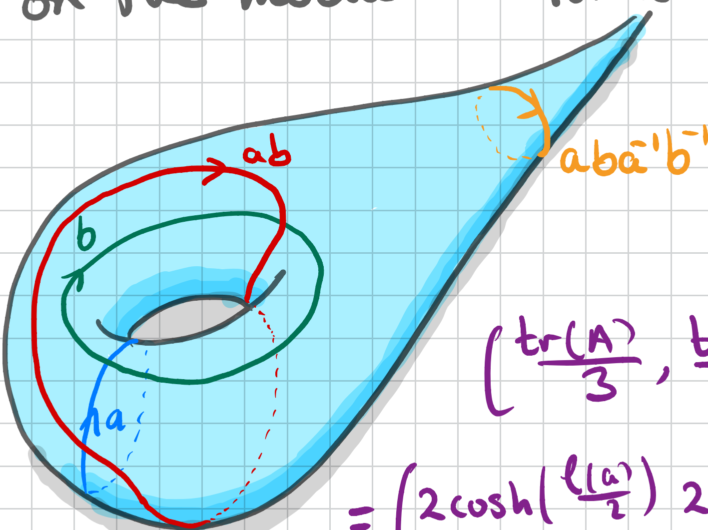
TCC: Hyperbolic surfaces, their length spectra and connections to Markov triples

Recall that we established a one-to-one correspondence between Markoff triples; i.e. triples $(x,y,z) \in \mathbb{Z}^3_{\geq 1}$ satisfying Markoff's equation:

$$x^2 + y^2 + z^2 = 3xyz,$$

and triples of simple closed geodesics on the modular torus:



$$\left(\frac{\mathrm{tr}(A)}{3}, \frac{\mathrm{tr}(B)}{3}, \frac{\mathrm{tr}(AB)}{3}\right)$$

$$= \left(\frac{2\cosh\left(\frac{\ell(a)}{2}\right)}{3}, \frac{2\cosh\left(\frac{\ell(b)}{2}\right)}{3}, \frac{2\cosh\left(\frac{\ell(ab)}{2}\right)}{3}\right)$$

We also discussed Markoff's uniqueness conjecture:

Conjecture (Frobenius)
Let (x,y,z), $x \le y \le z$, be a Markoff triple, then z uniquely determines x and y. That is, if (x,y,z) and (x',y',z), $x \le y \le z$ and $x' \le y' \le z$, are Markoff triples, then $x = x'$ and $y = y'$.

From now on, we will assume that all Markoff triples (x,y,z) are ordered $x \le y \le z$.

In 1975, Borosh performed a computer check and verified the conjecture for all Markoff triples (x,y,z) with $z \le 10^{105}$.

In the same work Borosh also noticed that

$$M(R) := |\{\text{Markoff triples } (x,y,z) \mid z \le R\}|$$

seemed to grow like $\log^2 R$.

Today we will present a proof due to Lang-Tan of the following result of Button and Schmutz

Theorem (Button, Schmutz)

Markoff's uniqueness conjecture is true when z is a prime power.

The proof will use hyperbolic geometry.

We shall also prove the following result of McShane-Rivin:

Theorem (McShane-Rivin)

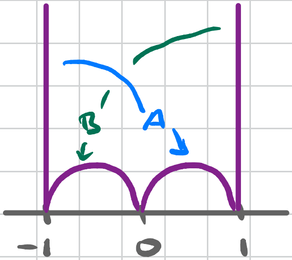
$\exists$ a constant $c>0$, so that

$$M(R) = c \cdot \log^2 R + O(\log R \log\log R).$$

They prove this by determining a similar count for

$$N(L) = \left| \left\{ \begin{array}{l} \text{simple closed} \\ \text{geodesics } \gamma \text{ on} \\ \text{the modular torus} \end{array} \middle| \ \ell(\gamma) \leq L \right\} \right|.$$

I. MARKHOFF MATRICES



Let $\Gamma = [PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})]$

$$= \left\langle A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \right\rangle$$

so that the modular torus is the hyperbolic surface $\mathbb{T} := {}_{\Gamma}\backslash \mathbb{H}^2$.

A matrix $M \in \Gamma$ is **primitive** if its corresponding geodesic on $\mathbb{T}$ is primitive; i.e. $M \neq N^n$ for some $N \in \Gamma$, $n \neq \pm 1$.

A primitive matrix M is a **Markoff matrix** if

i) its corresponding geodesic γ on $\mathbb{T}$ is simple;

ii) its fixed axis $\alpha_M \subseteq \mathbb{H}$ is a lift of γ with maximal height in $\mathbb{H}$.

N.B. M Markoff $\Leftrightarrow$ M^{-1} Markoff.

Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so that $T^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$. Recall

that $[B^{-1}, A^{-1}] = B^{-1}A^{-1}BA = -T^6$

We have the following:

Proposition

i) If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a Markoff matrix, then $|c|$ is a Markoff number and $\operatorname{tr} M = a+d = 3|c|$.

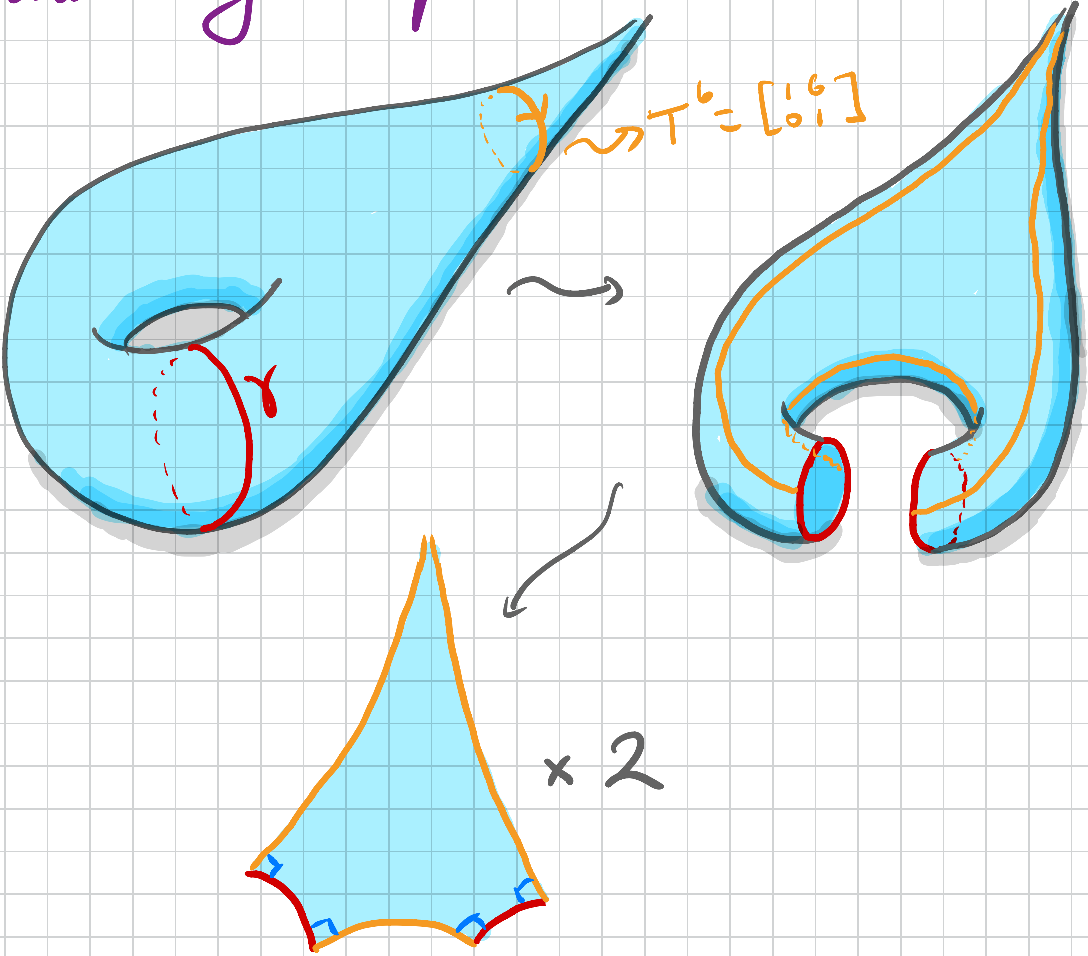
ii) Conversely, for any Markoff number c, $\exists$ a Markoff matrix M with $\operatorname{tr} M = 3c$ and $m_{21} = c$.

iii) Two Markoff matrices M and N correspond to the same simple closed geodesic in $\mathbb{T}$ $\Leftrightarrow$ $M^{\pm 1} = T^{3n} N T^{-3n}$ for some $n \in \mathbb{Z}$.

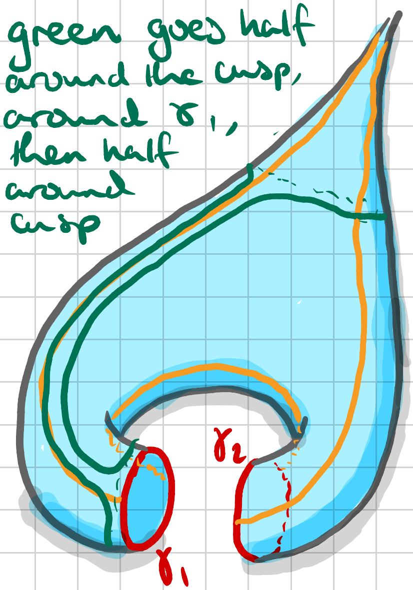
Proof

i) Suppose M is a Markoff matrix. Since M does not fix ∞ (i.e. it corresponds to a closed geodesic not bounding the cusp), $c \neq 0$. So, up to inverses, we can assume that $c > 0$.

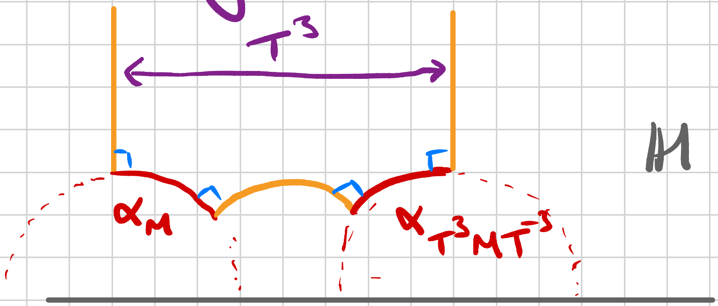
Cut the torus along the corresponding geodesic γ on $\mathbb{T}$, then once again along the perpendicular geodesics between the boundary components:



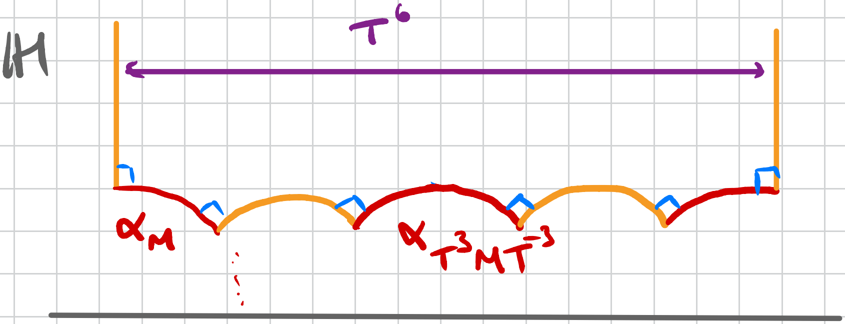
We get two hyperbolic right-angled pentagons.



The green curve is homotopic to γ_2 and so we can draw the right-angled pentagon in $\mathbb{H}$ as:



Since $T^6 \in \Gamma$ and fixes ∞, we can get a fundamental domain for Π by adding the translation of this pentagon to the right by 3:



By considering the loops around the boundary components of $\mathbb{T}\setminus\{\gamma\}$, we get

$$(T^{-6})(T^{3}MT^{-3})M = \pm I$$

$$\Leftrightarrow \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pm 1 & \pm 6 \\ 0 & \pm 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} a+3c & b+3d \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pm 1 & \pm 6 \\ 0 & \pm 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} a+3c & -3a-9c+b+3d \\ c & d-3c \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \pm 1 & \pm 6 \\ 0 & \pm 1 \end{pmatrix}$$

Considering the (2,1) entry forces $a+d=3c$, since $c\neq 0$. c is a Markoff number by the correspondence with simple closed geodesics.

ii) We have seen that Markoff numbers can be realised as $\frac{\operatorname{tr} M}{3}$ for some M

the matrix associated to a simple closed geodesic γ on $\mathbb{I}$; i.e. $\operatorname{tr} M = 3c$ for c the Markoff number. Conjugation preserves trace and so the Markoff matrix associated to γ has trace $3c$ and by i) has $(2,1)$ entry equal to $\pm c$.

iii) For the lift of γ corresponding to N to have maximal height, it must be an image of α_M or $T^3 \alpha_M T^{-3}$ by T^{6n} for some n. Hence, N must equal $T^{6n} M^{\pm 1} T^{-6n}$ or $T^{6n+3} M^{\pm 1} T^{-6n-3}$ $\square$.

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Markoff matrix. The endpoints of its geodesic axis are

$$z = \frac{a-d \pm \sqrt{\operatorname{tr}(M)^2 - 4}}{2c} .$$

Hence, the geodesic has height

$$\frac{\sqrt{\operatorname{tr}(M)^2 - 4}}{2c} = \frac{\sqrt{(3c)^2 - 4}}{2c} = \frac{1}{2}\sqrt{9 - \frac{4}{c^2}} .$$

Compare to the elements of the Markoff spectrum.

Let M' be the matrix obtained from M by interchanging diagonal entries. Since the $\mathrm{Isom}(\mathbb{I}) \cong D_6$ action on $\mathbb{I}$ is generated by reflection across $i\mathbb{R}$ and T, two Markoff matrices M and N correspond to the same geodesic up to isometry $\Leftrightarrow$

$$N^{\pm 1} = T^n M T^{-n} \quad \text{or} \quad N^{\pm 1} = T^n M' T^{-n}.$$

In such a case, say that $M \sim N$.

Markoff's uniqueness conjecture is then equivalent to saying $M \sim N \Leftrightarrow |m_{2,1}| = |n_{2,1}| \Leftrightarrow \operatorname{tr} M = \operatorname{tr} N$.

Define $M_k := \begin{pmatrix} 1 & k/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -k/c \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} a+k & b + \frac{k}{c}(d-a-k) \\ c & d-k \end{pmatrix}.$$

Then $M_k \in SL(2,\mathbb{Z}) \Leftrightarrow c \mid k(d-a-k)$.

Proposition Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Markoff matrix with $c>0$. Then $4 \nmid c$ and all the odd prime divisors of c are of the form $4m+1$, $m \in \mathbb{N}$.

Proof

Since $1 = ad - bc = (3c-d)d - bc$
$$= 3cd - d^2 - bc,$$
we have $d^2 \equiv -1 \bmod c$. But $x^2 \equiv -1 \bmod c$ is only solvable if c satisfies the conditions of the hypothesis. $\square$

Proposition Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a Markoff matrix and let $k \in \mathbb{Z}$. Then $\gcd(c, k, d-a-k)$ is 1 or 2. Hence, if $M_k \in SL(2,\mathbb{Z})$, then any exact odd prime power divisor p^m of c is relatively prime to k or $d-a-k$.

Proof Suppose that $\gcd(c, k, d-a-k) \neq 1$.

Let $f \neq 1$ divide c, k and $d-a-k$. Then f divides $d-a$. Since $a+d=3c$ and f divides c, f also divides $a+d$. Hence, f divides $2a=(a+d)-(d-a)$. Since $f|c$ and $f|2a$, $\underbrace{\gcd(a,c)=1}_{\text{since } ad-bc=1} \Rightarrow f|2 \Rightarrow f=2$. $\square$

II. UNIQUENESS FOR PRIME POWERS

We can now prove the following equivalent statement of the uniqueness conjecture for prime powers.

Theorem Suppose M and N are two Markoff matrices with $\mathrm{tr} M = \mathrm{tr} N = 3p^n$, where p is prime. Then $M \sim N$.

Proof

Let $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N=\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Up to inverses, we may assume that $c=c'>0$ and that

$$a+d=a'+d'=3c.$$

Hence, $\exists\, k \in \mathbb{Z}$ such that $a' = a+k$ and $d' = d-k$. Being Markoff matrices of the same trace and with axes of the same height, M and N are conjugate by some parabolic fixing ∞. In fact, we must have

$$N = M_k = \begin{pmatrix} a+k & b+\frac{k}{c}(d-a-k) \\ c & d-k \end{pmatrix}.$$

Since $N \in SL(2,\mathbb{Z})$, $c \mid k(d-a-k)$. Since $4 \nmid c$, $p=2 \Rightarrow c=2$ and the proof follows from the uniqueness of $(1,1,2)$. Hence, we may assume p is odd. By the previous proposition, p^n divides k or $d-a-k$. In the first case, $N = T^{\ell} M T^{-\ell}$ for some $\ell \in \mathbb{Z}$. In the latter, $N = T^{\ell} M' T^{-\ell}$ for some $\ell \in \mathbb{Z}$. Either way $M \sim N$.

$\square$

III. GEODESICS ON THE MODULAR TORUS

Recall the following results from two lectures ago:

- $\pi_1(\mathbb{T}) \cong F_2 = \langle a, b \rangle$

- $H_1(\mathbb{T}, \mathbb{Z}) := \pi_1(\mathbb{T})^{ab} = \langle a, b \mid [a,b] \rangle$
  $\cong \mathbb{Z}^2.$

  $\psi : F_2 \to \mathbb{Z}^2$
  $a \mapsto (1,0)$
  $b \mapsto (0,1)$

- $\mathrm{Out}(F_2) \cong \mathrm{Aut}(\mathbb{Z}^2) \cong GL(2, \mathbb{Z})$
  $\cong \mathrm{Mod}^{\pm}(\mathbb{T}).$

- $c \in F_2$ primitive (i.e. $\exists\, d$ s.t. $F_2 = \langle c, d \rangle$)
  $\longleftrightarrow$ γ simple closed curve on $\mathbb{T}$.

We will require the following lemma.

Lemma Let $p_1, p_2 \in F_2$ be primitive. Then $\psi(p_1) = \psi(p_2) \in \mathbb{Z}^2 \iff p_1$ is conjugate to p_2 in F_2.

Proof:

$\Leftarrow$: Let $p_1 = t p_2 t^{-1}$, then

$$\psi(p_1) = \psi(t p_2 t^{-1}) = \psi(t)\psi(p_2)\psi(t^{-1}) = \psi(p_2)\psi(t)\psi(t^{-1})$$
$$= \psi(p_2).$$

$\Rightarrow$: Since p_1 is primitive, $\exists\, q_1 \in F_2$ so that $F_2 = \langle p_1, q_1 \rangle$. Let $\alpha: F_2 \to F_2$ be an automorphism taking p_1 to p_2. Let $\bar{\alpha}: \mathbb{Z}^2 \to \mathbb{Z}^2$ be the induced automorphism of $\mathbb{Z}^2$. We have

$$\bar{\alpha}(\psi(p_1)) = \psi(\alpha(p_1)) = \psi(p_2) = \psi(p_1).$$

Hence, on the basis $\langle \psi(p_1), \psi(q_1) \rangle = \mathbb{Z}^2$, $\bar{\alpha} = \begin{pmatrix} 1 & l \\ 0 & \varepsilon \end{pmatrix}$ for some $l \in \mathbb{Z}$ and $\varepsilon = \pm 1$. That is, $\bar{\alpha}(\psi(q_1)) = \psi(p_1)^l \psi(q_1)^\varepsilon$.

Define $\beta : F_2 \to F_2$ by $\beta(p_1) = p_1$, $\beta(q_1) = p_1^{-\varepsilon l} q_1^{\varepsilon}$.
Then $\bar{\alpha}\bar{\beta} : \mathbb{Z}^2 \to \mathbb{Z}^2$ equals

$$\begin{pmatrix} 1 & l \\ 0 & \varepsilon \end{pmatrix} \begin{pmatrix} 1 & -\varepsilon l \\ 0 & \varepsilon \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So $\alpha\beta \in \ker(\mathrm{Aut}(F_2) \to \mathrm{Aut}(\mathbb{Z}^2)) = \mathrm{Inn}(F_2)$.
Hence, since $\alpha\beta(p_1) = p_2$, they are conjugate. $\square$

Corollary Every homotopy class of simple closed curves on $\mathbb{T}$ corresponds to a conjugacy class of primitive elements in $\pi_1(F_2)$ which corresponds to a unique generator of $\mathbb{Z}^2$ (i.e. a primitive vector $(m,n) \in \mathbb{Z}^2$, $\gcd(m,n) = 1$).

Given an element $h \in H_1(\mathbb{T}, \mathbb{Z})$, we would like to discuss its length. The following result allows us to do this.

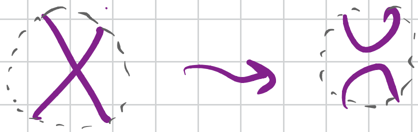
A **multicurve** is a map $\bigsqcup_{i=1}^{k} S^1 \to \mathbb{T}$. Image not necessarily disjoint.

A multicurve gives rise to an element of $H_1(\mathbb{T}, \mathbb{Z})$ and $(m,n) \in H_1(\mathbb{T}, \mathbb{Z})$ can be represented by a multicurve: $\bigsqcup_{i=1}^{m} a \sqcup \bigsqcup_{j=1}^{n} b \to \mathbb{T}$.

Theorem Let $h = (m,n) \in H_1(\mathbb{T}, \mathbb{Z})$ be a non-trivial element. Then $\exists$ a unique shortest multicurve representing h. If $h = d \cdot v$ with $d = \gcd(m,n)$ and v primitive, then h is represented by d copies of the unique geodesic γ_v corresponding to v.

<u>Proof</u>

h is represented by $d \cdot v$. So the shortest rep. of h has length at most $d \cdot \ell(\gamma_v)$. Let C be another multicurve representing h. Replace each component of C by its geodesic representative. This still represents h. If one component self-intersects or two components intersect, we can shorten C by smoothing:



Hence, all components are simple closed geodesics that are disjoint from one another. They must then be copies of the same simple closed geodesic. Hence, C must be $d \cdot \gamma_v$. This forces $d \cdot \gamma_v$ to be minimal and unique. $\square$

IV. A NORM ON HOMOLOGY

We now define a norm ℓ on $H_1(\mathbb{T}, \mathbb{Z})$. Let $\ell(h)$ be the length of the shortest multicurve representing h. By the previous theorem,

$$\ell(nh) = n\ell(h)$$

and

$$\ell(h+g) \leq \ell(h) + \ell(g).$$

union of two curves is not simple so smooth and shorten

So we can extend ℓ to $H_1(\mathbb{T}, \mathbb{Q})$ by linearity and to $H_1(\mathbb{T}, \mathbb{R})$ by continuity. This gives a pseudo-norm ℓ on $H_1(\mathbb{T}, \mathbb{R})$.

Consider the norm $\|\cdot\|_1$ on $H_1(\mathbb{T}, \mathbb{R})$ induced by $\|(m,n)\|_1 = |m| + |n|$ on $\mathbb{Z}^2$. This also equals the word length of the element of F_2 representing $(m,n) \in H_1(\mathbb{T}, \mathbb{Z})$.

Recall that no simple closed curve on $\mathbb{T}$ has height greater than $\frac{3}{2}$. We can therefore remove a neighbourhood of ∞ in $\mathbb{H}$ (and its images under Γ). The quotient is the compact torus with boundary.

F_2 acts on $\mathbb{H} \setminus \bigcup$ neighbourhoods properly discontinuously and cocompactly. Hence, by Milnor-Schwarz, F_2 with word length
$\frac{1}{K}\|\cdot\| - C \le d_X(\ ) \le K\|\cdot\| + C$
is quasi-isometric to $X := \mathbb{H} \setminus \bigcup$ neighbourhoods.
The simple closed geodesics avoid the truncated

parts and so have no additive errors.

In otherwords, we have

$$0 < c_1 < \frac{\ell(h)}{\|h\|_1} < c_2 < \infty.$$

This implies that ℓ is a norm whose unit ball B_ℓ is a compact convex subset of $H_1(\mathbb{T}, \mathbb{R}) \cong \mathbb{R}^2$.

V. THE FINAL COUNT

Finally, to count the simple closed curves on $\mathbb{T}$ with length $\leq L$, we count the primitive elements $h \in H_1(\mathbb{T}, \mathbb{Z})$ with $\ell(h) \leq L$. This equals the number of primitive elements in $L \cdot B_\ell$.

A standard argument (see the appendix) gives us that

$$\left|\{(m,n) \in L \cdot B_\ell \mid (m,n) = 1\}\right| = \frac{\operatorname{area}(B_\ell)}{\zeta(2)} \cdot L^2 + O(L \log L).$$

For a geodesic of length L the Markoff number has size

$$\frac{2\cosh\left(\frac{L}{2}\right)}{3} \sim e^{L}.$$

Hence,

$$\left|\{\text{Markoff triples } (x,y,z) \mid x \leq y \leq z \leq R\}\right|$$
$$= c \cdot \log^2 R + O(\log R \log\log R).$$

APPENDIX A. MILNOR-SCHWARZ

Theorem (Milnor-Schwarz lemma)

Let G be a group acting by isometries on a proper length space X properly discontinuously and cocompactly. Then G is finitely generated and for every finite generating set S of G and every $x_0 \in X$ the orbit

map $f : (G, d_S) \to (X, d_X)$
$$g \mapsto g \cdot x_0$$

is a quasi-isometry. That is, $\exists\, K \geq 1$ and $C > 0$, such that

$$\frac{1}{K} d_S(g,h) - C \leq d_X(f(g), f(h)) \leq K d_S(g,h) + C$$

$\forall g, h \in G$. d_S is the word metric determined by S.

APPENDIX B. LATTICE POINT COUNTS

First, consider a bounded convex domain $B \subseteq \mathbb{R}^2$.

Let $N(B) = |\mathbb{Z}^2 \cap B|$.

Let I_L be the set of unit squares $[0,1)^2$ in $\mathbb{Z}^2$ entirely contained in $L \cdot B$ and O_L be the set of unit squares of $\mathbb{Z}^2$ intersecting $\partial(L \cdot B)$.

Then $N(L) \leq |I_L| + |O_L|$ and

$|I_L| \leq \text{area}(L\cdot B) = \text{area}(B)\cdot L^2$.

Similarly,

$$N(L) \geq |I_L| \geq \text{area}(L\cdot B) - |O_L|$$
$$= \text{area}(B)\cdot L^2 - |O_L|.$$

Now, since B is convex and bounded every column of $\mathbb{Z}^2$ meets $\partial(LB)$ in a uniformly bounded number of squares but $L\cdot B$ covers $O(L)$ columns. Hence, $|O_L| = O(L)$ and we get

$$N(L) = \text{area}(B)\cdot L^2 + O(L).$$

Recall the definition of the Möbius function

$$\mu : \mathbb{N} \to \{-1, 0, 1\}$$

by

$$\mu(n) = \begin{cases} 1, & n = 1 \\ (-1)^k, & n = p_1 \cdots p_k \text{ a product of } k \text{ distinct primes} \\ 0, & p^2 \mid n \text{ for some prime } p. \end{cases}$$

It satisfies:

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1, & n = 1 \\ 0, & n > 1. \end{cases}$$

Indeed, if $n = p_1^{a_1} \cdots p_r^{a_r} \neq 1$, then $\mu(d) = 0$ unless $d = p_{i_1} \cdots p_{i_k}$ for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$. Hence,

$$\sum_{d \mid n} \mu(d) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} = 0.$$

Hence, for $(m,n) \in \mathbb{Z}^2$,

$$\sum_{d \mid \gcd(m,n)} \mu(d) = \begin{cases} 1, & \gcd(m,n) = 1 \\ 0, & \text{otherwise}. \end{cases}$$

That is, this sum detects primitivity.

Define

$$N_{prim}(\mathcal{B}) = \left| \{ (m,n) \in \mathbb{Z}^2 \cap \mathcal{B} \mid \gcd(m,n) = 1 \} \right|.$$

Then

$$N_{prim}(L \cdot \mathcal{B}) = \sum_{(m,n) \in \mathbb{Z}^2 \cap L\cdot\mathcal{B}} \; \sum_{d \mid \gcd(m,n)} \mu(d)$$

$$= \sum_{d \geq 1} \mu(d) \sum_{\substack{(m,n) \in \mathbb{Z}^2 \cap L\cdot\mathcal{B} \\ d \mid \gcd(m,n)}} 1 .$$

But $d \mid \gcd(m,n)$ for $(m,n) \in L \cdot \mathcal{B}$ if and only if $\left(\frac{m}{d}, \frac{n}{d}\right) \in \frac{L}{d} \cdot \mathcal{B}$.

Hence,

$$N_{prim}(L \cdot \mathcal{B}) = \sum_{d \geq 1} \mu(d) N\left(\frac{L}{d} \cdot \mathcal{B}\right)$$

$$= \sum_{d \geq 1} \mu(d) \left(\operatorname{area}(B) \frac{L^2}{d^2} + O\left(\frac{L}{d}\right)\right)$$

$$= \operatorname{area}(B) \cdot L^2 \cdot \sum_{d \geq 1} \frac{\mu(d)}{d^2} + \sum_{d \geq 1} \mu(d) \cdot O\left(\frac{L}{d}\right).$$

Now,

$$\sum_{d \geq 1} \frac{\mu(d)}{d^2} = \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)}$$

while the error term can be handled by a finite sum (since $N\left(\frac{L}{d} B\right) = 0$ (1) for d larger than some $L' = O(L)$) giving

for some $C > 0$.

$$\left| \sum_{d \leq O(L)} \mu(d) \cdot O\left(\frac{L}{d}\right) \right| \leq C \cdot \sum_{d \leq O(L)} \frac{L}{d}$$

$$= C \cdot L \cdot \sum_{d \leq O(L)} \frac{1}{d}$$

$$= C \cdot L \cdot O(\log L)$$

$$= O(L \log L).$$

So, we finally obtain

$$N_{prim}(L \cdot B) = \frac{\text{area}(B)}{\zeta(2)} \cdot L^2 + O(L \log L).$$