaces lie in the odd components. We will do this by calculating the spin parity of the surfaces with respect to representatives corresponding to the standard homology basis.

We first prove that the surface represented by permutation (3.3) has odd spin structure and thus represents $\mathcal{H}^{odd}(4)$. We realise the filling pair diagram as the curves γ and δ in Figure 3.5, and label the arcs of each curve between their intersections with the labels $\gamma_1, \ldots, \gamma_5$, and $\delta_1, \ldots, \delta_5$, respectively. Note that we only show the first label of each curve in the diagram. Next, choose the homology representatives $\{\alpha_i, \beta_i\}_{i=1}^{3}$ as in

Figure 3.5. Recall that, as in Lemma 2.2, we choose $\alpha_1 = \gamma$ and β_1 to be the curve of slope 1 with respect to the horizontal direction.

We now cut the surface open along the filling pair $\{\gamma, \delta\}$ to form the 20-gon shown in Figure 3.6. We have also included in the figure the leaves of the vertical and horizontal foliations given by the edges of the squares making up the surface. The index of a curve can then be calculated by keeping track of the number of these lines the curve crosses and in which direction. It is then easy to show that we have

$$\sum_{i=1}^{3} (\text{ind}(\alpha_i) + 1)(\text{ind}(\beta_i) + 1) \equiv 1 \bmod 2,$$

and so the canonical spin structure on the surface has odd spin parity.

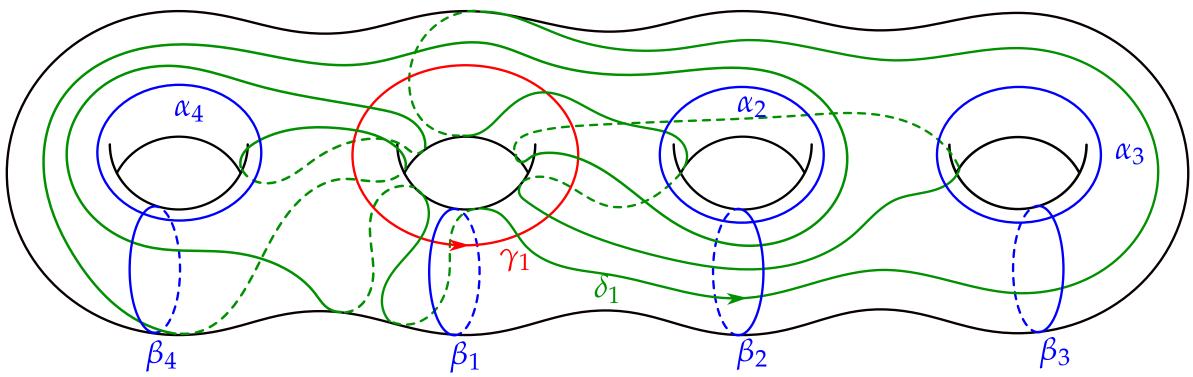

Figure 3.7: Realisation of filling pair diagram for $\mathcal{H}(6)$.

The filling pair diagram given by permutation (3.4) representing $\mathcal{H}(6)$ can be realised as in Figure 3.7. A similar calculation to that above shows that the canonical spin structure on this surface also has odd spin parity.

From this point onwards, for every additional increase of g in permutation (3.4), the polygonal decompositions given by the realisations of the filling pair diagrams vary in a predictable manner. This is because the 'final handle' on the surface, the handle associated to the final two vertices of the filling pair diagram, has the form of the handle on the left of Figure 3.7; that is, the handle containing α_4 and β_4. The change to the polygonal decompositions is then demonstrated by the changes between Figures 3.8 and 3.9.

We see that the standard homology representatives around the added genus, α_g and β_g, both have index 1 and so contribute 0 to the calculation of the spin parity modulo 2. Moreover, 8 sides are added to the polygon in one piece and so, since each additional side crossed requires a rotation by $\pi/2$, the index of a curve passing these sides will change by 2 and so the contribution to the calculation of the spin parity is changed by 0 modulo 2. Altogether, we have added 0 modulo 2 to the calculation of the spin parity and so, since the surface in Figure 3.7 had odd spin parity, the surface represented by permutation (3.4) lies in $\mathcal{H}^{odd}(2g-2)$.

Note also that all permutations in the proposition represent square-tiled surfaces with the minimum number of squares for the respective strata, namely $2g-1$. As such, the proposition has been proved. □

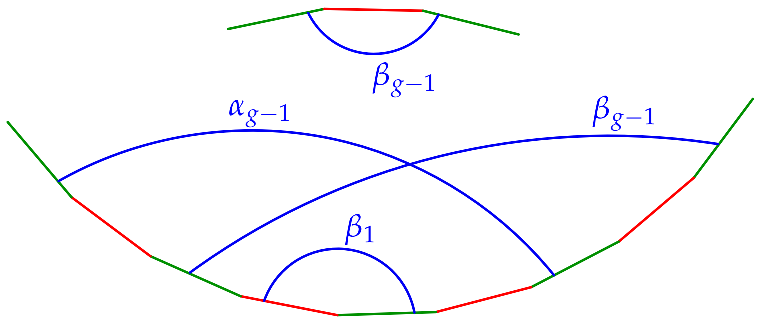

Figure 3.8: Part of the polygonal decomposition of surface of genus $g-1$ given by permutation (3.4).

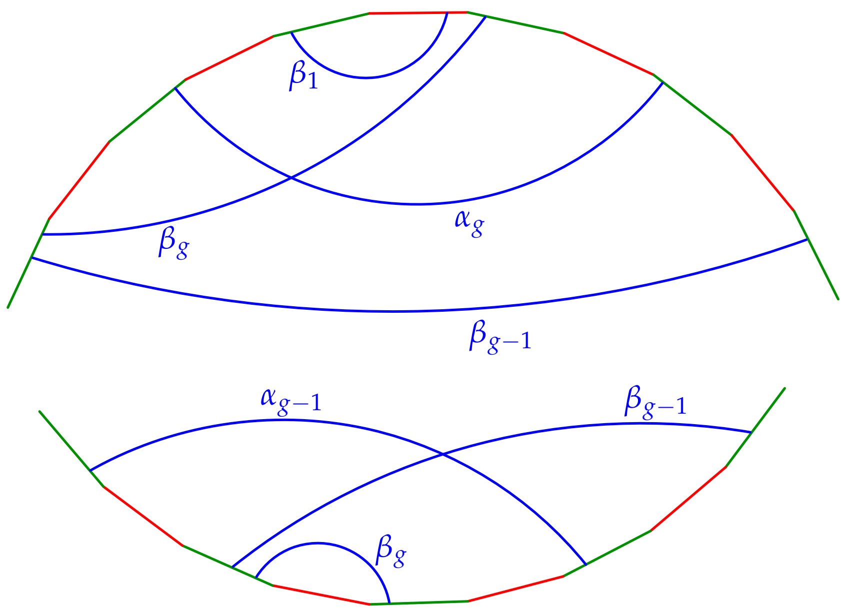

Figure 3.9: Part of the polygonal decomposition of surface of genus g given by permutation (3.4).

Since the surfaces given by Proposition 3.3 have the desired form we can use Lemmas 2.1 and 2.2 to produce 1,1-square-tiled surfaces in the odd components of all strata with even order zeros of order greater than or equal to 4.

The following proposition constructs surfaces in the even components of the strata $\mathcal{H}(2g-2)$.

Proposition 3.4. *The permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 6 & 5 & 3 & 1 & 4 & 0 \end{pmatrix}, \tag{3.5}$$

and, for $g \geq 5$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 7 & 6 & 5 & 3 & 9 & 4 & 11 & 8 & 13 & 10 \end{pmatrix.} \\ \begin{matrix} 11 & \cdots & 2g-4 & 2g-3 & 2g-2 & 2g-1 \\ 15 & \cdots & 2g-4 & 1 & 2g-2 & 0 \end{matrix}\Bigg) \tag{3.6}$$

represent 1,1-square-tiled surfaces in $\mathcal{H}^{even}(6)$ and $\mathcal{H}^{even}(2g-2)$, respectively. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

Proof. The proof is completely analogous to the proof of Proposition 3.3. That is, one can show that permutation (3.5) represents a 1,1-square-tiled surface in $\mathcal{H}^{even}(6)$ by calculating directly on the polygonal decomposition given by the filling pair. Applying the same induction used in the proof of Proposition 3.3 then shows that permutation (3.6) represents a 1,1-square-tiled surface in $\mathcal{H}^{even}(2g-2)$. □

Handling the non-existence of $\mathcal{H}^{even}(4)$

We now have the first instance in which we must construct an exceptional case separately. Note that using Lemmas 2.1 and 2.2, we can use surfaces given by Propositions 3.3 and 3.4 to produce 1,1-square-tiled surfaces in the even components of all strata with even order zeros of order greater than or equal to 4 apart from strata containing only zeros of order 4. This is because there is no even component in the stratum $\mathcal{H}(4)$. However, the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 10 & 7 & 5 & 8 & 1 & 9 & 6 & 4 & 3 & 0 \end{pmatrix}$$

represents a 1,1-square-tiled surface in $\mathcal{H}^{even}(4,4)$ and so we can use this to produce 1,1-square-tiled surfaces in the even components of these exceptional strata.

Handling the hyperellipticity of $\mathcal{H}(2)$ and non-existence of $\mathcal{H}^{even}(2,2)$

As we saw in the previous section, all surfaces in genus 2 are hyperelliptic and so 1,1-square-tiled surfaces require more than the minimum number of squares required for a square-tiled surface in their respective stratum. As such, we do not have a 1,1-square-tiled surface in $\mathcal{H}(2)$ that can be concatenated, as in Lemmas 2.1 and 2.2, to the even order surfaces we have produced in the propositions above. It is therefore necessary to produce strata containing zeros of order 2 separately. This is the content of the following propositions.

The first proposition produces 1,1-square-tiled surfaces in $\mathcal{H}^{odd}(2k,2)$, for $k \geq 3$. The proof is similar to the proof of Proposition 3.3.

Proposition 3.5. *The permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 5 & 4 & 6 & 3 & 8 & 10 & 7 & 1 & 9 & 0 \end{pmatrix}, \tag{3.7}$$

and, for $k \geq 4$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots & 2k-4 & 2k-3 \\ 2 & 5 & 4 & 7 & 3 & 9 & 6 & 11 & 8 & 13 & \cdots & 2k-4 & 2k \end{pmatrix.}$$
$$\begin{matrix} 2k-2 & 2k-1 & 2k & 2k+1 & 2k+2 & 2k+3 & 2k+4 \\ 2k-2 & 2k+2 & 2k+4 & 2k+1 & 1 & 2k+3 & 0 \end{matrix}\Bigg) \tag{3.8}$$

represent 1,1-square-tiled surfaces in $\mathcal{H}^{odd}(6,2)$ and $\mathcal{H}^{odd}(2k,2)$, respectively. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

Proof. We first begin by realising permutation (3.7) by the filling pair diagram in Figure 3.10.

Note that to produce this filling pair diagram we have added the 5 vertices shown in Figure 3.11 to the right-hand side of filling pair diagram for $\mathcal{H}^{odd}(4)$ in Figure 3.1. These additional vertices contribute another 10 edges to the diagram and so another 20 sides to the boundary components of the associated ribbon graph. It is easy to check that there is a boundary component with 12 sides corresponding to a zero of order 2, and that the 8 remaining additional sides are added to the boundary component that represented the zero of order 4 in the original ribbon graph. Hence, we have a second boundary component corresponding to a zero of order 6. That is, the filling pair diagram represents a 1,1-square-tiled surface in the stratum $\mathcal{H}(6,2)$.

The argument for permutation (3.8) is similar in that we add 5 vertices in the same way to the right-hand side of the filling pair diagram representing permutation (3.4). These have a single zero of order 2, as above, and a second zero of order two more than the order of the zero represented by permutation (3.4). It is easy to check that the resulting 1,1-square-tiled surface lies in the claimed stratum.

To check that the surfaces have the claimed spin parity, we investigate the effect that modifying the permutations of Proposition 3.3 to achieve permutations (3.7) and (3.8) has on the associated polygonal decompositions. This effect is demonstrated in Figure 3.12. We see that the 8 sides added to the polygon of the original surface are added in one piece and so, as was the case in Proposition 3.3, the index of any curve crossing these sides is changed by 2 and so changes the calculation of the spin parity by 0 modulo 2. We also observe that the homology representatives, $\alpha_{g+1}, \alpha_{g+2}, \beta_{g+1}$, and β_{g+2}, around the two additional genus all have index 1 and so together contribute 0 modulo 2 to the calculation of the spin parity. Therefore, the spin parity of the resulting surface is the same as the spin parity of the surface we started with which in this case is odd. That is, permutations (3.7) and (3.8) do indeed represent the odd components of their respective strata.

Finally, observe that the minimum number of squares required for square-tiled surfaces in $\mathcal{H}(6,2)$ and $\mathcal{H}(2k,2)$ are 10 and $2k+4$, respectively. As such, the 1,1-square-tiled surfaces we have produced have the minimum number of squares required for their respective strata. Hence the proposition is proved. □

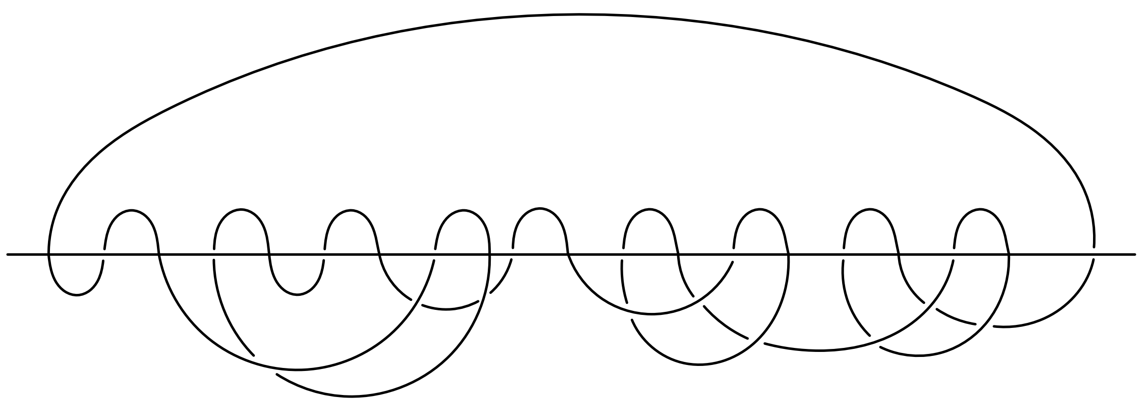

Figure 3.10: Filling pair diagram representing permutation (3.7).

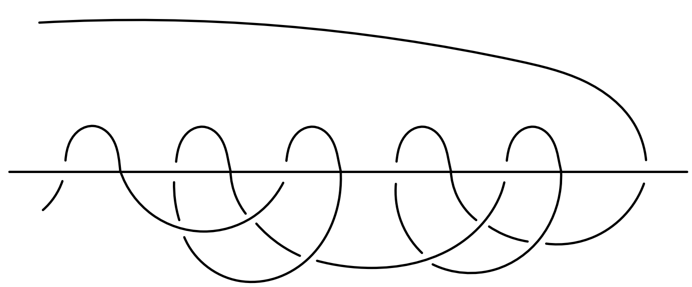

Figure 3.11: Filling pair diagram combinatorics for adding $\mathcal{H}(2)$.

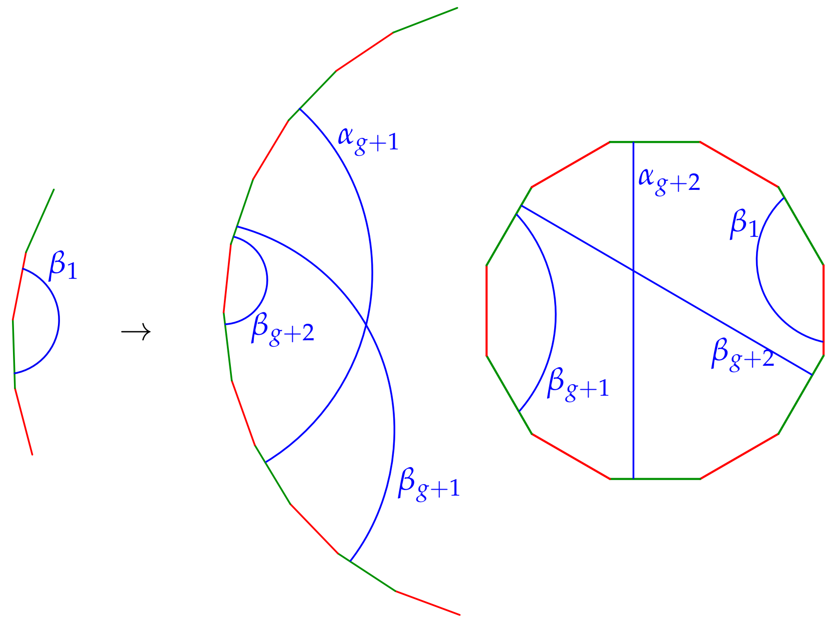

Figure 3.12: The effect on the polygonal decomposition of changing permutation (3.4) to permutation (3.8), where the genus of the resulting surface is $g+2$.

The following proposition produces 1,1-square-tiled surfaces in $\mathcal{H}^{even}(2k,2)$, for $k \geq 4$.

Proposition 3.6. *The permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 7 & 6 & 5 & 3 & 8 & 4 & 10 & 12 & 9 & 1 & 11 & 0 \end{pmatrix}, \tag{3.9}$$

and, for $k \geq 5$*,*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \cdots & 2k-4 & 2k-3 \\ 2 & 7 & 6 & 5 & 3 & 9 & 4 & 11 & 8 & 13 & 10 & 15 & \cdots & 2k-4 & 2k \end{pmatrix.$$
$$\begin{matrix} 2k-2 & 2k-1 & 2k & 2k+1 & 2k+2 & 2k+3 & 2k+4 \\ 2k-2 & 2k+2 & 2k+4 & 2k+1 & 1 & 2k+3 & 0 \end{matrix}\Bigg) \tag{3.10}$$

represent 1,1-square-tiled surfaces in $\mathcal{H}^{even}(8,2)$ *and* $\mathcal{H}^{even}(2k,2)$*, respectively. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.*

Proof. The proof is analogous to the proof of Proposition 3.5. That is, we add the filling pair diagram combinatorics shown in Figure 3.11 in the same way as above to the filling pair diagrams representing the permutations of Proposition 3.4. $\square$

As above, we have a number of exceptional cases not covered by these propositions.

These are resolved as follows. The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 3 & 1 & 5 & 0 \end{pmatrix}$$

represents $\mathcal{H}^{odd}(2,2)$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 8 & 6 & 9 & 4 & 1 & 3 & 5 & 7 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 9 & 8 & 7 & 6 & 3 & 5 & 1 & 4 & 0 \end{pmatrix}$$

represent $\mathcal{H}^{odd}(2,2,2)$ and $\mathcal{H}^{even}(2,2,2)$, respectively. The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 5 & 4 & 1 & 12 & 3 & 10 & 7 & 11 & 9 & 6 & 8 & 0 \end{pmatrix}$$

represents $\mathcal{H}^{even}(2,2,2,2)$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 5 & 8 & 3 & 6 & 4 & 1 & 7 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 1 & 8 & 7 & 5 & 3 & 6 & 0 \end{pmatrix}$$

represent $\mathcal{H}^{odd}(4,2)$ and $\mathcal{H}^{even}(4,2)$, respectively. The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 8 & 5 & 3 & 1 & 10 & 9 & 6 & 4 & 11 & 7 & 0 \end{pmatrix}$$

represents $\mathcal{H}^{even}(4,2,2)$. Finally, the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 10 & 9 & 8 & 6 & 3 & 5 & 1 & 4 & 7 & 0 \end{pmatrix}$$

represents $\mathcal{H}^{even}(6,2)$. Using these surfaces and the those given by the propositions proved in this section, we can produce 1,1-square-tiled surfaces in the even and odd components of all strata of Abelian differentials that have even order zeros. This completes the work of this section.

3.3 Odd order zeros

In this section, we will construct 1,1-square-tiled surfaces in all strata of Abelian differentials with odd order zeros. More specifically, if the strata is not connected, we will construct them in the nonhyperelliptic component. To do this, we must construct the base cases $\mathcal{H}(2j+1,2k+1)$. Similar to the difficulties caused by the hyperellipticity of

$\mathcal{H}(2)$, we must in this section also deal with the hyperellipticity of $\mathcal{H}(1,1)$.

Strata of the form $\mathcal{H}(2j+1, 2k+1)$

Before giving the proofs of the constructions in this section, we give examples of the methods used in Propositions 3.7 and 3.8 below. We will see how we can use the 1,1-square-tiled surfaces representing $\mathcal{H}^{odd}(4)$ and $\mathcal{H}^{odd}(6)$ constructed in Proposition 3.3 to construct 1,1-square-tiled surfaces in $\mathcal{H}(5,3)$, $\mathcal{H}(5,5)$, $\mathcal{H}(7,3)$ and $\mathcal{H}(9,3)$.

To construct a 1,1-square-tiled surface in $\mathcal{H}(5,3)$, we begin with two copies of the 1,1-square-tiled surface in $\mathcal{H}^{odd}(4)$ given by Proposition 3.3 as in the top of Figure 3.13. We then perform cylinder concatenation on these two surfaces to obtain the 1,1-square-tiled surface shown in the middle of Figure 3.13. Finally, we swap the square that was the first square of the second surface - the sixth square now - with the square to its right - now the seventh square. The resulting surface is still a 1,1-square-tiled surface and it can be checked that this surface lies in $\mathcal{H}(5,3)$. If we instead perform this operation with the first of the two surfaces being the 1,1-square-tiled surface in $\mathcal{H}^{odd}(6)$ given by Proposition 3.3 then it can be checked that the resulting surface lies in $\mathcal{H}(5,5)$. The investigation of the outcomes of this procedure in general is the content of Proposition 3.7.

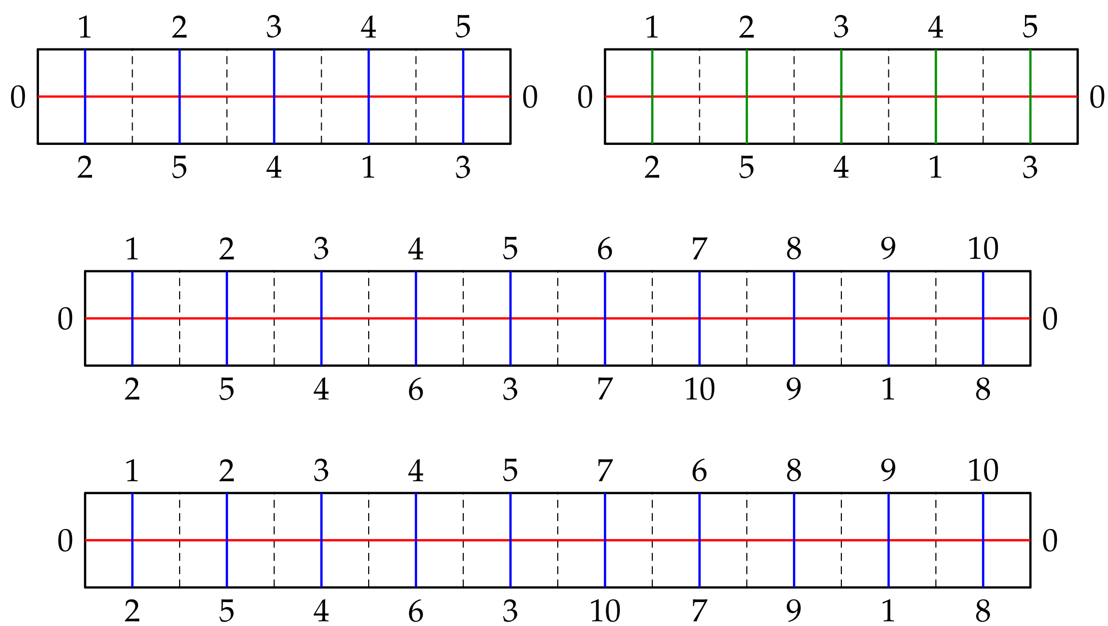

Figure 3.13: Construction of a 1,1-square-tiled surface in $\mathcal{H}(5,3)$.

Alternatively, if we start with the first surface being the 1,1-square-tiled surface in $\mathcal{H}^{odd}(6)$ and the second surface being the 1,1-square-tiled surface in $\mathcal{H}^{odd}(4)$, and after performing the cylinder concatenation instead swap the square that was the first square on the second surface with the square to its left, then we obtain a 1,1-square-tiled surface in $\mathcal{H}(7,3)$. Performing this operation with both surfaces being the 1,1-square-tiled surface in $\mathcal{H}^{odd}(6)$ results in a 1,1-square-tiled surface in $\mathcal{H}(9,3)$. The investigation of the outcomes of this procedure in general is the content of Proposition 3.8.

Proposition 3.7. *Suppose, for $j, k \geq 1$,*

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 4j-1 & 4j & 4j+1 \\ 2 & 5 & 4 & \cdots & 1 & \cdots & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 4k-1 & 4k & 4k+1 \\ 2 & 5 & 4 & \cdots & 1 & \cdots & 0 \end{pmatrix}$$

are, respectively, the permutation representatives for $\mathcal{H}^{odd}(4j)$ and $\mathcal{H}^{odd}(4k)$ given by Proposition 3.3. Then the permutation

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 4j-1 & 4j & 4j+1 & 4j+3 & 4j+2 \\ 2 & 5 & 4 & \cdots & 4j+2 & \cdots & 4j+6 & 4j+3 & 4j+5 \end{pmatrix}$$
$$\begin{pmatrix} \cdots & 4j+4k & 4j+4k+1 & 4j+4k+2 \\ \cdots & 1 & \cdots & 0 \end{pmatrix} \tag{3.11}$$

represents $\mathcal{H}(2(j+k)+1, 2(j+k)-1)$. Recall that this stratum is nonempty and connected. Moreover, if

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 4j+1 & 4j+2 & 4j+3 \\ 2 & 5 & 4 & \cdots & 1 & 4j+2 & 0 \end{pmatrix}$$

is the permutation representative for $\mathcal{H}^{odd}(4j+2)$ given by Proposition 3.3, then the permutation

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 4j+1 & 4j+2 & 4j+3 & 4j+5 & 4j+4 \\ 2 & 5 & \cdots & \cdots & 4j+4 & 4j+2 & 4j+8 & 4j+5 & 4j+7 \end{pmatrix}$$
$$\begin{pmatrix} \cdots & 4j+4k+2 & 4j+4k+3 & 4j+4k+4 \\ \cdots & 1 & \cdots & 0 \end{pmatrix} \tag{3.12}$$

represents $\mathcal{H}^{nonhyp}(2(j+k)+1, 2(j+k)+1)$. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

Proof. Permutations (3.11) and (3.12) are obtained by applying Lemma 2.1 to the permutations representing $\mathcal{H}^{odd}(4j)$ and $\mathcal{H}^{odd}(4k)$, and $\mathcal{H}^{odd}(4j+2)$ and $\mathcal{H}^{odd}(4k)$, respectively, and then, in the permutation with the 0s removed, permuting the columns with top entries $4j+2$ and $4j+3$, or the columns with top entries $4j+4$ and $4j+5$, respectively, before adding the 0s back in.

A more useful way of viewing this process is to use filling pair diagrams. The filling pair diagram of a surface produced as in Lemma 2.1 is the end to end concatenation of the filling pair diagrams of the constituent surfaces where the edge that would have returned to the top of vertex 1 on the filling pair diagram of the first surface is connected to the top of what was vertex 1 on the second surface and vice versa. After this, the column swap corresponds to a vertex transposition of the vertices that were the first two vertices of the filling pair diagram of the second surface. We will keep track of the boundary components of the associated ribbon graph to determine the stratum of the resulting surface.

The boundary components of the ribbon graph associated to the filling pair diagram of the surface obtained by Lemma 2.1 are shown in Figure 3.14. We read the diagram as follows. The two boundary components are represented by different line types. Following the orientation designated by the arrows, one counts the sides of the boundary components by starting at the outward arrow labelled by 1. We count this outgoing side. Then we continue to the next side labelled by 1 and of the same line type. If an incoming side has the same orientation (vertical or horizontal) as the outgoing side with the same label, then we do no count this incoming side, otherwise we do. We then add on the number of sides shown in brackets next to this incoming side. These numbers can be calculated by induction on the filling pair diagrams of Proposition 3.3. We continue to count sides until we reach the next outgoing side and repeat as above. This continues until we return to where we started, that is, the outgoing side labelled by 1.

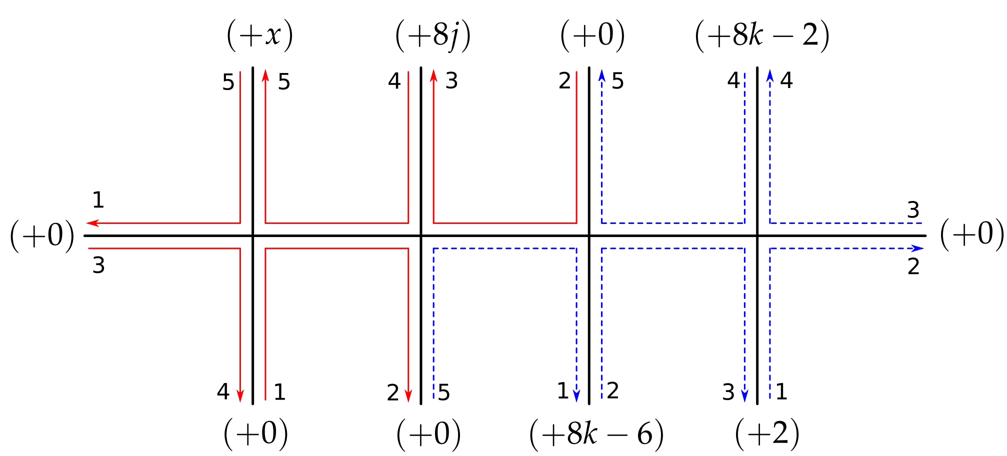

Figure 3.14: Boundary components of the ribbon graph before vertex transposition.

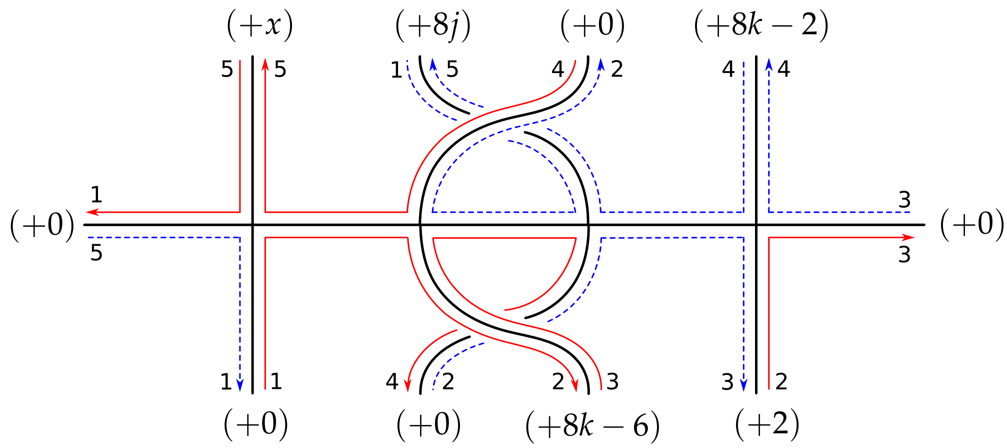

Figure 3.15: Boundary components of the ribbon graph after vertex transposition.

Note that in the diagram we have

$$x = \begin{cases} 8j-6, & \text{for } \mathcal{H}^{odd}(4j), \\ 8j+2, & \text{for } \mathcal{H}^{odd}(4j+2). \end{cases}$$

It is then easy to see that the boundary components do give rise to zeros of the correct orders.

The effect of the vertex transposition on the boundary components of the ribbon graph associated to the filling pair diagram is shown in Figure 3.15. We see that we have one boundary component with $8(j+k)+8$ sides corresponding to a zero of order $2(j+k)+1$, and a second boundary component with $8(j+k)$ sides corresponding to a zero of order $2(j+k)-1$, if $x = 8j-6$, or $8(j+k)+8$ sides corresponding to a zero of order $2(j+k)+1$, if $x = 8j+2$. As we have already shown that a 1,1-square-tiled surface in the hyperelliptic component of the stratum $\mathcal{H}(g-1, g-1)$ requires strictly more than the minimum numbers of squares. It is clear that the 1,1-square-tiled surfaces we have produced representing $\mathcal{H}(2(j+k)+1, 2(j+k)+1)$ are in the nonhyperelliptic component since these surfaces have the minimum number of squares required for their respective strata which completes the proof of the proposition. □

Choosing $k = 1$ in the above proposition, then choosing any $j \geq 1$ and applying the above construction using $\mathcal{H}^{odd}(4j)$ gives us 1,1-square-tiled surfaces in the strata $\mathcal{H}(2j+3, 2j+1)$, for $j \geq 1$. If instead we apply the above construction using $\mathcal{H}^{odd}(4j+2)$ then we produce 1,1-square-tiled surfaces in the nonhyperelliptic components of the strata $\mathcal{H}(2j+3, 2j+3)$, for $j \geq 1$. The following permutation, not produced by the above proposition, represents a 1,1-square-tiled surface in $\mathcal{H}^{nonhyp}(3,3)$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 8 & 6 & 5 & 7 & 4 & 1 & 3 & 0 \end{pmatrix}.$$

Proposition 3.8. *Suppose, for $j, k \geq 1$,*

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 4j+1 & 4j+2 & 4j+3 \\ 2 & 5 & 4 & \cdots & 1 & 4j+2 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 4k-1 & 4k & 4k+1 \\ 2 & 5 & 4 & \cdots & 1 & \cdots & 0 \end{pmatrix}$$

are, respectively, the permutation representatives for $\mathcal{H}^{odd}(4j+2)$ and $\mathcal{H}^{odd}(4k)$ given by Proposition 3.3. Then the permutation

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 4j+1 & 4j+2 & 4j+4 & 4j+3 & 4j+5 \\ 2 & 5 & 4 & \cdots & 4j+4 & 4j+5 & 4j+2 & 4j+8 & 4j+7 \\ & & & & & \cdots & 4j+4k+2 & 4j+4k+3 & 4j+4k+4 \\ & & & & & \cdots & 1 & \cdots & 0 \end{pmatrix} \tag{3.13}$$

represents $\mathcal{H}(2(2k+j)+1, 2j+1)$. Recall that this stratum is nonempty and connected. Moreover, if

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 4k+1 & 4k+2 & 4k+3 \\ 2 & 5 & 4 & \cdots & 1 & 4k+2 & 0 \end{pmatrix}$$

is the permutation representative for $\mathcal{H}^{odd}(4k+2)$ given by Proposition 3.3, then the permutation

$$\begin{pmatrix} 0 & 1 & 2 & \cdots & 4j+1 & 4j+2 & 4j+4 & 4j+3 & 4j+5 \\ 2 & 5 & 4 & \cdots & 4j+4 & 4j+5 & 4j+2 & 4j+8 & 4j+7 \\ & & & & & \cdots & 4j+4k+4 & 4j+4k+5 & 4j+4k+6 \\ & & & & & \cdots & 1 & 4j+4k+5 & 0 \end{pmatrix} \tag{3.14}$$

represents $\mathcal{H}(2(2k+j)+3, 2j+1)$. Recall that this stratum is nonempty and connected also. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

Proof. The proof is similar to the proof of Proposition 3.7. Indeed, permutations (3.13) and (3.14) are obtained by applying the cylinder concatenation of Lemma 2.1 to the permutations representing $\mathcal{H}^{odd}(4j+2)$ and $\mathcal{H}^{odd}(4k)$, and $\mathcal{H}^{odd}(4j+2)$ and $\mathcal{H}^{odd}(4k+2)$, respectively, and then, in the permutation with the 0s removed, permuting the columns with top entries $4j+3$ and $4j+4$, before adding the 0s back in.

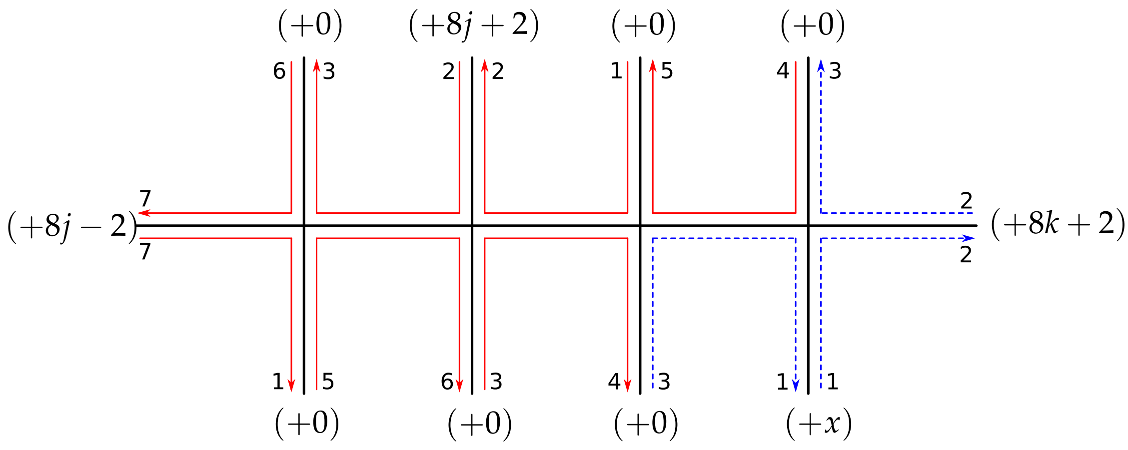

Figure 3.16: Boundary components of the ribbon graph before vertex transposition.

As above, a more useful way of viewing this process is to use filling pair diagrams. In this case, the column swap corresponds to a vertex transposition of the vertices that were the last vertex of the filling pair diagram of the first surface and the first vertex of the filling pair diagram of the second. We will again keep track of the boundary components of the associated ribbon graph.

The boundary components of the ribbon graph associated to the filling pair diagram of the surface obtained by Lemma 2.1 are shown in Figure 3.16. We read the diagram as in the proof of Proposition 3.7. Note that in the diagram

$$x = \begin{cases} 8k-2, & \text{for } \mathcal{H}^{odd}(4k), \\ 8k+6, & \text{for } \mathcal{H}^{odd}(4k+2). \end{cases}$$

It is then easy to see that the boundary components do give rise to zeros of the correct orders.

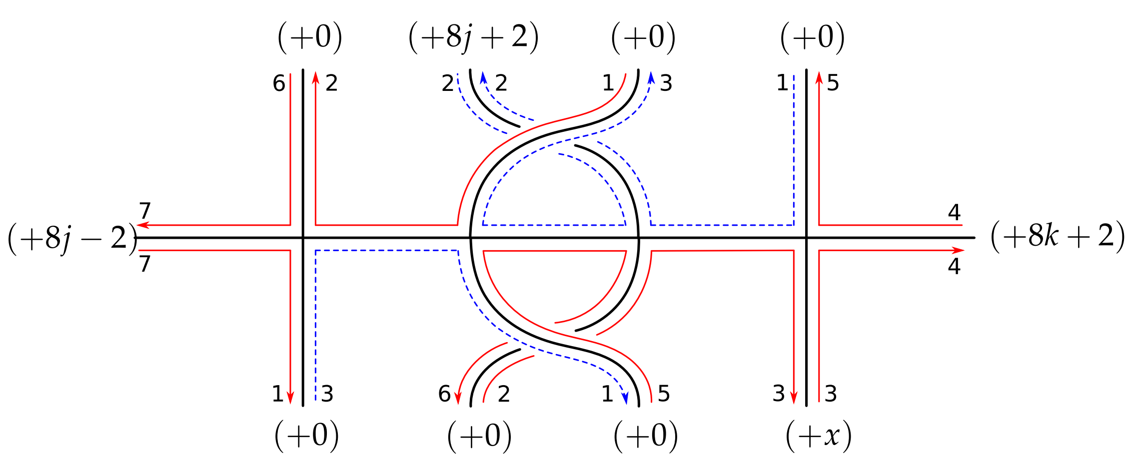

Figure 3.17: Boundary components of the ribbon graph before vertex transposition.

The effect of the vertex transposition on the boundary components of the ribbon graph associated to the filling pair diagram is shown in Figure 3.17. We see that we have one boundary component with $8j+8$ sides corresponding to a zero of order $2j+1$, and a second boundary component with $8(2k+j)+8$ sides corresponding to a zero of order $2(2k+j)+1$, if $x=8k-2$, or $8(2k+j)+16$ sides corresponding to a zero of order $2(2k+j)+3$, if $x=8k+6$. It is easy to check that these surfaces have the minimum number of squares required for their respective strata which completes the proof of the proposition. □

Observe that we have

$$2(2k+j)+1-(2j+1)=4k \quad \text{and} \quad 2(2k+j)+3-(2j+1)=4k+2,$$

and so, since we have $j,k \geq 1$, the above proposition allows us to construct 1,1-square-tiled surfaces in the strata $\mathcal{H}(2j+1+2n, 2j+1)$, for $j \geq 1$ and $n \geq 2$.

We have yet to construct 1,1-square-tiled surfaces in strata with zeros of order 1. We first construct such surfaces in strata with a pair of odd order zeros, only one of which is a zero of order 1.

Proposition 3.9. *The permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 5 & 4 & 9 & 3 & 8 & 6 & 1 & 10 & 7 & 0 \end{pmatrix} \tag{3.15}$$

and, for $k \geq 4$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots & 2k-4 & 2k-3 \\ 2 & 5 & 4 & 7 & 3 & 9 & 6 & 11 & 8 & 13 & \cdots & 2k-4 & 2k+3 \end{pmatrix.$$
$$\begin{matrix} 2k-2 & 2k-1 & 2k & 2k+1 & 2k+2 & 2k+3 & 2k+4 \\ 2k-2 & 2k+2 & 2k & 1 & 2k+4 & 2k+1 & 0 \end{matrix}\Bigg) \tag{3.16}$$

represent 1,1-square-tiled surfaces in $\mathcal{H}(7,1)$ and $\mathcal{H}(2k+1,1)$, respectively. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

Proof. The proof is similar to the proof of Proposition 3.5. Indeed, as before, we construct permutations (3.15) and (3.16) by adding 5 vertices to the right-hand sides of the filling pair diagrams representing permutations (3.3) and (3.4) of Proposition 3.3. Indeed, Figure 3.18 is the filling pair diagram representing permutation (3.15).

The 5 vertices we add to the filling pair diagram add 10 edges to the diagram and 20 sides to the boundary components of the associated ribbon graph. We observe that 8 of these sides form a single boundary component and correspond to a zero of order 1. The remaining 12 sides are added to the boundary component of the ribbon graph of the original filling pair diagram. That is, we increase the order of the associated zero by 3. It is easy to check that this is the order claimed in the statement of the proposition. Note also that all surfaces have the minimum number of squares required for square-tiled surfaces in their respective strata. □

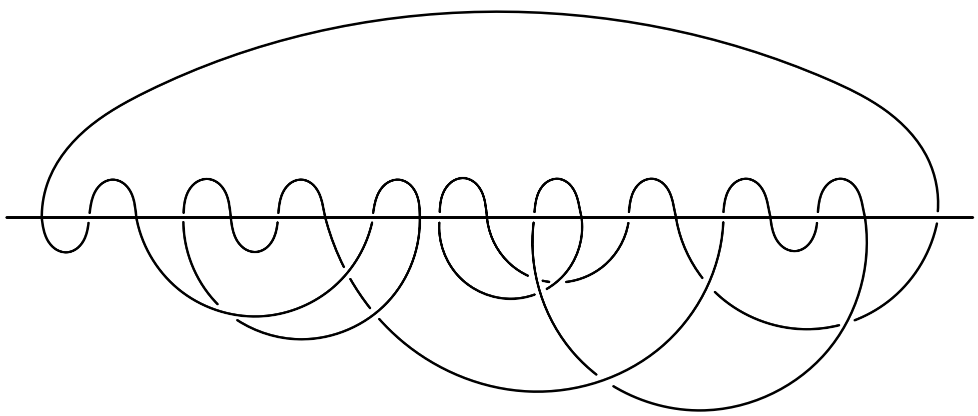

Figure 3.18: Filling pair diagram representing permutation (3.15)

The strata $\mathcal{H}(3,1)$ and $\mathcal{H}(5,1)$ are not covered by this proposition however the permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 1 & 6 & 4 & 3 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 7 & 3 & 1 & 8 & 6 & 5 & 0 \end{pmatrix}$$

represent 1,1-square-tiled surfaces in $\mathcal{H}(3,1)$ and $\mathcal{H}(5,1)$, respectively.

Handling the hyperellipticity of $\mathcal{H}(1,1)$

As in the previous section, the hyperellipticity of genus two again causes us difficulty. In this case, we have no 1,1-square-tiled surface in $\mathcal{H}(1,1)$ that we can use to build 1,1-square-tiled surfaces with the minimum number of squares. Observe that 1,1-square-tiled surfaces in the strata $\mathcal{H}(1,1,1,1)$ and $\mathcal{H}(1,1,1,1,1,1)$ are represented by the permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 6 & 5 & 3 & 1 & 8 & 4 & 7 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 8 & 1 & 5 & 11 & 7 & 3 & 10 & 6 & 12 & 9 & 4 & 0 \end{pmatrix},$$

respectively.

Suppose now that we wish to build a 1,1-square-tiled surface in a stratum with odd order zeros, including zeros of order 1. If we have four or more zeros of order 1, then we can build a 1,1-square-tiled surface from the surfaces given by the permutations in previous paragraph and from those constructed so far in this section. Moreover, if the stratum has one or two zeros of order 1 then we can again build a 1,1-square-tiled surface in such a stratum from the surfaces we have constructed in this section. This is also the case for a stratum with three zeros of order 1 and at least three other zeros. Indeed, in all such cases the zeros of order 1 that cannot be produced using the permutations of the previous paragraph can be paired with zeros of higher order and so the surface can be constructed from the surfaces we have already produced in this section.

We see then that the only strata with odd order zeros in which we cannot construct 1,1-square-tiled surfaces are $\mathcal{H}(2g-5,1,1,1)$, $g \geq 4$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 10 & 6 & 5 & 1 & 8 & 4 & 7 & 3 & 9 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 12 & 9 & 8 & 1 & 7 & 3 & 6 & 10 & 5 & 4 & 11 & 0 \end{pmatrix},$$

represent 1,1-square-tiled surface in $\mathcal{H}(3,1,1,1)$ and $\mathcal{H}(5,1,1,1)$, respectively. The remaining cases are constructed in the following proposition.

As mentioned in the introduction, the permutation representatives given in Proposition 3.10 below are examples demonstrating the complexity of finding 1,1-square-tiled surfaces by using a sequence of Rauzy moves on the permutation representatives given by Zorich. Indeed, these general forms were discovered by such a method and the sequences of Rauzy moves required, and hence the resulting permutation representatives, differed depending on the residue of $2g-5$ modulo 4. As such, we would not expect to be able to find a general method giving a sequence of Rauzy moves for an arbitrary stratum.

Proposition 3.10. *For $k \geq 1$, the permutation*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 4k+3 \\ 2 & 4k+10 & 4k+6 & 4k+5 & 1 & 4 & 3 & 6 & 5 & \cdots & 4k+2 \end{pmatrix}$$
$$\begin{matrix} 4k+4 & 4k+5 & 4k+6 & 4k+7 & 4k+8 & 4k+9 & 4k+10 \\ 4k+1 & 4k+8 & 4k+4 & 4k+7 & 4k+3 & 4k+9 & 0 \end{matrix} \tag{3.17}$$

represents a 1,1-square-tiled surface in the stratum $\mathcal{H}(4k+3,1,1,1)$. For $k \geq 2$, the permuta-

tion

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots & 4k-1 & 4k \\ 2 & 4k+8 & 4k+5 & 4k+4 & 1 & 4 & 3 & 6 & 5 & \cdots & 4k-2 & 4k-3 \end{pmatrix}$$

$$\begin{matrix} 4k+1 & 4k+2 & 4k+3 & 4k+4 & 4k+5 & 4k+6 & 4k+7 & 4k+8 \\ 4k+3 & 4k-1 & 4k+2 & 4k+6 & 4k+1 & 4k & 4k+7 & 0 \end{matrix} \qquad (3.18)$$

represents a 1,1-square-tiled surface in $\mathcal{H}(4k+1,1,1,1)$. All such surfaces achieve the minimal number of squares for their respective strata.

Proof. We prove the proposition for permutation (3.17) above. The proof for permutation (3.18) is similar. It is easy to check that the permutation does indeed represent the stratum $\mathcal{H}(4k+3,1,1,1)$. It also has a single horizontal cylinder. Removing the 0s from permutation (3.17) gives a single cycle as follows:

$$1 \to 2 \to 4k+10 \to 4k+9 \to 4k+3 \to 4k-1 \to 4k-5 \to \cdots$$

$$\to 3 \to 4k+6 \to 4k+8 \to 4k+7 \to 4k+4 \to 4k+2 \to 4k \to \cdots$$

$$\to 4 \to 4k+5 \to 4k+1 \to 4k-3 \to \cdots \to 5 \to 1.$$

Since we have a single cycle containing every symbol in the permutation, the surface has a single vertical cylinder and so we do indeed have a 1,1-square-tiled surface.

Observe also that the permutations represent square-tiled surfaces with the minimum number of squares necessary for their respective strata. □

This proposition now completes the construction of 1,1-square-tiled surfaces in all strata with zeros of odd order.

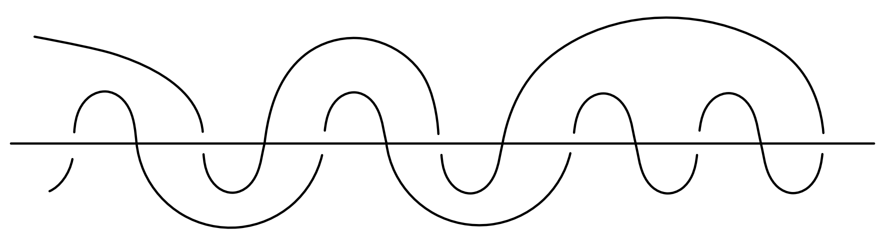

Figure 3.19: Filling pair diagram combinatorics for adding $\mathcal{H}(1,1)$.

Since similar methods will be used in several places in Chapter 6, we describe an alternative method of constructing 1,1-square-tiled surfaces in $\mathcal{H}(2g-5,1,1,1)$. In particular, we will use a method similar to that used to produce the 1,1-square-tiled surfaces in the proofs of Propositions 3.5 and 3.6. Indeed, one can check that if we add the vertices shown in Figure 3.19 to the right-hand side of the filling pair diagrams associated to the 1,1-square-tiled surfaces in $\mathcal{H}(2k+1,1)$ constructed above, then we obtain 1,1-square-tiled surfaces in $\mathcal{H}(2k+3,1,1,1)$. That is, we add two zeros of order 1 to the surface at the expense of increasing the order of the zero of order $2k+1$ by two. The reason that only the zero of order $2k+1$ is increased in order is discussed in the considerations of adding a zero of order 2 in the following section.

3.4 General strata

In this section, we complete the proof of Theorem 1.4 by constructing 1,1-square-tiled surfaces in general strata; that is, in strata with both even and odd order zeros. Recall that such strata are connected.

Using the surfaces we have constructed in the previous two sections, by Lemma 2.1, we can construct 1,1-square-tiled surfaces with the minimum number of squares in all strata except those whose only even order zero is of order 2, or those who have only two odd order zeros both of order 1. Again, this difficulty is caused by the hyperellipticity of genus two surfaces.

In the former case, if we can construct 1,1-square-tiled surfaces in all strata with two odd order zeros and a zero of order 2 then we can use the surfaces we constructed in the previous section to complete this case. We have one exception, that being $\mathcal{H}(2,1,1,1,1)$ however a 1,1-square-tiled surface in this stratum is represented by the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 7 & 11 & 6 & 3 & 9 & 5 & 1 & 8 & 4 & 10 & 0 \end{pmatrix}.$$

In the latter case, if we can construct 1,1-square-tiled surfaces in the strata $\mathcal{H}(2k,1,1)$, $k \geq 1$, then we can use surfaces we have already constructed to complete the remaining cases.

We will solve the latter case first because it is much simpler. This is the content of the following proposition.

Proposition 3.11. *For $k \geq 1$, the permutation*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & 4k+2 \\ 2 & 6 & 4 & 1 & 8 & 7 & 10 & 9 & \cdots & 4k+6 \end{pmatrix. \qquad (3.19)$$

$$\begin{matrix} 4k+3 & 4k+4 & 4k+5 & 4k+6 & 4k+7 \\ 4k+5 & 4k+7 & 5 & 3 & 0 \end{pmatrix}$$

represents a 1,1-square-tiled surface in $\mathcal{H}(4k+2,1,1)$. For $k \geq 2$, the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots & 4k-2 & 4k-1 \\ 2 & 7 & 4 & 1 & 9 & 8 & 11 & 10 & \cdots & 4k+3 & 4k+2 \end{pmatrix. \tag{3.20}$$

$$\begin{matrix} 4k & 4k+1 & 4k+2 & 4k+3 & 4k+4 & 4k+5 \\ 4k+5 & 5 & 4k+4 & 6 & 3 & 0 \end{pmatrix}$$

represents a 1,1-square-tiled surfaces in the stratum $\mathcal{H}(4k,1,1)$. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

Proof. We prove the proposition for permutation (3.19) above. The proof for permutation (3.20) is analogous. It is easy to check that the permutation does indeed represent the stratum $\mathcal{H}(4k+2,1,1)$. It clearly has a single horizontal cylinder. Removing the 0s from the permutation gives a single cycle as follows:

$$1 \to 2 \to 6 \to 7 \to 10 \to 11 \to \cdots$$

$$\to 4k+3 \to 4k+6 \to 5 \to 8 \to 9 \to 12 \to \cdots$$

$$\to 4k+4 \to 4k+5 \to 4k+7 \to 3 \to 4 \to 1.$$

Since we have a single cycle containing every symbol in the permutation, the surface has a single vertical cylinder and so does indeed represent a 1,1-square-tiled surface.

Observe also that the permutations represent square-tiled surfaces with the minimum number of squares required for their respective strata. □

We note that the permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 6 & 4 & 1 & 7 & 5 & 3 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 7 & 4 & 1 & 9 & 5 & 8 & 6 & 3 & 0 \end{pmatrix}$$

represent 1,1-square-tiled surfaces in $\mathcal{H}(2,1,1)$ and $\mathcal{H}(4,1,1)$, respectively. This solves the latter of the exceptional cases discussed above. A final exceptional case is the stratum $\mathcal{H}(2,2,1,1)$ which is represented by the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 4 & 9 & 7 & 3 & 8 & 5 & 1 & 10 & 6 & 0 \end{pmatrix}.$$

We remark that we could have also produced 1,1-square-tiled surfaces in the strata considered in Proposition 3.20 by adding the combinatorics of Figure 3.19 to the filling pair diagrams associated to the 1,1-square-tiled surfaces in $\mathcal{H}^{odd}(2k)$ that we produced in Proposition 3.3.

To construct 1,1-square-tiled surfaces in strata with two odd order zeros and one zero of order 2, we will revisit the technique we used in Propositions 3.5 and 3.6. That is, the addition of the 5 vertices in Figure 3.11 to the right-hand side of the filling pair diagram to produce the zero of order 2. However, we must proceed with more care than we did in the proof of Proposition 3.5 as, when we add these vertices to the filling pair diagram of a surface with two zeros, a phenomenon emerges that was not apparent in the case when we were adding these vertices to the filling pair diagram of a surface with a single zero. Indeed, this phenomenon explains why only the order of the zero of order $2k+1$ increased in the method of adding the combinatorics in Figure 3.19 to the filling pair diagrams for $\mathcal{H}(2k+1,1)$, as discussed at the end of the previous section.

Recall that when we added the 5 vertices to the filling pair diagram we added 20 sides to the boundary components of the associated ribbon graph, 12 of which formed a single boundary component associated to the zero of order 2 and the remaining 8 were added to the boundary component of the original ribbon graph increasing the order of the associated zero by two. The intricacy that arises in the case of adding these 5 vertices to a filling pair diagram whose ribbon graph has two boundary components is that the 8 sides that were added to the single boundary component before are added only to the boundary component that 'leaves' the right-hand side of filling pair diagram at the bottom. This can easily be seen by observing the combinatorics of the filling pair diagrams.

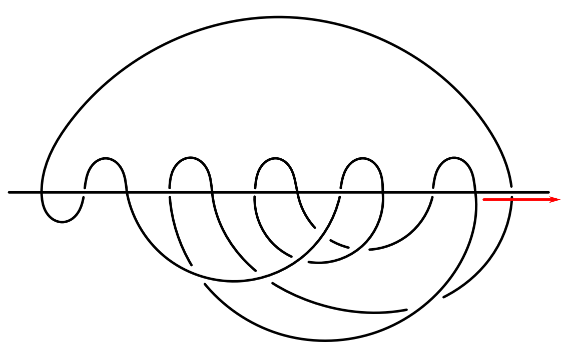

Figure 3.20: The boundary component corresponding to the zero of order 3 leaves the filling pair diagram at the bottom.

An example of this phenomenon is shown in Figures 3.20 and 3.21. We observe that the original filling pair diagram, shown in Figure 3.20, represents a 1,1-square-tiled surface in $\mathcal{H}(3,1)$ and that the boundary component that leaves the filling pair diagram on the bottom at the right is the boundary component corresponding to the zero of order 3. We would then expect that adding the 5 vertices to the filling pair diagram in order to add a zero of order 2 to the surface will increase the order of this zero by two and indeed one can see that the resulting filling pair diagram, shown in Figure 3.21, represents a 1,1-square-tiled surface in $\mathcal{H}(5,1,2)$.

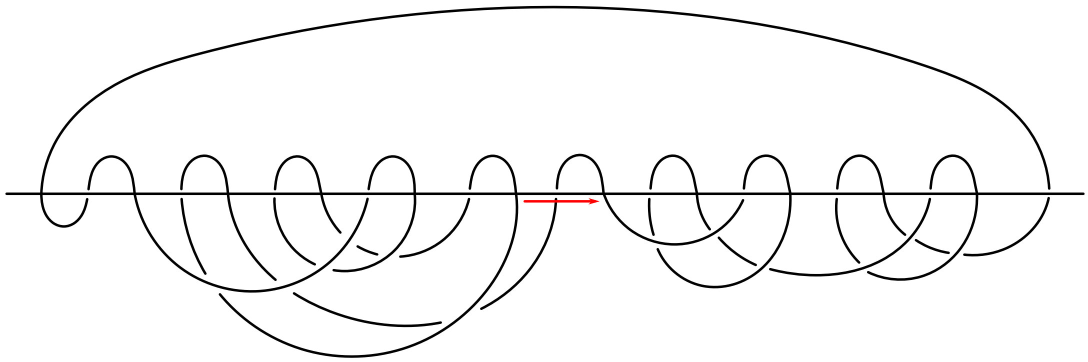

Figure 3.21: In the process of adding the zero of order 2, the 8 sides have been added to the boundary component that leaves on the bottom.

With this in mind, we need only keep track of which zero is associated to the boundary component that leaves on the bottom for the surfaces that we constructed in the previous section. It is easy to check that for the 1,1-square-tiled surfaces in the strata $\mathcal{H}(2k+1, 2k+1+2n)$, for $k \geq 1$ and $n \geq 2$, constructed in Proposition 3.8, that the boundary component that leaves on the bottom is the one associated to the zero of order $2k+1+2n$. Hence, we can construct 1,1-square-tiled surfaces in the strata $\mathcal{H}(2k+1, 2k+1+2n, 2)$, for $k \geq 1$ and $n \geq 3$. Using the 1,1-square-tiled surfaces in the strata $\mathcal{H}(2k+1, 2k+1)$, $k \geq 1$, constructed in (and after for $\mathcal{H}(3,3)$) Proposition 3.7, we can construct 1,1-square-tiled surfaces in the strata $\mathcal{H}(2k+3, 2k+1, 2)$, $k \geq 1$. Moreover, the 1,1-square-tiled surfaces in the strata $\mathcal{H}(2k+1, 1)$, $k \geq 1$, constructed in and after Proposition 3.9, have the boundary component that leaves on the bottom being the one associated to the zero of order $2k+1$, and so we can construct 1,1-square-tiled surfaces in the strata $\mathcal{H}(2k+3, 1, 2)$, $k \geq 1$. Finally, we observe that for the 1,1-square-tiled surfaces in the strata $\mathcal{H}(2k+3, 2k+1)$, $k \geq 1$, constructed in Proposition 3.7, the boundary component that leaves on the bottom is the one associated to the zero of order $2k+3$. Hence, we can construct 1,1-square-tiled surfaces in the strata $\mathcal{H}(2k+5, 2k+1, 2)$, $k \geq 1$.

The only strata not covered thus far are $\mathcal{H}(3,1,2)$ and $\mathcal{H}(2k+1, 2k+1, 2)$, $k \geq 1$. The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 6 & 8 & 3 & 7 & 4 & 1 & 9 & 5 & 0 \end{pmatrix}$$

represents a 1,1-square-tiled surface in the stratum $\mathcal{H}(3,1,2)$. Hence, we see that to complete the proof of Theorem 1.4 we must construct 1,1-square-tiled surfaces in the strata $\mathcal{H}(2k+3, 2k+1)$, $k \geq 0$, with the boundary component that leaves on the bottom being the one associated to the zero of order $2k+1$. Indeed, we would then be able to construct 1,1-square-tiled surfaces in the strata $\mathcal{H}(2k+3, 2k+3, 2)$, $k \geq 0$. This is completed by the following proposition and the permutation representative that follows.

Proposition 3.12. *The permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 6 & 4 & 10 & 8 & 3 & 1 & 9 & 7 & 5 & 0 \end{pmatrix}, \tag{3.21}$$

and, for $k \geq 2$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & \cdots & 4k+3 & 4k+4 & 4k+5 & 4k+6 \\ 2 & 6 & 4 & 10 & 8 & 3 & 12 & 9 & 7 & 5 & 14 & 11 & 16 & 13 & 18 & \cdots & 4k+3 & 1 & 4k+5 & 0 \end{pmatrix} \tag{3.22}$$

represent 1,1-square-tiled surfaces in $\mathcal{H}(5,3)$ and $\mathcal{H}(2k+3, 2k+1)$, respectively, with the boundary component that leaves the filling pair diagram on the bottom being associated to the zeros of order 3 and $2k+1$, respectively. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

Proof. A filling pair diagram representing permutation (3.21) is shown in Figure 3.22. It is easy to check that it represents a 1,1-square-tiled surface in $\mathcal{H}(5,3)$, and that, on the associated ribbon graph, the boundary component that leaves on the bottom is the one associated to zero of order 3.

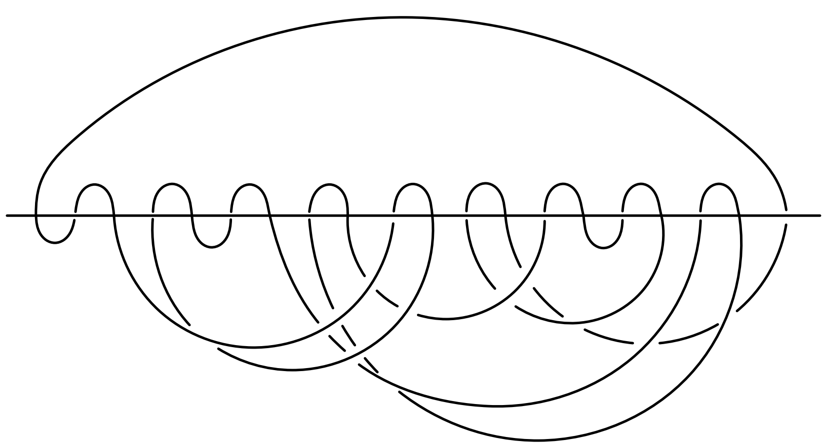

Figure 3.22: Filling pair diagram representing permutation (3.21)

Moreover, we see that a filling pair diagram representing permutation (3.22) is obtained from the filling pair diagram in Figure 3.22 by adding $4(k-1)$ vertices. These vertices add $16(k-1)$ sides to the boundary components of the associated ribbon graph in such a way that $8(k-1)$ sides are added to each boundary component. This corresponds to increasing the orders of the associated zeros by $2(k-1)$ and so the surface will have two zeros of orders $2(k-1)+3 = 2k+1$ and $2(k-1)+5 = 2k+3$. Hence, the surface is in $\mathcal{H}(2k+3, 2k+1)$, as claimed. It is also easy to check that after adding these vertices, the boundary component that leaves on the bottom is the one associated to the zero of order $2k+1$, as required. Moreover, these surfaces have the minimum number of squares necessary for their respective strata. □

Finally, for the only remaining case, we construct a 1,1-square-tiled surface in the stratum $\mathcal{H}(3,1)$ with the boundary component that leaves on the bottom being the one associated to the zero of order 1. Indeed, one can check that the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 5 & 1 & 4 & 3 & 0 \end{pmatrix}$$

represents such a surface. As such, the proof of Theorem 1.4 is complete.

Chapter 4

Ratio-optimising pseudo-Anosovs

This chapter contains the proof of Theorem 1.6. We begin by recalling some definitions related to Teichmüller space that we gave in more detail in the introduction.

4.1 Teichmüller preliminaries

Recall that the Teichmüller space $\mathcal{T}(S)$ of a closed surface S of genus $g \geq 2$ is the set of equivalence classes of pairs (X, φ), where X is a hyperbolic surface of genus g and $\varphi : S \to X$ is a homeomorphism. Two pairs (X_1, φ_1) and (X_2, φ_2) are equivalent if the change of marking map $\varphi_2 \circ \varphi_1^{-1}$ is isotopic to an isometry. As such, given an isotopy class $\boldsymbol{\alpha} = [\alpha]$ of an essential simple closed curve α on the surface S and a point $x \in \mathcal{T}(S)$, we can talk the length of $\boldsymbol{\alpha}$, $\ell_x(\boldsymbol{\alpha})$, in the hyperbolic metric determined by the point x. Teichmüller space carries a metric $d_{\mathcal{T}}$ called the Teichmüller metric and we denoted by $\mathcal{T}(S)$ the metric space $(\mathcal{T}(S), d_{\mathcal{T}})$. Given a pseudo-Anosov homeomorphism f, we defined the translation length of f on $\mathcal{T}(S)$ to be $\ell_{\mathcal{T}}(f) := \frac{1}{2}\log(K_f)$, where K_f is the dilatation of f.

We defined the curve graph $\mathcal{C}(S)$ of the surface S to be a graph whose vertices are isotopy classes of essential simple closed curves on the surface S, with two vertices joined by an edge if and only if they can be realised disjointly on S. Assigning length 1 to each edge, we equipped $\mathcal{C}(S)$ with the associated path metric $d_{\mathcal{C}}$ and denoted by $\mathcal{C}(S)$ the metric space $(\mathcal{C}(S), d_{\mathcal{C}})$. Given a pseudo-Anosov homeomorphism f, we defined the asymptotic translation length of f on $\mathcal{C}(S)$ to be

$$\ell_{\mathcal{C}}(f) := \liminf_{n\to\infty} \frac{d_{\mathcal{C}}(f^n(\boldsymbol{\alpha}), \boldsymbol{\alpha})}{n},$$

for any $\boldsymbol{\alpha} \in \mathcal{C}^0(S)$, which for a pseudo-Anosov is a strictly positive limit.

Recall that the $\mathrm{SO}(2,\mathbb{R})\backslash \mathrm{SL}(2,\mathbb{R})$-orbit of an Abelian differential (X, ω), or quadratic differential (X, q), gave an embedding of $\mathbb{H}$ into $\mathcal{T}(S)$. We called the image of this embedding the Teichmüller disk of the differential.

4.2 The systole map and Lipschitz constant

We now define the *systole map* $\mathrm{sys} : \mathcal{T}(S) \to \mathcal{C}(S)$ to be the coarsely-defined map that sends a point $x \in \mathcal{T}(S)$ to the isotopy class of the curve with shortest length in the hyperbolic metric determined by x, known as the systole. The map is only coarsely-defined as there can be multiple systoles on a surface, however the set of systoles of x is a set of diameter at most two in $\mathcal{C}(S)$. We will abuse notation and think of sys as a well-defined map. The study of this map played a key role in the work of Masur and Minsky in which they proved that the curve complex is δ-hyperbolic [40]. They showed in particular that the map is coarsely K-Lipschitz. That is, there exist $K > 0$ and $C \geq 0$ such that

$$d_{\mathcal{C}}(\mathrm{sys}(x), \mathrm{sys}(y)) \leq K \cdot d_{\mathcal{T}}(x, y) + C,$$

for all $x, y \in \mathcal{T}(S)$.

It is natural to ask what is the optimum Lipschitz constant, κ_g, defined by

$$\kappa_g := \inf\{K > 0 \mid \exists\, C \geq 0 \,\text{such that sys is coarsely K-Lipschitz}\},$$

and Gadre-Hironaka-Kent-Leininger determined that the ratio of κ_g to $1/\log(g)$ is bounded from above and below by two positive constants [23, Theorem 1.1]. Recall that, in such a case, we use the notation $\kappa_g \asymp 1/\log(g)$, and say that κ_g is *comparable* to $1/\log(g)$. To find an upper bound for κ_g, Gadre-Hironaka-Kent-Leininger gave a careful version of the proof of Masur-Minsky that sys is coarsely Lipschitz. They then constructed pseudo-Anosov homeomorphisms for which the ratio $\ell_{\mathcal{C}}(f)/\ell_{\mathcal{T}}(f) \asymp 1/\log(g)$, where $\ell_{\mathcal{C}}(f)$ and $\ell_{\mathcal{T}}(f)$ are the asymptotic translation lengths of f in $\mathcal{C}(S)$ and $\mathcal{T}(S)$, respectively. A lower bound for κ_g then followed by noting that, for any pseudo-Anosov homeomorphism f, we have

$$\kappa_g \geq \frac{\ell_{\mathcal{C}}(f)}{\ell_{\mathcal{T}}(f)}.$$

4.3 Constructing ratio-optimising pseudo-Anosov homeomorphisms

Using a Thurston construction on filling pairs, Aougab-Taylor constructed a larger family of pseudo-Anosov homeomorphisms for which $\tau(f) := \ell_{\mathcal{T}}(f)/\ell_{\mathcal{C}}(f)$ was bounded above by a function $F(g) \asymp \log(g)$ [4, Theorem 1.1]. Such homeomorphisms are said to be *ratio-optimising*. Moreover, they proved that there exists a Teichmüller disk $\mathcal{D} \cong \mathbb{H} \subset \mathcal{T}(S)$ such that there are infinitely many conjugacy classes of primitive ratio-optimising pseudo-Anosovs f with the invariant axis of f being contained in $\mathcal{D}$. Recall that a group element g of a group G is said to be *primitive* if there does not exist a $h \in G$ such that $g = h^k$ for $|k| > 1$.

To construct these pseudo-Anosovs, Aougab-Taylor began with a filling pair $\{\alpha, \beta\}$ on the surface S. They then took high powers, independent of the genus of the surface, of the Dehn twists about each curve and showed that the Bass-Serre tree of the free group generated by these elements quasi-isometrically embeds in $\mathcal{C}(S)$ [4, Proposition 3.1]. This then allowed them to bound the asymptotic translation length $\ell_{\mathcal{C}}$ of elements of this group in terms of their syllable length. Recall that the *syllable length* of a reduced word $w = a^{k_1}b^{k_2}\cdots a^{k_l}$ in terms of the two generators a and b is defined to be $|w|_s := l$. They also bounded $\ell_{\mathcal{T}}$ in terms of the syllable length of the element and the geometric intersection number, $i(\alpha, \beta)$, of the filling pair. From this they were able to deduce that, for pseudo-Anosov elements of this free group,

$$\tau(f) \leq \log(D \cdot i(\alpha, \beta)),$$

where D is a constant independent of the genus of S. Ratio-optimising pseudo-Anosovs were then constructed by using filling pairs for which $i(\alpha, \beta) \asymp g$.

Recall that a filling pair $\{\alpha, \beta\}$ on a surface S, with all intersections occurring with the same orientation, determines an Abelian differential on that surface. We will denote the Teichmüller disk of this Abelian differential by $\mathcal{D}(\alpha, \beta)$. The ratio-optimising pseudo-Anosovs produced from this filling pair will stabilise $\mathcal{D}(\alpha, \beta)$ and, moreover, their invariant axis will be contained in $\mathcal{D}(\alpha, \beta)$. Aougab-Taylor used the hyperbolicity of $\mathcal{C}(S)$ and the acylindricity of the action of $\mathrm{Mod}(S)$ on $\mathcal{C}(S)$ to show that in fact there are infinitely many conjugacy classes of primitive ratio-optimising pseudo-Anosovs constructed from this filling pair that have this property. We remark that their theorem deals with the general case of filling pairs that determine quadratic differentials on a punctured surface $S_{g,p}$. Here we are specialising to the case of Abelian differentials on closed surfaces.

4.4 Proof of Theorem 1.6

Fix g and let $\mathscr{C}$ be any connected component of any stratum of $\mathcal{H}$. By Theorem 1.4, we can find a 1,1-square-tiled surface in $\mathscr{C}$. The core curves, α and β, of the vertical and horizontal cylinders of this surface form a filling pair and so we can construct pseudo-Anosovs from this filling pair using the above technique of Aougab-Taylor. For any such pseudo-Anosov, we have

$$\tau(f) \leq \log(D \cdot i(\alpha, \beta)) \leq \log(D \cdot (4g-2)) \asymp \log(g),$$

since the greatest number of squares required for a 1,1-square-tiled surface of genus g, and so the greatest intersection number of the associated filling pair, is given by the connected component $\mathcal{H}^{hyp}(g-1, g-1)$ which requires $4g-2$ squares. Hence we have that the pseudo-Anosovs are ratio-optimising. Moreover, as above, we have infinitely many conjugacy classes of primitive ratio-optimising pseudo-Anosovs having their invariant axis contained in the Teichmüller disk determined by this 1,1-square-tiled surface. As such, we have completed the proof of Theorem 1.6.

Note that this extends the abundance result of Aougab-Taylor. That is, not only are there infinitely many conjugacy classes of primitive ratio-optimising pseudo-Anosovs in a Teichmüller disk of $\mathcal{T}(S)$ but this Teichmüller disk can be taken to be the Teichmüller disk of an Abelian differential from any connected component of any stratum of $\mathcal{H}$.

For a discussion of the applications of 1,1-pillowcase covers, the quadratic analogue of 1,1-square-tiled surfaces, to the construction of ratio-optimisers we refer the reader to Section 7.2.

Chapter 5

Filling pairs on punctured surfaces

In this chapter we present our results relating to filling pairs on punctured surfaces. That is, we will prove Theorem 1.7 and Theorem 1.8. We arrived at the Theorem 1.7 while investigating whether one could use the generalised filling permutations introduced in Subsection 5.1.1 to build 1,1-square-tiled surfaces. We discovered that these would not be adequate for our purposes but were able to complete the determination of the minimal geometric intersection numbers for filling pairs on punctured surfaces. Theorem 1.8, proved in Section 5.2, arose as a natural observation of the properties of the filling pairs given by 1,1-square-tiled surfaces.

5.1 Minimally intersecting filling pairs on the punctured surface of genus two

Let $S_{g,p}$ be an orientable surface of genus g with p punctures. Recall that a pair of essential simple closed curves in minimal position on the surface $S_{g,p}$ is a filling pair if the complement of their union is a disjoint collection of disks and once punctured disks. We define $i_{g,p}$ to be the minimal geometric intersection number of a filling pair on the surface $S_{g,p}$. For closed surfaces, Aougab-Huang calculated the values of $i_{g,0}$ in order to count the number of mapping class group orbits of minimally intersecting filling pairs [1, Theorem 1.1]. This count then allowed them to estimate the growth rate of the number of global minima of a topological Morse function they defined on the moduli space of Riemann surfaces of genus g [1, Theorem 1.3].

Aougab-Taylor extended the calculations of $i_{g,p}$ to certain cases of $p \neq 0$ [3, Lemma 3.1]. This allowed them to construct geodesic rays in the curve graph realising an optimal intersection property [3, Theorem 1.2], answering a question of Margalit. Moreover, the filling pairs realising these values of $i_{g,p}$ enabled Aougab-Taylor to construct the ratio-optimising pseudo-Anosov homeomorphisms discussed in Chapter 4.

The values of $i_{g,p}$ determined so far can be summarised as follows.

Theorem ([1, Section 2], [3, Lemma 3.1]). *The values of $i_{g,p}$ are the following:*

(1) *If $g \neq 2, 0$ and $p = 0$, then $i_{g,p} = 2g - 1$;*

(2) *If $g \neq 2, 0$ and $p \geq 1$, then $i_{g,p} = 2g + p - 2$;*

(3) *If $g = 0$ and $p \geq 6$ is even, then $i_{g,p} = p - 2$, and if $p \geq 5$ is odd then $i_{g,p} = p - 1$;*

(4) *If $g = 2$ and $p \leq 2$, then $i_{g,p} = 4$;*

(5) *If $g = 2$ and $p \geq 2$ is even, then $i_{g,p} = 2g + p - 2$. Otherwise, if $p \geq 3$ is odd, then $2g + p - 2 \leq i_{g,p} \leq 2g + p - 1$.*

In this section, we complete the list by demonstrating the following.

Theorem 5.1. *Let $g = 2$ and $p \geq 3$ be odd, then $i_{g,p} = 2g + p - 2$.*

That is, we construct filling pairs on $S_{2,p}$ that realise the lower bound in part (5) of the theorem above. To prove the existence of such filling pairs, we generalise the construction of filling permutations given by Nieland in unpublished work [46, Theorem 2.1], which are themselves generalisations of the filling permutations introduced by Aougab-Huang [1, Lemma 2.2]. We use these to produce a minimally intersecting filling pair on $S_{2,3}$, and then apply the double-bigon inductive method used by Aougab-Taylor [3, Proof of Lemma 3.1] to extend to all odd $p \geq 3$.

5.1.1 Generalised filling permutations

We will make use of a generalisation, extending the construction given by Nieland [46, Theorem 2.1], of the filling permutations used by Aougab-Huang in their determinations of $i_{g,0}$ [1, Lemma 2.2]. The notation and result that follows is a simple extension of the works of Aougab-Huang and Nieland, however our generalised construction allows us to work with filling pairs on punctured surfaces.

Let $\{\alpha, \beta\}$ be a filling pair on the surface $S_{g,p}$ and let $n = i(\alpha, \beta)$. Fix orientations for the curves α and β and choose one of the intersection points $x \in \alpha \cap \beta$. Starting at x, and following the orientation of α, number the arcs of α between consecutive intersection points in order to obtain the set $\{\alpha_1, \ldots, \alpha_n\}$. Similarly, and possibly choosing a different intersection point $y \in \alpha \cap \beta$, construct the set $\{\beta_1, \ldots, \beta_n\}$. Let the set $A = A_{\alpha,\beta}$ be defined by

$$A := \{\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n, \alpha_1^{-1}, \beta_1^{-1}, \ldots, \alpha_n^{-1}, \beta_n^{-1}\},$$

and identify this set with the set $\{1, 2, \ldots, 4n\}$.

We now define a *filling permutation* $\sigma = \sigma_{\alpha,\beta} \in \Sigma_{4n}$ associated to the filling pair $\{\alpha, \beta\}$ as follows. Firstly, cut $S_{g,p}$ along $\alpha \cup \beta$ to form a collection of $n+2-2g$ many polygons with sides labelled by α and β. Orienting the polygons clockwise, we obtain a labelling of the sides of the polygons by the elements of A. We now define σ in the following way. If, going around the sides of the polygons in a clockwise direction, the edge labelled by the j^{th} element of A is followed by the edge labelled by the k^{th} element of A, then we define $\sigma(j) = k$. We see then that σ is an element of the symmetric group Σ_{4n} consisting of $n+2-2g$ many cycles.

We will also be interested in two more elements of Σ_{4n} that have geometric significance. Firstly, we define the permutation $Q = Q_{\alpha,\beta}$ by

$$Q = (1, 2, \ldots, 4n)^{2n}.$$

Observe that this permutation sends j to k if and only if the j^{th} and k^{th} elements of A are the inverses of one another. Secondly, we define the permutation $\tau = \tau_{\alpha,\beta}$ by

$$\tau = (1, 3, 5, \ldots, 2n-1)(2, 4, 6, \ldots, 2n)(4n-1, 4n-3, \ldots, 2n+1)(4n, 4n-2, \ldots, 2n+2).$$

In this case, τ corresponds to sending an arc of one of the curves to the following arc in the same curve with the same orientation.

Note that we will say that a permutation is parity reversing if it sends odd numbers to even numbers and even numbers to odd numbers.

The following lemma generalises the results of Aougab-Huang [1, Lemma 2.2] and Nieland [46, Theorem 2.1], which dealt with the cases $p = 0$, $i(\alpha, \beta) = 2g-1$, and $p = 0$, $i(\alpha, \beta) = n \geq i_{g,0}$, respectively. This amounts to ensuring that any bigons, equivalently 2-cycles of the filling permutation, are dealt with appropriately.

Lemma 5.2. *Let α and β be a filling pair on $S_{g,p}$ with $i(\alpha, \beta) = n \geq i_{g,p}$. Then $\sigma = \sigma_{\alpha,\beta}$ satisfies the equation*

$$\sigma Q \sigma = \tau.$$

Conversely, a parity reversing permutation $\sigma \in \Sigma_{4n}$ consisting of $n+2-2g$ cycles and no more than p number of 2-cycles, and satisfying the above equation, defines a filling pair on $S_{g,p}$ with intersection number n.

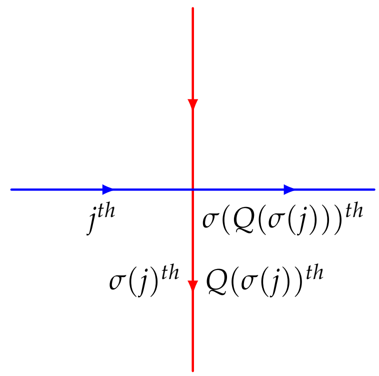

Figure 5.1: The filling permutation equation around a vertex.

Proof. Let $j \in \{1, 2, \dots, 4n\}$, then the edge labelled by the j^{th} element of A is followed by the edge labelled by the $\sigma(j)^{th}$ element of A. As discussed above, we then have that $Q(\sigma(j))$ is the inverse element in A of $\sigma(j)$. Finally, the edge labelled by the $\sigma(Q(\sigma(j)))^{th}$ element of A is the edge following the edge labelled by the $Q(\sigma(j))^{th}$ element of A. As can be seen in Figure 5.1, this is in fact the edge labelled by the arc that follows the arc labelled by j in the same curve. That is, the composition has the same action as the action of τ.

Conversely, suppose that $\sigma \in \Sigma_{4n}$ satisfies the conditions of the lemma. Since σ is parity reversing, each of the cycles in σ is of even length. Associate to each cycle of σ a polygon with the same number of sides. Puncture every 2-gon and then puncture any of the remaining polygons at most once until all p punctures have been placed. This is possible since $n \geq i_{g,p}$ guarantees that $n + 2 - 2g \geq p$. Furthermore, since the number of 2-cycles in σ was at most p, we do not have any unpunctured bigons that could reduce the intersection number of the resulting curves.

Label each polygon cyclically in a clockwise direction with the elements of the associated cycle of σ. Now relabel each side with the corresponding element of A and glue the polygons together by gluing each edge to the edge labelled with its inverse. Since every edge occurs once with each orientation, the resulting surface is closed with p punctures.

In the construction so far, we have $n + 2 - 2g$ faces and $2n$ edges, so we have $V - E + F = 2 - 2g$ if and only if we have n equivalence classes of vertices under the gluing of the polygons. Applying the filling permutation equation in Figure 5.1, we see that four edges will glue together to give a single vertex. In other words, under the gluing, the $4n$ vertices of the polygons form equivalences classes of size 4 and so we have n equivalence classes under the gluing. Hence the resulting surface is $S_{g,p}$.

Finally, another application of the filling permutation equation proves that the α-arcs and β-arcs glue to form a filling pair on $S_{g,p}$ with geometric intersection number equal to n. □

5.1.2 Filling pair construction

The filling permutations introduced in the previous section now give us a way to construct filling pairs on punctured surfaces satisfying specific conditions on their intersection number. We now make use of this tool to construct the filling pairs we require to prove Theorem 5.1.

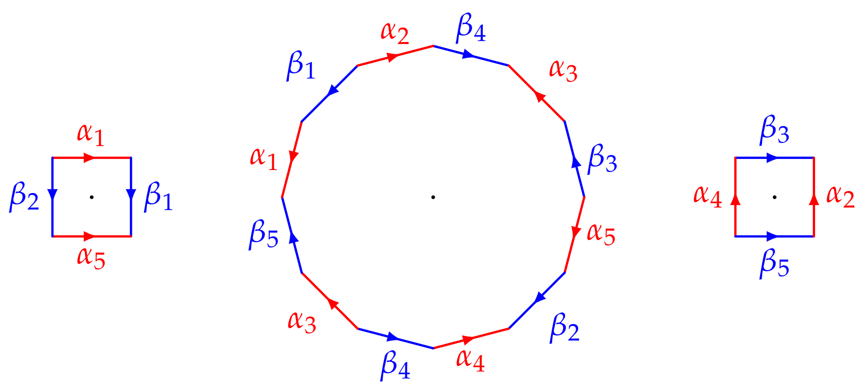

Figure 5.2: Polygonal decomposition of $S_{2,3}$ associated to the filling permutation σ.

We begin by constructing an appropriate filling pair on the surface $S_{2,3}$. We require that the intersection number of the filling pair is equal to $2g + p - 2 = 5$ and so we must find a permutation $\sigma \in \Sigma_{20}$ consisting of $5 + 2 - 2g = 3$ cycles and satisfying the conditions of Lemma 5.2. Indeed, the permutation

$$\sigma = (1, 2, 19, 14)(3, 8, 15, 16, 9, 4, 17, 18, 5, 10, 11, 12)(6, 13, 20, 7)$$

satisfies the lemma, and the polygonal decomposition of $S_{2,3}$ determined by the associated filling pair is shown in Figure 5.2.

We now apply the double-bigon inductive method used by Aougab-Taylor [3, Proof of Lemma 3.1] to complete the proof of Theorem 5.1. Namely, suppose that we have a filling pair on the surface $S_{2,p}$ with intersection number equal to $2g + p - 2$. Choose an intersection point of the two curves, form two bigons as in Figure 5.3, and puncture each of these bigons. We now have a filling pair on $S_{2,p+2}$ with intersection number equal to $2g + (p + 2) - 2$, as required. Since we have constructed a suitable filling pair on $S_{2,3}$, by induction, we have completed the proof of the theorem.

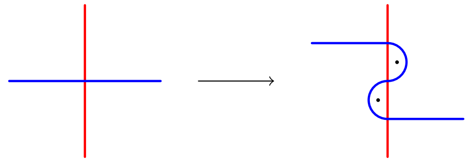

Figure 5.3: Double-bigon inductive method.

We conclude with the observation that the filling pair in Figure 5.4 is a minimally intersecting filling pair on $S_{0,4}$ having intersection number equal to $2 = p - 2$. As such, the complete list of values of $i_{g,p}$ can be summarised as follows.

Theorem 5.3. *The values of $i_{g,p}$ are the following:*

(1) If $g \neq 2, 0$ and $p = 0$, then $i_{g,p} = 2g - 1$;

(2) If $g \neq 2, 0$ and $p \geq 1$, then $i_{g,p} = 2g + p - 2$;

(3) If $g = 0$ and $p \geq 4$, then $i_{g,p} = p - 2$ if p is even, and $i_{g,p} = p - 1$ if p is odd;

(4) If $g = 2$ and $p \leq 2$, then $i_{g,p} = 4$;

(5) If $g = 2$ and $p \geq 2$, then $i_{g,p} = 2g + p - 2$.

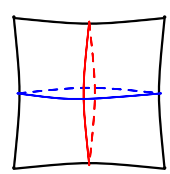

Figure 5.4: A minimally intersecting filling pair on $S_{0,4}$.

We briefly discuss the reason that such methods were insufficient for our construction of 1,1-square-tiled surfaces in Chapter 3. Indeed, this would amount to solving the permutation equation in Lemma 5.2 while simultaneously controlling the cycle decomposition of the permutation σ and satisfying a further algebraic condition corresponding to the filling pair having algebraic intersection number equal to geometric intersection number. Furthermore, it is not clear how one would be able to detect which connected component contained the Abelian differential corresponding to the associated filling pair.

5.2 Algebraic intersection number equal to geometric intersection number

With Theorem 5.3 in mind, one can ask for $g \geq 1$ whether $i_{g,p}$ can be realised as the algebraic intersection number, $\widehat{i}(\alpha, \beta)$, of an oriented filling pair $\{\alpha, \beta\}$. Aougab-Menasco-Nieland [2] answered this question for the case of $i_{g,0}$; that is, for minimally intersecting filling pairs on closed surfaces. Moreover, they were interested in counting the number of mapping class group orbits of such filling pairs. Their method involved algebraically constructing 1,1-square-tiled surfaces with the minimum number of squares in the stratum $\mathcal{H}(2g-2)$, which they call square-tiled surfaces with connected leaves. The core curves of the cylinders of such surfaces give rise to filling pairs with algebraic intersection number equal to $i_{g,0}$.

For $n \geq i_{g,p}$, by a *compatible decomposition* of the surface $S_{g,p}$ into $n+2-2g$ many $4k$-gons, we mean a decomposition of the surface into $4k$-gons $P_1, \ldots, P_{n+2-2g}$ such that, if P_i is a $4k_i$-gon, then $\sum(k_i - 1) = 2g-2$.

Observe that a filling pair on the surface $S_{g,p}$ with $\widehat{i}(\alpha, \beta) = i(\alpha, \beta) = n \geq i_{g,p}$ divides the surface into a collection of $n+2-2g$ many $4k$-gons forming a compatible decomposition. Conversely, given an appropriate choice of orientation, the core curves of a 1,1-square-tiled surface with n squares and $n+2-2g$ many zeros, of orders greater than or equal to zero, form a filling pair with $\widehat{i}(\alpha, \beta) = i(\alpha, \beta) = n$ dividing the surface into $n+2-2g$ many $4k$-gons with a zero of order $k-1$ giving rise to a $4k$-gon. These $4k$-gons also form a compatible decomposition.

Note that the square torus can be represented by the permutation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and that this permutation can be combined with a 1,1-square-tiled surface of genus g, by cylinder concatenation as in Lemma 2.1, to produce another 1,1-square-tiled surface of genus g. This process will add a zero of order 0 to the surface and one additional square.

For $g \geq 2$ and $p \geq 0$, let $n \geq i_{g,p}$ and choose a compatible decomposition of the surface $S_{g,p}$ into $n+2-2g$ many $4k$-gons, as described above. There will be a number, less than or equal to $2g-2$, of these $4k$-gons having $k \geq 2$. Let $k_1, \ldots, k_m$ be the list of these k values. By Theorem 1.4, we can choose a 1,1-square-tiled surface in the stratum $\mathcal{H}(k_1 - 1, \ldots, k_m - 1)$. Adding $(n+2-2g-m)$ zeros of order 0 to this surface, using the method described in the previous paragraph, and choosing orientations appropriately, we will have a 1,1-square-tiled surface such that the core curves of the cylinders form a filling pair $\{\alpha, \beta\}$ with $\widehat{i}(\alpha, \beta) = i(\alpha, \beta) = n$ which, after adding p punctures to distinct complementary regions of the filling pair, gives rise to the specified polygonal decomposition of $S_{g,p}$.

In the case $g = 1$, we need only combine, as in Lemma 2.1, the permutation for the square torus above with itself n times and then add p punctures to the surface in distinct complementary regions of the filling pair. This completes the proof of Theorem 1.8.

Chapter 6

Extension to quadratic strata

In this chapter, we present results in the direction of extending the work of Chapter 3 to the case of the moduli space of quadratic differentials on a Riemann surface. We will begin in Section 6.1 by presenting the necessary background on quadratic differentials, their moduli spaces, and the classification of connected components of strata. In Section 6.2 we generalise Lemma 2.1 to 1,1-pillowcase covers, the analogue of 1,1-square-tiled surfaces in this setting. We then give the construction of 1,1-pillowcase covers in all hyperelliptic components in Section 6.3, before constructing 1,1-pillowcase covers in all connected components of all quadratic strata of genus greater than or equal to two having no poles. These constructions are presented in Sections 6.4 to 6.6. Finally, in Sections 6.7 and 6.8, we construct 1,1-pillowcase covers in genus one and genus zero, respectively. In genus zero, this is equivalent to the construction of special planar graphs called meanders.

6.1 Quadratic differentials

Here we give an introduction to quadratic differentials, building up to the classification given by Lanneau [36,37] of the connected components of the strata of the moduli space of quadratic differentials on a Riemann surface. For more details we refer the reader to the references of Strebel [48] and Masur-Tabachnikov [42], as well as the paper of Lanneau [37] and the references therein.

6.1.1 Quadratic differentials and half-translation surfaces

A *quadratic differential* q on a compact Riemann surface X of genus $g \geq 0$ is a global section of the symmetric square of the canonical line bundle $\Omega(X)$. That is, in local coordinates q is given by $f(z)dz^2$. Note that the global square of an Abelian differential gives rise to a quadratic differential on a Riemann surface. We define $\mathcal{Q}_g$ to be the quotient by the action of the mapping class group of the set of pairs (X, q), where X is a closed connected Riemann surface of genus g and q, not the global square of an Abelian differential, is a non-zero meromorphic quadratic differential on X having at most simple poles. We then have that the moduli space of integrable meromorphic quadratic differentials on a Riemann surface X is a disjoint union

$$\{\omega^2 \,|\, \omega \in \mathcal{H}_g\} \bigsqcup \mathcal{Q}_g.$$

The structure of the first set is given by that of $\mathcal{H}_g$, and so in this chapter we focus our attention on $\mathcal{Q}_g$. We will drop the subscript g if the genus is clear from the context.

Half-translation surfaces

Let S be a topological surface of genus g. By a *half-translation atlas on S*, we will mean an atlas of charts $\{\varphi_\alpha : U_\alpha \to V_\alpha\}_\alpha$ with $U_\alpha \subset S$, $V_\alpha \subset \mathbb{C}$ open sets, $\bigcup_\alpha U_\alpha = S \setminus \Sigma$, for a finite set of points $\Sigma \subset S$, such that all transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are given by *half-translations* $z \mapsto \pm z + c$. Two half-translation atlases are equivalent if their union is also a half-translation atlas, and an equivalence class of half-translation atlases is called a *half-translation structure*. A surface S equipped with a maximal translation atlas will be called a *half-translation surface*. In the literature, a half-translation surface may be called a *flat surface*. We claim that a half-translation surface is equivalent to a Riemann surface equipped with a quadratic differential.

The argument is similar to the one given for Abelian differentials and translation surfaces in Chapter 1. Let $\Sigma \subset X$ denote the zeros of q, and choose $u \in X \setminus \Sigma$. Then one can define local coordinates in a neighbourhood of u by integrating $\sqrt{q}$. The transition maps are then given by half-translations. Conversely, let S be a half-translation surface, then S is endowed with a Riemann surface structure and the local pullback of dz^2 on $\mathbb{C}$ gives rise to a well defined quadratic differential on the Riemann surface. Indeed, note that $dz^2 = d(\pm z + c)^2$.

Given a quadratic differential (X, q), the pullback of the flat metric on $\mathbb{C}$ gives rise to a flat metric on $X \setminus \Sigma$. Each zero of q order k gives rise to a cone-type singularity of the flat metric with cone-angle equal to $(k+2)\pi$. The metric has holonomy in $\{\pm \mathrm{Id}\}$. Moreover, the induced Riemannian structure allows us to make sense of horizontal and vertical directions.

Pillowcase covers

Similar to the case of translation surfaces, a half-translation surface can also be realised by a collection of polygons in $\mathbb{C}$ with pairs of parallel sides of equal length identified by half-translations such that the quotient is a closed connected oriented surface. If a quadratic differential can be realised in this way by identifying the sides of a collection of unit squares, then we will call such a surface a *pillowcase cover* since such surfaces can be realised as covers of the four-times punctured sphere (the pillowcase, see Figure 5.4). Pillowcase covers differ from square-tiled surfaces as top sides of squares can be identified with top sides of squares which was prohibited in the constructions of square-tiled surfaces. Indeed, such an identification requires a half-translation.

As we did for translation surfaces, we can also discuss cylinders inside a half-translation surface and these will be defined as before; that is, as maximally embedded flat annuli. If a pillowcase cover has a single vertical cylinder and a single horizontal cylinder then we shall call it a *1,1-pillowcase cover*. The core curves of the cylinders of a 1,1-pillowcase cover form a filling pair on the underlying surface. The filling pair will not have geometric intersection number equal to algebraic intersection number (for any choice of orientation) otherwise the quadratic differential would be the global square of an Abelian differential. Conversely, a filling pair with geometric intersection number not equal to algebraic intersection number will give rise to a 1,1-pillowcase cover.

We remark that the core curves of 1,1-square-tiled surfaces were forced to be non-separating. This is because sides could only be identified by translation and so all sides on one side of a core curve must also occur on the other side. In the case of 1,1-pillowcase covers, the core curves need not be separating. Indeed, one, both, or neither of the curves may be separating.

Stratification of $\mathcal{Q}$

By the Riemann-Roch theorem, we have that the sum of the orders of the zeros of a quadratic differential on a Riemann surface of genus g is equal to $4g-4$. We define the stratum $\mathcal{Q}(k_1,\ldots,k_n) \subset \mathcal{Q}$, with $k_i \geq 1$ or $k_i = -1$ and $\sum_{i=1}^{n} k_i = 4g-4$, to be the subset of $\mathcal{Q}$ consisting of quadratic differentials with n distinct zeros of orders $k_1,\ldots,k_n$. Each stratum is a complex orbifold of complex dimension $2g+n-2$. Masur-Smillie [41] showed that all strata are non-empty apart from $\mathcal{Q}(\varnothing)$ and $\mathcal{Q}(1,-1)$ in genus one, and $\mathcal{Q}(3,1)$ and $\mathcal{Q}(4)$ in genus two.

The emptiness of these strata will present us with difficulty in our construction of 1,1-pillowcase covers. Indeed, we will not have 1,1-pillowcase covers in these strata that can be used to produce 1,1-pillowcase covers in strata of higher complexity with zeros of these orders. As such, we will have to build 1,1-pillowcase covers in such strata separately. Furthermore, the combinatorics in the quadratic case is much more delicate than in the Abelian case. For instance, we are dealing now with partitions of multiples of 4 and so the number of basic cases we have to construct in the inductive method is increased. Moreover, with the addition of poles, there are now infinitely many strata in each genus.

6.1.2 Classification of connected components

The classification of the connected components of strata of quadratic differentials was completed by Lanneau [36,37], with a small correction by Chen-Möller [10]. Outside of a small number of exceptional strata in low genus, hyperellipticity is sufficient to determine the number of connected components. Indeed, the classification is as follows.

Theorem 6.1 ([37], Theorem 1.1). *Let $g \geq 5$, then the strata*

$$\mathcal{Q}(4(g-k)-6, 4k+2), 0 \leq k \leq g-2,$$
$$\mathcal{Q}(2(g-k)-3, 2(g-k)-3, 4k+2), 0 \leq k \leq g-1, \text{ and}$$
$$\mathcal{Q}(2(g-k)-3, 2(g-k)-3, 2k+1, 2k+1), -1 \leq k \leq g-2$$

have two connected components: one is hyperelliptic and the other is not.

All other strata are non-empty and connected.

For lower genera, we have the following.

Theorem 6.2 ([37], Theorem 1.2; [10], Theorem 1.2). *If $g = 0, 1$, then all strata are non-empty and connected apart from the strata $\mathcal{Q}(\varnothing)$, and $\mathcal{Q}(1,-1)$ which are empty.*

If $g = 2$, $\mathcal{Q}(6,-1,-1)$ and $\mathcal{Q}(3,3,-1,-1)$ have two connected components: one hyperelliptic and the other not. The strata $\mathcal{Q}(4)$ and $\mathcal{Q}(3,1)$ are empty, and all other strata are non-empty and connected.

If $g = 3$, any stratum having a hyperelliptic component has two components: one hyperelliptic and the other not. The strata $\mathcal{Q}(9,-1)$, $\mathcal{Q}(6,3,-1)$, and $\mathcal{Q}(3,3,3,-1)$ have two non-hyperelliptic connected components. All other strata are not empty and connected.

If $g = 4$, the strata $\mathcal{Q}(12)$, $\mathcal{Q}(9,3)$, $\mathcal{Q}(6,6)$, $\mathcal{Q}(6,3,3)$, and $\mathcal{Q}(3,3,3,3)$ have two non-hyperelliptic connected components. The strata $\mathcal{Q}(9,3)$, $\mathcal{Q}(6,6)$, $\mathcal{Q}(6,3,3)$, and $\mathcal{Q}(3,3,3,3)$ also have a hyperelliptic connected component. Any other stratum having a hyperelliptic connected component has two components: one hyperelliptic and the other not. All other strata are non-empty and connected.

The fact that all half-translation surfaces in the genus two strata $\mathcal{Q}(2,1,1)$ and $\mathcal{Q}(1^4)$ are hyperelliptic will present us with difficulty in our construction of 1,1-pillowcase covers in strata of higher complexity.

Hyperellipticity

Similar to the case for translation surfaces, we will say that a half-translation surface (X, q) is hyperelliptic if there exists an isometric involution $\tau : X \to X$, known as a hyperelliptic involution, that induces a ramified double cover $\pi : X \to S_{0,2g+2}$ from X to the $(2g+2)$-times punctured sphere. The hyperelliptic components contain those hyperelliptic half-translations surfaces (X, q) for which there exists a quadratic differential

q_0 on the sphere such that $\pi^* q_0 = q$. Similar to what we saw for translation surfaces, there exists a double covering construction that takes a quadratic differential (X_0, q_0) on the sphere and gives a quadratic differential (X, q) on a higher genus Riemann surface. In the cases

$$\mathcal{Q}(2(g-k)-4, 2k, -1^{2g}) \to \mathcal{Q}(4(g-k)-6, 4k+2)$$

for $g \geq 2$ and $0 \leq k \leq g-2$,

$$\mathcal{Q}(2(g-k)-3, 2k, -1^{2g+1}) \to \mathcal{Q}(2(g-k)-3, 2(g-k)-3, 4k+2)$$

for $g \geq 1$ and $0 \leq k \leq g-1$, and

$$\mathcal{Q}(2(g-k)-3, 2k+1, -1^{2g+2}) \to \mathcal{Q}(2(g-k)-3, 2(g-k)-3, 2k+1, 2k+1)$$

for $g \geq 1$ and $-1 \leq k \leq g-2$, the maps given by this construction are immersions. Again the connectedness of genus zero strata, the equality of dimension, and the ergodicity of the geodesic flow give that the images must be connected components. We will denote the hyperelliptic components of such strata by $\mathcal{Q}^{hyp}(k_1, \ldots, k_n)$.

Exceptional strata

As we saw in Theorem 6.2 above, there exist a number of exceptional strata in low genera that are not connected but for which neither of these connected components are hyperelliptic. The non-connectedness of the strata $\mathcal{Q}(12)$, $\mathcal{Q}(9, -1)$, $\mathcal{Q}(6, 3, -1)$, and $\mathcal{Q}(3, 3, 3, -1)$ was first observed by Zorich by a direct calculation [57, Proposition 13]. These two connected components are called the regular and irregular connected components and are denoted by $\mathcal{Q}^{reg}(k_1, \ldots, k_n)$ and $\mathcal{Q}^{irr}(k_1, \ldots, k_n)$, respectively. The classification of connected components given by Lanneau [37, Theorem 1.2] was then slightly corrected by Chen-Möller who observed that the strata $\mathcal{Q}(9, 3)$, $\mathcal{Q}(6, 6)$, $\mathcal{Q}(6, 3, 3)$, and $\mathcal{Q}(3, 3, 3, 3)$ also have regular and irregular connected components as well as a hyperelliptic connected component [10, Theorem 1.2].

6.2 Generalisation of Lemma 2.1

In this section, we introduce the notion of a generalised permutation representative. This is an object analogous to the permutation representatives used for Abelian differentials above. We will then generalise Lemma 2.1 which will allow us to combine 1,1-pillowcase covers in a similar way to the construction given in Chapter 3 for 1,1-square-tiled surfaces.

6.2.1 Generalised permutation representatives

Recall that the vertical flow on a translation surface induced an interval exchange transformation on a choice of horizontal transversal (Figure 2.1). The permutation describing this interval exchange transformation was then taken to be a permutation representative for the connected component of the stratum containing the translation surface.

We can perform a similar construction on half-translation surfaces. However, since the holonomy of a half-translation surface is non-trivial, the vertical flow induces a *linear involution* (sometimes called a *non-classical interval exchange transformation*) on a choice of horizontal transversal. Such a transformation permutes subintervals of the transversal, but subintervals on the same side of the transversal can be sent to one another with an orientation reversing flip. Linear involutions were introduce by Danthony-Nogueira [18,19].

Indeed, consider the vertical flow on the half-translation surface in Figure 6.1. It induces a linear involution on the horizontal transversal T. The interval on the top side of T below the side labelled 0 returns on the bottom side of T under the side labelled 3. This is similar to the translation surface situation. However, the interval on the top side of T below the left-most side labelled 1 returns (with a flip of orientation) again to the top side of T under the right-most side labelled by 1, and something similar happens for the sides labelled by 4. In this situation, we represent this map by the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 3 \\ 3 & 2 & 4 & 4 & 0 \end{pmatrix}.$$

Since this matrix has symbols that occur twice in the same row it is not a permutation and, as such, in the literature such matrices are called *generalised permutations*. See the works of Zorich [57] and Boissy-Lanneau [8] for more details.

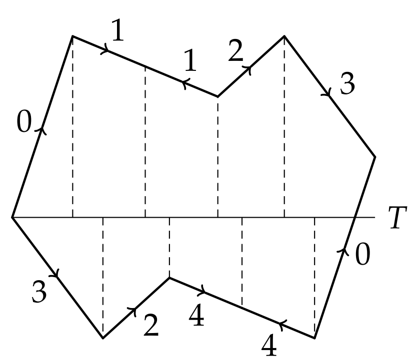

Figure 6.1: The first return map to the horizontal transversal T under the vertical flow induces a non-classical interval exchange transformation.

Analogous to the translation surface situation, there is a theory of Rauzy moves and Rauzy classes for generalised permutations. Indeed, by Boissy-Lanneau [8] there is a correspondence between connected components and extended Rauzy classes of certain generalised permutations, and we shall still call a choice of generalised permutation a *permutation representative* for the connected component.

Similar to the case of square-tiled surfaces, if a permutation representative has rows of the same length with the first symbol of the first row equal to the final symbol of the second row then we can construct a pillowcase cover with a single horizontal cylinder. See the half-translation surfaces in Figure 6.2. However, since a generalised permutation is not a classical permutation, we cannot resort to the cycle structure of a modified permutation to detect vertical cylinders. Instead, we must check the combinatorics by hand. Observe, that the surface on the right of Figure 6.2 has one vertical cylinder while the surface on the left does not.

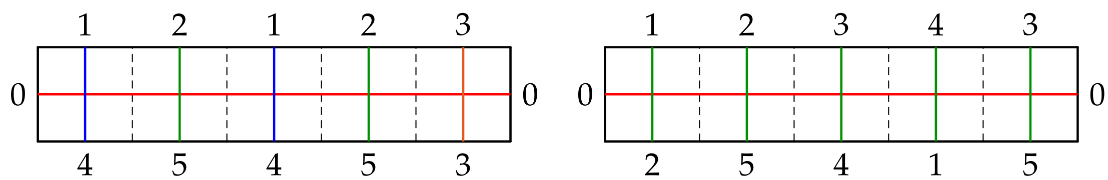

Figure 6.2: Two pillowcase covers in $\mathcal{Q}(8)$ with a single horizontal cylinder. The surface on the right also has a single vertical cylinder while the one on the left has three vertical cylinders.

Zorich constructed permutation representatives that can be used to build pillowcase covers with a single horizontal cylinder for all connected components of quadratic strata [57]. As was the case for Abelian differentials above, one cannot easily obtain representatives of 1,1-pillowcase covers from these representatives as extended Rauzy classes grow too greatly in complexity and so the hope of being able to find such representatives by searching the Rauzy class would be naive. Indeed, there are many representatives given by Zorich that do not have the same number of symbols in both rows and so one would have to first perform a series of Rauzy moves to balance the rows before then searching for a single vertical cylinder. Instead, we will perform an inductive construction similar to the one given in Chapter 3.

6.2.2 Filling pair diagrams

The notion of filling pair diagram easily generalises to the case of 1,1-pillowcase covers. Here however, the edges leaving the top of a vertex may enter another vertex at either the top or the bottom. Recall that for an Abelian differential the edges leaving the top of a vertex had to enter at the bottom of another. This is because permutation representatives for Abelian differentials are classical permutations. The method of constructing half-translation surfaces from the ribbon graph given by the filling pair diagram is analogous. That is, one adds a saddle of cone-angle $(k+2)\pi$ to every boundary component with $2k+4$ sides. It is also easy to produce the generalised permutation corresponding to a given filling pair diagram and vice versa. Indeed, the filling pair diagram corresponding to the permutation representative

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 \\ 2 & 5 & 4 & 1 & 5 & 0 \end{pmatrix} \tag{6.1}$$

is shown in Figure 6.3.

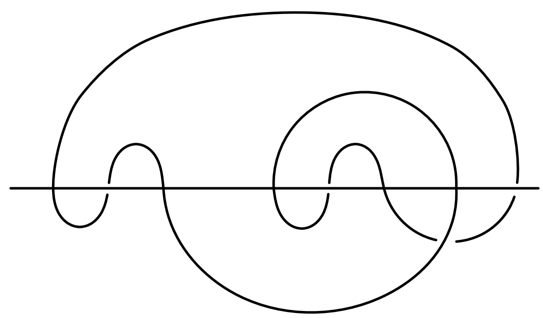

Figure 6.3: A filling pair diagram corresponding to the generalised permutation (6.1).

6.2.3 Combination lemma

We now present a generalisation of Lemma 2.1 that will allow us to combine 1,1-pillowcase covers to produce another 1,1-pillowcase cover of higher complexity. The first thing to observe is that cylinder concatenation, as described in Section 2.3, can be performed on 1,1-pillowcase covers of the appropriate form. The following lemma then describes the properties of the resulting surface. In the statement, we use $\mathcal{S}$ as a placeholder for $\mathcal{H}$ or $\mathcal{Q}$, and σ as a placeholder for ω or q.

Lemma 6.3. *Suppose that (X, σ) and (X', σ') represent 1,1-square-tiled surfaces (or 1,1-pillowcase covers) in the strata $\mathcal{S}_g(k_1, \ldots, k_n)$ and $\mathcal{S}'_{g'}(l_1, \ldots, l_m)$, with $2g+n-2$ and $2g'+m-2$ squares, respectively. Further, suppose that both X and X' have the form required to be used in the cylinder concatenation method; that is, the bottom of the first square is glued to the top of the second. Then the 1,1-square-tiled surface (or 1,1-pillowcase cover) obtained from these* *two surfaces by cylinder concatenation lies in the stratum*

$$\begin{aligned} \mathcal{H}_{g+g'-1}(k_1,\dots,k_n,l_1,\dots,l_m), &\quad \text{if } \mathcal{S}=\mathcal{S}'=\mathcal{H}, \\ \mathcal{Q}_{g+g'-1}(2k_1,\dots,2k_n,l_1,\dots,l_m), &\quad \text{if } \mathcal{S}=\mathcal{H} \text{ and } \mathcal{S}'=\mathcal{Q}, \\ \mathcal{Q}_{g+g'-1}(k_1,\dots,k_n,2l_1,\dots,2l_m), &\quad \text{if } \mathcal{S}=\mathcal{Q} \text{ and } \mathcal{S}'=\mathcal{H}, \text{ or} \\ \mathcal{Q}_{g+g'-1}(k_1,\dots,k_n,l_1,\dots,l_m), &\quad \text{if } \mathcal{S}=\mathcal{S}'=\mathcal{Q}. \end{aligned}$$

Moreover, the surface consists of $2(g+g'-1)+(n+m)-2$ *squares.*

Proof. The proof is analogous to the proof of Lemma 2.1. Indeed, the only assumption necessary for the proof of Lemma 2.1 is that the bottoms of the first squares in the surfaces are glued to the tops of the second. The remainder of the construction has no dependence on the manner (translation or half-translation) in which the remaining sides are identified. Therefore, the resulting surface is equipped with either an Abelian or quadratic differential depending on these identifications. The final two claims of the lemma are then easily verified. □

We present an example of this in Figure 6.4. The surface on the left is a 1,1-pillowcase cover in $\mathcal{Q}(8)$ and the surface on the right is a 1,1-square-tiled surface in $\mathcal{H}(4)$. The resulting surface is a 1,1-pillowcase cover lying in the stratum $\mathcal{Q}(8,8)$, as predicted by Lemma 6.3.

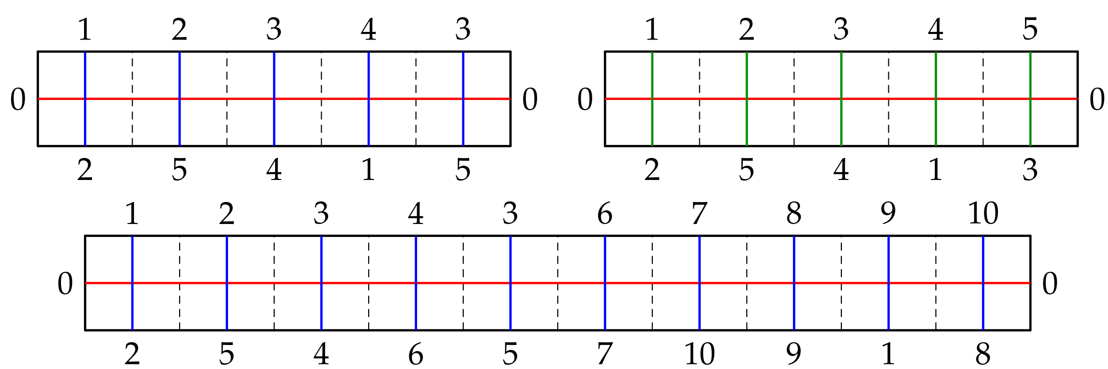

Figure 6.4: Cylinder concatenation construction as in Lemma 6.3.

6.2.4 Outline of proof

Recall the proof method used in Chapter 3. We first constructed hyperelliptic 1,1-square-tiled surfaces by hand and then, using Lemmas 2.1 and 2.2, applied an induction method to build 1,1-square-tiled surfaces in an arbitrary stratum. For this, we had to construct 1,1-square-tiled surfaces in strata of the form $\mathcal{H}(2k)$ and $\mathcal{H}(2j+1,2k+1)$. Furthermore, we had to handle the hyperellipticity of the genus two strata $\mathcal{H}(2)$ and

$\mathcal{H}(1,1)$ by finding ways to add zeros of these orders to 1,1-square-tiled surfaces we had already constructed.

For quadratic strata, we will proceed in a similar manner. Again, we begin by constructing by hand 1,1-pillowcase covers in the hyperelliptic components. For an arbitrary stratum $\mathcal{Q}(k_1,\ldots,k_n)$ of genus at least two, we will apply an induction method using Lemma 6.3. That is, we will build 1,1-pillowcase covers in lower genus strata that can be combined using Lemma 6.3 to give a 1,1-pillowcase cover in $\mathcal{Q}(k_1,\ldots,k_n)$. We will only construct 1,1-pillowcase covers in quadratic strata with no poles.

In the case of Abelian differentials, it was sufficient to build 1,1-square-tiled surface in strata with a single even order zero, or a pair of odd order zeros. This was because the orders of the zeros formed a partition of $2g-2$. In the quadratic case, the orders of the zeros form a partition of $4g-4$. As such, our induction method requires us to build 1,1-pillowcase covers in strata of the form $\mathcal{Q}(4k)$, $\mathcal{Q}(4j+2,4k+2)$, $\mathcal{Q}(4j+2,4k+1,4l+1)$, $\mathcal{Q}(4j+2,4k+3,4l+3)$, $\mathcal{Q}(4j+1,4k+1,4l+1,4m+1)$, and $\mathcal{Q}(4j+3,4k+3,4l+3,4m+3)$. Indeed, observe that such strata cannot be built from strata of lower complexity.

Similar to the case of Abelian differentials, we will have trouble caused by the hyperellipticity of the genus two strata $\mathcal{Q}(2,1,1)$ and $\mathcal{Q}(1,1,1,1)$. Moreover, the strata $\mathcal{Q}(3,1)$ and $\mathcal{Q}(4)$ are empty. We will therefore have to modify the induction method above to handle such cases.

For genus one, we build 1,1-pillowcase covers in strata of the form $\mathcal{Q}(2k,-1^{2k})$, $\mathcal{Q}(k_1,k_2,-1^{2k+1})$, $\mathcal{Q}(2k+1,-1^{2k+1})$, and $\mathcal{Q}(k_1,k_2,-1^{2k+1})$. These can be combined using Lemma 6.3 to build a 1,1-pillowcase cover in an arbitrary genus one stratum.

Finally, for genus zero, we are forced to develop a new inductive method. Indeed, the conditions of Lemma 6.3 can only be satisfied if the surfaces in question have positive genus. Realising that the filling pair diagram arising from a genus zero quadratic differential is a realisation of a special planar graph called a *meander*, we provide a method for combining two meanders of a specific form in such a way that we obtain a new meander whose complimentary regions are related to those of the meanders from which it is built. For every genus zero stratum with a single zero of positive order, we construct a meander that can be combined in this way and so a meander for an arbitrary genus zero stratum can be built by inductively applying this combination method.

6.3 Hyperelliptic components

We begin by recalling the permutation representatives for hyperelliptic components given by Lanneau [37, Section 4.1]. We will however use a variation of the notation given by Zorich [57, Section 3.6].

Theorem 6.4 ([37], Section 4.1). *For any pair of non-negative integer parameters r and s, the generalised permutation*

$$\begin{pmatrix} 0 & A & 1 & 2 & \dots & s & A & s+1 & \dots & s+r \\ s+r & \dots & s+1 & B & s & \dots & 2 & 1 & B & 0 \end{pmatrix}$$

represents a pillowcase cover with one horizontal cylinder in the hyperelliptic component. The stratum is determined as follows.

s	r	*Stratum*
$2j+1$	$2k+1$	$\mathcal{Q}(4j+2, 4k+2)$
$2j+1$	$2k$	$\mathcal{Q}(4j+2, 2k-1, 2k-1)$
$2j$	$2k+1$	$\mathcal{Q}(2j-1, 2j-1, 4k+2)$
$2j$	$2k$	$\mathcal{Q}(2j-1, 2j-1, 2k-1, 2k-1)$

The sequences $1, \dots, s$ and $s+1, \dots, s+r$ are assumed to be empty if s and r are zero, respectively.

In fact, one can take representatives of the form

$$\begin{pmatrix} 0 & A & 1 & 2 & \dots & s & A & s+1 & \dots & s+r \\ s & \dots & 2 & 1 & B & s+r & \dots & s+1 & B & 0 \end{pmatrix}$$

where the non-zero symbols in the bottom row have been cyclically permuted. As we did for hyperelliptic Abelian differentials, we will produce representatives for hyperelliptic 1,1-pillowcase covers by splitting symbols in these permutation representatives. This is the content of the following proposition.

Proposition 6.5. *For any pair of non-negative integer parameters j and k, the generalised permutation with top row*

$$0,\ A,\ 1,\ 2,\ \dots,\ 4j,\ 4j+1,\ A,\ 4j+2,\ 4j+3,\ \dots,\ 4j+4k+1,\ 4j+4k+2$$

and bottom row

$$4j,\ 4j+1, \dots,\ 2j+2,\ 2j+3,\ 2j+1,\ 2j-1,\ 2j,\ \dots,$$

$$1,\ 2,\ B,\ 4j+4k+1,\ 4j+4k+2, \dots,\ 4j+2k+3,\ 4j+2k+4,$$

$$4j+2k+2,\ 4j+2k,\ 4j+2k+1,\ \dots,\ 4j+2,\ 4j+3,\ B,\ 0$$

represents a 1,1-pillowcase cover in $\mathcal{Q}^{hyp}(4j+2, 4k+2)$. The generalised permutation with top row

$$0,\ A,\ 1,\ 2,\ \dots,\ 4j,\ 4j+1,\ A,\ 4j+2,\ 4j+3,\ \dots,\ 4j+4k-1,\ 4j+4k$$

and bottom row

$$4j,\ 4j+1, \ldots,\ 2j+2,\ 2j+3,\ 2j+1,\ 2j-1,\ 2j,\ \ldots,$$

$$1,\ 2,\ B,\ 4j+4k-1,\ 4j+4k, \ldots,\ 4j+2k+1,\ 4j+2k+2,$$

$$4j+2k,\ 4j+2k-2,\ 4j+2k-1,\ \ldots,\ 4j+2,\ 4j+3,\ B,\ 0$$

represents a 1,1-pillowcase cover in $\mathcal{Q}^{hyp}(4j+2, 2k-1, 2k-1)$. *The generalised permutation with top row*

$$0,\ A,\ 1,\ 2,\ \ldots,\ 4j-2,\ 4j-1,\ A,\ 4j,\ 4j+1,\ \ldots,\ 4j+4k-1,\ 4j+4k$$

and bottom row

$$4j-2,\ 4j-1, \ldots,\ 2j,\ 2j+1,\ 2j-1,\ 2j-3,\ 2j-2,\ \ldots,$$

$$1,\ 2,\ B,\ 4j+4k-1,\ 4j+4k, \ldots,\ 4j+2k+1,\ 4j+2k+2,$$

$$4j+2k,\ 4j+2k-2,\ 4j+2k-1,\ \ldots,\ 4j,\ 4j+1,\ B,\ 0$$

represents a 1,1-pillowcase cover in $\mathcal{Q}^{hyp}(2j-1, 2j-1, 4k+2)$. *The generalised permutation with top row*

$$0,\ A,\ 1,\ 2,\ \ldots,\ 4j-2,\ 4j-1,\ A,\ 4j,\ 4j+1,\ \ldots,\ 4j+4k-3,\ 4j+4k-2$$

and bottom row

$$4j-2,\ 4j-1, \ldots,\ 2j,\ 2j+1,\ 2j-1,\ 2j-3,\ 2j-2,\ \ldots,$$

$$1,\ 2,\ B,\ 4j+4k-3,\ 4j+4k-2, \ldots,\ 4j+2k-1,\ 4j+2k,$$

$$4j+2k-2,\ 4j+2k-4,\ 4j+2k-3,\ \ldots,\ 4j,\ 4j+1,\ B,\ 0$$

represents a 1,1-pillowcase cover in $\mathcal{Q}^{hyp}(2j-1, 2j-1, 2k-1, 2k-1)$.

Before presenting the proof of this proposition, we briefly describe how these permutations are obtained from those discussed above by splitting a number of the symbols. Indeed, the permutation for $\mathcal{Q}^{hyp}(4j+2, 4k+2)$ is obtained by splitting all of the non-zero symbols apart from the symbols $A, B, j+1 = \frac{s+1}{2}$, and $2j+k+2 = s + \frac{r+1}{2}$. The permutations for the remaining strata are obtained similarly.

Proof of Proposition 6.5. We will present the proof of the case $\mathcal{Q}^{hyp}(4j+2, 4k+2)$. The remaining cases follow similarly.

From our discussion above, we see that the permutation does indeed represent the claimed connected component. Indeed, we have only split symbols in the permutation. It is clear that the associated pillowcase cover has one horizontal cylinder. As such, we need only check that the associated pillowcase cover has a single vertical cylinder.

Starting in the leftmost square and travelling upwards, the cycle of symbols hit is as follows:

$$A \to 4j \to 2 \to 4j-2 \to 4 \to \cdots \to 2j+2 \to 2j \to 2j+1 \to 2j-1 \to$$

$$2j+3 \to 2j-3 \to \cdots \to 3 \to 4j-1 \to 1 \to 4j+1 \to B \to 4j+4k+2 \to$$

$$4j+2 \to 4j+4k \to 4j+4 \to \cdots \to 4j+2k+4 \to 4j+2k+2 \to 4j+2k+3 \to$$

$$4j+2k+1 \to 4j+2k+5 \to 4j+2k-1 \to \cdots \to 4j+4k-1 \to 4j+3 \to$$

$$4j+4k+1 \to A.$$

We see that every symbol is contained in this cycle and so we do indeed have a single vertical cylinder. □

Recall that in contrast to the situation with 1,1-square-tiled surfaces, the core curves of a 1,1-pillowcase cover can be non-separating (as in the Abelian case) or separating. The following proposition claims that the above permutations realise the minimal number of squares for a hyperelliptic 1,1-pillowcase cover whose core curves are both non-separating.

Proposition 6.6. *The minimal number of squares required for a 1,1-pillowcase cover representing a hyperelliptic component whose cylinders have non-separating core curves are as follows.*

Component	*min. num. of squares*
$\mathcal{Q}^{hyp}(4j+2,4k+2)$	$4j+4k+4=4g-4$
$\mathcal{Q}^{hyp}(4j+2,2k-1,2k-1)$	$4j+4k+2=4g-2$
$\mathcal{Q}^{hyp}(2j-1,2j-1,4k+2)$	$4j+4k+2=4g-2$
$\mathcal{Q}^{hyp}(2j-1,2j-1,2k-1,2k-1)$	$4j+4k=4g$

Proof. The proof is completely analogous to the proof of Proposition 3.2. Indeed, the proof of Proposition 3.2 depended only on the cylinders of the surface being non-separating curves fixed by the hyperelliptic involution. In which case, the minimum number of squares was shown to be $4g-4$ when all of the zeros were fixed by the involution. This is the case for the component $\mathcal{Q}^{hyp}(4j+2,4k+2)$. For the component $\mathcal{Q}^{hyp}(4j+2,2k-1,2k-1)$ (resp. $\mathcal{Q}^{hyp}(2j-1,2j-1,4k+2)$), the zeros of order $2j-1$ (resp. $2k-1$) are sent to one another by the involution. As we saw in the proof of Proposition 3.2, this situation forces an extra two squares, and so the minimum number of squares for these components is $4g-2$. Similarly, for the component

$\mathcal{Q}^{hyp}(2j-1, 2j-1, 2k-1, 2k-1)$, the zeros of order $2j-1$ are sent to each other and the zeros of order $2k-1$ are sent to each other under the hyperelliptic involution, which forces an additional two squares and so the minimum number of squares required for this component is $4g$. □

Therefore, the minimum number of squares required for a 1,1-pillowcase cover in a hyperelliptic connected component is bounded above by $4g$. Computational evidence suggests that the number of squares required increases when one or both of the core curves are separating. That is, we conjecture the following.

Conjecture 6.7. *When they exist, the number of squares required for 1,1-pillowcase covers in hyperelliptic components having one or both of the core curves being separating is greater than the number of squares required when both of the core curves are non-separating.*

6.4 Even order zeros

In this section, we will construct 1,1-pillowcase covers in all non-hyperelliptic components of strata of $\mathcal{Q}$ having zeros of even order and no poles. That is, we will construct them in the non-hyperelliptic components of strata of the form $\mathcal{Q}(2k_1, \ldots, 2k_n)$, with $k_i \geq 1$ and $\sum_{i=1}^{n} k_i = 2g-2$. We will do this by modifying the 1,1-square-tiled surfaces we constructed in Chapter 3. Since quadratic strata correspond to partitions of multiples of 4, the base cases we require are strata of the form $\mathcal{Q}(4k)$ and $\mathcal{Q}(4j+2, 4k+2)$.

Strata of the from $\mathcal{Q}(4k)$

Firstly, suppose that we have a filling pair diagram representing a 1,1-square-tiled surface in $\mathcal{H}(k_1, \ldots, k_n)$. If we are able to change the orientation of one of the intersections without changing the number of and number of sides of the boundary components then the associated 1,1-pillowcase cover will lie in the stratum $\mathcal{Q}(2k_1, \ldots, 2k_n)$. In the language of squares and not filling pair diagrams, if we are able to turn a single square in a 1,1-square-tiled surface upside-down without changing the number of cone-points and their cone angles, then the resulting surface will be a 1,1-pillowcase cover with zeros of double the order.

For an explicit example, recall that the surface in Figure 6.5 is a 1,1-square-tiled surface in $\mathcal{H}(4)$. Flipping (turning upside-down) the rightmost square of this surface, one can check that the number of cone-points remains the same and the cone-angles are fixed. The resulting 1,1-pillowcase cover lies in $\mathcal{Q}(8)$ and is shown in Figure 6.6.

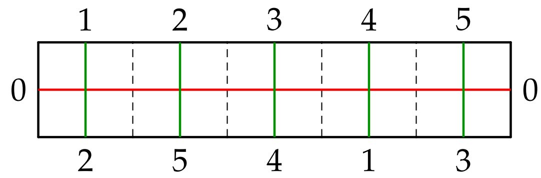

Figure 6.5: A 1,1-square-tiled surface in $\mathcal{H}(4)$.

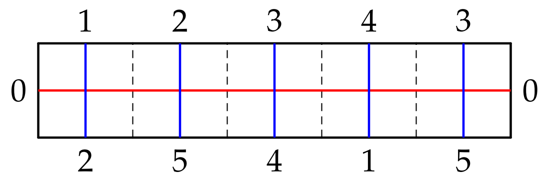

Figure 6.6: A 1,1-pillowcase cover in $\mathcal{Q}(8)$.

We claim that this construction (flipping the fifth square) works for all of the 1,1-square-tiled surfaces lying in $\mathcal{H}^{odd}(2k)$, $k \geq 2$, given by Proposition 3.3.

Lemma 6.8. *Flipping the fifth square of the 1,1-square-tiled surfaces given in Proposition 3.3 does not change the number of cone-points or the associated cone-angles.*

Proof. We prove this by directly observing the effect of the flip on the associated filling pair diagram. This is shown in Figure 6.7. One can see that there is still only one boundary component after the flip and so there is still only one cone-point on the surface occurring with the same cone-angle. □

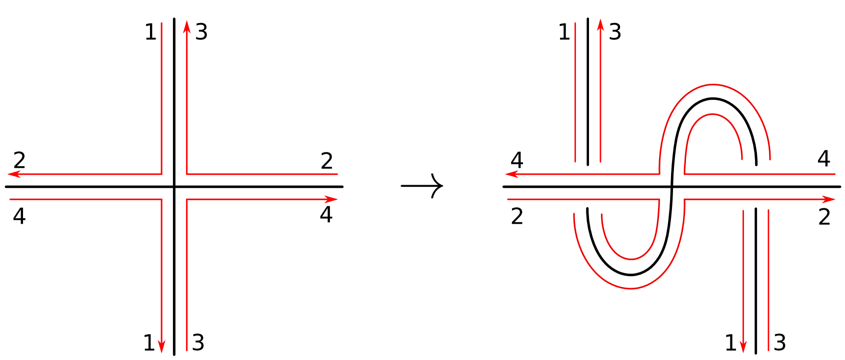

Figure 6.7: Effect on the filling pair diagram of flipping the fifth square.

As such, we have proved the following proposition.

Proposition 6.9. *The generalised permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 \\ 2 & 5 & 4 & 1 & 5 & 0 \end{pmatrix}, \tag{6.2}$$

and, for $g \geq 4$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 6 & 7 & 8 & 9 & \cdots & 2g-4 & 2g-3 & 2g-2 & 2g-1 \\ 2 & 5 & 4 & 7 & 5 & 9 & 6 & 11 & 8 & 13 & \cdots & 2g-4 & 1 & 2g-2 & 0 \end{pmatrix} \tag{6.3}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(8)$ and $\mathcal{Q}(4g-4)$, respectively. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

It can be checked that the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 6 & 7 \\ 2 & 5 & 4 & 7 & 5 & 1 & 6 & 0 \end{pmatrix}$$

given by Proposition 6.9 for the stratum $\mathcal{Q}(12)$ lies in the component $\mathcal{Q}^{reg}(12)$. The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 5 & 4 \\ 2 & 6 & 1 & 5 & 7 & 6 & 7 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in the component $\mathcal{Q}^{irr}(12)$.

Strata of the form $\mathcal{Q}(4j+2, 4k+2)$

To construct 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+2, 4k+2)$, we will mimic the constructions of 1,1-square-tiled surfaces in the strata $\mathcal{H}(2j+1, 2k+1)$ given in the proofs of Propositions 3.7 and 3.8. Indeed, one can check that the combinatorics of Figures 3.14 and 3.16 are preserved if the 1,1-square-tiled surfaces in $\mathcal{H}^{odd}(4j)$ and $\mathcal{H}^{odd}(4j+2)$ are replaced by the 1,1-pillowcase covers in $\mathcal{Q}(8j)$ and $\mathcal{Q}(8j+4)$ given by Proposition 6.9. We will describe the constructions of Propositions 3.7 and 3.8 as right-swaps and left-swaps respectively. Indeed, in each construction the first square on the second surface is, after the two surfaces have been combined, swapped with the square to its right or to its left, respectively.

For $j, k \geq 1$, consider a 1,1-pillowcase cover X_1 in $\mathcal{Q}(8j)$ given by Proposition 6.9, and a 1,1-square-tiled surface X_2 in $\mathcal{H}^{odd}(4k)$, given by Proposition 3.3. One can then check that the 1,1-pillowcase cover obtained from these two by the right-swap method of Proposition 3.7 lies in $\mathcal{Q}(4(j+k)+2, 4(j+k)-2)$. If instead X_1 is a 1,1-pillowcase cover in $\mathcal{Q}(8j+4)$ given by Proposition 6.9, then the 1,1-pillowcase cover obtained from X_1 and X_2 by the right-swap method of Proposition 3.7 lies in $\mathcal{Q}(4(j+k)+2, 4(j+$

$k) + 2)$. Moreover, these surfaces have the minimum number of squares necessary for their respective strata. Hence, we can construct 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+6, 4j+2)$ and $\mathcal{Q}(4j+6, 4j+6)$, for $j \geq 1$.

Now instead, for $j, k \geq 1$, consider a 1,1-pillowcase cover X_1 in $\mathcal{Q}(8j+4)$ given by Proposition 6.9, and a 1,1-square-tiled surface X_2 in $\mathcal{H}^{odd}(4k)$, given by Proposition 3.3. We then have that the 1,1-pillowcase cover obtained from these two by the left-swap method of Proposition 3.8 lies in $\mathcal{Q}(4(2k+j)+2, 4j+2)$. If instead X_2 is a 1,1-square-tiled surface in $\mathcal{H}^{odd}(4k+2)$ given by Proposition 3.3, then the 1,1-pillowcase cover obtained from X_1 and X_2 by the left-swap method of Proposition 3.8 lies in $\mathcal{Q}(4(2k+j)+6, 4j+2)$. Moreover, these surfaces have the minimum number of squares necessary for their respective strata. Hence, we can construct 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+2+4n, 4j+2)$, for $j \geq 1$ and $n \geq 2$.

The smallest of the two zeros in the strata constructed above is strictly greater than 2 and so we do not yet have 1,1-pillowcase covers in strata of the form $\mathcal{Q}(4k+2, 2)$. To deal with these cases, we have the following.

Proposition 6.10. *The permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 6 & 7 & 8 & 9 & 10 \\ 2 & 5 & 4 & 9 & 5 & 8 & 6 & 1 & 10 & 7 & 0 \end{pmatrix} \tag{6.4}$$

and, for $k \geq 4$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 6 & 7 & 8 & 9 & \cdots & 2k-4 & 2k-3 \\ 2 & 5 & 4 & 7 & 5 & 9 & 6 & 11 & 8 & 13 & \cdots & 2k-4 & 2k+3 \end{pmatrix.$$
$$\left. \begin{matrix} 2k-2 & 2k-1 & 2k & 2k+1 & 2k+2 & 2k+3 & 2k+4 \\ 2k-2 & 2k+2 & 2k & 1 & 2k+4 & 2k+1 & 0 \end{matrix} \right) \tag{6.5}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(14, 2)$ and $\mathcal{Q}(4k+2, 2)$, respectively. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

Proof. We observe that when flipping the fifth square in the 1,1-square-tiled surfaces lying in the strata $\mathcal{H}(2k+1, 1)$, $k \geq 3$, given by Proposition 3.9, the number of sides of the boundary components are not changed. As such, the resulting 1,1-pillowcase covers lie in the strata $\mathcal{Q}(4k+2, 2)$, for $k \geq 3$ and are represented by the above permutations. □

From the above, we are missing 1,1-pillowcase covers in the strata $\mathcal{Q}(6, 2)$, $\mathcal{Q}(10, 2)$, $\mathcal{Q}(6, 6)$ and $\mathcal{Q}(2, 2)$. The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 4 & 5 \\ 2 & 6 & 4 & 3 & 5 & 6 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in the stratum $\mathcal{Q}(6,2)$ with the boundary component corresponding to the zero of order 6 leaving the right on the bottom.

The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 5 & 4 & 6 \\ 2 & 7 & 6 & 8 & 1 & 7 & 5 & 8 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in the stratum $\mathcal{Q}(10,2)$ with the boundary component corresponding to the zero of order 10 leaving the right on the bottom.

The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 1 & 3 \\ 2 & 4 & 3 & 4 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in the stratum $\mathcal{Q}(2,2)$. We give this permutation here despite having constructed a representative for $\mathcal{Q}(2,2)$ in the previous section because this permutation has the appropriate form to be used in the cylinder concatenation method.

The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 & 6 & 7 \\ 2 & 7 & 8 & 5 & 6 & 4 & 1 & 8 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 3 \\ 2 & 7 & 1 & 6 & 4 & 8 & 5 & 8 & 0 \end{pmatrix}$$

represent $\mathcal{Q}^{reg}(6,6)$ and $\mathcal{Q}^{irr}(6,6)$, respectively.

Observe that all of the 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+2, 4k+2)$ that we have produced in this section lie in the nonhyperelliptic components. Indeed, we demonstrated in Section 6.3 that 1,1-pillowcase covers with non-separating core curves in the hyperelliptic components require at least $4g-4$ squares which is strictly more than the number exhibited by those constructed in this section.

Handling the emptiness of $\mathcal{Q}(4)$

Recall that the stratum $Q(4)$ is empty. In particular, there is no 1,1-pillowcase cover that we can use to build 1,1-pillowcase covers in strata with zeros of order 4. We can use the same technique that we did to construct 1,1-square-tiled surfaces in Abelian strata with zeros of order 2. Recall that we did this by adding the combinatorics shown in Figure 6.8 to the right-hand side of the filling pair diagram. In the process, this added 8 sides to the boundary component that left the filling pair diagram on the bottom which, in the case of Abelian strata, added two to the order of the associated zero. In the quadratic case, this will add a zero of order 4 to the surface and add 4 to the order of the zero associated to the boundary component leaving the bottom of the filling pair diagram.

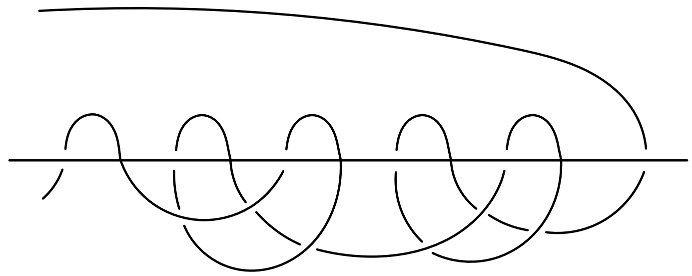

Figure 6.8: Filling pair diagram combinatorics for adding $\mathcal{Q}(4)$.

We first have the following proposition.

Proposition 6.11. *The permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 6 & 7 & 8 & 9 & 10 \\ 2 & 5 & 4 & 6 & 5 & 8 & 10 & 7 & 1 & 9 & 0 \end{pmatrix}, \tag{6.6}$$

and, for $k \geq 4$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 6 & 7 & 8 & 9 & \cdots & 2k-4 & 2k-3 \\ 2 & 5 & 4 & 7 & 5 & 9 & 6 & 11 & 8 & 13 & \cdots & 2k-4 & 2k \\ & & & & 2k-2 & 2k-1 & 2k & 2k+1 & 2k+2 & 2k+3 & 2k+4 \\ & & & & 2k-2 & 2k+2 & 2k+4 & 2k+1 & 1 & 2k+3 & 0 \end{pmatrix} \tag{6.7}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(12,4)$ and $\mathcal{Q}(4k,4)$, respectively. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

Proof. The proof is similar to the proof of Proposition 3.5 but makes use of the permutations given by Proposition 6.23 and the combinatorics from Figure 6.8. □

The missing strata $\mathcal{Q}(4,4)$ and $\mathcal{Q}(8,4)$ are represented by the permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 1 & 5 \\ 2 & 6 & 4 & 6 & 5 & 3 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 5 \\ 2 & 7 & 1 & 8 & 3 & 8 & 4 & 6 & 0 \end{pmatrix},$$

respectively.

Similar to the argument given for the strata $\mathcal{H}(2j+1, 2k+1, 2)$ in Section 3.4, we now consider which zeros are associated to the boundary components of the filling pair diagrams of the 1,1-pillowcase covers we constructed in the strata $\mathcal{Q}(4j+2, 4k+2)$.

One can check that for the 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+2, 4j+2+4n)$, for $j \geq 1$ and $n \geq 2$, the boundary component that leaves on the bottom is the one associated to the zero of order $4j+2+4n$. Hence, we can construct 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+2, 4j+2+4n, 4)$, for $j \geq 1$ and $n \geq 3$. Using the 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+2,4j+2)$, $j \geq 1$, we can construct 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+6,4j+2,4)$, $j \geq 1$. Moreover, the 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+2,2)$, $j \geq 1$, have the boundary component that leaves on the bottom being the one associated to the zero of order $4j+2$, and so we can construct 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+6,2,4)$, $j \geq 1$. Finally, we observe that for the 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+6,4j+2)$, $j \geq 1$, the boundary component that leaves on the bottom is the one associated to the zero of order $4j+6$. Hence, we can construct 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+10,4j+2,4)$, $j \geq 1$.

We are missing the strata $\mathcal{Q}(4j+2,4j+2,4)$ for $j \geq 0$. For $j \geq 2$, we have the following.

Proposition 6.12. *The permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 & 7 & 8 & 9 & 10 \\ 2 & 6 & 4 & 10 & 8 & 6 & 1 & 9 & 7 & 5 & 0 \end{pmatrix}, \tag{6.8}$$

and, for $k \geq 2$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 & 7 & 8 & 9 & 10 & 11 \\ 2 & 6 & 4 & 10 & 8 & 6 & 12 & 9 & 7 & 5 & 14 & 11 \\ & & & & & & 12 & 13 & 14 & \cdots & 4k+3 & 4k+4 & 4k+5 & 4k+6 \\ & & & & & & 16 & 13 & 18 & \cdots & 4k+3 & 1 & 4k+5 & 0 \end{pmatrix} \tag{6.9}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(10,6)$ and $\mathcal{Q}(4k+6,4k+2)$, respectively, with the boundary component that leaves the filling pair diagram on the bottom being associated to the zeros of order 6 and $4k+2$, respectively. Moreover, these surfaces have the minimum number of squares necessary for their respective strata.

Proof. The proof is analogous to the proof of Proposition 3.12. Indeed, one can check that we have flipped the sixth square of the 1,1-square-tiled surfaces constructed in Proposition 3.12 and that this does not change the number of sides of the boundary components or which boundary component left the filling pair diagram on the bottom. □

Using these permutations, we can therefore construct 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+2,4j+2,4)$, for $j \geq 2$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 1 & 3 & 5 \\ 2 & 6 & 7 & 6 & 5 & 4 & 7 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 4 & 9 & 3 \\ 2 & 10 & 1 & 6 & 8 & 9 & 11 & 10 & 11 & 5 & 7 & 0 \end{pmatrix}$$

represent 1,1-pillowcase covers in the strata $\mathcal{Q}(2,2,4)$ and $\mathcal{Q}(6,6,4)$, respectively.

Finally, the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 7 & 8 \\ 2 & 9 & 7 & 1 & 9 & 5 & 8 & 6 & 3 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in $\mathcal{Q}(4,4,4)$, and this completes the work of this section.

6.5 Odd order zeros

In this section we construct 1,1-pillowcase covers in every connected component of every quadratic stratum having zeros of odd order and no poles. Since strata correspond to partitions of multiples of 4, the base cases we require are strata of the form $\mathcal{Q}(4j+1,4k+3)$, $\mathcal{Q}(4j+1,4k+1,4l+1,4m+1)$ and $\mathcal{Q}(4j+3,4k+3,4l+3,4m+3)$.

Strata of the form $\mathcal{Q}(4j+1,4k+3)$

We begin by constructing 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+1,4k+3)$. For a large number of cases, this is carried out in the following proposition which makes further use of the technique of flipping squares that we introduced in the previous section.

Proposition 6.13. *Let $j,k \geq 1$.*

Let X_1 be a 1,1-square-tiled surface in the stratum $\mathcal{H}(2(j+k)+1,2(j+k)-1)$ given by Proposition 3.7. Then flipping the third and fourth squares of X_1 gives a 1,1-pillowcase cover in the stratum $\mathcal{Q}(4(j+k)+3,4(j+k)-3)$, while flipping the third, fourth and fifth squares gives a 1,1-pillowcase cover in the stratum $\mathcal{Q}(4(j+k)+1,4(j+k)-1)$.

Let X_2 be a 1,1-square-tiled surface in the stratum $\mathcal{H}(2(j+k)+1,2(j+k)+1)$ given by Proposition 3.7. Then flipping the third and fourth, or third, fourth and fifth squares of X_2 gives a 1,1-pillowcase cover in the stratum $\mathcal{Q}(4(j+k)+3,4(j+k)+1)$.

Let X_3 be a 1,1-square-tiled surface in the stratum $\mathcal{H}(2j+1+4k,2j+1)$ given by Proposition 3.8. Then flipping the third and fourth squares of X_3 gives a 1,1-pillowcase cover in the stratum $\mathcal{Q}(4j+1+8k,4j+3)$, while flipping the third, fourth and fifth squares gives a 1,1-pillowcase cover in the stratum $\mathcal{Q}(4j+3+8k,4j+1)$.

Let X_4 be a 1,1-square-tiled surface in the stratum $\mathcal{H}(2j+1+4k+2,2j+1)$ given by the Proposition 3.8. Then flipping the third and fourth squares of X_4 gives a 1,1-pillowcase cover in the stratum $\mathcal{Q}(4j+5+8k,4j+3)$, while flipping the third, fourth and fifth squares gives a 1,1-pillowcase cover in the stratum $\mathcal{Q}(4j+7+8k,4j+1)$.

All of the 1,1-pillowcase covers produced have the minimal number of squares required for a 1,1-pillowcase cover in their respective strata.

Proof. We give the proof for the case of flipping the third and fourth squares of the surface X_1. The other cases can be proved similarly.

We begin by observing that the combinatorics of the filling pair diagram around the third and fourth vertices are as shown in Figure 6.9. This diagram is read in the same way as the diagrams used in the proofs of Propositions 3.7 and 3.8.

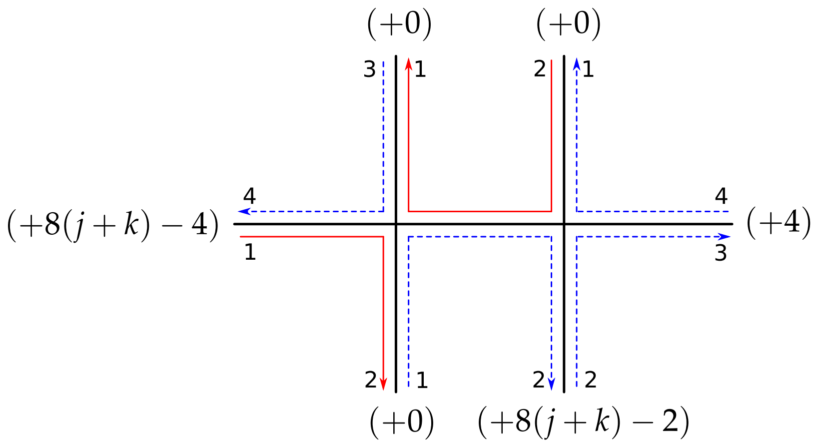

Figure 6.9: Combinatorics around vertices 3 and 4 before they are flipped.

After flipping these squares, the combinatorics of the filling pair diagram around the third and fourth vertices are as shown in Figure 6.10. We have one boundary component with $8(j+k)-2$ sides corresponding to a zero of order $4(j+k)-3$, and a second boundary component with $8(j+k)+10$ sides corresponding to a zero of order $4(j+k)+3$. That is, the resulting 1,1-pillowcase cover lies in the stratum $\mathcal{Q}(4(j+k)+3, 4(j+k)-3)$ as claimed. $\square$

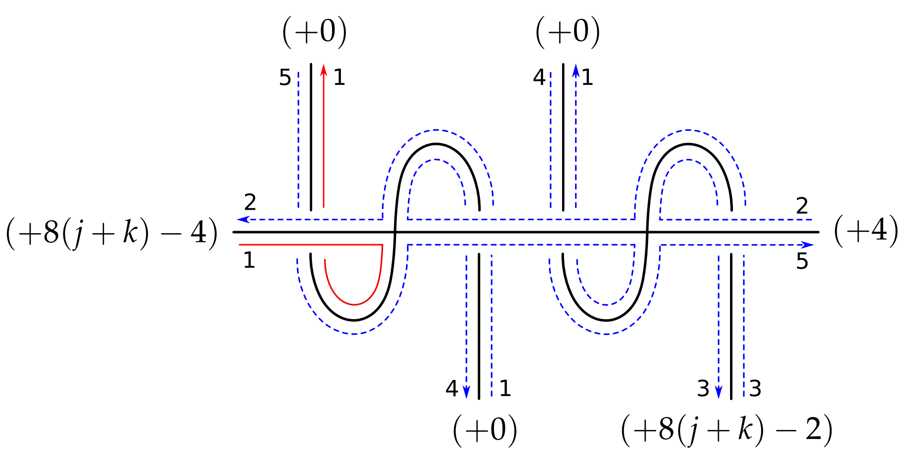

Figure 6.10: Combinatorics around vertices 3 and 4 after they are flipped.

Fixing $k=1$ in the cases of X_1 and X_2, we can produce 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+7, 4j+1)$, $\mathcal{Q}(4j+5, 4j+3)$, and $\mathcal{Q}(4(j+1)+3, 4(j+1)+1)$, for $j \geq 1$.

We are however missing the stratum $\mathcal{Q}(7,5)$ for which the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 1 & 6 & 7 \\ 2 & 8 & 6 & 3 & 7 & 5 & 8 & 4 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover.

We also remark that for the surfaces X_1, X_3, and X_4, after the flips we have the boundary component leaving the filling pair diagram on the bottom being the one associated to the zero of larger order. For X_2, flipping the third and fourth squares gives the boundary component leaving the filling pair diagram on the bottom being the one associated to the zero of order $4(j+k)+1$, while flipping the third, fourth and fifth squares gives the boundary component leaving the filling pair diagram on the bottom being the one associated to the zero of order $4(j+k)+3$.

For strata $\mathcal{Q}(4k+1,3)$, $k \geq 3$, we have the following.

Proposition 6.14. *The permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 3 & 4 \\ 2 & 5 & 6 & 4 & 5 & 6 & 0 \end{pmatrix}, \tag{6.10}$$

and, for $k \geq 1$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 8 & 3 & 4 & 7 & 8 & 9 & 10 \\ 2 & 5 & 6 & 4 & 5 & 6 & 10 & 7 & 12 & 9 & 14 \end{pmatrix.\atop \begin{matrix} 11 & 12 & \cdots & 2k+3 & 2k+4 & 2k+5 & 2k+6 \\ 11 & 16 & \cdots & 2k+3 & 1 & 2k+5 & 0 \end{matrix}\Biggr) \tag{6.11}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(5,3)$ and $\mathcal{Q}(4k+5,3)$, respectively. Moreover, these surfaces have the minimum number of squares required for their respective strata.

Proof. The proof is similar to the proof of Proposition 3.3. That is, we start with the filling pair diagram associated to permutation 6.10 and modify it as in Proposition 3.3 to increase the zero of order 5 as required. □

The permutation representative given for $\mathcal{Q}(9,3)$ in the above proposition represents $\mathcal{Q}^{reg}(9,3)$. The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 6 \\ 2 & 7 & 1 & 8 & 7 & 5 & 8 & 4 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in $\mathcal{Q}^{irr}(9,3)$, respectively.

In the following proposition, proved analogously, we construct 1,1-pillowcase covers in the strata $\mathcal{Q}(4k+3,1)$ for $k \geq 1$. Recall that the stratum $\mathcal{Q}(3,1)$ is empty.

Proposition 6.15. *The permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 \\ 2 & 6 & 5 & 1 & 4 & 6 & 0 \end{pmatrix}, \tag{6.12}$$

and, for $k \geq 1$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 & 7 & 8 & 9 & 10 & & & & & & \\ 2 & 6 & 5 & 8 & 4 & 6 & 10 & 7 & 12 & 9 & 14 & & & & & & \\ & & & & & 11 & 12 & \cdots & 2k+3 & 2k+4 & 2k+5 & 2k+6 \\ & & & & & 11 & 16 & \cdots & 2k+3 & 1 & 2k+5 & 0 \end{pmatrix} \tag{6.13}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(7,1)$ *and* $\mathcal{Q}(4k+7,1)$, *respectively. Moreover, these surfaces have the minimum number of squares required for their respective strata.*

Strata of the form $\mathcal{Q}(k_1, k_2, k_3, k_4)$, $k_i \equiv 1 \bmod 4$ or $k_i \equiv 3 \bmod 4$

We now construct 1,1-pillowcase covers in strata of the form $\mathcal{Q}(4j+1, 4k+1, 4l+1, 4m+1)$ and $\mathcal{Q}(4j+3, 4k+3, 4l+3, 4m+3)$. This is the most delicate construction we have considered so far. Indeed, we require the ability to increase the orders of any one of the four zeros of the surface. We will achieve this by arranging particular combinatorics within the filling pair diagrams that can be modified to leave only local changes. We first observe the effects of the following ways of modifying filling pair diagrams.

The first allows us to take a zero of order k to a zero of order $k+4j$ for $j \geq 2$ and is demonstrated in Figure 6.11. The boundary component corresponding to the zero of order k is shown in red.

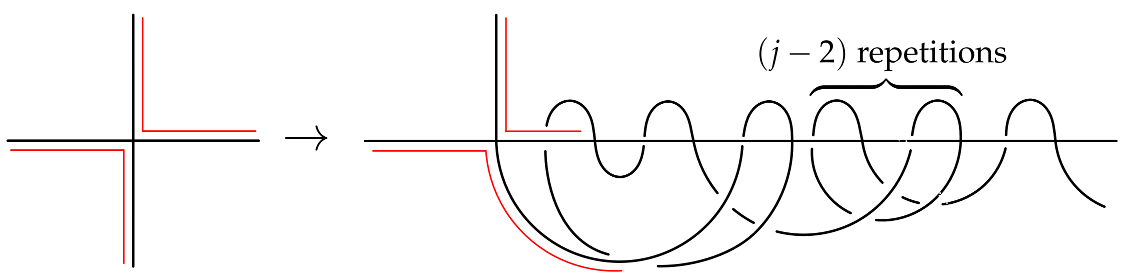

Figure 6.11: Filling pair diagram modification taking a zero of order k to a zero of order $k+4j$.

Indeed, one can show that $8j$ sides are added to the boundary component under this operation. The combinatorics of the second and third right-most vertices are repeated $(j-2)$ times. That is, the combinatorics explicitly demonstrated in Figure 6.11 is that of $j=3$.

The second method allows us to take two zeros of orders k_1 and k_2 to zeros of orders $k_1 + 4$ and $k_2 + 4$ and is demonstrated in Figure 6.12. The boundary components of the two zeros are shown in solid red and dashed blue, respectively.

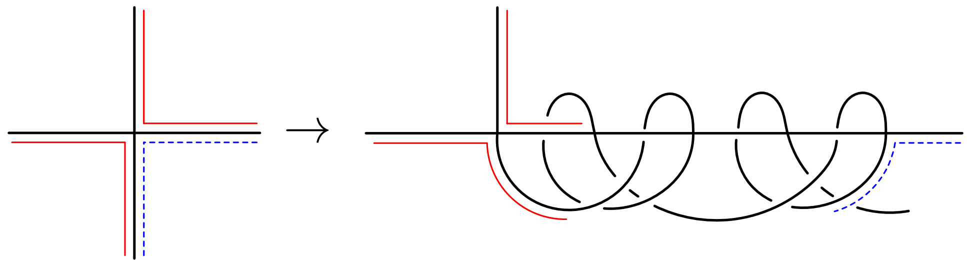

Figure 6.12: Filling pair diagram modification taking two zeros of orders k_1 and k_2 to zeros of order $k_1 + 4$ and $k_2 + 4$.

One can check that 8 sides are added to each of the boundary components.

We use these modifications in the following propositions.

Proposition 6.16. *By applying the filling pair diagram modifications of Figures 6.11 and 6.12 to the filling pair diagrams associated to the permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 6 & 8 \\ 2 & 5 & 4 & 9 & 10 & 7 & 9 & 3 & 8 & 10 & 0 \end{pmatrix} \tag{6.14}$$

for $\mathcal{Q}(3,3,3,3)$*, and*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 & 8 \\ 2 & 10 & 11 & 5 & 11 & 7 & 9 & 12 & 4 & 6 & 12 & 3 & 0 \end{pmatrix} \tag{6.15}$$

for $\mathcal{Q}(7,3,3,3)$*, we can construct 1,1-pillowcase covers in all strata of the form* $\mathcal{Q}(4j+3, 4k+3, 4l+3, 4m+3)$*, for* $j,k,l,m \geq 0$*.*

By applying the filling pair diagram modifications of Figures 6.11 and 6.12 to the filling pair diagrams associated to the permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 4 & 10 & 5 & 6 & 11 \\ 2 & 12 & 7 & 13 & 8 & 1 & 11 & 3 & 13 & 10 & 14 & 12 & 9 & 14 & 0 \end{pmatrix} \tag{6.16}$$

for $\mathcal{Q}(5,5,5,5)$*, and*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 9 \\ 2 & 16 & 8 & 15 & 14 & 5 & 16 & 7 & 13 & 12 & 6 & 11 & 10 & 3 & 1 & 4 & 0 \end{pmatrix} \tag{6.17}$$

for $\mathcal{Q}(9,5,5,5)$*, we can construct 1,1-pillowcase covers in all strata of the form* $\mathcal{Q}(4j+1, 4k+$

$1, 4l+1, 4m+1)$, for $j, k, l, m \geq 1$.

Proof. One can check that the boundary component for each zero of order 3 or of order 5 in the filling pair diagrams associated to the above permutations has the combinatorics shown on the left of Figure 6.11. Moreover, one can check that the boundary components for the zeros in the filling pair diagrams associated to permutations (6.14) and (6.16) can be paired up so that each pair has the combinatorics shown on the left of Figure 6.12. It can also be checked that two of the boundary components associated to zeros of order 3 and of order 5 in the filling pair diagrams associated to permutations (6.15) and (6.17), respectively, have the combinatorics shown on the left of Figure 6.12. We can therefore perform the modifications demonstrated in Figures 6.11 and 6.12.

Applying these modifications to the filling pair diagrams associated to permutations (6.14) and (6.16), we can produce 1,1-pillowcase covers in $\mathcal{Q}(4j+3, 4k+3, 4l+3, 4m+3)$, $\mathcal{Q}(4j+5, 4k+5, 4l+5, 4m+5)$, $\mathcal{Q}(4j+3, 4k+3, 7, 7)$ and $\mathcal{Q}(4j+5, 4k+5, 9, 9)$, for $j, k, l, m \neq 1$, and in $\mathcal{Q}(7, 7, 7, 7)$ and $\mathcal{Q}(9, 9, 9, 9)$.

Applying these modifications to the filling pair diagrams associated to the permutations (6.15) and (6.17), we can produce 1,1-pillowcase covers in $\mathcal{Q}(4j+3, 4k+3, 4l+3, 7)$, $\mathcal{Q}(4j+5, 4k+5, 4l+5, 9)$, $\mathcal{Q}(4j+3, 7, 7, 7)$ and $\mathcal{Q}(4j+5, 9, 9, 9)$, for $j, k, l \neq 1$. This completes the proof of the proposition. □

The permutation representative given for $\mathcal{Q}(3, 3, 3, 3)$ in the proposition above represents a 1,1-pillowcase cover in $\mathcal{Q}^{reg}(3, 3, 3, 3)$, while the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 7 \\ 2 & 10 & 4 & 8 & 1 & 10 & 6 & 9 & 5 & 3 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in $\mathcal{Q}^{irr}(3, 3, 3, 3)$.

Proposition 6.17. *By applying the filling pair diagram modifications of Figures 6.11 and 6.12 to the filling pair diagrams associated to the permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 7 \\ 2 & 7 & 8 & 6 & 4 & 1 & 5 & 8 & 0 \end{pmatrix} \tag{6.18}$$

for $\mathcal{Q}(5, 1, 1, 1)$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 1 & 6 & 3 & 7 & 8 \\ 2 & 9 & 10 & 5 & 8 & 6 & 4 & 7 & 9 & 10 & 0 \end{pmatrix} \tag{6.19}$$

for $\mathcal{Q}(5, 5, 1, 1)$, and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 7 & 8 & 9 & 10 & 8 \\ 2 & 11 & 7 & 10 & 9 & 12 & 1 & 6 & 11 & 3 & 5 & 12 & 0 \end{pmatrix} \tag{6.20}$$

for $\mathcal{Q}(9,5,1,1)$*, we can construct 1,1-pillowcase covers in strata of the form* $\mathcal{Q}(4j+1,1,1,1)$ *for* $j \geq 1$*,* $\mathcal{Q}(4j+1,4k+1,1,1)$ *for* $j,k \geq 1$*, and* $\mathcal{Q}(4j+1,4k+1,4l+1,1)$ *for* $j,k,l \geq 1$*.*

Proof. The proof is analogous to the proof of Proposition 6.16. □

Handling the emptiness of $\mathcal{Q}(3,1)$ and the hyperellipticity of $\mathcal{Q}(1,1,1,1)$

Recall that the stratum $\mathcal{Q}(3,1)$ is empty and that the minimum number of squares required for a 1,1-pillowcase cover in $\mathcal{Q}(1,1,1,1)$ with non-separating core curves was $4g = 8$ which is strictly greater than the theoretical minimum of $2g+n-2 = 6$. We must therefore describe how to build strata with odd order zeros that should be built from these strata.

First let us suppose that we wish to construct 1,1-pillowcase covers in strata of the form $\mathcal{Q}(4j+1,4k+3,3,1)$. If $j,k \geq 1$, then we can construct such surfaces by concatenating 1,1-pillowcase covers from $\mathcal{Q}(4j+1,3)$ and $\mathcal{Q}(4k+3,1)$ which we have already constructed.

The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 5 \\ 2 & 8 & 4 & 1 & 6 & 3 & 8 & 7 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 5 & 6 & 7 & 8 & 9 \\ 2 & 5 & 7 & 10 & 9 & 8 & 10 & 1 & 4 & 6 & 0 \end{pmatrix}$$

represent 1,1-pillowcase covers in the stratum $\mathcal{Q}(3,3,1,1)$, and $\mathcal{Q}(7,3,1,1)$, respectively. For strata of the form $\mathcal{Q}(7+4j,3,1,1)$, $j \geq 1$, one can concatenate the combinatorics shown in Figure 6.13 with the filling pair diagrams associated to the 1,1-pillowcase covers in the strata $\mathcal{Q}(7+4(j-1),1)$ constructed above.

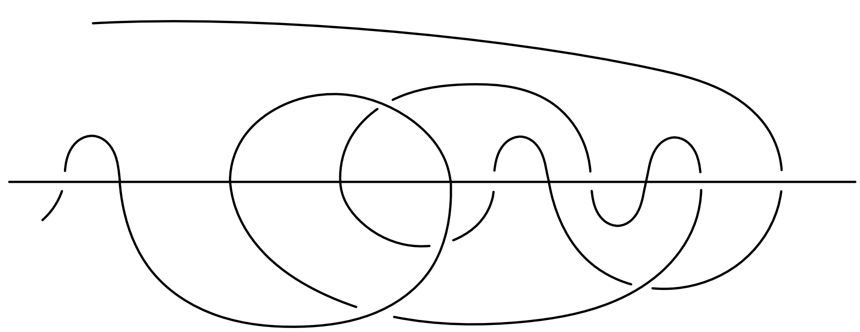

Figure 6.13: Filling pair diagram combinatorics for adding $\mathcal{Q}(3,1)$.

The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 5 & 9 \\ 2 & 10 & 4 & 8 & 6 & 3 & 1 & 7 & 9 & 10 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in the stratum $\mathcal{Q}(5,3,3,1)$. We can construct 1,1-pillowcase covers in strata of the form $\mathcal{Q}(5+4k,3,3,1)$, $k \geq 1$, by concatenating the combinatorics in Figure 6.13 with the filling pair diagrams associated to 1,1-pillowcase covers in the strata $\mathcal{Q}(5+4(k-1),3)$ constructed above.

The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 3 & 8 \\ 2 & 11 & 5 & 12 & 7 & 1 & 11 & 9 & 6 & 12 & 10 & 4 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in $\mathcal{Q}(3^3,1^3)$.

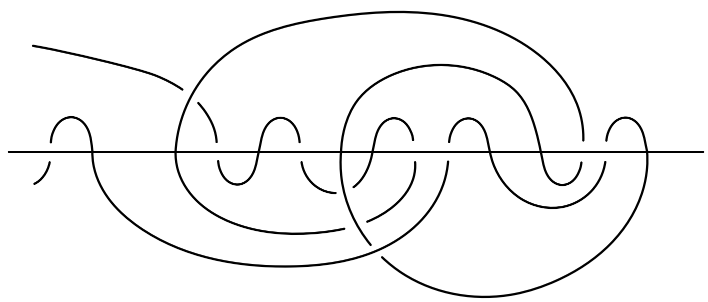

Figure 6.14: Filling pair diagram combinatorics for adding $\mathcal{Q}(1,1,1,1)$.

For $j,k \geq 1$, we can construct a 1,1-pillowcase cover in $\mathcal{Q}(4j+1,4k+3,1^4)$ by concatenating 1,1-pillowcase covers from $\mathcal{Q}(4j+1,1,1,1)$ and $\mathcal{Q}(4k+3,1)$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 4 & 7 & 5 \\ 2 & 8 & 9 & 10 & 8 & 3 & 7 & 6 & 9 & 10 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 3 & 9 & 5 & 10 \\ 2 & 9 & 6 & 11 & 10 & 12 & 8 & 4 & 7 & 12 & 1 & 11 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 3 & 9 & 5 & 10 \\ 2 & 9 & 6 & 11 & 12 & 8 & 1 & 10 & 4 & 11 & 7 & 12 & 0 \end{pmatrix}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(3,1^5)$, $\mathcal{Q}(7,1^5)$, and $\mathcal{Q}(5,3,1^4)$, respectively. For, $j,k \geq 1$, we can construct 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+5,3,1^4)$ and $\mathcal{Q}(4k+7,1^5)$ by concatenating the combinatorics shown in Figure 6.14 to the filling pair diagrams associated to the 1,1-pillowcase covers in $\mathcal{Q}(4(j-1)+5,3)$ and $\mathcal{Q}(4(k-1)+$

$7,1)$, respectively.

The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 4 & 5 & 10 & 11 \\ 2 & 12 & 9 & 11 & 13 & 7 & 14 & 6 & 13 & 1 & 3 & 14 & 8 & 12 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 7 & 8 & 9 & 10 & 5 & 6 & 10 & 11 & 12 \\ 2 & 13 & 9 & 11 & 14 & 15 & 8 & 13 & 3 & 7 & 15 & 16 & 1 & 14 & 12 & 16 & 0 \end{pmatrix}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(3^2,1^6)$ and $\mathcal{Q}(3,1^9)$, respectively.

We now consider how to add $\mathcal{Q}(3,1)$ and $\mathcal{Q}(1,1,1,1)$ to strata of the form $\mathcal{Q}(4j+1,4k+1,4l+1,4m+1)$ and $\mathcal{Q}(4j+3,4k+3,4l+3,4m+3)$.

For $j,k,l,m \geq 0$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+5,4k+5,4l+1,4m+1,3,1)$ by concatenating 1,1-pillowcase covers from $\mathcal{Q}(4j+5,4l+1,4m+1,1)$ and $\mathcal{Q}(4k+5,3)$. We have already constructed 1,1-pillowcase covers in strata of the form $\mathcal{Q}(4j+5,3,1^4)$, $j \geq 0$.

For $j \geq 1$ and $k,l,m \geq 0$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+3,4k+3,4l+3,4m+3,3,1)$ by concatenating 1,1-pillowcase covers from the strata $\mathcal{Q}(4j+3,1)$ and $\mathcal{Q}(3,4k+3,4l+3,4m+3)$. The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 5 & 8 & 9 & 10 & 11 & 10 & 12 \\ 2 & 13 & 14 & 9 & 4 & 1 & 6 & 14 & 11 & 13 & 3 & 12 & 8 & 7 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in $\mathcal{Q}(3^5,1)$.

For $j,k,l,m \geq 0$, we can construct a 1,1-pillowcase cover in the stratum $\mathcal{Q}(4j+5,4k+5,4l+1,4m+1,1^4)$ by concatenating 1,1-pillowcase covers from $\mathcal{Q}(4k+5,4l+1,4m+1,1)$ and $\mathcal{Q}(4j+5,1,1,1)$. The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 6 & 10 & 11 & 7 & 9 \\ 2 & 12 & 13 & 11 & 8 & 1 & 10 & 13 & 14 & 5 & 4 & 14 & 12 & 3 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in $\mathcal{Q}(5,1^7)$. For $j \geq 1$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+5,1^7)$ by concatenating the combinatorics in Figure6.14 with the filling pair diagrams associated to 1,1-pillowcase covers in $\mathcal{Q}(4(j-1)+5,1,1,1)$ which we have already constructed. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 & 6 & 7 & 8 & 9 & 10 & 7 \\ 2 & 6 & 1 & 11 & 10 & 8 & 12 & 5 & 4 & 12 & 9 & 11 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 7 & 8 & 9 & 10 & 11 & 8 & 12 & 9 & 11 & 13 & 14 \\ 2 & 7 & 12 & 10 & 13 & 15 & 16 & 5 & 17 & 1 & 18 & 15 & 14 & 17 & 6 & 4 & 16 & 18 & 0 \end{pmatrix}$$

represents 1,1-pillowcase covers in $\mathcal{Q}(1^8)$ and $\mathcal{Q}(1^{12})$, respectively.

For $j,k,l,m \geq 0$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+3, 4k+3, 4l+3, 4m+3, 1^4)$ by concatenating 1,1-pillowcase cover from $\mathcal{Q}(4j+3,1)$, $\mathcal{Q}(4k+3,1)$, $\mathcal{Q}(4l+3,1)$, and $\mathcal{Q}(4m+3,1)$. We must make use of the permutations constructed above if any of j,k,l or m are equal to zero.

6.6 General strata

In this section, we will construct 1,1-pillowcase covers in strata having both even and odd order zeros and no poles.

Strata of the form $\mathcal{Q}(4j+2, 4k+1, 4l+1)$ and $\mathcal{Q}(4j+2, 4k+3, 4l+3)$

We will begin by constructing 1,1-pillowcase covers in strata of the form $\mathcal{Q}(4j+2, 4k+1, 4l+1)$ and $\mathcal{Q}(4j+2, 4k+3, 4l+3)$. We will make use of the filling pair diagram modifications considered in Figures 6.11 and 6.12 of the previous section.

Proposition 6.18. *By applying the filling pair diagram modifications of Figures 6.11 and 6.12 to the filling pair diagrams associated to the permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 & 6 \\ 2 & 6 & 5 & 7 & 1 & 7 & 4 & 0 \end{pmatrix} \tag{6.21}$$

for $\mathcal{Q}(2,5,1)$*,*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 1 & 4 & 6 & 7 \\ 2 & 8 & 6 & 5 & 8 & 9 & 7 & 9 & 3 & 0 \end{pmatrix} \tag{6.22}$$

for $\mathcal{Q}(2,5,5)$*,*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 \\ 2 & 6 & 4 & 7 & 1 & 7 & 5 & 0 \end{pmatrix} \tag{6.23}$$

for $\mathcal{Q}(6,1,1)$*, and*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 4 & 6 & 7 & 8 \\ 2 & 8 & 7 & 5 & 3 & 6 & 9 & 1 & 9 & 0 \end{pmatrix} \tag{6.24}$$

for $\mathcal{Q}(6,5,1)$*, we can construct 1,1-pillowcase covers in all strata of the form* $\mathcal{Q}(4j+2, 4k+1, 4l+1)$*, for* $j,k,l \geq 0$*, with* $(j,k,l) \neq (0,0,0)$*.*

By applying the filling pair diagram modifications of Figures 6.11 and 6.12 to the filling pair diagrams associated to the permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 \\ 2 & 5 & 7 & 1 & 6 & 3 & 7 & 0 \end{pmatrix} \tag{6.25}$$

for $\mathcal{Q}(2,3,3)$*,*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 4 \\ 2 & 8 & 7 & 3 & 9 & 1 & 6 & 5 & 9 & 0 \end{pmatrix} \tag{6.26}$$

for $\mathcal{Q}(2,7,3)$*,*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 7 & 8 \\ 2 & 9 & 1 & 9 & 6 & 8 & 7 & 5 & 3 & 0 \end{pmatrix} \tag{6.27}$$

for $\mathcal{Q}(6,3,3)$*,*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 7 & 8 & 9 & 10 \\ 2 & 11 & 5 & 11 & 1 & 8 & 7 & 10 & 9 & 4 & 6 & 0 \end{pmatrix} \tag{6.28}$$

for $\mathcal{Q}(6,7,3)$*, and*

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 3 & 8 & 9 & 10 & 11 & 12 \\ 2 & 12 & 4 & 6 & 13 & 5 & 13 & 9 & 11 & 10 & 8 & 1 & 7 & 0 \end{pmatrix} \tag{6.29}$$

for $\mathcal{Q}(10,7,3)$*, we can construct 1,1-pillowcase covers in all strata of the form* $\mathcal{Q}(4j+2, 4k+3, 4l+3)$*, for* $j,k,l \geq 0$*.*

Proof. The proof is analogous to the proof of Proposition 6.16 in that the boundary components associated to the zeros of odd order in the filling pair diagrams associated to the above permutations have the combinatorics shown in the left of Figures 6.11 and 6.12 and so can be modified as in these figures to produce the desired strata. □

The permutation representative given for $\mathcal{Q}(6,3,3)$ in the proposition above represents a 1,1-pillowcase cover in $\mathcal{Q}^{reg}(6,3,3)$, while the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 5 & 6 & 7 & 8 \\ 2 & 8 & 6 & 9 & 1 & 4 & 7 & 5 & 9 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in $\mathcal{Q}^{irr}(6,3,3)$.

Handling the emptiness of $\mathcal{Q}(4)$ and $\mathcal{Q}(3,1)$, and the hyperellipticity of $\mathcal{Q}(2,1,1)$ and $\mathcal{Q}(1,1,1,1)$

As in the previous two sections, we must now deal with fact that $\mathcal{Q}(4)$ and $\mathcal{Q}(3,1)$ are empty, and that 1,1-pillowcase covers in $\mathcal{Q}(2,1,1)$ and $\mathcal{Q}(1,1,1,1)$ require more than the theoretical minimum number of squares.

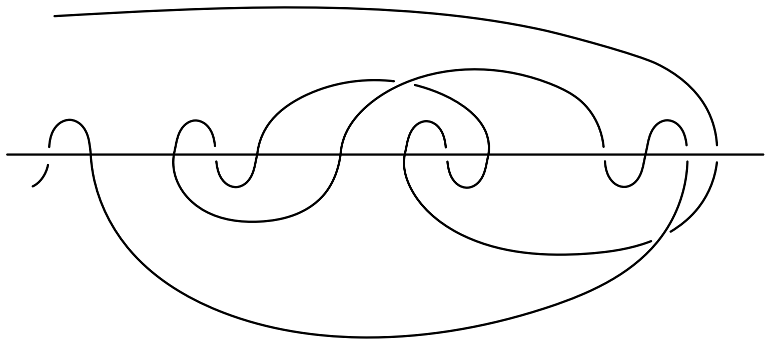

Figure 6.15: Filling pair diagram combinatorics for adding $\mathcal{Q}(2,1,1)$.

We begin by constructing 1,1-pillowcase covers in strata $\mathcal{Q}(4j,3,1)$, $\mathcal{Q}(4j,1,1,1,1)$, and $\mathcal{Q}(4j,2,1,1)$, and in strata $\mathcal{Q}(4j+2,4k+2,3,1)$, $\mathcal{Q}(4j+2,4k+2,1,1,1,1)$, and $\mathcal{Q}(4j+2,4k+2,2,1,1)$. In Section 6.4, we discussed how to add a zero of order 4 to the 1,1-pillowcase covers we had already built in that section by adding the combinatorics shown in Figure 6.8 to the associated filling pair diagrams. Performing the same construction with the combinatorics given in Figures 6.13, 6.14, and 6.15, will give 1,1-pillowcase covers in all of the above strata apart from $\mathcal{Q}(4,3,1)$, $\mathcal{Q}(4,1,1,1,1)$, $\mathcal{Q}(4,2,1,1)$, $\mathcal{Q}(8,3,1)$, $\mathcal{Q}(8,1,1,1,1)$, $\mathcal{Q}(8,2,1,1)$, $\mathcal{Q}(2,2,3,1)$, $\mathcal{Q}(2^2,1^4)$, $\mathcal{Q}(2^3,1^2)$, $\mathcal{Q}(6,6,3,1)$, $\mathcal{Q}(6,6,1,1,1,1)$, and $\mathcal{Q}(6,6,2,1,1)$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 & 6 \\ 2 & 6 & 5 & 7 & 1 & 4 & 7 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 6 \\ 2 & 9 & 5 & 1 & 8 & 7 & 4 & 9 & 3 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 5 \\ 2 & 8 & 1 & 7 & 6 & 4 & 8 & 3 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 7 & 8 \\ 2 & 9 & 8 & 3 & 7 & 5 & 9 & 1 & 6 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 7 & 5 & 6 & 8 \\ 2 & 8 & 9 & 10 & 3 & 7 & 11 & 9 & 1 & 10 & 11 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 3 & 8 & 9 \\ 2 & 8 & 10 & 1 & 10 & 9 & 4 & 7 & 6 & 5 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 4 \\ 2 & 6 & 8 & 1 & 7 & 5 & 3 & 8 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 7 \\ 2 & 10 & 9 & 8 & 10 & 3 & 6 & 1 & 5 & 4 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 5 & 8 \\ 2 & 7 & 9 & 1 & 4 & 8 & 9 & 6 & 3 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 1 & 6 & 7 & 5 & 8 & 9 & 10 \\ 2 & 9 & 4 & 8 & 6 & 11 & 7 & 10 & 12 & 11 & 12 & 3 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 4 & 12 & 9 \\ 2 & 13 & 7 & 14 & 11 & 5 & 13 & 3 & 8 & 1 & 12 & 6 & 10 & 14 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 5 & 6 & 7 & 8 & 9 & 10 & 8 & 11 \\ 2 & 12 & 6 & 13 & 7 & 11 & 13 & 10 & 9 & 1 & 12 & 5 & 4 & 0 \end{pmatrix}$$

respectively represent 1,1-pillowcase covers in $\mathcal{Q}(4,3,1)$, $\mathcal{Q}(4,1,1,1,1)$, $\mathcal{Q}(4,2,1,1)$, $\mathcal{Q}(8,3,1)$, $\mathcal{Q}(8,1,1,1,1)$, $\mathcal{Q}(8,2,1,1)$, $\mathcal{Q}(2,2,3,1)$, $\mathcal{Q}(2^2,1^4)$, $\mathcal{Q}(2^3,1^2)$, $\mathcal{Q}(6,6,3,1)$, $\mathcal{Q}(6,6,1,1,1,1)$, and $\mathcal{Q}(6,6,2,1,1)$. We also require the permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 7 & 8 & 7 \\ 2 & 9 & 10 & 5 & 9 & 1 & 6 & 8 & 4 & 10 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 3 & 5 & 4 & 6 & 7 & 8 & 9 \\ 2 & 5 & 6 & 8 & 10 & 7 & 10 & 9 & 11 & 1 & 11 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 5 & 8 & 9 & 10 & 9 \\ 2 & 11 & 4 & 8 & 3 & 11 & 1 & 12 & 7 & 6 & 12 & 10 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 5 & 11 & 12 & 13 & 8 \\ 2 & 14 & 7 & 1 & 4 & 11 & 15 & 10 & 6 & 14 & 3 & 13 & 9 & 15 & 12 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 5 & 7 & 9 \\ 2 & 10 & 4 & 8 & 6 & 1 & 3 & 10 & 11 & 9 & 11 & 0 \end{pmatrix}$$

respectively representing 1,1-pillowcase covers in the strata $\mathcal{Q}(4^2,3,1)$, $\mathcal{Q}(4,3^2,1^2)$, $\mathcal{Q}(4^2,1^4)$, $\mathcal{Q}(4,1^8)$, and $\mathcal{Q}(4^2,2,1^2)$.

We now consider constructing 1,1-pillowcase covers in strata of the form $\mathcal{Q}(4j+1,4k+3,4)$. By checking which zero is associated to the boundary component that leaves the bottom of the filling pair diagrams for the 1,1-pillowcase covers in $\mathcal{Q}(4j+1,4k+3)$ constructed in Section 6.5, and by adding a zero of order 4 to these surfaces using the combinatorics of Figure 6.8, we are able to construct 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+1,4k+3,4)$ apart from $\mathcal{Q}(7,1,4)$, $\mathcal{Q}(5,3,4)$, $\mathcal{Q}(7,5,4)$, and $\mathcal{Q}(9,7,4)$ which are represented by the permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 6 & 8 \\ 2 & 9 & 8 & 9 & 3 & 7 & 5 & 1 & 4 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 7 & 8 \\ 2 & 8 & 5 & 7 & 9 & 1 & 3 & 6 & 9 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 & 6 & 7 & 5 & 8 & 6 \\ 2 & 9 & 8 & 9 & 10 & 4 & 7 & 11 & 1 & 11 & 10 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 12 & 5 & 13 & 7 & 9 & 1 & 11 & 13 & 10 & 3 & 12 & 8 & 0 \end{pmatrix},$$

respectively.

For $j,k \geq 1$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+1,4k+3,2,1,1)$ by concatenating 1,1-pillowcase covers from $\mathcal{Q}(4k+3,1)$ and $\mathcal{Q}(2,4j+1,1)$. For $j \geq 1$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+7,2,1^3)$ and $\mathcal{Q}(4j+5,3,2,1^2)$ by concatenating the combinatorics in Figure 6.15 with the filling pair diagrams associated to 1,1-pillowcase covers in $\mathcal{Q}(4(j-1)+7,1)$ and $\mathcal{Q}(4(j-1)+5,1)$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 5 & 8 \\ 2 & 9 & 1 & 6 & 4 & 8 & 9 & 3 & 7 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 4 & 10 \\ 2 & 11 & 8 & 7 & 11 & 1 & 3 & 10 & 6 & 9 & 5 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 7 \\ 2 & 5 & 11 & 9 & 8 & 11 & 6 & 4 & 3 & 1 & 10 & 0 \end{pmatrix}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(3,2,1^3)$, $\mathcal{Q}(7,2,1^3)$, and $\mathcal{Q}(5,3,2,1^2)$, respectively. We also require the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 8 & 5 & 11 \\ 2 & 7 & 6 & 12 & 1 & 13 & 4 & 11 & 12 & 3 & 13 & 10 & 9 & 0 \end{pmatrix}$$

representing a 1,1-pillowcase cover in $\mathcal{Q}(3^2,2,1^4)$.

We now construct 1,1-pillowcase covers in strata of the form $\mathcal{Q}(4j+3,4k+3,4l+3,4m+3,4)$, $\mathcal{Q}(4j+1,4k+1,4l+1,4m+1,4)$, $\mathcal{Q}(4j+3,4k+3,4l+3,4m+3,2,1,1)$, and $\mathcal{Q}(4j+1,4k+1,4l+1,4m+1,2,1,1)$.

The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 9 & 10 & 11 & 8 \\ 2 & 10 & 12 & 9 & 13 & 7 & 12 & 4 & 13 & 3 & 11 & 6 & 5 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 9 & 8 & 10 & 11 & 6 & 12 \\ 2 & 10 & 4 & 13 & 12 & 5 & 14 & 11 & 15 & 13 & 1 & 3 & 14 & 15 & 9 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 3 & 12 & 8 & 10 & 13 & 14 \\ 2 & 15 & 11 & 5 & 12 & 14 & 16 & 9 & 1 & 15 & 17 & 13 & 16 & 17 & 6 & 4 & 7 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 & 6 & 7 & 8 & 9 & 10 & 11 & 9 & 12 & 13 & 14 & 5 & 12 & 6 \\ 2 & 15 & 13 & 15 & 16 & 17 & 8 & 18 & 1 & 17 & 19 & 16 & 18 & 7 & 11 & 4 & 14 & 10 & 19 & 0 \end{pmatrix}$$

respectively represent 1,1-pillowcase covers in the strata $\mathcal{Q}(3,3,3,3,4)$, $\mathcal{Q}(7,3,3,3,4)$, $\mathcal{Q}(5,5,5,5,4)$, and $\mathcal{Q}(9,5,5,5,4)$ that have the form necessary to be modified as in Proposition 6.16 to produce 1,1-pillowcase covers in strata of the form $\mathcal{Q}(4j+3,4k+3,4l+3,4m+3,4)$ and $\mathcal{Q}(4j+5,4k+5,4l+5,4m+5,4)$, for $j,k,l,m \geq 0$.

The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 4 & 8 & 9 & 6 \\ 2 & 10 & 9 & 11 & 7 & 5 & 3 & 8 & 10 & 1 & 11 & 0 \end{pmatrix}$$

represents $\mathcal{Q}(5,1,1,1,4)$ and can be modified as in Proposition 6.17 to produce a 1,1-pillowcase cover in $\mathcal{Q}(5,5,5,1,4)$. The permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 7 & 10 & 11 & 12 \\ 2 & 11 & 5 & 12 & 13 & 3 & 8 & 13 & 1 & 4 & 6 & 10 & 9 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in $\mathcal{Q}(5,5,1,1,4)$ and can be modified to produce 1,1-pillowcase covers in strata of the form $\mathcal{Q}(4j+5,4k+5,1,1,4)$, $j,k \geq 2$, and in $\mathcal{Q}(9,9,1,1,4)$. All remaining strata of the form $\mathcal{Q}(4j+1,4k+1,4l+1,4m+1,4)$ containing zeros of order 1 can be constructed by concatenating the combinatorics of Figure 6.8 with the filling pair diagrams for 1,1-pillowcase covers that we have already constructed in the previous section.

To produce 1,1-pillowcase covers in $\mathcal{Q}(4j+3,4k+3,4l+3,4m+3,2,1,1)$ we can concatenate 1,1-pillowcase covers from $\mathcal{Q}(4j+3,4k+3,1,1)$ and $\mathcal{Q}(2,4l+3,4m+3)$. Similarly, for $j,k \geq 1$ and $l,m \geq 0$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+1,4k+1,4l+1,4m+1,2,1,1)$ by concatenating 1,1-pillowcase covers from the strata

$\mathcal{Q}(4j+1, 4l+1, 4m+1, 1)$ and $\mathcal{Q}(2, 4k+1, 1)$. For strata of the form $\mathcal{Q}(4j+5, 1^3, 2, 1, 1)$, with $j \geq 1$, one can concatenate the combinatorics from Figure 6.15 with the filling pair diagrams associated to 1,1-pillowcase covers in $\mathcal{Q}(4(j-1)+5, 1, 1, 1)$, while the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 7 & 8 & 9 & 10 & 8 & 11 \\ 2 & 11 & 12 & 10 & 9 & 13 & 6 & 5 & 13 & 12 & 1 & 3 & 7 & 0 \end{pmatrix}$$

represents a 1,1-pillowcase cover in $\mathcal{Q}(5, 1^3, 2, 1, 1)$.

Finally, we consider how to add a single zero of order 4, zeros of order 3 and 1, four zeros of order 1, or a zero of order two and two zeros of order 1 to strata of the form $\mathcal{Q}(4j+2, 4k+1, 4l+1)$ and $\mathcal{Q}(4j+2, 4k+3, 4l+3)$.

We will begin by constructing 1,1-pillowcase covers in $\mathcal{Q}(4j+2, 4k+1, 4j+1, 4)$ and $\mathcal{Q}(4j+2, 4k+3, 4l+3, 4)$. Indeed, the permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 6 \\ 2 & 4 & 8 & 3 & 9 & 7 & 5 & 10 & 1 & 10 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 3 & 7 & 8 & 6 & 4 & 8 \\ 2 & 9 & 10 & 11 & 1 & 12 & 9 & 10 & 11 & 7 & 12 & 5 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 1 & 7 & 8 \\ 2 & 9 & 4 & 10 & 6 & 8 & 9 & 5 & 10 & 3 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 & 6 & 7 & 8 & 9 & 10 & 5 \\ 2 & 6 & 4 & 11 & 7 & 9 & 8 & 10 & 11 & 12 & 1 & 12 & 0 \end{pmatrix}$$

respectively represent 1,1-pillowcase covers in $\mathcal{Q}(2,5,1,4)$, $\mathcal{Q}(2,5,5,4)$, $\mathcal{Q}(6,1,1,4)$ and $\mathcal{Q}(6,5,1,4)$ that can be modified as in Proposition 6.18 to produce 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+2, 4k+1, 4l+1)$. Note that we have already produced a 1,1-pillowcase cover in $\mathcal{Q}(2,1,1,4)$. Similarly, the permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 6 & 8 & 9 \\ 2 & 5 & 1 & 9 & 8 & 4 & 10 & 7 & 10 & 3 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 6 & \\ 2 & 11 & 1 & 9 & 5 & 12 & 4 & 12 & 11 & 10 & 8 & 7 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 9 & 5 & 6 \\ 2 & 10 & 11 & 10 & 11 & 3 & 8 & 4 & 9 & 12 & 1 & 12 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 4 & 10 & 11 & 7 & 11 \\ 2 & 5 & 10 & 3 & 12 & 13 & 14 & 13 & 8 & 14 & 9 & 1 & 12 & 6 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 10 & 3 & 12 & 11 & 13 \\ 2 & 7 & 5 & 14 & 15 & 1 & 15 & 16 & 13 & 4 & 9 & 6 & 12 & 16 & 8 & 14 & 0 \end{pmatrix}$$

respectively represent 1,1-pillowcase covers in $\mathcal{Q}(2,3,3,4)$, $\mathcal{Q}(2,7,3,4)$, $\mathcal{Q}(6,3,3,4)$, $\mathcal{Q}(6,7,3,4)$ and $\mathcal{Q}(10,7,3,4)$ that can be modified as in Proposition 6.18 to produce 1,1-pillowcase covers in $\mathcal{Q}(4j+2,4k+3,4l+3,4)$.

For $j,k \geq 1$, we can produce 1,1-pillowcase covers in the strata $\mathcal{Q}(4j+6,1,1,3,1)$ and $\mathcal{Q}(2,4k+5,1,3,1)$ by adding the combinatorics of Figure 6.13 to the filling pair diagrams associated to 1,1-pillowcase covers in $\mathcal{Q}(4(j-1)+6,1,1)$ and $\mathcal{Q}(2,4(k-1)+5,1)$, respectively. For $k,l \geq 1$ we can construct 1,1-pillowcase covers in $\mathcal{Q}(2,4k+1,4l+1,3,1)$ by concatenating 1,1-pillowcase covers from $\mathcal{Q}(2,4k+1,1)$ and $\mathcal{Q}(3,4l+1)$. For $j,k \geq 0$ and $l \geq 1$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+6,4k+1,4l+1,3,1)$ by concatenating 1,1-pillowcase covers from $\mathcal{Q}(4j+6,4k+1,1)$ and $\mathcal{Q}(3,4l+1)$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 4 & 7 & 8 & 9 & 10 \\ 2 & 6 & 5 & 8 & 11 & 10 & 11 & 9 & 1 & 3 & 7 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 3 & 6 & 7 & 8 & 9 & 10 \\ 2 & 6 & 1 & 9 & 8 & 5 & 4 & 11 & 7 & 10 & 11 & 0 \end{pmatrix}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(6,1,1,3,1)$ and $\mathcal{Q}(2,5,1,3,1)$, respectively. Recall that we have already constructed a 1,1-pillowcase cover in $\mathcal{Q}(2,1,1,3,1)$.

For $j \geq 1$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+6,3,3,3,1)$ by adding the combinatorics of Figure 6.13 to the filling pair diagrams associated to 1,1-pillowcase covers in $\mathcal{Q}(4(j-1)+6,3,3)$. For $j,k \geq 0$ and $l \geq 1$, we can construct 1,1-pillowcase covers in strata of the form $\mathcal{Q}(4j+2,4k+3,4l+3,3,1)$ by concatenating 1,1-pillowcase covers from $\mathcal{Q}(4j+2,4k+3,3)$ and $\mathcal{Q}(4l+3,1)$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 4 & 10 \\ 2 & 11 & 6 & 5 & 9 & 7 & 1 & 10 & 8 & 11 & 3 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 5 & 9 & 10 & 11 & 4 \\ 2 & 9 & 1 & 3 & 10 & 12 & 13 & 12 & 8 & 6 & 13 & 7 & 11 & 0 \end{pmatrix}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(2,3,3,3,1)$ and $\mathcal{Q}(6,3,3,3,1)$, respectively.

Similarly, for $j,k \geq 1$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+6,1^6)$ and $(2,4k+5,1^5)$ by adding the combinatorics of Figure 6.14 to the filling pair diagrams associated to 1,1-pillowcase covers in $\mathcal{Q}(4(j-1)+6,1,1)$ and $\mathcal{Q}(2,4(k-1)+5,1)$. For $k,l \geq 1$ we can construct 1,1-pillowcase covers in $\mathcal{Q}(2,4k+1,4l+1,1^4)$ by concatenating 1,1-pillowcase covers from $\mathcal{Q}(2,4k+1,1)$ and $\mathcal{Q}(4l+1,1,1,1)$. For $j,k \geq 0$ and $l \geq 1$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+6,4k+1,4l+1,1^4)$ by concatenating 1,1-pillowcases covers from $\mathcal{Q}(4j+6,4k+1,1)$ and $\mathcal{Q}(4l+1,1,1,1)$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 4 & 11 & 8 \\ 2 & 12 & 6 & 13 & 10 & 5 & 12 & 3 & 7 & 1 & 11 & 9 & 13 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 7 & 9 & 10 & 5 & 11 \\ 2 & 12 & 13 & 1 & 12 & 3 & 10 & 6 & 9 & 4 & 11 & 13 & 8 & 0 \end{pmatrix}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(6,1^6)$ and $\mathcal{Q}(2,5,1^5)$, respectively. We also require the permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 4 & 8 & 5 & 7 \\ 2 & 9 & 10 & 11 & 9 & 3 & 8 & 6 & 1 & 10 & 11 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 3 & 11 & 12 & 13 & 9 & 14 & 12 \\ 2 & 11 & 1 & 7 & 15 & 16 & 5 & 17 & 15 & 8 & 14 & 13 & 10 & 4 & 16 & 17 & 6 & 0 \end{pmatrix}$$

respectively representing $\mathcal{Q}(2,1^6)$ and $\mathcal{Q}(2,1^{10})$.

For $k \geq 1$ and $l \geq 0$, we can construct 1,1-pillowcase covers $\mathcal{Q}(2,4k+3,4l+3,1^4)$ by adding the combinatorics in Figure 6.14 to the filling pair diagrams associated to 1,1-pillowcase covers in $\mathcal{Q}(2,4(k-1)+3,4l+3)$. For $j,k,l \geq 0$, we can construct 1,1-pillowcase covers in $\mathcal{Q}(4j+6,4k+3,4l+3,1^4)$ by concatenating 1,1-pillowcase covers from $\mathcal{Q}(4j+6,1,1)$ and $\mathcal{Q}(4k+3,4l+3,1,1)$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 6 & 9 & 11 \\ 2 & 5 & 10 & 7 & 12 & 3 & 13 & 1 & 8 & 11 & 13 & 4 & 12 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 6 & 12 & 13 & 12 & 5 & 11 \\ 2 & 14 & 13 & 7 & 9 & 15 & 16 & 15 & 10 & 1 & 4 & 14 & 3 & 8 & 17 & 16 & 17 & 0 \end{pmatrix}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(2,3,3,1^4)$ and $\mathcal{Q}(2,7,7,1^4)$, respectively.

Finally, 1,1-pillowcase covers in $\mathcal{Q}(4j+2,4k+1,4l+1,2,1,1)$ and $\mathcal{Q}(4j+2,4k+3,4l+3,2,1,1)$ can be constructed by concatenating a 1,1-pillowcase cover from $\mathcal{Q}(4j+$

2,2) with one from $\mathcal{Q}(4k+1,4l+1,1,1)$ or $\mathcal{Q}(4k+3,4l+3,1,1)$, respectively.

This completes the work of this section and allows us to construct 1,1-pillowcase covers in all connected components of all strata of quadratic differentials in genus at least two and with no poles.

6.7 Genus one

In this section, we construct 1,1-pillowcase covers in every non-empty stratum of genus one half-translation surfaces. We begin by constructing 1,1-pillowcase covers in strata that have an even number of poles. Recall that the stratum $\mathcal{Q}(\varnothing)$ is empty. This does not present us with a problem as we would not have needed to use a 1,1-pillowcase cover in this stratum to construct 1,1-pillowcase covers of higher complexity. For an even, strictly greater than zero, number of poles we have the following.

Proposition 6.19. *For $n \geq 2$ even and $k \geq 1$ odd, the permutations*

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 4 & 6 & \cdots & n & n & 1 \\ 2 & 3 & 3 & 5 & 5 & 7 & \cdots & n+1 & n+1 & 0 \end{pmatrix} \tag{6.30}$$

and

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 4 & 6 & \cdots & k+1 & k+3 \\ 2 & 3 & 3 & 5 & 5 & 7 & \cdots & k+2 & k+3 \\ & & & & & & & & & k+4 & k+4 & k+6 & \cdots & n+1 & n+1 & 1 \\ & & & & & & & & & k+5 & k+5 & k+7 & \cdots & n+2 & n+2 & 0 \end{pmatrix} \tag{6.31}$$

represent 1,1-pillowcase covers in $\mathcal{Q}(n,-1^n)$ and $\mathcal{Q}(n-k,k,-1^n)$, respectively. Moreover, these surfaces have the minimum number of squares required for their respective strata.

Proof. It is easy to check that the permutations represent 1,1-pillowcase covers in the claimed strata and have the minimum number of squares required. □

Using these surfaces, we can construct 1,1-pillowcase covers in any stratum of the form $\mathcal{Q}(k_1,\ldots,k_n,-1^\kappa)$, where $k_i \geq 1$ and $\kappa = \sum k_i$ is even.

When the number of poles is odd however, the situation is less straightforward. Indeed, as the following proposition demonstrates, there exist strata for which the theoretical minimum number of squares required for a 1,1-pillowcase cover cannot be achieved. Further still, for the first time we witness the phenomenon of having the minimum number of squares required for a 1,1-pillowcase cover to have the form necessary to be used in the cylinder concatenation method being different to the minimum number required for an arbitrary 1,1-pillowcase cover. These behaviours arise in certain strata because the necessary poles cannot be arranged on the surface without giving rise to either another pole or another zero.

Recall that the theoretical minimum number of squares for a genus one stratum $\mathcal{Q}(k_1, \ldots, k_n, -1^{2k+1})$ is $2g + (n + 2k + 1) - 2 = n + 2k + 1$. We have the following.

Proposition 6.20. *The minimum number of squares required for a 1,1-pillowcase cover in the strata $\mathcal{Q}(2k+1, -1^{2k+1})$ and $\mathcal{Q}(k+1, k, -1^{2k+1})$ is $2k+3$ and $2k+4$, respectively. Both of these values are greater than the theoretical minimum for each stratum.*

Moreover, the minimum number of squares required for a 1,1-pillowcase cover that has the appropriate form to be used in the cylinder concatenation method in the strata $\mathcal{Q}(2k+1, -1^{2k+1})$ and $\mathcal{Q}(k_1, k_2, -1^{2k+1})$ is $2k+5$ and $2k+4$, respectively.

Proof. The theoretical minimum number of squares for $\mathcal{Q}(2k+1, -1^{2k+1})$ is $2k+2$. Consider a line of $2k+2$ squares forming a single horizontal cylinder. We require $2k+1$ poles $k+1$ of which must lie on one side of the horizontal cylinder, say the top, and k must lie on the other. The top side must therefore consist of $k+1$ pairs of symbols, each pair giving rise to a pole. There must therefore be k pairs of symbols on the bottom side giving rise to the k remaining poles. To achieve a single vertical cylinder we must place these k symbols in a row offset by one from the symbols on the top side. However, this forces the two remaining symbols on the bottom side to be adjacent creating an additional pole. As such, the minimum number of squares must be at least $2k+3$. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 1 \\ 3 & 4 & 5 & 5 & 4 & 0 \end{pmatrix}$$

and, for $k \geq 2$,

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 4 & 4 & \cdots & k+2 & k+2 & 1 \\ 3 & k+3 & k+4 & k+4 & k+3 & k+5 & k+5 & \cdots & 2k+3 & 2k+3 & 0 \end{pmatrix}$$

represent 1,1-pillowcase cover in $\mathcal{Q}(3, -1^3)$ and $\mathcal{Q}(2k+1, -1^{2k+1})$, respectively. These permutations realise $2k+3$ squares.

The theoretical minimum number of squares for $\mathcal{Q}(k_1, k_2, -1^{2k+1})$ is $2k+3$. As above, consider $2k+3$ squares forming a single horizontal cylinder. We must have $k+1$ poles on one side, say the top, and k poles on the bottom. As such, there must be $k+1$ pairs of symbols on the top side forming the $k+1$ poles and one spare symbol. One can then check that the order of the zero lying on the top of the surface has order at least $k+2$. As such, we cannot produce a 1,1-pillowcase cover in $\mathcal{Q}(k+1, k, -1^{2k+1})$

with $2k+3$ squares. For $j \geq 1$ and $n \geq 0$, the permutation

$$\left(\begin{matrix} 0 & 1 & 2 & 3 & 3 & 4 \\ j+4+n & j+5+n & j+5+n & j+6+n & j+6+n & j+7+n \end{matrix}\right.$$
$$\begin{matrix} \cdots & j+2 & j+2 & j+3 & j+3 & j+4 \\ \cdots & 2j+5+n & 2j+5+n & j+4+n & 2 & 2j+6+n \end{matrix}$$
$$\left.\begin{matrix} j+4 & j+5 & \cdots & j+3+n & j+3+n & 1 \\ 2j+6+n & 2j+7+n & \cdots & 2j+2n+5 & 2j+2n+5 & 0 \end{matrix}\right)$$

represents a 1,1-pillowcase cover in $\mathcal{Q}(j, j+3+2n, -1^{2j+2n+3})$. These permutations realise $2k+3$ squares for 1,1-pillowcase covers in strata of the form $\mathcal{Q}(k_1, k_2, -1^{2k+1})$ apart from $\mathcal{Q}(k+1, k, -1^{2k+1})$.

Suppose now that we want to construct a 1,1-pillowcase cover that can be used in the cylinder concatenation method. Consider a row of $2k+3$ squares. The labels on the top side must start with the symbols 1 and 2, and the first symbol on the bottom must be a 2. The remaining $2k+2$ symbols on the bottom must be pairs of symbols forming $k+1$ poles. It is easy to check that there is no way to place k pairs of symbols forming k poles on the top row in such a way that achieves a single vertical cylinder without also forming an additional pole. So we must need at least $2k+4$ squares. For $j, n \geq 0$, the permutations

$$\left(\begin{matrix} 0 & 1 & 2 & 3 & 3 & 4 \\ 2 & 2j+5+n & 2j+5+n & 2j+6+n & 2j+6+n & 2j+7+n \end{matrix}\right.$$
$$\begin{matrix} \cdots & j+1 & j+2 & j+2 & j+3 & j+3 \\ \cdots & 3j+4+n & j+4 & 3j+5+n & 3j+6+n & 3j+6+n \end{matrix}$$
$$\left.\begin{matrix} j+4 & j+5 & \cdots & 2j+4+n & 2j+4+n & 1 \\ 3j+5+n & 3j+7+n & \cdots & 4j+2n+6 & 4j+2n+6 & 0 \end{matrix}\right)$$

and

$$\left(\begin{matrix} 0 & 1 & 2 & 3 & 3 & 4 \\ 2 & 2j+6+n & 2j+6+n & 2j+7+n & 2j+7+n & 2j+8+n \end{matrix}\right.$$
$$\begin{matrix} \cdots & j+2 & j+3 & j+4 & j+5 & j+5 \\ \cdots & 3j+6+n & 3j+7+n & 3j+7+n & j+3 & 3j+8+n \end{matrix}$$
$$\left.\begin{matrix} j+4 & j+6 & \cdots & 2j+5+n & 2j+5+n & 1 \\ 3j+8+n & 3j+9+n & \cdots & 4j+2n+8 & 4j+2n+8 & 0 \end{matrix}\right)$$

represent 1,1-pillowcase covers in $\mathcal{Q}(2j+1, 2j+2+2n, -1^{4j+2n+3})$ and $\mathcal{Q}(2j+2, 2j+3+2n, -1^{4j+2n+5})$, respectively. These have the form necessary to be used in the cylinder concatenation method. Note that the stratum $\mathcal{Q}(k+1, k, -1^{2k+1})$ is included in this list.

A similar argument can be used to show that there is no way to achieve a 1,1-pillowcase cover in $\mathcal{Q}(2k+1, -1^{2k+1})$ using $2k+4$ squares that has the form necessary to be used in the cylinder concatenation method. The permutations

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 1 & 4 & 4 \\ 2 & 5 & 6 & 7 & 7 & 6 & 5 & 0 \end{pmatrix}$$

and, for $k \geq 2$,

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 3 & 4 & 4 & \cdots \\ 2 & k+4 & k+4 & k+5 & k+5 & k+6 & k+6 & \cdots \end{pmatrix.$$
$$\begin{matrix} k & k+1 & k+1 & k+2 & k+2 & 1 & k+3 & k+3 \\ 2k+2 & 2k+3 & 2k+4 & 2k+5 & 2k+5 & 2k+4 & 2k+3 & 0 \end{matrix}\Bigg)$$

represent 1,1-pillowcase covers in $\mathcal{Q}(3, -1^3)$ and $\mathcal{Q}(2k+1, -1^{2k+1})$, respectively. These have the form to be used in the cylinder concatenation method and this completes the proof. □

Suppose now that we wish to construct a 1,1-pillowcase cover in a genus one stratum with an odd number of poles. If the stratum has one or two zeros, then this is covered by the proposition above. If we have at least three zeros then we can proceed as follows. If there is at least one zero of even order, then begin with a 1,1-pillowcase cover given by the proposition above in the stratum containing this even order zero and one zero of odd order from the remaining zeros. We can then concatenate this surface with a 1,1-pillowcase cover from the stratum containing the remaining zeros (which is a stratum with an even number of poles). The resulting 1,1-pillowcase cover will lie in the desired stratum and will have one more than the theoretical minimum number of squares. If all of the zeros are of odd order then we instead proceed as follows. If there exists at least one zero of order at least 3, then we concatenate the 1,1-pillowcase cover given by the above proposition representing the stratum containing a single zero of this order with a 1,1-pillowcase cover representing the stratum with the remaining zeros. This surface will have three more than the theoretical minimum number of squares. If all of the zeros are of order 1 and there are at least five zeros, then we can concatenate the permutation

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 4 & 5 & 6 & 6 & 7 & 1 \\ 2 & 8 & 9 & 9 & 10 & 10 & 8 & 3 & 5 & 7 & 0 \end{pmatrix}$$

representing $\mathcal{Q}(1^5, -1^5)$ with a 1,1-pillowcase cover in $\mathcal{Q}(1^{2k}, -1^{2k})$, for the appropriate k. The resulting 1,1-pillowcase cover achieves the theoretical minimum number of squares. If there are only three zeros then we use the permutation

$$\begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 & 3 \\ 4 & 5 & 6 & 4 & 5 & 6 & 0 \end{pmatrix}$$

representing a 1,1-pillowcase cover in $\mathcal{Q}(1^3, -1^3)$. We remark that there is not a 1,1-pillowcase cover in $\mathcal{Q}(1^3, -1^3)$ with six squares that can be used in the cylinder concatenation method.

We have therefore demonstrated that there exist 1,1-pillowcase covers in all non-empty genus one strata requiring at most three more than the theoretical minimum number of squares.

6.8 Genus zero

In this section, we describe a method for constructing 1,1-pillowcase covers in genus zero strata. We remark that this requires an approach distinct from the cylinder concatenation method used above. Indeed, the condition that the surfaces involved in the cylinder concatenation method must have the bottom of the first square identified with the top of the second forces the surfaces involved to have non-zero genus. As such, we describe a different technique for combining genus zero 1,1-pillowcase covers.

We first observe that a filling pair diagram for a genus zero 1,1,-pillowcase cover is an example of a special type of planar graph called a *meander*. Examples are shown in Figure 6.16. These objects have been well studied in a variety of settings – including in physics where there are connections to polymer chain folding and Feynman diagrams – and their enumeration has proven to be a particularly challenging problem. We refer the reader to the works of Lando-Zvonkin [35], Di Francesco-Golinelli-Guitter [16, 17], and Jensen [30] for more details. The enumeration of meanders was applied to the calculation of the volumes of genus zero quadratic strata by Delecroix-Goujard-Zograf-Zorich [15].

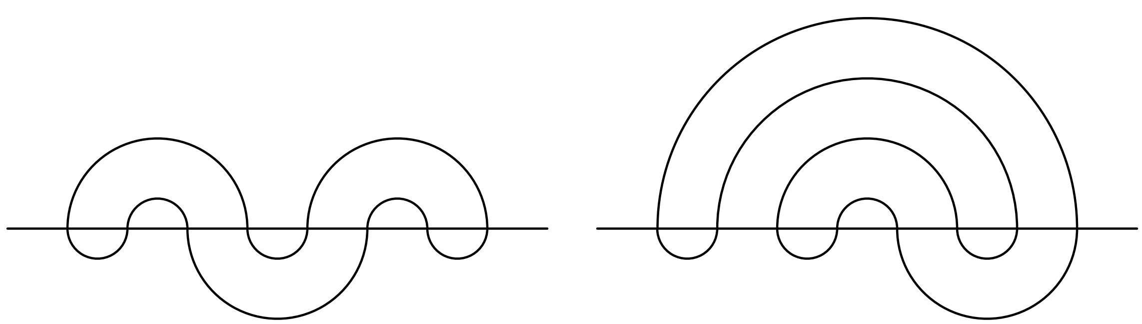

Figure 6.16: Two meanders representing $\mathcal{Q}(1, -1^5)$.

Using the terminology of Delecroix-Goujard-Zograf-Zorich [15], we will say that a meander has a *maximal arc* if the top (or bottom) of the first vertex is joined to the top (or bottom) of the final vertex. For example, the meander on the right of Figure 6.16 has a maximal arc while the meander on the left does not.

We now describe the combination method we will use for genus zero 1,1-pillowcase covers. Indeed, suppose that we have a meander containing two adjacent bigons as shown in Figure 6.17. Furthermore, suppose that we have a second meander which has a maximal arc with a bigon at one end of this arc as shown in Figure 6.18. Then observe that we can add the maximal arc meander into the bigon configuration in the first meander in such a way that we lose two bigons in each meander but preserve all of the remaining combinatorics. Specifically, we remove the bigons A and A' from each meander, then join the blue vertices to each other and the red vertices similarly while including the remainder of the second meander into bigon B of the first. As we mentioned above, in this process we lose two bigons from each meander but the remaining combinatorics are preserved. That is, if the first meander represents a 1,1-pillowcase cover in $\mathcal{Q}(k_1, \ldots, k_n, -1^{\kappa+4})$ and the second a 1,1-pillowcase cover in $\mathcal{Q}(l_1, \ldots, l_m, -1^{\lambda+4})$, then the resulting meander represents a 1,1-pillowcase cover in $\mathcal{Q}(k_1, \ldots, k_n, l_1, \ldots, l_m, -1^{\kappa+\lambda+4})$, where $\kappa = \sum k_i$ and $\lambda = \sum l_i$.

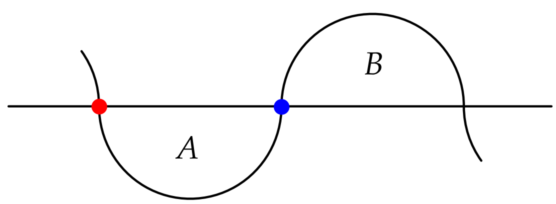

Figure 6.17: Adjacent bigon configuration.

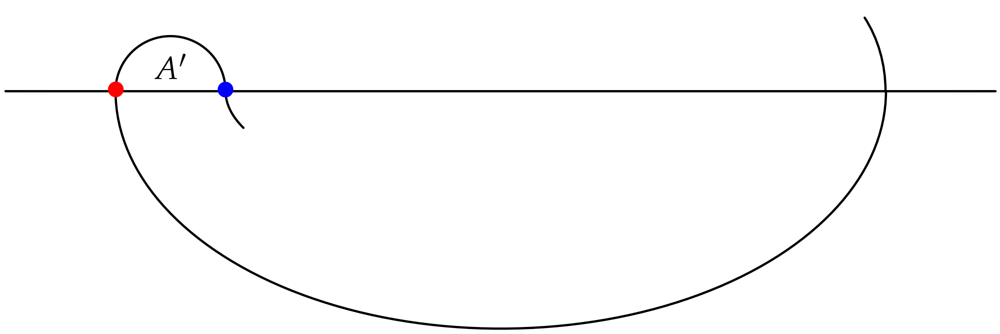

Figure 6.18: Maximal arc and bigon configuration.

For example, observe that the meander on the right of Figure 6.16 representing $\mathcal{Q}(1, -1^5)$ has adjacent bigons at its centre and also has the maximal arc and bigon configuration. If we perform this construction with both meanders being this meander (one copy being reflected across the horizontal), then we obtain the meander in Figure 6.19 representing $\mathcal{Q}(1,1,-1^6)$. The first copy is shown in blue and the second in red. The dashed bigons are those removed from each meander during the construction. That is, the blue dashed bigon represents the bigon A and the red dashed bigon represents the bigon A'. The region that was bigon B is also labelled.

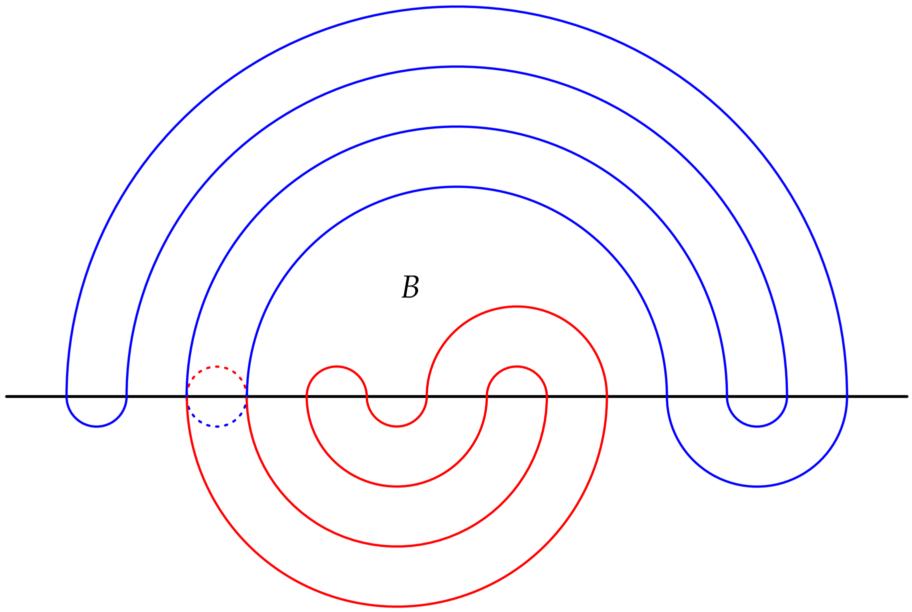

Figure 6.19: A meander representing $\mathcal{Q}(1,1,-1^6)$.

With this construction in mind, we will construct 1,1-pillowcase covers in the strata $\mathcal{Q}(k,-1^{k+4})$, for $k \geq 1$, that contain adjacent bigons, and a maximal arc and bigon.

Proposition 6.21. *For $k \geq 1$, the permutations*

$$\left(\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \cdots \\ 4k+1 & 4k+1 & 4k+2 & 4k+3 & 4k+3 & 4k+2 & 4k+4 & 4k+5 & 4k+5 & 4k+4 & \cdots \end{matrix}\right.$$
$$\begin{matrix} 4k-6 & 4k-5 & 4k-4 & 4k-3 & 4k-2 & 4k-1 & 4k & 4k & 4k-1 & 4k-2 & \cdots \\ 6k-2 & 6k-1 & 6k-1 & 6k-2 & 6k & 6k & 6k+1 & 6k+2 & 6k+2 & 6k+1 & \cdots \end{matrix}$$
$$\left.\begin{matrix} 13 & 12 & 11 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 8k-5 & 8k-4 & 8k-4 & 8k-5 & 8k-3 & 8k-2 & 8k-2 & 8k-3 & 8k-1 & 8k & 8k & 8k-1 & 0 \end{matrix}\right)$$

and

$$\left(\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ 4k+3 & 4k+4 & 4k+4 & 4k+3 & 4k+5 & 4k+6 & 4k+6 & 4k+5 & \cdots \end{matrix}\right.$$
$$\begin{matrix} 4k-4 & 4k-3 & 4k-2 & 4k-1 & 4k & 4k+1 & 4k+2 & 4k+2 & 4k+1 & 4k & \cdots \\ 6k+1 & 6k+2 & 6k+2 & 6k+1 & 6k+3 & 6k+3 & 6k+4 & 6k+5 & 6k+5 & 6k+4 & \cdots \end{matrix}$$
$$\left.\begin{matrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 8k+2 & 8k+3 & 8k+3 & 8k+2 & 8k+4 & 8k+4 & 0 \end{matrix}\right)$$

represent 1,1-pillowcase covers in $\mathcal{Q}(2k-1,-1^{2k+3})$ and $\mathcal{Q}(2k,-1^{2k+4})$, respectively. Moreover, the associated meanders have the combinatorics shown in Figures 6.17 and 6.18.

Proof. We begin with the meander shown in Figure 6.20 representing a 1,1-pillowcase cover in $\mathcal{Q}(1,-1^5)$. Then the 1,1-pillowcase covers in all successive strata are achieved by repeating the folding method that takes the meander in Figure 6.20 to the meander shown in Figure 6.21 representing a 1,1-pillowcase cover in $\mathcal{Q}(2,-1^6)$. One can see that at their centre they have adjacent bigons as in Figure 6.17 and also have the maximal arc and bigon structure of Figure 6.18. □

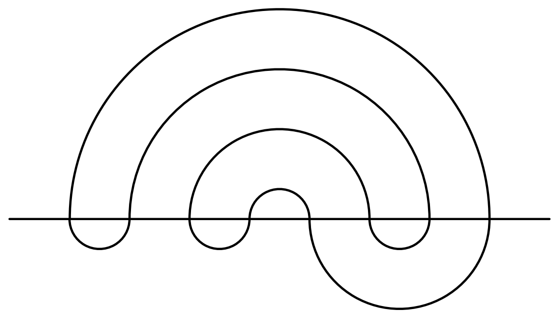

Figure 6.20: A meander representing $\mathcal{Q}(1,-1^5)$.

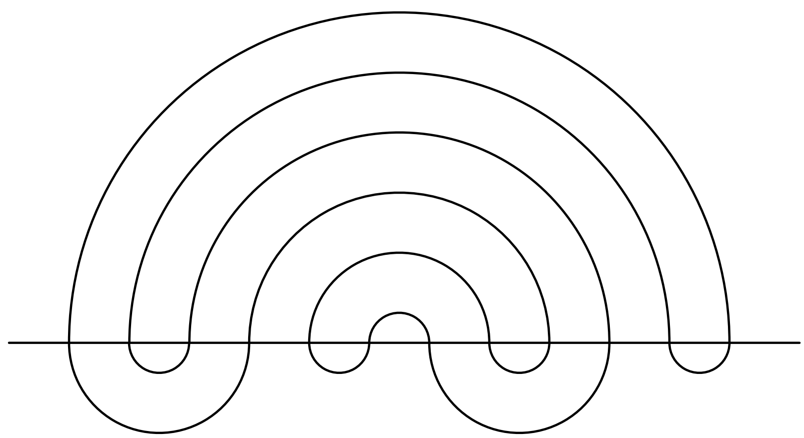

Figure 6.21: A meander representing $\mathcal{Q}(2,-1^6)$.

The final claim of the proposition means that these meanders have the ability to be combined in the method described above. As such, we have the following result.

Corollary 6.22. *Let $\mathcal{Q}(k_1,\ldots,k_n,-1^{\kappa+4})$, for $\kappa=\sum k_i$, be a genus zero stratum. Then there exists a 1,1-pillowcase cover in this stratum with $4\kappa+2n+2$ squares.*

We remark that this construction is certainly not minimal in the number of squares. Indeed, the following proposition and corollary demonstrate that for certain strata this construction is far from the minimal possible.

Proposition 6.23. *Let $k \geq 2$ be even. The minimum number of squares required for a 1,1-pillowcase cover in the stratum $\mathcal{Q}(\frac{k}{2}, \frac{k}{2}, -1^{k+4})$ is $k+4$.*

Let $k \geq 1$ be odd. The minimum number of squares required for a 1,1-pillowcase cover in the stratum $\mathcal{Q}(\frac{k+1}{2}, \frac{k-1}{2}, -1^{k+4})$, and the strata $\mathcal{Q}(\frac{k+1}{2}, \frac{k-3}{2}, 1, -1^{k+4})$, $\mathcal{Q}(\frac{k+1}{2}, \frac{k-5}{2}, 2, -1^{k+4})$, $\ldots$, $\mathcal{Q}(\frac{k+1}{2}, \lceil \frac{k-1}{4} \rceil, \lfloor \frac{k-1}{4} \rfloor, -1^{k+4})$ is $k+7$.

Proof. For $k \geq 2$, the meander shown in Figure 6.22 maximises the number of bigons for $k+4$ intersections. Since it represents a 1,1-pillowcase cover in $\mathcal{Q}(\frac{k}{2}, \frac{k}{2}, -1^{k+4})$, we must have that the minimum number of squares required for a 1,1-pillowcase cover in $\mathcal{Q}(\frac{k}{2}, \frac{k}{2}, -1^{k+4})$ is $k+4$.

For $k \geq 1$ odd, we have just seen that the maximum number of bigons that can be produced from $k+5$ intersections is $k+5$. If we want to produce fewer than this number of bigons from $k+5$ intersections we are forced to lose at least two bigons. Hence, the next highest number of bigons we can produce is $k+3$ which is less than we require and so $k+5$ intersections are not sufficient. The meander with $k+7$ crossings in Figure 6.23 represents a 1,1-pillowcase cover in $\mathcal{Q}(\frac{k+1}{2}, \frac{k-1}{2}, -1^{k+4})$. The remaining strata are then represented by the 1,1-pillowcase covers associated to the meanders obtained from the one in Figure 6.23 by consecutively moving the second bigon from the left one step further to the right. $\square$

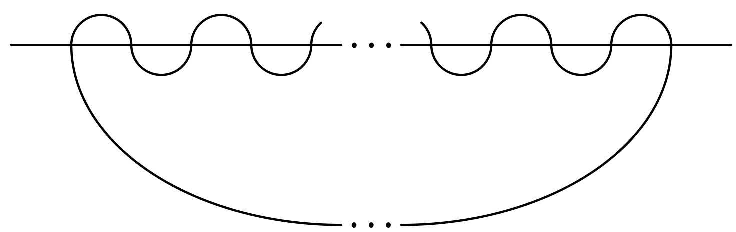

Figure 6.22: A meander representing $\mathcal{Q}(\frac{k}{2}, \frac{k}{2}, -1^{k+4})$.

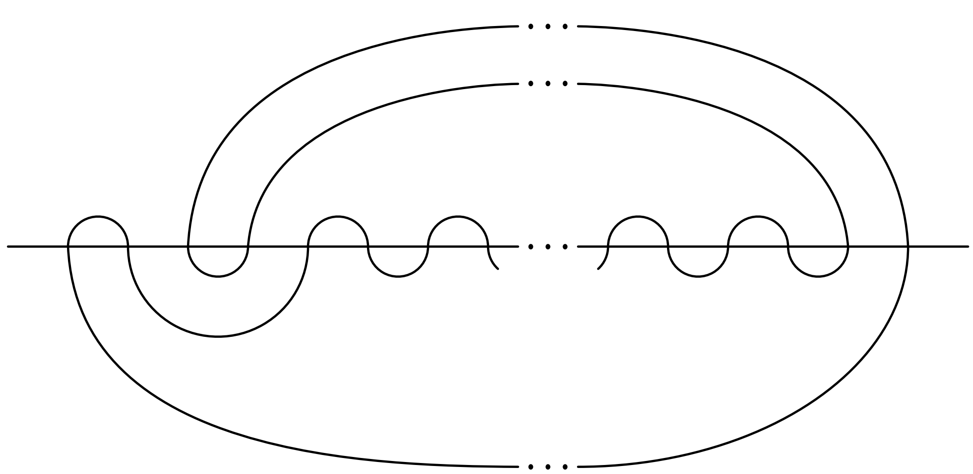

Figure 6.23: A meander representing $\mathcal{Q}(\frac{k+1}{2}, \frac{k-1}{2}, -1^{k+4})$.

Observe that the meanders in the above proof for the stratum $\mathcal{Q}(\frac{k}{2}, \frac{k}{2}, -1^{k+4})$ contained no regions with four boundary components - that is, no zeros of order zero - and they also have the form to be combined as described above. As such, we have the following corollary.

Corollary 6.24. *The minimum number of squares required for a 1,1-pillowcase cover in the stratum $\mathcal{Q}(k_1^2, \ldots, k_n^2, -1^{\kappa+4})$, where $k_i \geq 1$ are not necessarily distinct and $\kappa = 2\sum k_i$, is $\kappa + 2n + 2$.*

We finish this section with a brief discussion of how much work may be left to realise the minimum number of squares in each genus zero stratum. We first remark that computational evidence leads us to conjecture that the number of squares achieved by Proposition 6.21 for the stratum $\mathcal{Q}(k, -1^{k+4})$ is indeed minimal. Moreover, we observe that the meanders in Proposition 6.23 for k odd can be combined to improve upon the bounds in Corollary 6.22 for certain strata. However, there are examples for which this construction is still not minimal. As such, further investigation is required to determine the minimum number of squares required for each stratum. We expect there to be a connection between how 'symmetric' the orders of the zeros are and the number of squares required for a 1,1-pillowcase cover, where by symmetric we mean how well can the zeros be arranged into pairs of the same order. Indeed, Corollary 6.24 realises the case where all zeros can be paired in this manner and, in a sense, the case of Proposition 6.21 having only one zero is the furthest from this. A geometric motivation for this conjecture is the fact that the curves in a meander are separating curves on the sphere and so the boundary component corresponding to a given zero can only lie on one side of the horizontal curve. In the case of Corollary 6.24, equal order zeros placed on opposite sides of the horizontal curve are able to be produced simultaneously.

Chapter 7

Conclusion

In this short chapter, we discuss directions for future research and present some open questions.

7.1 Remaining quadratic strata

The next natural step in continuing the work of Chapter 6 is to begin adding poles to strata. If we wished to add poles to the strata of genus greater than or equal to two, then we have essentially two ways of doing this. Either the poles we add are 'cancelled out' by also adding zeros to the surface of appropriate orders, or poles are added and the 'cancelling' is achieved by increasing the orders of existing zeros.

The former case is equivalent to combining the surface by cylinder concatenation with a 1,1-pillowcase cover from an appropriate genus one stratum. However, as we saw in Section 6.7, there exist strata in genus one who do not have 1,1-pillowcase covers in the minimum number of squares. As such, we will be required to find a new way of adding poles and zeros of this type.

The latter case will be more combinatorially challenging. Indeed, it is equivalent to adding bigons to the filling pair diagram in such a way that only a specified subset of the original complementary regions are changed in the process. As such, this process will be more delicate than the constructions considered so far.

As discussed above, determining the minimum number of squares for genus zero 1,1-pillowcase covers still requires much work. A natural question to ask is the following.

Question 7.1. *Do the constructions of genus zero 1,1-pillowcase covers in Chapter 6 have applications to the enumeration of meanders?*

Finally, for genus at least one, finding the minimum number of squares required for 1,1-pillowcase covers in a given connected component of a stratum with both of the core curves being separating curves, if such a surface is possible, is still open.

Question 7.2. *Given a connected component $\mathscr{C}$ of a stratum $\mathcal{Q}(k_1, \ldots, k_n)$, does there exist a 1,1-pillowcase cover in $\mathscr{C}$ with both of the core curves being separating curves? If so, what is the minimum number of squares required to construct such a 1,1-pillowcase cover?*

Observe that the method of cylinder concatenation we have used in this thesis requires the filling curves to both be non-separating. As such, an entirely new method must be developed here.

7.2 Ratio-optimisers

We already have an extension of our Theorem 1.6 to the quadratic strata we produced in Chapter 6. In particular, for genus at least two, since the greatest number of squares required for a connected component of a stratum was the $4g$ squares required for $\mathcal{Q}^{hyp}(2j-1, 2j-1, 2k-1, 2k-1)$, we see that we have the following.

Theorem 7.3. *Given any connected component of any non-empty stratum of quadratic differentials with no poles and genus at least two, there exist infinitely many conjugacy classes of primitive ratio-optimising pseudo-Anosov homeomorphisms whose invariant axis is contained in the Teichmüller disk of a quadratic differential in that connected component.*

The discussion above about realising 1,1-pillowcase covers whose core curves are both separating also has interesting applications in the direction of ratio-optimisers. Let $\Gamma = \pi_1(S)$ and let Γ_i be the i^{th} term of its lower central series. So $\Gamma_1 = \Gamma$ and, for $i \geq 1$, $\Gamma_{i+1} = [\Gamma, \Gamma_i]$. The action of $\mathrm{Mod}(S)$ on Γ preserves Γ_i and so there is a well-defined action of $\mathrm{Mod}(S)$ on Γ/Γ_i. Johnson [32] defined a filtration of the mapping class group, now called the *Johnson filtration*, where the i^{th} term of the filtration denoted by $\mathcal{I}^i(S)$ is the kernel of the action of $\mathrm{Mod}(S)$ on Γ/Γ_{i+1}. In particular, $\mathcal{I}^0(S) = \mathrm{Mod}(S)$, $\mathcal{I}^1(S) = \mathcal{I}(S)$ is the well-studied Torelli group, and $\mathcal{I}^2(S) = \mathcal{K}(S)$ is the Johnson kernel. The textbook of Farb-Margalit [21, Chapter 6] contains more details on this filtration. Aougab-Taylor [4, Theorem 1.2] demonstrated that ratio-optimisers built from filling pairs where both curves are separating can lie arbitrarily deep in the Johnson filtration. As such, we have the following question.

Question 7.4. *Given a connected component $\mathscr{C}$ of a non-empty stratum $\mathcal{Q}(k_1, \ldots, k_n)$. Do there exist ratio-optimising pseudo-Anosov homeomorphisms stabilising the Teichmüller disk of a quadratic differential $q \in \mathscr{C}$, and lying arbitrarily deep in the Johnson filtration?*

7.3 Orbits of 1,1-square-tiled surfaces

Here we discuss some questions in a different direction from the main problems of this thesis. In particular, we will be concerned with the $\mathrm{SL}(2,\mathbb{Z})$-orbits of 1,1-square-tiled surfaces.

SL $(2, \mathbb{Z})$-orbits in $\mathcal{H}^{hyp}(2)$

A square-tiled surface is said to be *primitive* if it is not a proper branched cover of another square-tiled surface other than the square-torus.

Recall that the group $\mathrm{SL}(2,\mathbb{R})$ acts on translation surfaces by acting directly on the polygons in the plane. The subgroup $\mathrm{SL}(2,\mathbb{Z})$ sends primitive square-tiled surfaces to primitive square-tiled surfaces and so in particular it makes sense to discuss the $\mathrm{SL}(2,\mathbb{Z})$-orbit of a primitive square-tiled surface as a collection of square-tiled surfaces. Furthermore, the action preserves the number of squares.

In the case of $\mathcal{H}^{hyp}(2)$, the $\mathrm{SL}(2,\mathbb{Z})$-orbits of primitive square-tiled surfaces were classified in the works of Hubert-Lelièvre [26] and McMullen [44]. They showed that for $n = 3$ squares there is a single $\mathrm{SL}(2,\mathbb{Z})$-orbit, and for $n \geq 4$ there is one $\mathrm{SL}(2,\mathbb{Z})$-orbit if n is even, and two $\mathrm{SL}(2,\mathbb{Z})$-orbits if n is odd. In the case of $n \geq 5$ odd, the two orbits are called the A- and B-orbits.

We briefly recall an algebraic way of describing square-tiled surfaces. Firstly, number each square in the surface from 1 to n. We then define two elements h and v of the symmetric group Σ_n as follows. The image of i under the element h is the number of the square that is glued to the right of the square numbered i. The image of i under the element v is the number of the square that is glued to the top of the square numbered i. The square-tiled surface is then represented by the pair (h, v) up to simultaneous conjugation of h and v. For example, the surface in Figure 7.1, is represented by the pair (h, v), where $h = (1, 2, 3)$ and $v = (1, 3)(2)$. We refer the reader to the thesis of Zmiaikou [54] for more details on this representation of square-tiled surfaces.

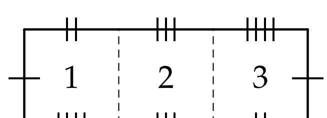

Figure 7.1: A square-tiled surface represented by $h = (1, 2, 3)$ and $v = (1, 3)(2)$.

To give a connection to the permutation representatives used in this thesis, suppose that we have a permutation representative

$$\Pi = \begin{pmatrix} 0 & 1 & 2 & \cdots & n \\ \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

giving rise to a square-tiled surface with one horizontal cylinder. If we label the squares as above – that is, 1 to n form left to right – then we obtain $h = (1, 2, \ldots, n)$ and v is given by the inverse of the permutation obtained from Π by removing the 0s from the top and bottom rows. Indeed, for the surface in Figure 7.1, we can take

$$\Pi = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

and we obtain

$$v = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}^{-1} = (1, 3)(2).$$

Zmiaikou showed that for n odd, the two $\mathrm{SL}(2, \mathbb{Z})$-orbits can be determined by the group generated by h and v, called the monodromy group of the square-tiled surface. In particular, the A-orbits correspond to this group being the full symmetric group Σ_n, while the B-orbits correspond to this group being the alternating group A_n.

Since a 1,1-square-tiled surface in $\mathcal{H}^{hyp}(2)$ will have h and v both being n-cycles, we see that 1,1-square-tiled surfaces must lie in the B-orbit for $n \geq 5$ odd.

HLK-invariants

A square-tiled surface is said to be *reduced* if its lattice of periods is equal to $\mathbb{Z} \oplus i\mathbb{Z}$. In other words, reduced square-tiled surfaces are square-tiled surfaces that do not cover a square-torus of area greater than one with one branch point.

Consider now a reduced square-tiled surface (X, ω) in $\mathcal{H}^{hyp}(2g-2)$. Following the work of Kani [33] and Hubert-Lelièvre [26] in genus two, and Matheus-Möller-Yoccoz [43] in genus three, it is natural to partition the fixed points of the hyperelliptic involution as follows. Recall that X is a branched cover of the square torus $\mathbb{T}^2$, $p : X \to \mathbb{T}^2$. The fixed points of the hyperelliptic involution τ on X are sent by p to the fixed points on $\mathbb{T}^2$ of the map $z \mapsto -z$. These fixed points are $(0,0)$, $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{2})$.

The aim is to define an $\mathrm{SL}(2, \mathbb{Z})$-invariant of (X, ω). First note that the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{T}^2$ fixes the point $(0,0)$ and permutes the points $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{2})$, acting by the symmetric group Σ_3. As such, the HLK-invariant of (X, ω) denoted by $\ell(X, \omega)$ is defined to be the tuple $(l_0, \{l_1, l_2, l_3\})$, where the number l_0 is the number of fixed points of τ distinct from the zero that lie above $(0,0)$, and the numbers l_1, l_2, and l_3, are the number of fixed points of τ lying above the points $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{2})$, respectively. This is an unordered set as these points can be permuted by the action of $\mathrm{SL}(2, \mathbb{Z})$, however we will write them in decreasing order

From the alternative proof of Proposition 3.2 in Appendix A, we can see that a reduced 1,1-square-tiled surface in $\mathcal{H}^{hyp}(2g-2)$ has HLK-invariant $(2g-3, \{2, 2, 0\})$ if n

is even, or $(2g-2,\{1,1,1\})$ if n is odd.

In the case of $\mathcal{H}^{hyp}(4)$, it is a conjecture of Delecroix-Lelièvre [43, Conjecture 6.8] that for strictly greater than 8 squares the HLK-invariant is a strong $\mathrm{SL}(2,\mathbb{Z})$-invariant. This would then imply that all reduced 1,1-square-tiled surfaces in $\mathcal{H}^{hyp}(4)$ are contained in the same $\mathrm{SL}(2,\mathbb{Z})$-orbit. It is natural to ask the following question.

Question 7.5. *Are all reduced 1,1-square-tiled surfaces in $\mathcal{H}^{hyp}(2g-2)$ contained in the same $\mathrm{SL}(2,\mathbb{Z})$-orbit?*

Appendix A

Alternative proof of Proposition 3.2

Here we will give an alternative proof of Proposition 3.2.

Saddle connections and Weierstrass points

A *saddle connection* on a translation surface is a flat geodesic segment whose endpoints are cone-points and whose interior contains no cone-points. For example, the three horizontal saddle connections on the surface in Figure A.1 are shown in red, blue and green.

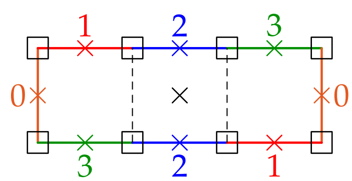

Figure A.1: The saddle connections and Weierstrass points of a square-tiled surface in $\mathcal{H}^{hyp}(2)$.

For a hyperelliptic translation surface with one horizontal cylinder as shown in Figure A.1, the saddle connections on the top of the cylinder lie in a cyclically reversed order on the bottom of the cylinder, and are sent to themselves with the reverse orientation by the hyperelliptic involution on the surface. As such, each horizontal saddle connection has a Weierstrass point at its centre. Similarly, when the surface is realised as above, the vertical saddle connection corresponding to the left and right sides contains a Weierstrass point at its centre. The remaining Weierstrass points correspond to the centre of the polygon that is the union of the squares and, only in the case of $\mathcal{H}^{hyp}(2g-2)$, the zero of the Abelian differential. See for example the surface in Figure A.1. The Weierstrass points are shown as crosses apart from the zero which is denoted by a square. More generally, every horizontal saddle connection will have a Weierstrass point at its centre and there will be a further two Weierstrass points in the interior of the horizontal cylinder which will lie on the core curve.

Proof of Proposition 3.2

We will now present an alternative proof of Proposition 3.2. First consider a 1,1-square-tiled surface (X, ω) in $\mathcal{H}^{hyp}(2g-2)$. The horizontal cylinder of X is flipped by the hyperelliptic involution and so the horizontal saddle connections above the horizontal cylinder lie in a cyclically reversed order on the bottom. As above, each of the horizontal saddle connections will have a Weierstrass point at its centre. For a square-tiled surface with one horizontal cylinder in $\mathcal{H}^{hyp}(2g-2)$, there are $2g-1$ horizontal saddle connections.

The core curve of the vertical cylinder of X passes through the center of the top side of every square. As discussed above for the horizontal cylinder, the vertical cylinder will also contain two Weierstrass points in its interior. Therefore, there can be at most two horizontal saddle connections of odd length. Indeed, the Weierstrass point at the centre of these saddle connections will lie at the centre of the top side of a square, and hence lie on the core curve of the vertical cylinder.

If there are no horizontal saddle connections of odd length, then every one of the $2g-1$ horizontal saddle connections has even length at least two and so the minimum number of squares is $4g-2$. In this case, the surface is not reduced.

If there is one horizontal saddle connection of odd length, then this saddle connection has length at least one and the remaining $2g-2$ horizontal saddle connections have even length at least two. Therefore, the minimum number of squares is $1 \times 1 + (2g-2) \times 2 = 4g-3$. In this case, the second Weierstrass point on the core curve of the vertical cylinder must lie in the centre of a square. Hence it lies on the core curve of the horizontal cylinder. Since there are already two Weierstrass points on the core curve of the vertical cylinder, the second Weierstrass point on the core curve of the horizontal cylinder must lie at the centre of a vertical saddle connection.

If there are two horizontal saddle connections of odd length, then these saddle connections have length at least one and the remaining $2g-3$ horizontal saddle connections have even length at least two. Therefore, the minimum number of squares is $2 \times 1 + (2g-3) \times 2 = 4g-4$. The Weierstrass points at the centre of the horizontal saddle connections of odd length are the two Weierstrass points lying on the core curve of the vertical cylinder. The two Weierstrass points lying on the core curve of the horizontal cylinder therefore lie at the centres of two vertical saddle connections.

We have shown that the minimum number of squares required for a 1,1-square-tiled surface in $\mathcal{H}^{hyp}(2g-2)$ is $4g-4$. A similar argument on the $2g$ horizontal saddle connections of a 1,1-square-tiled surface in $\mathcal{H}^{hyp}(g-1, g-1)$ gives that the minimum number of squares required in this case is $4g-2$.

Appendix B

Python code

The code below realises the construction of 1,1-square-tiled surfaces given by Theorem 1.4. It has been submitted to and included in the `surface_dynamics` python package [13] for use with the open source `SageMath` mathematical software [49].

All of the methods that are not included in the Abelian stratum classes make use of the following `surface_dynamics` methods.

```
from surface_dynamics.interval_exchanges.constructors import GeneralizedPermutation
from surface_dynamics.flat_surfaces.abelian_strata import AbelianStratum
```

Cylinder check

The reader will notice that the following method is called throughout the examples included within each method below. It checks whether a permutation representative represents a 1,1-square-tiled surface.

```
def cylinder_check(perm):
    r"""
    Checks for a single vertical cylinder and a single horizontal cylinder.

    INPUT::

        - ``perm`` - a permutation representative of an Abelian stratum

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import cylinder_check

        sage: C = AbelianStratum(4)
        sage: perm_1 = C.permutation_representative()
        sage: perm_1
        0 1 2 3 4 5
        3 2 5 4 1 0
        sage: cylinder_check(perm_1)
        False
        sage: perm_2 = C.single_cylinder_representative()
        sage: perm_2
```

```
        0 1 2 3 4 5
        2 5 4 1 3 0
        sage: cylinder_check(perm_2)
        True
        sage: perm_3 = iet.GeneralizedPermutation('a b', 'b a')
        sage: perm_3
        a b
        b a
        sage: cylinder_check(perm_3)
        True
        sage: perm_4 = iet.GeneralizedPermutation([0,3,2,1],[1,3,2,0])
        sage: perm_4
        0 3 2 1
        1 3 2 0
        sage: cylinder_check(perm_4)
        True
        sage: perm_5 = iet.GeneralizedPermutation('1 2 3 4', '4 3 1 2')
        sage: cylinder_check(perm_5)
        False
        sage: perm_6 = iet.GeneralizedPermutation('A B C D', 'B C D A')
        sage: cylinder_check(perm_6)
        True

    """
    from sage.combinat.permutation import Permutation
    from surface_dynamics.flat_surfaces.origamis.origami import Origami

    if len(perm[0]) != len(perm[1]) or perm[0][0] != perm[1][-1]:
        return False
    else:
        alph = perm.alphabet()
        perm.alphabet(len(perm[0]))
        u = Permutation(perm[1][:-1]).inverse()
        r = tuple(range(1,len(perm[0])))
        O = Origami(r,u)
        RO = O.vertical_twist().horizontal_twist(-1).vertical_twist()
        if RO.num_cylinders() == 1:
            perm.alphabet(alph)
            return True
        else:
            perm.alphabet(alph)
            return False
```

Cylinder Concatenation

The following method realises the cylinder concatenation procedure of Lemma 2.1.

```
def cylinder_concatenation(perm_1, perm_2, alphabet=None):
    r"""
    Combines two single cylinder permutation representatives.

    Combines two single cylinder permutation representatives of connected components
    of Abelian strata to produce another single cylinder representative of a
    different stratum.

    Such a method was described by Jeffreys [Jef19].

```

```
    INPUT:

        - ``perm_1``, ``perm_2`` - two single cylinder permutation representatives.

        - ``alphabet`` - alphabet or ``None`` (defaut: ``None``):
        whether you want to specify an alphabet for your representative.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

    We first take two single cylinder permutation representatives for the odd
    components of H_3(4)^odd and H_4(6)^odd.

        sage: perm_1 = AbelianStratum(4).odd_component().single_cylinder_representative()
        sage: perm_1
        0 1 2 3 4 5
        2 5 4 1 3 0
        sage: perm_1.stratum_component() == AbelianStratum(4).odd_component()
        True
        sage: cylinder_check(perm_1)
        True
        sage: perm_2 = AbelianStratum(6).odd_component().single_cylinder_representative()
        sage: perm_2
        0 1 2 3 4 5 6 7
        2 5 4 7 3 1 6 0
        sage: perm_2.stratum_component() == AbelianStratum(6).odd_component()
        True
        sage: cylinder_check(perm_2)
        True

    We check that the cylinder_concatenation of these permutations produces
    a single cylinder permutation representative of the connected component
    H_6(6,4)^odd.

        sage: perm_3 = cylinder_concatenation(perm_1,perm_2)
        sage: perm_3
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 5 4 6 3 7 10 9 12 8 1 11 0
        sage: perm_3.stratum_component() == AbelianStratum(6,4).odd_component()
        True
        sage: cylinder_check(perm_3)
        True

    We now instead take the cylinder_concatenation of perm_1 with a single cylinder
    permutation representative of the connected component H_4(6)^even. We see
    that the resulting permutation is a single cylinder permutation representative
    of the connected component H_6(6,4)^even.

        sage: perm_4 = AbelianStratum(6).even_component().single_cylinder_representative()
        sage: perm_5 = cylinder_concatenation(perm_1,perm_4,Alphabet(name='lower'))
        sage: perm_4
        0 1 2 3 4 5 6 7
        2 7 6 5 3 1 4 0
        sage: perm_4.stratum_component() == AbelianStratum(6).even_component()
        True
        sage: cylinder_check(perm_4)
```

```
        True
        sage: perm_5
        a b c d e f g h i j k l m
        c f e g d h m l k i b j a
        sage: perm_5.stratum_component() == AbelianStratum(6,4).even_component()
        True
        sage: cylinder_check(perm_5)
        True

    """
    from sage.combinat.words.alphabet import Alphabet
    from sage.rings.semirings.non_negative_integer_semiring import NN

    alph = Alphabet(NN)
    perm_1.alphabet(alph)
    perm_2.alphabet(alph)
    length_1 = len(perm_1[0])-1
    length_2 = len(perm_2[0])-1
    top_row = [i for i in range(length_1+length_2+1)]
    bot_row1 = perm_1[1][:-1]
    bot_row2 = perm_2[1][:-1]
    for i in range(length_1):
        if bot_row1[i] == 1:
            bot_row1[i] += length_1
    for j in range(length_2):
        if not bot_row2[j] == 1:
            bot_row2[j] += length_1
    bot_row = bot_row1 + bot_row2 + [0]
    perm = GeneralizedPermutation(top_row,bot_row)
    if not alphabet == None:
        perm.alphabet(alphabet)
    return perm
```

Hyperelliptic components

The following method was added to the Abelian stratum hyperelliptic component class (`HypAbelianStratumComponent`) of the `surface_dynamics` package. It produces the permutation representatives given by Proposition 3.1. The error messages are required because SageMath recognises zeros of order zero. That is, since the 1,1-square-tiled surfaces we produced in $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1,g-1)$ respectively required $2g-3$ and $2g-2$ additional squares than the theoretical minimum, in SageMath they instead lie in the strata $\mathcal{H}^{hyp}(2g-2,0^{2g-3})$ and $\mathcal{H}^{hyp}(g-1,g-1,0^{2g-2})$, respectively.

```
    def single_cylinder_representative(self, alphabet=None):
        r"""
        Returns a single cylinder permutation representative.

        Returns a permutation representative of a square-tiled surface in this
        component having a single vertical cylinder and a single horizontal cylinder.

        Such representatives were constructed for every stratum of Abelian
        differentials by Jeffreys [Jef19].

        INPUT::
```

```

        - ``alphabet`` - alphabet or ``None`` (defaut: ``None``):
        whether you want to specify an alphabet for your representative.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import cylinder_check

        sage: cc = AbelianStratum(2,0).hyperelliptic_component()
        sage: p = cc.single_cylinder_representative(alphabet=Alphabet(name='upper'))
        sage: p
        A B C D E
        E D B C A
        sage: p.stratum_component() == cc
        True
        sage: cylinder_check(p)
        True
        sage: cc = AbelianStratum({3:2,0:6}).hyperelliptic_component()
        sage: p = cc.single_cylinder_representative()
        sage: p
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
        14 12 13 10 11 8 9 7 5 6 3 4 1 2 0
        sage: p.stratum_component() == cc
        True
        sage: cylinder_check(p)
        True
        sage: cc = AbelianStratum(2).hyperelliptic_component()
        sage: cc.single_cylinder_representative()
        Traceback (most recent call last):
          ...
        ValueError: no 1,1-square-tiled surfaces in this connected component try again with
 H_2(2, 0)^hyp
        sage: cc = AbelianStratum({3:2,0:5}).hyperelliptic_component()
        sage: cc.single_cylinder_representative()
        Traceback (most recent call last):
          ...
        ValueError: no 1,1-square-tiled surfaces in this connected component try again with
 H_4(3^2, 0^6)^hyp
    """
    stratum = self.stratum()
    genus = stratum.genus()
    nb_fk_zeros = stratum.nb_fake_zeros()
    nb_real_zeros = stratum.nb_zeros()-nb_fk_zeros
    add_fk_zeros = nb_fk_zeros - 2*genus+4-nb_real_zeros

    from surface_dynamics.interval_exchanges.constructors import GeneralizedPermutation
    from surface_dynamics.flat_surfaces.single_cylinder import cylinder_concatenation

    if nb_real_zeros == 1 and add_fk_zeros < 0:
        raise ValueError("no 1,1-square-tiled surfaces in this connected component try
again with %s^hyp" %(str(AbelianStratum({2*genus-2:1,0:2*genus-3}))))
    elif nb_real_zeros == 2 and add_fk_zeros < 0:
        raise ValueError("no 1,1-square-tiled surfaces in this connected component try
again with %s^hyp" %(str(AbelianStratum({genus-1:2,0:2*genus-2}))))
    elif nb_real_zeros == 0:
        fk_zeros_perm = GeneralizedPermutation([0],[0])
        mk_pt_perm = GeneralizedPermutation([0,1],[1,0])
```

```
            for i in range(nb_fk_zeros):
                fk_zeros_perm = cylinder_concatenation(fk_zeros_perm,mk_pt_perm)
            if not alphabet == None:
                fk_zeros_perm.alphabet(alphabet)
            return fk_zeros_perm
        else:
            top_row = [i for i in range(0,4*genus-3+2*(nb_real_zeros-1)+add_fk_zeros)]
            bot_row = [4*genus-4+2*(nb_real_zeros-1)+add_fk_zeros]
            for i in range(4*genus-6+2*(nb_real_zeros-1)+add_fk_zeros,2*genus-2+add_fk_zeros,-2):
                bot_row = bot_row + [i,i+1]
            bot_row = bot_row + [2*genus-1+i for i in range(add_fk_zeros+1)]
            for i in range(2*genus-3,-1,-2):
                bot_row = bot_row + [i,i+1]
            bot_row = bot_row + [0]
            perm = GeneralizedPermutation(top_row,bot_row)
            if not alphabet == None:
                perm.alphabet(alphabet)
            return perm
```

Odd components

The following method was added to the Abelian stratum odd component class (`OddAbelianStratumComponent`) of the `surface_dynamics` package.

```
    def single_cylinder_representative(self, alphabet=None):
        r"""
        Returns a single cylinder permutation representative.

        Returns a permutation representative of a square-tiled surface in this
        component having a single vertical cylinder and a single horizontal cylinder.

        Such representatives were constructed for every stratum of Abelian
        differentials by Jeffreys [Jef19].

        INPUT::

            - ``alphabet`` - alphabet or ``None`` (defaut: ``None``):
            whether you want to specify an alphabet for your representative.

        EXAMPLES::

            sage: from surface_dynamics import *
            sage: from surface_dynamics.flat_surfaces.single_cylinder import cylinder_check

            sage: cc = AbelianStratum(4).odd_component()
            sage: p = cc.single_cylinder_representative(alphabet=Alphabet(name='upper'))
            sage: p
            A B C D E F
            C F E B D A
            sage: p.stratum_component() == cc
            True
            sage: cylinder_check(p)
            True
            sage: cc = AbelianStratum(6,2).odd_component()
            sage: p = cc.single_cylinder_representative()
            sage: p
```

```
            0 1 2 3 4 5 6 7 8 9 10
            2 5 4 6 3 8 10 7 1 9 0
            sage: p.stratum_component() == cc
            True
            sage: cylinder_check(p)
            True

        """
        from surface_dynamics.flat_surfaces.single_cylinder import cylinder_concatenation
        from surface_dynamics.flat_surfaces.single_cylinder import no_two_odd
        from surface_dynamics.flat_surfaces.single_cylinder import one_two_odd
        from surface_dynamics.flat_surfaces.single_cylinder import even_twos_odd
        from surface_dynamics.flat_surfaces.single_cylinder import odd_twos_odd
        from surface_dynamics.interval_exchanges.constructors import GeneralizedPermutation

        zeros = self.stratum().zeros()
        real_zeros = [z for z in zeros if z != 0]

        fk_zeros_perm = GeneralizedPermutation([0],[0])
        mk_pt_perm = GeneralizedPermutation([0,1],[1,0])
        for i in range(self.stratum().nb_fake_zeros()):
            fk_zeros_perm = cylinder_concatenation(fk_zeros_perm,mk_pt_perm)

        two_count = real_zeros.count(2)
        if two_count == 0:
            perm = cylinder_concatenation(fk_zeros_perm,no_two_odd(real_zeros))
        elif two_count == 1 :
            perm = cylinder_concatenation(fk_zeros_perm,one_two_odd(real_zeros))
        elif two_count >= 2 and two_count%2 == 0:
            perm = cylinder_concatenation(fk_zeros_perm,even_twos_odd(real_zeros,two_count))
        else:
            perm = cylinder_concatenation(fk_zeros_perm,odd_twos_odd(real_zeros,two_count))

        if not alphabet == None:
            perm.alphabet(alphabet)
        return perm
```

It makes use of the following method which produces the permutation representatives given by Proposition 3.3.

```
def even_zero_odd(num):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having
    a single vertical cylinder and a single horizontal cylinder in the odd
    component of the Abelian stratum with a single zero of the given order.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``num`` - an even integer at least four.

    EXAMPLES::

        sage: from surface_dynamics import *
```

```
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = even_zero_odd(4)
        sage: perm
        0 1 2 3 4 5
        2 5 4 1 3 0
        sage: perm.stratum_component() == AbelianStratum(4).odd_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = even_zero_odd(6)
        sage: perm
        0 1 2 3 4 5 6 7
        2 5 4 7 3 1 6 0
        sage: perm.stratum_component() == AbelianStratum(6).odd_component()
        True
        sage: cylinder_check(perm)
        True

    """
    genus = (num+2)//2
    if genus == 3:
        top_row = [0,1,2,3,4,5]
        bot_row = [2,5,4,1,3,0]
        return GeneralizedPermutation(top_row,bot_row)
    else:
        top_row = [i for i in range(2*genus)]
        bot_row = [2,5,4,7,3]
        for i in range(9,2*genus+1,2):
            bot_row += [i,i-3]
        bot_row += [1,2*genus-2,0]
        return GeneralizedPermutation(top_row,bot_row)
```

These are then concatenated in the following method.

```
def no_two_odd(real_zeros):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the odd component of an
    Abelian stratum having no zeros of order two.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``real_zeros`` - a list of even positive integers none of which
        are equal to two.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = no_two_odd([6,4])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
```

```
        2 5 4 7 3 8 6 9 12 11 1 10 0
        sage: perm.stratum_component() == AbelianStratum(6,4).odd_component()
        True
        sage: cylinder_check(perm)
        True

    """
    perm = even_zero_odd(real_zeros[0])
    if len(real_zeros) == 1:
        return perm
    else:
        for i in range(1,len(real_zeros)):
            perm = cylinder_concatenation(perm,even_zero_odd(real_zeros[i]))
        return perm
```

The difficulties of adding zeros of order two are then dealt with by the following methods. They carry out the constructions discussed in and after Proposition 3.5.

```
def one_two_odd(real_zeros):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the odd component of an
    Abelian stratum having one zero of order two.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``real_zeros`` - a list of even positive integers one of which is equal to two.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = one_two_odd([4,2])
        sage: perm
        0 1 2 3 4 5 6 7 8
        2 5 8 3 6 4 1 7 0
        sage: perm.stratum_component() == AbelianStratum(4,2).odd_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = one_two_odd([8,6,2])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
        2 5 4 7 3 8 6 10 12 9 13 11 14 17 16 19 15 1 18 0
        sage: perm.stratum_component() == AbelianStratum(8,6,2).odd_component()
        True
        sage: cylinder_check(perm)
        True

    """
    real_zeros.remove(2)
    if set(real_zeros) == {4}:
```

```
        perm = GeneralizedPermutation([0,1,2,3,4,5,6,7,8],[2,5,8,3,6,4,1,7,0])
        if len(real_zeros) == 1:
            return perm
        else:
            return cylinder_concatenation(perm,no_two_odd(real_zeros[1:]))
    else:
        perm_1 = even_zero_odd(real_zeros[0]-2)
        length_1 = len(perm_1[0])-1
        top_row_1 = perm_1[0]
        bot_row_1 = perm_1[1][:-1]

        for i in range(length_1):
            if bot_row_1[i] == 1:
                bot_row_1[i] += length_1

        top_row_2 = [i+length_1 for i in range(1,6)]
        bot_row_2 = [3+length_1,5+length_1,2+length_1,1,4+length_1,0]
        top_row = top_row_1 + top_row_2
        bot_row = bot_row_1 + bot_row_2
        perm = GeneralizedPermutation(top_row,bot_row)

        if len(real_zeros) == 1:
            return perm
        else:
            return cylinder_concatenation(perm,no_two_odd(real_zeros[1:]))

def even_twos_odd(real_zeros,two_count):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the odd component of an
    Abelian stratum having an even, at least two, number of zeros of order two.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``real_zeros`` - a list of even positive integers an even number of which
        are equal to two.

        - ``two_count`` - a positive integer equal to the number of twos in ``real_zeros``.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = even_twos_odd([2,2],2)
        sage: perm
        0 1 2 3 4 5 6
        2 4 6 3 1 5 0
        sage: perm.stratum_component() == AbelianStratum(2,2).odd_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = even_twos_odd([4,2,2,2,2],4)
```

```
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17
        2 4 6 3 7 5 8 10 12 9 13 11 14 17 16 1 15 0
        sage: perm.stratum_component() == AbelianStratum({4:1,2:4}).odd_component()
        True
        sage: cylinder_check(perm)
        True

    """
    for i in range(two_count):
        real_zeros.remove(2)

    odd_2_2 = GeneralizedPermutation([0,1,2,3,4,5,6],[2,4,6,3,1,5,0])
    twos_perm = odd_2_2

    for i in range((two_count-2)//2):
        twos_perm = cylinder_concatenation(twos_perm,odd_2_2)

    if len(real_zeros) == 0:
        return twos_perm
    else:
        return cylinder_concatenation(twos_perm,no_two_odd(real_zeros))

def odd_twos_odd(real_zeros,two_count):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the odd component of an
    Abelian stratum having an odd, at least three, number of zeros of order two.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``real_zeros`` - a list of even positive integers an odd number of which
        are equal to two.

        - ``two_count`` - a positive integer equal to the number of twos in ``real_zeros``.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = odd_twos_odd([2,2,2],3)
        sage: perm
        0 1 2 3 4 5 6 7 8 9
        2 8 6 9 4 1 3 5 7 0
        sage: perm.stratum_component() == AbelianStratum(2,2,2).odd_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = odd_twos_odd([4,2,2,2,2,2],5)
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
        2 8 6 9 4 10 3 5 7 11 13 15 12 16 14 17 20 19 1 18 0
```

```
        sage: perm.stratum_component() == AbelianStratum({4:1,2:5}).odd_component()
        True
        sage: cylinder_check(perm)
        True

    """
    for i in range(two_count):
        real_zeros.remove(2)

    odd_2_2 = GeneralizedPermutation([0,1,2,3,4,5,6],[2,4,6,3,1,5,0])

    odd_2_2_2 = GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9],[2,8,6,9,4,1,3,5,7,0])

    twos_perm = odd_2_2_2

    for i in range((two_count-3)//2):
        twos_perm = cylinder_concatenation(twos_perm,odd_2_2)

    if len(real_zeros) == 0:
        return twos_perm
    else:
        return cylinder_concatenation(twos_perm,no_two_odd(real_zeros))
```

Even components

The following method was added to the Abelian stratum even component class (`EvenAbelianStratumComponent`) of the `surface_dynamics` package.

```
    def single_cylinder_representative(self, alphabet=None):
        r"""
        Returns a single cylinder permutation representative.

        Returns a permutation representative of a square-tiled surface in this
        component having a single vertical cylinder and a single horizontal cylinder.

        Such representatives were constructed for every stratum of Abelian
        differentials by Jeffreys [Jef19].

        INPUT::

            - ``alphabet`` - alphabet or ``None`` (defaut: ``None``):
            whether you want to specify an alphabet for your representative.

        EXAMPLES::

            sage: from surface_dynamics import *
            sage: from surface_dynamics.flat_surfaces.single_cylinder import cylinder_check

            sage: cc = AbelianStratum(6).even_component()
            sage: p = cc.single_cylinder_representative(alphabet=Alphabet(name='lower'))
            sage: p
            a b c d e f g h
            c h g f d b e a
            sage: p.stratum_component() == cc
            True
            sage: cylinder_check(p)
            True
```

```
            sage: cc = AbelianStratum(4,4).even_component()
            sage: p = cc.single_cylinder_representative()
            sage: p
            0 1 2 3 4 5 6 7 8 9 10
            2 10 7 5 8 1 9 6 4 3 0
            sage: p.stratum_component() == cc
            True
            sage: cylinder_check(p)
            True

        """
        from surface_dynamics.flat_surfaces.single_cylinder import cylinder_concatenation
        from surface_dynamics.flat_surfaces.single_cylinder import no_two_even
        from surface_dynamics.flat_surfaces.single_cylinder import one_two_even
        from surface_dynamics.flat_surfaces.single_cylinder import two_twos_even
        from surface_dynamics.flat_surfaces.single_cylinder import even_twos_even
        from surface_dynamics.flat_surfaces.single_cylinder import odd_twos_even
        from surface_dynamics.interval_exchanges.constructors import GeneralizedPermutation

        zeros = self.stratum().zeros()
        real_zeros = [z for z in zeros if z != 0]

        fk_zeros_perm = GeneralizedPermutation([0],[0])
        mk_pt_perm = GeneralizedPermutation([0,1],[1,0])
        for i in range(self.stratum().nb_fake_zeros()):
            fk_zeros_perm = cylinder_concatenation(fk_zeros_perm,mk_pt_perm)

        two_count = real_zeros.count(2)
        if two_count == 0:
            perm = cylinder_concatenation(fk_zeros_perm,no_two_even(real_zeros))
        elif two_count == 1:
            perm = cylinder_concatenation(fk_zeros_perm,one_two_even(real_zeros))
        elif two_count == 2:
            perm = cylinder_concatenation(fk_zeros_perm,two_twos_even(real_zeros))
        elif two_count > 2 and two_count%2 == 0:
            perm = cylinder_concatenation(fk_zeros_perm,even_twos_even(real_zeros,two_count))
        else:
            perm = cylinder_concatenation(fk_zeros_perm,odd_twos_even(real_zeros,two_count))

        if not alphabet == None:
            perm.alphabet(alphabet)
        return perm
```

It makes use of the following method which produces the permutation representatives given by Proposition 3.4.

```
def even_zero_even(num):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having
    a single vertical cylinder and a single horizontal cylinder in the even
    component of the Abelian stratum with a single zero of the given order.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::
```

```

        - ``num`` - an even integer at least six.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = even_zero_even(6)
        sage: perm
        0 1 2 3 4 5 6 7
        2 7 6 5 3 1 4 0
        sage: perm.stratum_component() == AbelianStratum(6).even_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = even_zero_even(8)
        sage: perm
        0 1 2 3 4 5 6 7 8 9
        2 7 6 5 3 9 4 1 8 0
        sage: perm.stratum_component() == AbelianStratum(8).even_component()
        True
        sage: cylinder_check(perm)
        True

    """
    genus = (num+2)//2
    if genus == 4:
        top_row = [0,1,2,3,4,5,6,7]
        bot_row = [2,7,6,5,3,1,4,0]
        return GeneralizedPermutation(top_row,bot_row)
    else:
        top_row = [i for i in range(2*genus)]
        bot_row = [2,7,6,5,3,9,4]
        for i in range(11,2*genus+1,2):
            bot_row = bot_row + [i,i-3]
        bot_row = bot_row + [1,2*genus-2,0]
        return GeneralizedPermutation(top_row,bot_row)
```

These are then concatenated in the following method. Note that this method must deal with the issues causes by the non-existence of an even component in the stratum $\mathcal{H}(4)$.

```
def no_two_even(real_zeros):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the even component of
    an Abelian stratum having no zeros of order two.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``real_zeros`` - a list of even positive integers none of which
        are equal to two.
```

```

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = no_two_even([4,4])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10
        2 10 7 5 8 1 9 6 4 3 0
        sage: perm.stratum_component() == AbelianStratum(4,4).even_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = no_two_even([6,4])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 7 6 5 3 8 4 9 12 11 1 10 0
        sage: perm.stratum_component() == AbelianStratum(6,4).even_component()
        True

    """
    if set(real_zeros) == {4}:
        four_count = real_zeros.count(4)
        even_4_4 = GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9,10],[2,10,7,5,8,1,9,6,4,3,0])
        real_zeros.remove(4)
        real_zeros.remove(4)

        if real_zeros != []:
            odd_perm = AbelianStratum(real_zeros).odd_component().single_cylinder_representative()
            return cylinder_concatenation(even_4_4,odd_perm)
        else:
            return even_4_4
    else:
        perm = even_zero_even(real_zeros[0])
        if len(real_zeros) == 1:
            return perm
        else:
            odd_perm = AbelianStratum(real_zeros[1:]).odd_component().single_cylinder_representative()
            return cylinder_concatenation(perm,odd_perm)
```

Similar to the above, the difficulties of adding zeros of order two are then dealt with by the following methods. They carry out the constructions discussed in and after Proposition 3.6.

```
def one_two_even(real_zeros):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the even component of
    an Abelian stratum having one zero of order two.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

```

```
    INPUT::

        - ``real_zeros`` - a list of even positive integers one of which is equal to two.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = one_two_even([4,2])
        sage: perm
        0 1 2 3 4 5 6 7 8
        2 4 1 8 7 5 3 6 0
        sage: perm.stratum_component() == AbelianStratum(4,2).even_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = one_two_even([6,2])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10
        2 10 9 8 6 3 5 1 4 7 0
        sage: perm.stratum_component() == AbelianStratum(6,2).even_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = one_two_even([8,6,2])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
        2 7 6 5 3 8 4 10 12 9 13 11 14 17 16 19 15 1 18 0
        sage: perm.stratum_component() == AbelianStratum(8,6,2).even_component()
        True
        sage: cylinder_check(perm)
        True

    """
    if real_zeros == [6,2]:
        top_row = [0,1,2,3,4,5,6,7,8,9,10]
        bot_row = [2,10,9,8,6,3,5,1,4,7,0]
        return GeneralizedPermutation(top_row,bot_row)
    elif real_zeros == [4,2]:
        top_row = [0,1,2,3,4,5,6,7,8]
        bot_row = [2,4,1,8,7,5,3,6,0]
        return GeneralizedPermutation(top_row,bot_row)
    else:
        real_zeros.remove(2)
        if set(real_zeros) == {4}:
            perm = GeneralizedPermutation([0,1,2,3,4,5,6,7,8],[2,4,1,8,7,5,3,6,0])
            odd_perm = AbelianStratum(real_zeros[1:]).odd_component().single_cylinder_representative()
            return cylinder_concatenation(perm,odd_perm)
        elif set(real_zeros) == {6} or set(real_zeros) == {6,4}:
            perm = GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9,10],[2,10,9,8,6,3,5,1,4,7,0])
            odd_perm = AbelianStratum(real_zeros[1:]).odd_component().single_cylinder_representative()
            return cylinder_concatenation(perm,odd_perm)
        else:
            perm_1 = even_zero_even(real_zeros[0]-2)
            length_1 = len(perm_1[0])-1
```

```
            top_row_1 = perm_1[0]
            bot_row_1 = perm_1[1][:-1]
            for i in range(length_1):
                if bot_row_1[i] == 1:
                    bot_row_1[i] += length_1
            top_row_2 = [i+length_1 for i in range(1,6)]
            bot_row_2 = [3+length_1,5+length_1,2+length_1,1,4+length_1,0]
            top_row = top_row_1 + top_row_2
            bot_row = bot_row_1 + bot_row_2
            perm = GeneralizedPermutation(top_row,bot_row)
            if len(real_zeros) == 1:
                return perm
            else:
                odd_perm = AbelianStratum(real_zeros[1:]).odd_component().
    single_cylinder_representative()
                return cylinder_concatenation(perm,odd_perm)

def two_twos_even(real_zeros):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the even component of
    an Abelian stratum having two zeros of order two.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``real_zeros`` - a list of even positive integers two of which are equal to two.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = two_twos_even([4,2,2])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11
        2 8 5 3 1 10 9 6 4 11 7 0
        sage: perm.stratum_component() == AbelianStratum(4,2,2).even_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = two_twos_even([6,2,2])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13
        2 7 6 5 3 8 4 9 11 13 10 1 12 0
        sage: perm.stratum_component() == AbelianStratum(6,2,2).even_component()
        True
        sage: cylinder_check(perm)
        True

    """
    real_zeros.remove(2)
    real_zeros.remove(2)
    if set(real_zeros) == {4}:
```

```
        perm = GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9,10,11],[2,8,5,3,1,10,9,6,4,11,7,0])
        if len(real_zeros) == 1:
            return perm
        else:
            odd_perm = AbelianStratum(real_zeros[1:]).odd_component().
    single_cylinder_representative()
            return cylinder_concatenation(perm,odd_perm)
    else:
        odd_2_2 = GeneralizedPermutation([0,1,2,3,4,5,6],[2,4,6,3,1,5,0])
        perm = cylinder_concatenation(even_zero_even(real_zeros[0]),odd_2_2)
        if len(real_zeros) == 1:
            return perm
        else:
            odd_perm = AbelianStratum(real_zeros[1:]).odd_component().
    single_cylinder_representative()
            return cylinder_concatenation(perm,odd_perm)

def even_twos_even(real_zeros,two_count):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the even component of
    an Abelian stratum having an even, at least four, number of zeros of order two.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``real_zeros`` - a list of even positive integers an even number of which
        are equal to two.

        - ``two_count`` - a positive integer equal to the number of twos in ``real_zeros``.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = even_twos_even([2,2,2,2],4)
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 5 4 1 12 3 10 7 11 9 6 8 0
        sage: perm.stratum_component() == AbelianStratum(2,2,2,2).even_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = even_twos_even([4,2,2,2,2,2,2],6)
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23
        2 5 4 13 12 3 10 7 11 9 6 8 14 16 18 15 19 17 20 23 22 1 21 0
        sage: perm.stratum_component() == AbelianStratum({4:1,2:6}).even_component()
        True
        sage: cylinder_check(perm)
        True

    """
```

```
    for i in range(two_count):
        real_zeros.remove(2)
    odd_2_2 = GeneralizedPermutation([0,1,2,3,4,5,6],[2,4,6,3,1,5,0])
    even_2_2_2_2 = GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9,10,11,12],[2,5,4,1,12,3,10,7,11,
    9,6,8,0])
    twos_perm = even_2_2_2_2
    for i in range((two_count-4)//2):
        twos_perm = cylinder_concatenation(twos_perm,odd_2_2)
    if len(real_zeros) == 0:
        return twos_perm
    else:
        odd_perm = AbelianStratum(real_zeros).odd_component().single_cylinder_representative()
        return cylinder_concatenation(twos_perm,odd_perm)

def odd_twos_even(real_zeros,two_count):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the even component of
    an Abelian stratum having an odd, at least three, number of zeros of order two.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``real_zeros`` - a list of even positive integers an even number of which
        are equal to two.

        - ``two_count`` - a positive integer equal to the number of twos in ``real_zeros``.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = odd_twos_even([2,2,2],3)
        sage: perm
        0 1 2 3 4 5 6 7 8 9
        2 9 8 7 6 3 5 1 4 0
        sage: perm.stratum_component() == AbelianStratum(2,2,2).even_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = odd_twos_even([4,2,2,2,2,2],5)
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
        2 9 8 7 6 3 5 10 4 11 13 15 12 16 14 17 20 19 1 18 0
        sage: perm.stratum_component() == AbelianStratum({4:1,2:5}).even_component()
        True
        sage: cylinder_check(perm)
        True

    """
    for i in range(two_count):
        real_zeros.remove(2)
    odd_2_2 = GeneralizedPermutation([0,1,2,3,4,5,6],[2,4,6,3,1,5,0])
```

```
    even_2_2_2 = GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9],[2,9,8,7,6,3,5,1,4,0])
    twos_perm = even_2_2_2
    for i in range((two_count-3)//2):
        twos_perm = cylinder_concatenation(twos_perm,odd_2_2)
    if len(real_zeros) == 0:
        return twos_perm
    else:
        odd_perm = AbelianStratum(real_zeros).odd_component().single_cylinder_representative()
        return cylinder_concatenation(twos_perm,odd_perm)
```

General components

The following method was added to the Abelian stratum general component class (`AbelianStratumComponent`) of the `surface_dynamics` package.

```
    def single_cylinder_representative(self, alphabet=None):
        r"""
        Returns a single cylinder permutation representative.

        Returns a permutation representative of a square-tiled surface in this
        component having a single vertical cylinder and a single horizontal cylinder.

        Such representatives were constructed for every stratum of Abelian
        differentials by Jeffreys [Jef19].

        INPUT::

            - ``alphabet`` - alphabet or ``None`` (defaut: ``None``):
            whether you want to specify an alphabet for your representative.

        EXAMPLES::

            sage: from surface_dynamics import *
            sage: from surface_dynamics.flat_surfaces.single_cylinder import cylinder_check

            sage: cc = AbelianStratum(1,1,1,1).unique_component()
            sage: p = cc.single_cylinder_representative()
            sage: p
            0 1 2 3 4 5 6 7 8
            2 6 5 3 1 8 4 7 0
            sage: p.stratum_component() == cc
            True
            sage: cylinder_check(p)
            True
            sage: cc = AbelianStratum(2,1,1).unique_component()
            sage: p = cc.single_cylinder_representative()
            sage: p
            0 1 2 3 4 5 6 7
            2 6 4 1 7 5 3 0
            sage: p.stratum_component() == cc
            True
            sage: cylinder_check(p)
            True
            sage: cc = AbelianStratum(3,3).non_hyperelliptic_component()
            sage: p = cc.single_cylinder_representative(alphabet=Alphabet(name='lower'))
            sage: p
            a b c d e f g h i
```

```
            c i g f h e b d a
            sage: p.stratum_component() == cc
            True
            sage: cylinder_check(p)
            True

        """
        from surface_dynamics.flat_surfaces.single_cylinder import cylinder_concatenation
        from surface_dynamics.flat_surfaces.single_cylinder import only_even_2
        from surface_dynamics.flat_surfaces.single_cylinder import only_odds_11
        from surface_dynamics.flat_surfaces.single_cylinder import odd_zeros_one_one
        from surface_dynamics.interval_exchanges.constructors import GeneralizedPermutation

        zeros = self.stratum().zeros()
        real_zeros = [z for z in zeros if z != 0]
        odd_zeros = [z for z in real_zeros if z%2 == 1]
        even_zeros = [z for z in real_zeros if z%2 == 0]

        fk_zeros_perm = GeneralizedPermutation([0],[0])
        mk_pt_perm = GeneralizedPermutation([0,1],[1,0])
        for i in range(self.stratum().nb_fake_zeros()):
            fk_zeros_perm = cylinder_concatenation(fk_zeros_perm,mk_pt_perm)

        if even_zeros == [2]:
            perm = only_even_2(odd_zeros)
        elif odd_zeros == [1,1]:
            perm = only_odds_11(even_zeros)
        else:
            if even_zeros != []:
                even_perm = AbelianStratum(even_zeros).odd_component().single_cylinder_representative()
            else:
                even_perm = GeneralizedPermutation([0],[0])
            odd_perm = odd_zeros_one_one(odd_zeros)
            perm = cylinder_concatenation(even_perm,odd_perm)

        perm = cylinder_concatenation(fk_zeros_perm,perm)

        if not alphabet == None:
            perm.alphabet(alphabet)

        return perm
```

Recall that there are difficulties in the general case when the only even order zero has order two, or when the only odd order zeros both have order one.

```
def only_even_2(odd_zeros):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the Abelian stratum
    having zeros of the given odd orders and a single zero of order two.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::
```

```

        - ``odd_zeros`` - an even length list of odd positive integers.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = only_even_2([1,1])
        sage: perm
        0 1 2 3 4 5 6 7
        2 6 4 1 7 5 3 0
        sage: perm.stratum_component() == AbelianStratum(2,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = only_even_2([1,1,1,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
        2 6 4 8 7 5 3 9 13 12 10 1 15 11 14 0
        sage: perm.stratum_component() == AbelianStratum({2:1,1:6}).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = only_even_2([1,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11
        2 7 11 6 3 9 5 1 8 4 10 0
        sage: perm.stratum_component() == AbelianStratum(2,1,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = only_even_2([1,1,1,1,1,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19
        2 7 11 6 3 9 5 12 8 4 10 13 17 16 14 1 19 15 18 0
        sage: perm.stratum_component() == AbelianStratum({2:1,1:8}).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = only_even_2([3,3,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
        2 6 4 8 7 5 3 9 15 13 12 14 11 1 10 0
        sage: perm.stratum_component() == AbelianStratum(3,3,2,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = only_even_2([3,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13
        2 6 4 8 7 5 3 9 12 1 13 11 10 0
        sage: perm.stratum_component() == AbelianStratum(3,2,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = only_even_2([5,3,3,3])
        sage: perm
```

```
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21
        2 8 6 5 7 4 9 3 11 13 10 14 12 15 21 19 18 20 17 1 16 0
        sage: perm.stratum_component() == AbelianStratum(5,3,3,3,2).unique_component()
        True
        sage: cylinder_check(perm)
        True

    """
    if odd_zeros.count(1) == len(odd_zeros) and odd_zeros.count(1) % 4 == 2:
        perm = GeneralizedPermutation([0,1,2,3,4,5,6,7],[2,6,4,1,7,5,3,0])
        if len(odd_zeros) == 2:
            return perm
        else:
            odd_zeros.remove(1)
            odd_zeros.remove(1)
            one_count = odd_zeros.count(1)
            return cylinder_concatenation(perm,even_ones_odds(odd_zeros,one_count))
    elif odd_zeros.count(1) == len(odd_zeros) and odd_zeros.count(1) % 4 == 0:
        perm = GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9,10,11],[2,7,11,6,3,9,5,1,8,4,10,0])
        if len(odd_zeros) == 4:
            return perm
        else:
            odd_zeros.remove(1)
            odd_zeros.remove(1)
            odd_zeros.remove(1)
            odd_zeros.remove(1)
            one_count = odd_zeros.count(1)
            return cylinder_concatenation(perm,even_ones_odds(odd_zeros,one_count))
    elif odd_zeros.count(1) == 2 and len(odd_zeros) == 4 :
        perm = GeneralizedPermutation([0,1,2,3,4,5,6,7],[2,6,4,1,7,5,3,0])
        return cylinder_concatenation(perm,no_ones_odds(odd_zeros[:2]))
    else:
        if len(odd_zeros) == 4 and odd_zeros.count(1) == 3:
            perm = GeneralizedPermutation([0,1,2,3,4,5,6,7],[2,6,4,1,7,5,3,0])
            return cylinder_concatenation(perm,one_one_odds([odd_zeros[0],1]))
        else:
            pair_zeros = odd_zeros[:2]
            odd_zeros = odd_zeros[2:]
            dif = abs(pair_zeros[0]-pair_zeros[1])
            if 1 in pair_zeros:
                if pair_zeros == [3,1]:
                    perm = GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9],[2,6,8,3,7,4,1,9,5,0])
                    if len(odd_zeros) == 0:
                        return perm
                    else:
                        one_count = odd_zeros.count(1)
                        return cylinder_concatenation(perm,even_ones_odds(odd_zeros,one_count))
                else:
                    perm_1 = one_one_odds([pair_zeros[0]-2,1])
                    length_1 = len(perm_1[0])-1
                    top_row_1 = perm_1[0]
                    bot_row_1 = perm_1[1][:-1]
                    for i in range(length_1):
                        if bot_row_1[i] == 1:
                            bot_row_1[i] += length_1
                    top_row_2 = [i+length_1 for i in range(1,6)]
                    bot_row_2 = [3+length_1,5+length_1,2+length_1,1,4+length_1,0]
                    top_row = top_row_1 + top_row_2
```

```
                    bot_row = bot_row_1 + bot_row_2
                    perm = GeneralizedPermutation(top_row,bot_row)
                    if len(odd_zeros) == 0:
                        return perm
                    else:
                        one_count = odd_zeros.count(1)
                        return cylinder_concatenation(perm,even_ones_odds(odd_zeros,one_count))
            elif dif > 0:
                pair_zeros[0] += -2
                perm_1 = no_ones_odds(pair_zeros)
                length_1 = len(perm_1[0])-1
                top_row_1 = perm_1[0]
                bot_row_1 = perm_1[1][:-1]
                for i in range(length_1):
                    if bot_row_1[i] == 1:
                        bot_row_1[i] += length_1
                top_row_2 = [i+length_1 for i in range(1,6)]
                bot_row_2 = [3+length_1,5+length_1,2+length_1,1,4+length_1,0]
                top_row = top_row_1 + top_row_2
                bot_row = bot_row_1 + bot_row_2
                perm = GeneralizedPermutation(top_row,bot_row)
                if len(odd_zeros) == 0:
                    return perm
                else:
                    perm_odd = odd_zeros_one_one(odd_zeros)
                    return cylinder_concatenation(perm,perm_odd)
            else:
                pair_zeros[1] += -2
                perm_1 = min_on_bot(pair_zeros)
                length_1 = len(perm_1[0])-1
                top_row_1 = perm_1[0]
                bot_row_1 = perm_1[1][:-1]
                for i in range(length_1):
                    if bot_row_1[i] == 1:
                        bot_row_1[i] += length_1
                top_row_2 = [i+length_1 for i in range(1,6)]
                bot_row_2 = [3+length_1,5+length_1,2+length_1,1,4+length_1,0]
                top_row = top_row_1 + top_row_2
                bot_row = bot_row_1 + bot_row_2
                perm = GeneralizedPermutation(top_row,bot_row)
                if len(odd_zeros) == 0:
                    return perm
                else:
                    odd_perm = odd_zeros_one_one(odd_zeros)
                    return cylinder_concatenation(perm,odd_perm)

def only_odds_11(even_zeros):
    r"""
    Returns a single cylinder permutation representative.


    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the Abelian stratum
    having zeros of the given even orders and two zeros of order one.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

```

```
    INPUT::

        - ``even_zeros`` - a list of even positive integers.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = only_odds_11([2])
        sage: perm
        0 1 2 3 4 5 6 7
        2 6 4 1 7 5 3 0
        sage: perm.stratum_component() == AbelianStratum(2,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = only_odds_11([2,2])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10
        2 4 9 7 3 8 5 1 10 6 0
        sage: perm.stratum_component() == AbelianStratum(2,2,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = only_odds_11([4,2])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 6 4 8 7 5 3 9 12 11 1 10 0
        sage: perm.stratum_component() == AbelianStratum(4,2,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = only_odds_11([6])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11
        2 6 4 1 8 7 10 9 11 5 3 0
        sage: perm.stratum_component() == AbelianStratum(6,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = only_odds_11([4,4])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
        2 7 4 10 9 5 8 6 3 11 14 13 1 12 0
        sage: perm.stratum_component() == AbelianStratum(4,4,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = only_odds_11([8,6])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
        2 7 4 14 9 8 11 10 13 5 12 6 3 15 18 17 20 16 1 19 0
        sage: perm.stratum_component() == AbelianStratum(8,6,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True

```

```
    """
    if set(even_zeros) == {2} and len(even_zeros) % 2 == 1:
        perm = GeneralizedPermutation([0,1,2,3,4,5,6,7],[2,6,4,1,7,5,3,0])
        if len(even_zeros) == 1:
            return perm
        else:
            even_zeros.remove(2)
            even_perm = AbelianStratum(even_zeros).odd_component().single_cylinder_representative()
            return cylinder_concatenation(perm,even_perm)
    elif set(even_zeros) == {2} and len(even_zeros) % 2 == 0:
        perm = GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9,10],[2,4,9,7,3,8,5,1,10,6,0])
        if len(even_zeros) == 2:
            return perm
        else:
            even_zeros.remove(2)
            even_zeros.remove(2)
            even_perm = AbelianStratum(even_zeros).odd_component().single_cylinder_representative()
            return cylinder_concatenation(perm,even_perm)
    elif even_zeros.count(2) == 1 and len(even_zeros) == 2:
        perm = GeneralizedPermutation([0,1,2,3,4,5,6,7],[2,6,4,1,7,5,3,0])
        even_perm = AbelianStratum(even_zeros[0]).odd_component().single_cylinder_representative()
        return cylinder_concatenation(perm,even_perm)
    else:
        num = even_zeros[0]
        if num % 4 == 2:
            top_row = [i for i in range(num+6)]
            bot_row = [2,6,4,1]
            for i in range(8,num+6,2):
                bot_row += [i,i-1]
            bot_row += [num+5,5,3,0]
            perm = GeneralizedPermutation(top_row,bot_row)
            if len(even_zeros) == 1:
                return perm
            else:
                even_perm = AbelianStratum(even_zeros[1:]).odd_component().single_cylinder_representative()
                return cylinder_concatenation(perm,even_perm)
        else:
            if num == 4:
                perm = GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9],[2,7,4,1,9,5,8,6,3,0])
            else:
                top_row = [i for i in range(num+6)]
                bot_row = [2,7,4,1]
                for i in range(9,num+5,2):
                    bot_row += [i,i-1]
                bot_row += [num+5,5,num+4,6,3,0]
                perm = GeneralizedPermutation(top_row,bot_row)
            if len(even_zeros) == 1:
                return perm
            else:
                even_perm = AbelianStratum(even_zeros[1:]).odd_component().single_cylinder_representative()
                return cylinder_concatenation(perm,even_perm)
```

The case of one even order zero having order two requires the following method which realises Proposition 3.12.

```
def min_on_bot(zero_pair):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the Abelian stratum
    having a pair of zeros of the given odd orders which differ by two.

    The permutations have a particular form required for the construction
    of other representatives.

    Such representatives were constructed by Jeffreys [Jef19].

    INPUT::

        - ``zero_pair`` - a list of two odd positive integers at least one and
        differing by two.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = min_on_bot([3,1])
        sage: perm
        0 1 2 3 4 5 6
        2 6 5 1 4 3 0
        sage: perm.stratum_component() == AbelianStratum(3,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = min_on_bot([5,3])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10
        2 6 4 10 8 3 1 9 7 5 0
        sage: perm.stratum_component() == AbelianStratum(5,3).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = min_on_bot([7,5])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
        2 6 4 10 8 3 12 9 7 5 14 11 1 13 0
        sage: perm.stratum_component() == AbelianStratum(7,5).unique_component()
        True
        sage: cylinder_check(perm)
        True

    """
    if zero_pair == [3,1]:
        return GeneralizedPermutation([0,1,2,3,4,5,6],[2,6,5,1,4,3,0])
    elif zero_pair == [5,3]:
        return GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9,10],[2,6,4,10,8,3,1,9,7,5,0])
    else:
        num = max(zero_pair)
        top_row = [i for i in range(2*num+1)]
        bot_row = [2,6,4,10,8,3,12,9,7,5]
```

```
        for i in range(14,2*num+2,2):
            bot_row = bot_row + [i,i-3]
        bot_row = bot_row + [1,2*num-1,0]
        return GeneralizedPermutation(top_row,bot_row)
```

The construction of permutation representatives of strata with only odd order zeros is carried out by the following method.

```
def odd_zeros_one_one(odd_zeros):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in an Abelian stratum
    having odd order zeros of the given orders.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``odd_zeros`` - an even length list of odd positive integers.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = odd_zeros_one_one([5,5])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 5 4 7 3 9 6 12 8 11 1 10 0
        sage: perm.stratum_component() == AbelianStratum(5,5).non_hyperelliptic_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = odd_zeros_one_one([5,1])
        sage: perm
        0 1 2 3 4 5 6 7 8
        2 4 7 3 1 8 6 5 0
        sage: perm.stratum_component() == AbelianStratum(5,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = odd_zeros_one_one([5,3,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
        2 4 7 3 9 8 6 5 10 13 1 14 12 11 0
        sage: perm.stratum_component() == AbelianStratum(5,3,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = odd_zeros_one_one([5,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 12 9 8 1 7 3 6 10 5 4 11 0
        sage: perm.stratum_component() == AbelianStratum(5,1,1,1).unique_component()
        True
```

```
        sage: cylinder_check(perm)
        True
        sage: perm = odd_zeros_one_one([5,3,1,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
        2 6 5 3 9 8 4 7 10 13 12 15 11 18 14 17 1 16 0
        sage: perm.stratum_component() == AbelianStratum(5,3,1,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = odd_zeros_one_one([5,1,1,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
        2 4 7 3 9 8 6 5 10 14 13 11 1 16 12 15 0
        sage: perm.stratum_component() == AbelianStratum(5,1,1,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True

    """
    one_count = odd_zeros.count(1)
    if one_count == 0:
        return no_ones_odds(odd_zeros)
    elif one_count == 1:
        return one_one_odds(odd_zeros)
    elif one_count == 2:
        return two_ones_odds(odd_zeros)
    elif one_count == 3:
        return three_ones_odds(odd_zeros)
    elif one_count >= 4 and one_count % 2 == 0:
        return even_ones_odds(odd_zeros,one_count)
    else:
        return odd_ones_odds(odd_zeros,one_count)
```

Different methods are called depending on the number of zeros of order one. The following method is called if there are no zeros of order one.

```
def no_ones_odds(odd_zeros):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in an Abelian stratum with odd
    order zeros and no zeros of order 1.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``odd_zeros`` - an even length list of odd positive integers none of which
        are equal to one.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

```

```
        sage: perm = no_ones_odds([7,3])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 5 4 8 3 7 9 6 12 11 1 10 0
        sage: perm.stratum_component() == AbelianStratum(7,3).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = no_ones_odds([5,3,3,3])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
        2 5 4 7 3 10 6 9 11 8 12 18 16 15 17 14 1 13 0
        sage: perm.stratum_component() == AbelianStratum(5,3,3,3).unique_component()
        True
        sage: cylinder_check(perm)
        True

    """
    if len(odd_zeros) == 2 and abs(odd_zeros[0]-odd_zeros[1]) <= 2:
        return odds_right_swap(odd_zeros)
    elif len(odd_zeros) == 2 and abs(odd_zeros[0]-odd_zeros[1]) > 2:
        return odds_left_swap(odd_zeros)
    else:
        perm = no_ones_odds(odd_zeros[:2])
        for i in range(2,len(odd_zeros),2):
            perm = cylinder_concatenation(perm,no_ones_odds(odd_zeros[i:i+2]))
        return perm
```

This method makes use of the following methods which carry out the constructions of Propositions 3.7 and 3.8, respectively.

```
def odds_right_swap(zero_pair):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the Abelian stratum
    having a pair of zeros of the given odd orders.

    Performs a column swap on another permutation to achieve this.

    Such a method was described by Jeffreys [Jef19].

    INPUT::

        - ``zero_pair`` - a list of two odd positive integers at least three and
        differing by zero or two.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = odds_right_swap([3,3])
        sage: perm
        0 1 2 3 4 5 6 7 8
        2 8 6 5 7 4 1 3 0
        sage: perm.stratum_component() == AbelianStratum(3,3).non_hyperelliptic_component()
```

```
        True
        sage: cylinder_check(perm)
        True
        sage: perm = odds_right_swap([5,5])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 5 4 7 3 9 6 12 8 11 1 10 0
        sage: perm.stratum_component() == AbelianStratum(5,5).non_hyperelliptic_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = odds_right_swap([5,3])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10
        2 5 4 7 3 10 6 9 1 8 0
        sage: perm.stratum_component() == AbelianStratum(5,3).unique_component()
        True
        sage: cylinder_check(perm)
        True

    """
    if zero_pair == [3,3]:
        return GeneralizedPermutation([0,1,2,3,4,5,6,7,8],[2,8,6,5,7,4,1,3,0])
    else:
        dif = abs(zero_pair[0]-zero_pair[1])
        if dif == 0:
            j = (min(zero_pair)-3)//2
        else:
            j = (min(zero_pair)-1)//2
        perm_1 = AbelianStratum(4*j+2-dif).odd_component().single_cylinder_representative()
        perm_2 = AbelianStratum(4).odd_component().single_cylinder_representative()
        perm = cylinder_concatenation(perm_1,perm_2)
        top_row = perm[0][1:]
        bot_row = perm[1][:-1]
        top_row = top_row[:-5]+[top_row[-4],top_row[-5]]+top_row[-3:]
        bot_row = bot_row[:-5]+[bot_row[-4],bot_row[-5]]+bot_row[-3:]
        top_row = [0]+top_row
        bot_row = bot_row+[0]
        perm_3 = GeneralizedPermutation(top_row,bot_row)
        perm_3.alphabet(len(perm_3[0]))
        return perm_3

def odds_left_swap(zero_pair):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in the Abelian stratum
    having a pair of zeros of the given odd orders.

    Performs a column swap on another permutation to achieve this.

    Such a method was described by Jeffreys [Jef19].

    INPUT::

        - ``zero_pair`` - a list of two odd positive integers at least three and
        differing by at least four.
```

```

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = odds_left_swap([7,3])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 5 4 8 3 7 9 6 12 11 1 10 0
        sage: perm.stratum_component() == AbelianStratum(7,3).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = odds_left_swap([11,5])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
        2 5 4 7 3 9 6 12 8 11 13 10 16 15 18 14 1 17 0
        sage: perm.stratum_component() == AbelianStratum(11,5).unique_component()
        True
        sage: cylinder_check(perm)
        True

    """
    dif = abs(zero_pair[0]-zero_pair[1])
    j = (min(zero_pair)-1)//2
    perm_1 = AbelianStratum(4*j+2).odd_component().single_cylinder_representative()
    perm_2 = AbelianStratum(dif).odd_component().single_cylinder_representative()
    perm = cylinder_concatenation(perm_1,perm_2)
    swap_point = len(perm_2[0])-1
    top_row = perm[0][1:]
    bot_row = perm[1][:-1]
    top_row = top_row[:-(swap_point+1)]+[top_row[-(swap_point)],top_row[-(swap_point+1)]]+
    top_row[-(swap_point-1):]
    bot_row = bot_row[:-(swap_point+1)]+[bot_row[-(swap_point)],bot_row[-(swap_point+1)]]+
    bot_row[-(swap_point-1):]
    top_row = [0]+top_row
    bot_row = bot_row+[0]
    perm_3 = GeneralizedPermutation(top_row,bot_row)
    perm_3.alphabet(len(perm_3[0]))
    return perm_3
```

The remaining cases are dealt with by the following methods which construct the permutation representatives given in and after Propositions 3.9 and 3.10.

```
def one_one_odds(odd_zeros):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in an Abelian stratum with odd
    order zeros and one zero of order 1.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

```

```
        - ``odd_zeros`` - an even length list of odd positive integers one of which
        is equal to one.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = one_one_odds([3,1])
        sage: perm
        0 1 2 3 4 5 6
        2 5 1 6 4 3 0
        sage: perm.stratum_component() == AbelianStratum(3,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = one_one_odds([5,1])
        sage: perm
        0 1 2 3 4 5 6 7 8
        2 4 7 3 1 8 6 5 0
        sage: perm.stratum_component() == AbelianStratum(5,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = one_one_odds([7,3,3,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
        2 5 4 9 3 8 6 11 10 7 12 18 16 15 17 14 1 13 0
        sage: perm.stratum_component() == AbelianStratum(7,3,3,1).unique_component()
        True
        sage: cylinder_check(perm)
        True

    """
    num = odd_zeros[0]
    if num == 3:
        perm = GeneralizedPermutation([0,1,2,3,4,5,6],[2,5,1,6,4,3,0])
        odd_zeros.remove(1)
        if len(odd_zeros) == 1:
            return perm
        else:
            return cylinder_concatenation(perm,no_ones_odds(odd_zeros[1:]))
    elif num == 5:
        perm = GeneralizedPermutation([0,1,2,3,4,5,6,7,8],[2,4,7,3,1,8,6,5,0])
        odd_zeros.remove(1)
        if len(odd_zeros) == 1:
            return perm
        else:
            return cylinder_concatenation(perm,no_ones_odds(odd_zeros[1:]))
    else:
        odd_zeros.remove(1)
        perm_1 = AbelianStratum(num-3).odd_component().single_cylinder_representative()
        length_1 = len(perm_1[0])-1
        top_row_1 = perm_1[0]
        bot_row_1 = perm_1[1][:-1]
        for i in range(length_1):
            if bot_row_1[i] == 1:
                bot_row_1[i] = 4+length_1
```

```
        top_row_2 = [i+length_1 for i in range(1,6)]
        bot_row_2 = [3+length_1,1+length_1,1,5+length_1,2+length_1,0]
        top_row = top_row_1 + top_row_2
        bot_row = bot_row_1 + bot_row_2
        perm = GeneralizedPermutation(top_row,bot_row)
        if len(odd_zeros) == 1:
            return perm
        else:
            return cylinder_concatenation(perm,no_ones_odds(odd_zeros[1:]))

def two_ones_odds(odd_zeros):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in an Abelian stratum with odd
    order zeros and two zeros of order 1.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``odd_zeros`` - an even length list of odd positive integers two of which
        are equal to one.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = two_ones_odds([3,3,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 5 7 6 4 3 8 11 1 12 10 9 0
        sage: perm.stratum_component() == AbelianStratum(3,3,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = two_ones_odds([5,5,3,3,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24
        2 4 7 3 9 8 6 5 10 12 15 11 17 16 14 13 18 24 22 21 23 20 1 19 0
        sage: perm.stratum_component() == AbelianStratum(5,5,3,3,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True

    """
    odd_zeros.remove(1)
    odd_zeros.remove(1)
    perm = cylinder_concatenation(one_one_odds([odd_zeros[0],1]),one_one_odds([odd_zeros[1],1])
    )
    if len(odd_zeros) == 2:
        return perm
    else:
        return cylinder_concatenation(perm,no_ones_odds(odd_zeros[2:]))
```

```
def three_ones_odds(odd_zeros):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in an Abelian stratum with odd
    order zeros and three zeros of order 1.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``odd_zeros`` - an even length list of odd positive integers three of which
        are equal to one.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = three_ones_odds([3,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10
        2 10 6 5 1 8 4 7 3 9 0
        sage: perm.stratum_component() == AbelianStratum(3,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = three_ones_odds([5,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 12 9 8 1 7 3 6 10 5 4 11 0
        sage: perm.stratum_component() == AbelianStratum(5,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = three_ones_odds([7,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14
        2 14 10 9 1 4 3 6 5 12 8 11 7 13 0
        sage: perm.stratum_component() == AbelianStratum(7,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = three_ones_odds([9,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
        2 16 13 12 1 4 3 6 5 11 7 10 14 9 8 15 0
        sage: perm = three_ones_odds([3,3,3,1,1,1])
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
        2 5 7 6 4 3 8 11 13 12 10 9 14 17 1 18 16 15 0
        sage: perm.stratum_component() == AbelianStratum(3,3,3,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True

```

```
    """
    if len(odd_zeros) > 4:
        num = odd_zeros[0]
        odd_zeros.remove(1)
        odd_zeros.remove(num)
        perm = one_one_odds([num,1])
        return cylinder_concatenation(perm,two_ones_odds(odd_zeros))
    else:
        num = odd_zeros[0]
        res_4 = num % 4
        if num == 3:
            return GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9,10],[2,10,6,5,1,8,4,7,3,9,0])
        elif num == 5:
            return GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9,10,11,12],[2,12,9,8,1,7,3,6,10,5
    ,4,11,0])
        elif res_4 == 3:
            top_row = [i for i in range(num+8)]
            bot_row = [2,num+7,num+3,num+2,1]
            for i in range(4,num+1,2):
                bot_row += [i,i-1]
            bot_row += [num+5,num+1,num+4,num,num+6,0]
            return GeneralizedPermutation(top_row,bot_row)
        else:
            top_row = [i for i in range(num+8)]
            bot_row = [2,num+7,num+4,num+3,1]
            for i in range(4,num-1,2):
                bot_row += [i,i-1]
            bot_row += [num+2,num-2,num+1,num+5,num,num-1,num+6,0]
            return GeneralizedPermutation(top_row,bot_row)

def even_ones_odds(odd_zeros,one_count):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in an Abelian stratum with odd
    order zeros and an even, at least four, number of zeros of order 1.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

    INPUT::

        - ``odd_zeros`` - an even length list of odd positive integers an even number of which
        are equal to one.

        - ``one_count`` - a positive integer equal to the number of ones in ``real_zeros``.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = even_ones_odds([1,1,1,1],4)
        sage: perm
        0 1 2 3 4 5 6 7 8
        2 6 5 3 1 8 4 7 0
        sage: perm.stratum_component() == AbelianStratum(1,1,1,1).unique_component()
```

```
        True
        sage: cylinder_check(perm)
        True
        sage: perm = even_ones_odds([1,1,1,1,1,1],6)
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12
        2 8 1 5 11 7 3 10 6 12 9 4 0
        sage: perm.stratum_component() == AbelianStratum(1,1,1,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = even_ones_odds([5,3,1,1,1,1],4)
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
        2 6 5 3 9 8 4 7 10 13 12 15 11 18 14 17 1 16 0
        sage: perm.stratum_component() == AbelianStratum(5,3,1,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = even_ones_odds([3,3,1,1,1,1,1,1],6)
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
        2 8 13 5 11 7 3 10 6 12 9 4 14 20 18 17 19 16 1 15 0
        sage: perm.stratum_component() == AbelianStratum(3,3,1,1,1,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True

    """
    for i in range(one_count):
        odd_zeros.remove(1)
    four_ones = GeneralizedPermutation([0,1,2,3,4,5,6,7,8],[2,6,5,3,1,8,4,7,0])
    six_ones = GeneralizedPermutation([0,1,2,3,4,5,6,7,8,9,10,11,12],[2,8,1,5,11,7,3,10,6,12,9,
    4,0])
    if one_count % 4 == 0:
        perm = four_ones
        for i in range((one_count-4)//4):
            perm = cylinder_concatenation(perm,four_ones)
    else:
        perm = six_ones
        for i in range((one_count-6)//4):
            perm = cylinder_concatenation(perm,four_ones)
    if len(odd_zeros) == 0:
        return perm
    else:
        return cylinder_concatenation(perm,no_ones_odds(odd_zeros))

def odd_ones_odds(odd_zeros,one_count):
    r"""
    Returns a single cylinder permutation representative.

    Returns a permutation representative of a square-tiled surface having a single
    vertical cylinder and a single horizontal cylinder in an Abelian stratum with odd
    order zeros and an odd, at least five, number of zeros of order 1.

    Such representatives were constructed for every stratum of Abelian
    differentials by Jeffreys [Jef19].

```

```
    INPUT::

        - ``odd_zeros`` - an even length list of odd positive integers an odd number of which
        are equal to one.

        - ``one_count`` - a positive integer equal to the number of ones in ``real_zeros``.

    EXAMPLES::

        sage: from surface_dynamics import *
        sage: from surface_dynamics.flat_surfaces.single_cylinder import *

        sage: perm = odd_ones_odds([5,1,1,1,1,1],5)
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16
        2 4 7 3 9 8 6 5 10 14 13 11 1 16 12 15 0
        sage: perm.stratum_component() == AbelianStratum(5,1,1,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True
        sage: perm = odd_ones_odds([3,1,1,1,1,1,1,1],7)
        sage: perm
        0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18
        2 5 7 6 4 3 8 14 1 11 17 13 9 16 12 18 15 10 0
        sage: perm.stratum_component() == AbelianStratum(3,1,1,1,1,1,1,1).unique_component()
        True
        sage: cylinder_check(perm)
        True

    """
    for i in range(one_count-1):
        odd_zeros.remove(1)
    even_ones = [1 for i in range(one_count-1)]
    return cylinder_concatenation(one_one_odds(odd_zeros),even_ones_odds(even_ones,one_count-1)
    )
```

Origami method

The following method returns the 1,1-square-tiled surface permutation representative as an origami; that is, as a pair of permutations in the symmetric group on the number of squares.

```
    def single_cylinder_origami(self):
        r"""
        Returns an origami associated to a single cylinder permutation representative.

        Returns an origami in this connected component having a single vertical
        cylinder and a single horizontal cylinder.

        Examples::

            sage: from surface_dynamics import *

            sage: cc = AbelianStratum(4).odd_component()
            sage: O = cc.single_cylinder_origami()
            sage: O
```

```
            (1,2,3,4,5)
            (1,4,3,5,2)
            sage: O.stratum_component() == cc
            True
            sage: cc = AbelianStratum(5,3).unique_component()
            sage: O = cc.single_cylinder_origami()
            sage: O
            (1,2,3,4,5,6,7,8,9,10)
            (1,9,8,10,6,7,4,3,5,2)
            sage: O.stratum_component() == cc
            True
            sage: cc = AbelianStratum(4,2).even_component()
            sage: O = cc.single_cylinder_origami()
            sage: O
            (1,2,3,4,5,6,7,8)
            (1,3,7,5,6,8,4,2)
            sage: O.stratum_component() == cc
            True

        """
        from surface_dynamics.flat_surfaces.origamis.origami import Origami

        perm = self.single_cylinder_representative()
        t0 = tuple([i for i in range(1,len(perm[0]))])
        t1 = [1]
        for i in range(len(perm[1])-2):
            ind = perm[1].index(t1[i])
            t1.append(ind+1)
        t1 = tuple(t1)
        return Origami(t0,t1)
```

Stratum methods

The following methods can be called on an Abelian Stratum.

```
    def single_cylinder_representative(self, alphabet=None):
        r"""
        Returns a single cylinder permutation representative.

        Returns a permutation representative of a square-tiled surface in this
        component having a single vertical cylinder and a single horizontal cylinder.

        Such representatives were constructed for every stratum of Abelian
        differentials by Jeffreys [Jef19].

        INPUT::

            - ``alphabet`` - alphabet or ``None`` (defaut: ``None``):
            whether you want to specify an alphabet for your representative.

        EXAMPLES::

            sage: from surface_dynamics import *
            sage: from surface_dynamics.flat_surfaces.single_cylinder import cylinder_check

            sage: C = AbelianStratum(2,0)
            sage: p = C.single_cylinder_representative()
```

```
            sage: p
            0 1 2 3 4
            4 3 1 2 0
            sage: p.stratum() == C
            True
            sage: cylinder_check(p)
            True
            sage: C = AbelianStratum(3,1)
            sage: p = C.single_cylinder_representative(alphabet=Alphabet(name='lower'))
            sage: p
            a b c d e f g
            c f b g e d a
            sage: p.stratum() == C
            True
            sage: cylinder_check(p)
            True
            sage: C = AbelianStratum(2)
            sage: C.single_cylinder_representative()
            Traceback (most recent call last):
               ...
            ValueError: no 1,1-square-tiled surfaces in this stratum try again with H_2(2, 0)
            sage: C = AbelianStratum(1,1)
            sage: C.single_cylinder_representative()
            Traceback (most recent call last):
               ...
            ValueError: no 1,1-square-tiled surfaces in this stratum try again with H_2(1^2, 0^2)
        """
        genus = self.genus()
        nb_real_zeros = self.nb_zeros()-self.nb_fake_zeros()

        if genus == 2 and nb_real_zeros == 1 and self.nb_fake_zeros() < 1:
            raise ValueError("no 1,1-square-tiled surfaces in this stratum try again with H_2(2, 0)")
        elif genus == 2 and nb_real_zeros == 2 and self.nb_fake_zeros() < 2:
            raise ValueError("no 1,1-square-tiled surfaces in this stratum try again with H_2(1^2, 0^2)")
        else:
            return self.one_component().single_cylinder_representative(alphabet)

    def single_cylinder_origami(self):
        r"""
        Returns an origami associated to a single cylinder permutation representative.

        Returns an origami in this connected component having a single vertical
        cylinder and a single horizontal cylinder.

        Examples::

            sage: from surface_dynamics import *

            sage: C = AbelianStratum(4)
            sage: O = C.single_cylinder_origami()
            sage: O
            (1,2,3,4,5)
            (1,4,3,5,2)
            sage: O.stratum() == AbelianStratum(4)
            True
```

```
            sage: C = AbelianStratum(2,0)
            sage: O = C.single_cylinder_origami()
            sage: O
            (1,2,3,4)
            (1,3,2,4)
            sage: O.stratum() == AbelianStratum(2)
            True

        """
        return self.one_component().single_cylinder_origami()
```

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