TCC: Hyperbolic surfaces, their length spectra and connections to Markov triples

Lecture 6: Markoff triples and geodesics in hyperbolic tori

Today's goal is to establish a link between integral solutions to the Markoff equation

$$x^2+y^2+z^2=3xyz$$

and the lengths of simple closed geodesics on the once-punctured hyperbolic torus

$$T = [PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})] \backslash \mathbb{H} .$$

This torus is called the modular torus since it is a 6-fold cyclic cover of the modular curve $PSL(2,\mathbb{Z}) \backslash \mathbb{H}$.

I. DIOPHANTINE APPROXIMATION

The Markoff equation first arose in the study of Diophantine approximation. Here, we ask:

Given an irrational $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, how well can we approximate α by $\frac{p}{q} \in \mathbb{Q}$?

But, what do we mean by "how well"?

Theorem (Dirichlet, 1842)
For any $\alpha \in \mathbb{R} / \mathbb{Q}$, $\exists$ infinitely many $\frac{p}{q} \in \mathbb{Q}$ such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2} .$$

<u>Proof</u> Let $N \in \mathbb{N}$. Consider $x_k = k\alpha - \lfloor k\alpha \rfloor \in (0,1)$ for $0 \leq k \leq N$. Let $I_j = [\frac{j}{N}, \frac{j+1}{N}]$, $0 \leq j \leq N-1$. By the pigeonhole principle $\exists$ $m < n$ such that

$$|x_n - x_m| < \frac{1}{N}.$$

Hence,

$$\left| \alpha - \frac{\lfloor n\alpha \rfloor - \lfloor m\alpha \rfloor}{(n-m)} \right| < \frac{1}{(n-m)N} < \frac{1}{(n-m)^2}.$$

Can we improve on $\frac{1}{q^2}$? Maybe $\frac{1}{2q^2}$?

Answer: A little.

Theorem (Hurwitz's Theorem, Hurwitz 1891 / Korkine–Zolotareff 1873)

For any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}\, q^2}$$

has infinitely many solutions $\frac{p}{q} \in \mathbb{Q}$.

For any $\varepsilon > 0$, the inequality

$$\left| \frac{1+\sqrt{5}}{2} - \frac{p}{q} \right| < \frac{1}{(\sqrt{5}+\varepsilon)\, q^2}$$

has only finitely many solutions.

We will move the proof to an appendix.

So, given $\alpha \in \mathbb{R}/\mathbb{Q}$ it is natural to ask what is the largest constant $\ell(\alpha)$ for which

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\ell(\alpha) q^2}$$

has infinitely many solutions $\frac{p}{q} \in \mathbb{Q}$?

Eg. $\ell\left(\frac{1+\sqrt{5}}{2}\right) = \sqrt{5}$.

Technically, $\ell(\alpha) := \limsup_{p,q \to \infty} \frac{1}{|q^2\alpha - pq|}$

We collect these values into a set called the Lagrange spectrum defined as

$$\mathcal{L} := \{\ell(\alpha) \mid \alpha \in \mathbb{R}/\mathbb{Q},\ \ell(\alpha) < \infty\}$$

There is a similar approximation for real indefinite binary quadratic forms

$$f(x,y) = ax^2 + bxy + cy^2,\ a,b,c \in \mathbb{R}$$

where we ask how close to 0 can

we get using $(x,y) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. We get a set called the Markoff spectrum defined by

$$M := \left\{ \frac{\sqrt{b^2 - 4ac}}{\inf\limits_{\substack{(x,y)\in\mathbb{Z}^2 \\ (x,y)\neq(0,0)}} |f(x,y)|} \;\middle|\; \begin{array}{l} f(x,y) \text{ real indefinite} \\ \text{binary quadratic form} \\ b^2 - 4ac > 0. \end{array} \right\}$$

We have $L, M \subseteq (0, \infty)$.

In 1880, Markoff proved the following.

Theorem (Markoff, 1880)

$L \cap (-\infty, 3) = M \cap (-\infty, 3) = \{k_1 < k_2 < k_3 < \cdots\}$ is a discrete countable set accumulating at 3 and with

$$k_i = \sqrt{9 - \frac{4}{m_i^2}}$$

where m_i is the i^{th} Markoff number and a Markoff number is the largest number in a triple $(x,y,z) \in \mathbb{N}^3$ satisfying

$$x^2+y^2+z^2 = 3xyz.$$

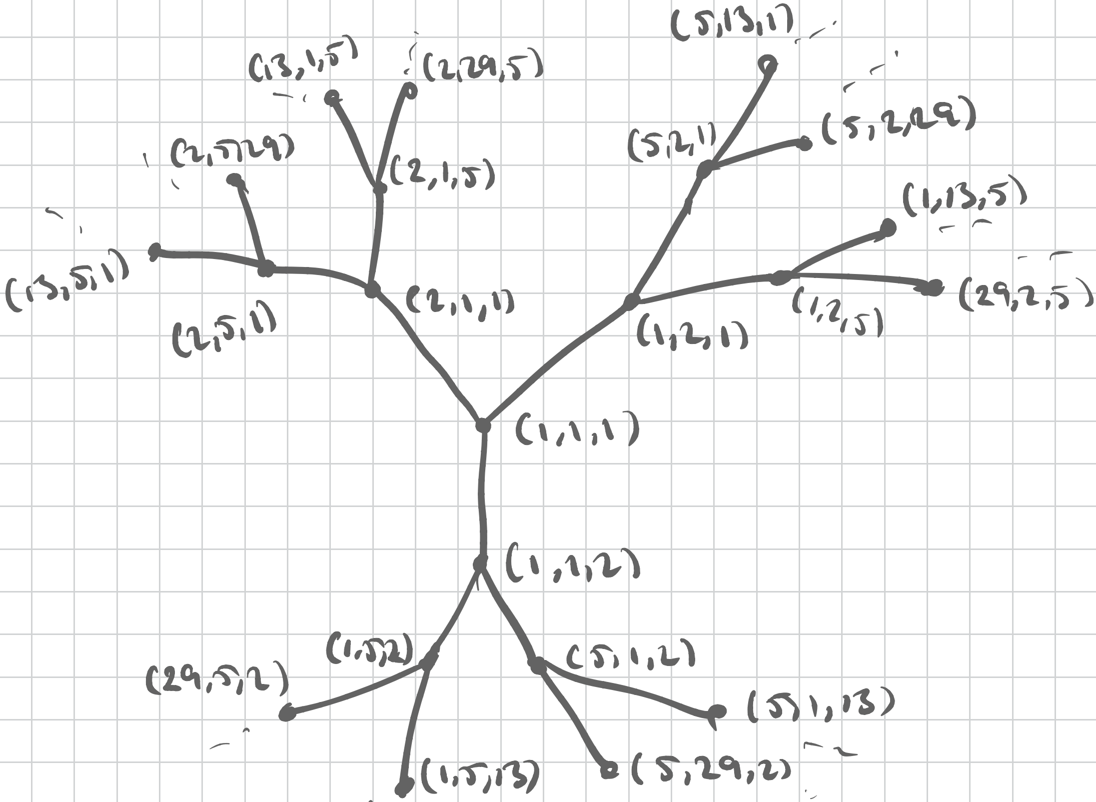
E.g. $(1,1,1)$ satisfies this equation and is the smallest non-zero solution. So $m_1 = 1 \Rightarrow k_1 = \sqrt{9 - \frac{4}{1^2}} = \sqrt{5}$, which is the constant from Hurwitz's Theorem.

In fact, Markoff proved that every non-zero solution can be reached from $(1,1,1)$ by applying moves of the form

$$(x,y,z) \mapsto \begin{cases} (3yz-x, y, z) \\ (x, 3xz-y, z) \\ (x, y, 3xy-z) \end{cases}$$

"Vieta involutions".

So, we get a tree of solutions



There is an action of $D_{12} = S_3 \times \mathbb{Z}/2\mathbb{Z}$ quotienting this tree to:

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & & (1,13,34) \\
 & & & (1,5,13) \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & (5,13,194) \\
(1,1,1) \arrow[r, dash] & (1,1,2) \arrow[r, dash] & (1,2,5) \arrow[uur, dash] \arrow[ddr, dash] & & \\
 & & & & (5,29,433) \\
 & & & (2,5,29) \arrow[ur, dash] \arrow[dr, dash] & \\
 & & & & (2,29,169)
\end{tikzcd}$$

One of the most famous conjectures in this setting is:

Conjecture (Frobenius, "Markoff's Uniqueness Conjecture")

Let (x, y, z) be a Markoff triple with $x \leq y \leq z$, then (x, y, z) is uniquely determined by z.

We shall see that this is equivalent to the following:

Conjecture

Let γ, γ' be two simple closed geodesics on the modular torus T of the same length. Then $\exists \, \phi \in \mathrm{Isom}(T) \cong D_{12}$ with $\phi(\gamma) = \gamma'$.

II. ANOTHER TRACE IDENTITY

Recall from the previous lecture the identities:

(i) $\mathrm{tr}(X) = \mathrm{tr}(X^{-1})$

$X = Y$

$\Rightarrow \mathrm{tr}(X^2) = \mathrm{tr}(X)^2 - 2$.

(ii) $\mathrm{tr}(X) = \mathrm{tr}(YXY^{-1})$

(iii) $\mathrm{tr}(XY) = \mathrm{tr}(X)\,\mathrm{tr}(Y) - \mathrm{tr}(XY^{-1})$.

$\forall\, X, Y \in SL(2,\mathbb{R})$.

We will use these to prove the Fricke-Klein identity:

$$\mathrm{tr}(A)^2 + \mathrm{tr}(B)^2 + \mathrm{tr}(AB)^2 = \mathrm{tr}(A)\,\mathrm{tr}(B)\,\mathrm{tr}(AB) + \mathrm{tr}(ABA^{-1}B^{-1}) + 2.$$

Note the resemblence to Markoff's equation.

<u>Proof</u> Let $a = \mathrm{tr}(A)$, $b = \mathrm{tr}(B)$, $c = \mathrm{tr}(AB)$.

Apply (iii) with $X = AB$ and $Y = A^{-1}B$ to get

$$\begin{aligned} \mathrm{tr}(ABA^{-1}B) &= \mathrm{tr}(AB)\,\mathrm{tr}(A^{-1}B) - \mathrm{tr}(ABB^{-1}A) \\ &= \mathrm{tr}(AB)\,\mathrm{tr}(A^{-1}B) - \mathrm{tr}(A^2). \end{aligned}$$

$$= c\,\mathrm{tr}(A^{-1}B) - (\mathrm{tr}(A)^2 - 2)$$

$$= c\,\mathrm{tr}(A^{-1}B) - a^2 + 2.$$

Now again for $X = A^{-1}$, $Y = B$:

$$\mathrm{tr}(A^{-1}B) = \mathrm{tr}(A^{-1})\,\mathrm{tr}(B) - \mathrm{tr}(A^{-1}B^{-1})$$

$$= ab - \mathrm{tr}(AB) = ab - c.$$

So, we have

$$\mathrm{tr}(ABA^{-1}B) = abc - c^2 - a^2 + 2.$$

Finally, applying (iii) with $X = ABA^{-1}$, $Y = B$ we get

$$\mathrm{tr}(ABA^{-1}B) = \mathrm{tr}(ABA^{-1})\,\mathrm{tr}(B) - \mathrm{tr}(ABA^{-1}B^{-1})$$

$$= b^2 - \mathrm{tr}(ABA^{-1}B^{-1}).$$

Hence, we get

$$a^2 + b^2 + c^2 = abc + \mathrm{tr}(ABA^{-1}B^{-1}) + 2$$

as required. $\square$

In 1954, Cohn was the first to notice that if we find matrices $A, B \in SL(2,\mathbb{R})$ with $\operatorname{tr}(ABA^{-1}B^{-1}) = -2$ then setting

$$x = \frac{\operatorname{tr}(A)}{3}, \quad y = \frac{\operatorname{tr}(B)}{3} \text{ and } z = \frac{\operatorname{tr}(AB)}{3}$$

we have

$$x^2+y^2+z^2 = \frac{\operatorname{tr}(A)^2}{9} + \frac{\operatorname{tr}(B)^2}{9} + \frac{\operatorname{tr}(AB)^2}{9}$$

$$= \frac{1}{9}\Big(\operatorname{tr}(A)\operatorname{tr}(B)\operatorname{tr}(AB) + \underbrace{\operatorname{tr}(ABA^{-1}B^{-1})}_{=-2} + 2\Big)$$

$$= 3 \cdot \frac{\operatorname{tr}(A)}{3} \cdot \frac{\operatorname{tr}(B)}{3} \cdot \frac{\operatorname{tr}(AB)}{3}$$

$$= 3xyz.$$

So, we obtain solutions to Markoff's equation provided $\operatorname{tr}(A), \operatorname{tr}(B), \operatorname{tr}(AB) \in 3\mathbb{N}$.

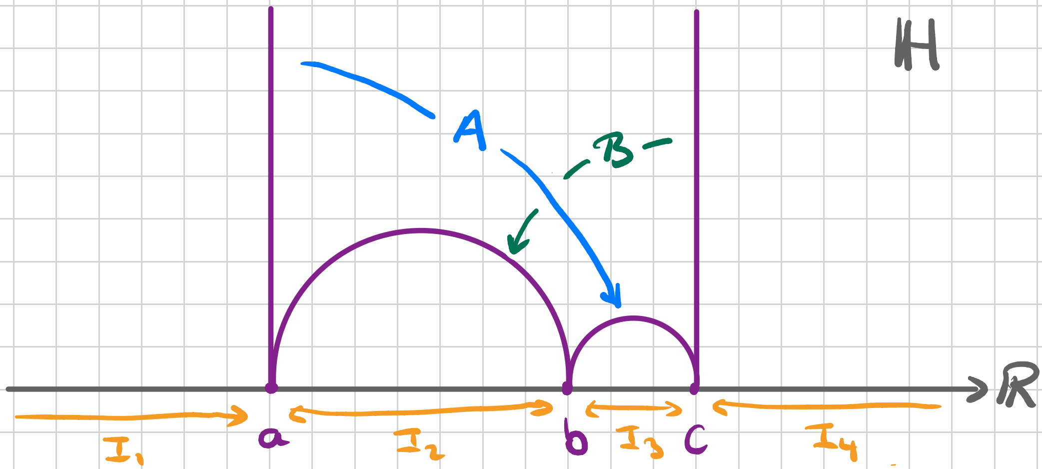
If we assume $A, B \in SL(2,\mathbb{Z})$, then

$$\operatorname{tr}(A)^2 + \operatorname{tr}(B)^2 + \operatorname{tr}(AB)^2 = \operatorname{tr}(A)\cdot\operatorname{tr}(B)\cdot\operatorname{tr}(AB)$$

already forces this. (Consider values modulo 3).

III. THE MODULAR TORUS

Consider a hyperbolic quadrilateral in $\mathbb{H}$ whose vertices lie in $\partial\mathbb{H}$. We may assume that one vertex lies at ∞:



Let $A, B \in PSL(2,\mathbb{R})$ be elements identifying the sides, as shown. We require that

$$A(a) = b, \quad A(\infty) = c, \quad B(\infty) = a, \quad B(c) = b.$$

Let us further suppose that $a, b, c \in \mathbb{Q}$ and that $A, B \in PSL(2,\mathbb{Z})$.

Notice that

$$B^{-1}A^{-1}BA(\infty) = B^{-1}A^{-1}B(c) = B^{-1}A^{-1}(b) = B^{-1}(a) = \infty.$$

Hence,

$$B^{-1}A^{-1}BA = \begin{pmatrix} \pm 1 & k \\ 0 & \pm 1 \end{pmatrix} =: K$$

By considering the action of A and B on the intervals $I_1, \ldots, I_4$, we can show that the quadrilateral is a fundamental domain for $\langle A, B \rangle \leq PSL(2,\mathbb{R})$.

Letting $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$ (i.e. choosing a lift in $SL(2,\mathbb{R})$) the bottom left entries of

$$BA = ABK$$

imply

$$\underbrace{b_{21}a_{11} + b_{22}a_{21}}_{M} = \pm \underbrace{(a_{21}b_{11} + a_{22}b_{21})}_{N}$$

while

$$-\frac{1}{MN} = (c-b)(b-a) > 0 \Rightarrow M = -N.$$

Hence,

$$K = \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix} \Rightarrow \operatorname{tr}(ABA^{-1}B^{-1}) = -2$$

In the quotient $\langle A,B\rangle\backslash\mathbb{H}$, a, b, c and ∞ are identified to give a torus with one cusp.

Let $C = AB$ and consider the triples (A,B,C) and $(x,y,z) = \left(\frac{\mathrm{tr}(A)}{3}, \frac{\mathrm{tr}(B)}{3}, \frac{\mathrm{tr}(C)}{3}\right)$.

Replacing (A,B,C) by $(B^{-1}A^{-1}, A, B^{-1})$ sends (x,y,z) to (z,x,y) but preserves Markoff's equation.

Similarly,

$$(A,B,C) \mapsto (B, A, BA)$$

sends

$$(x,y,z) \mapsto (y,x,z).$$

While

$$(A,B,C) \mapsto (B^{-1}A^{-1}B^{-1}, B, B^{-1}A^{-1})$$

affects

$$(x,y,z) \mapsto (3yz-x, y, z).$$

N.B. $\mathrm{tr}(B^{-1}A^{-1}B^{-1}) = \mathrm{tr}(B^{-1})\,\mathrm{tr}(A^{-1}B^{-1}) - \mathrm{tr}(A)$

In particular, by Markoff's result, we can assume that we have

$$\mathrm{tr}(A) = \mathrm{tr}(B) = \mathrm{tr}(C) = 3 .$$

Now, taking the traces of $B^{-1}A^{-1}B = KA^{-1}$

$$= \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

implies

$$3 = a_{11} + a_{22} = -ka_{21} - a_{22} - a_{11} = -ka_{21} - 3$$

$$\Rightarrow k \mid 6 .$$

∞ and c are Farey adjacent

$\infty = \frac{1}{0}$

$\infty = \frac{1}{0}$ $c = \frac{c}{1}$

$A(\infty) = C$

$A(\infty) = \frac{a_{11}}{a_{21}} = \frac{a_{11}}{1}$.

If $k = -6$, we get $a_{21} = 1$ and a similar calculation gives $b_{21} = 1$.

This forces a and b to be "Farey adjacent"*, and b and c $\left(\frac{p}{q} \text{ adjacent to } \frac{r}{s} \Leftrightarrow |ps - rq| = 1\right)$.

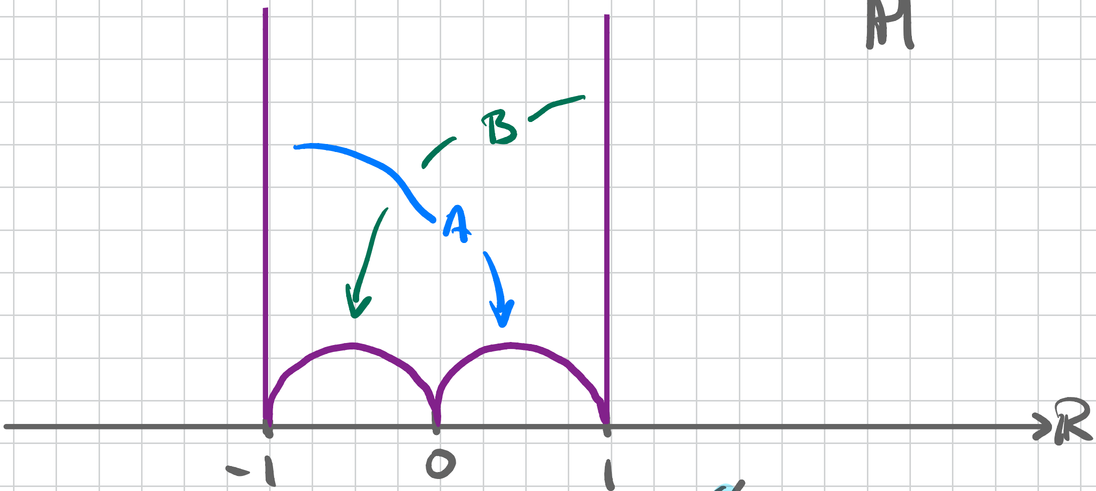
*see appendix

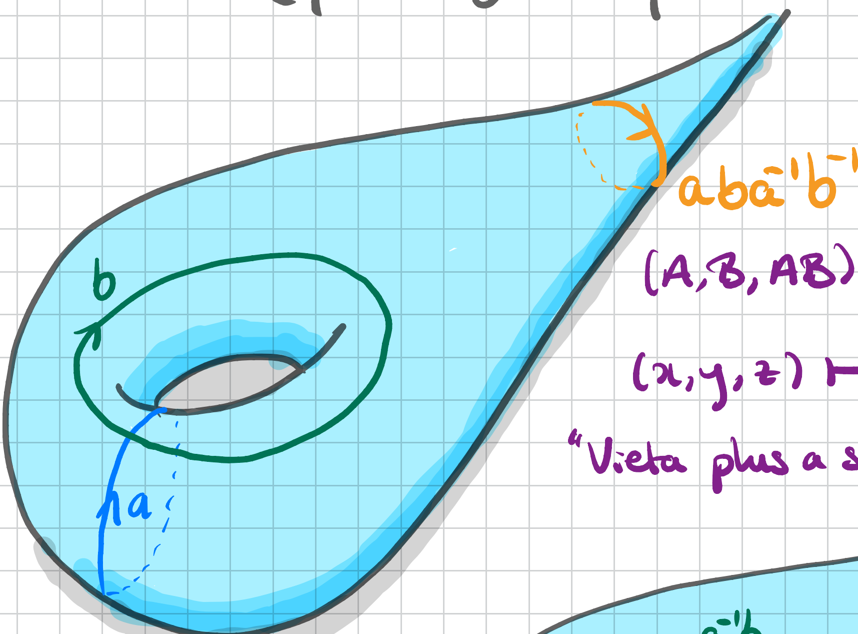
So, up to Möbius, we can take $a = -1$, $b = 0$, $c = 1$. This forces

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},\ B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}.$$

It can be checked that no other choice of $k \mid 6$ works.

M

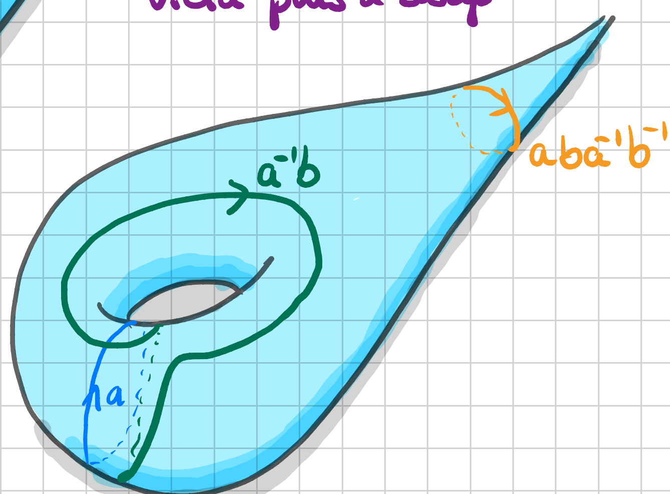




$(A, B, AB) \mapsto (A, A^{-1}B, B)$

$(x, y, z) \mapsto (x, 3xy - z, y)$

"Vieta plus a swap"



Dehn twist around a (a^{-1} technically)

The modular torus is then

$$T = \langle A, B \rangle \backslash \mathbb{H} .$$

It can be shown that

$$\langle A, B \rangle \cong F_2 \cong \pi_1(T)$$

and that

$$\langle A, B \rangle \cong [PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})]$$

i.e. the commutator subgroup.
We will see later that

$$[PSL(2,\mathbb{Z}) : \langle A, B \rangle] = 6$$

so that T is a 6-fold cover of the modular curve $PSL(2,\mathbb{Z}) \backslash \mathbb{H}$.

IV. ISOMETRY GROUPS

Let $X = \Gamma \backslash \mathbb{H}$ be a hyperbolic surface and let $\Gamma \cdot x \in X$.

Consider an element $A \in PSL(2,\mathbb{R})$.

We get a well-defined action of A on X if and only if for all $\gamma \in \Gamma$ we have

$$\Gamma(A \cdot x) = \Gamma(A \cdot \gamma x)$$

$\iff \forall \gamma \in \Gamma, \exists \gamma' \in \Gamma$ such that $\gamma' A = A\gamma$

$\iff \forall \gamma \in \Gamma, \exists \gamma' \in \Gamma$ such that $\gamma' = A\gamma A^{-1}$

$\iff A \in N_{PSL(2,\mathbb{R})}(\Gamma)$ (normaliser).

So we have:

Proposition

$$\mathrm{Isom}^+(X) \cong N_{PSL(2,\mathbb{R})}(\Gamma) / \Gamma$$

$$\mathrm{Isom}(X) = \mathrm{Isom}^+(X) \rtimes \mathbb{Z}/2\mathbb{Z}.$$

We determined earlier that the modular torus T is given by

$$[PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})] \backslash \mathbb{H}.$$

We have the following group theoretic facts (see appendix):

1. $PSL(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$

2. $N_{PSL(2,\mathbb{R})}([PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})]) \cong PSL(2,\mathbb{Z}).$

Hence,

Proposition

$$\mathrm{Isom}^+(T) \cong \mathbb{Z}/6\mathbb{Z}, \quad \mathrm{Isom}(T) \cong D_{12}.$$

Proof

By 2., and the previous proposition,

$$\mathrm{Isom}^+(T) \cong PSL(2,\mathbb{Z}) \Big/ [PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})]$$

$$\cong PSL(2,\mathbb{Z})^{ab}.$$

By 1., this is then $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$.

Hence, $\mathrm{Isom}(T) \cong \mathbb{Z}/6 \rtimes \mathbb{Z}/2\mathbb{Z} \cong D_{12}$.

$\square$

Isom(T) acts on the generators A and B of $[PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})]$ by permutations of (A, B, AB) and inversions.

So triples of traces can be ordered by acting by isometries.

V. SIMPLE CLOSED CURVES

We have

$$\pi_1(T) \cong F_2 \cong \langle A, B \rangle \cong [PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})].$$

An element $\gamma \in F_2 = \langle a, b \rangle$ is said to be primitive if $\exists\, \phi \in \mathrm{Aut}(F_2)$ such that $\phi(\gamma) = a$.

Recall that we can choose a and b to represent simple closed curves on T.

Theorem

$$\left\{ \begin{array}{c} \text{primitive elements} \\ \text{of } \pi_1(T) \text{ up} \\ \text{to conjugation} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{homotopy classes} \\ \text{of simple closed curve} \\ \text{on } T \end{array} \right\}$$

<u>Proof</u>

We require two lemmas.

Lemma

There are no separating simple closed curves on T.

Proof of lemma

Cut along the curve to get two components. Take the component without the cusp. Double along the boundary to get a compact hyperbolic surface. This surface has area at least 4π by Gauss-Bonnet. So the doubled component already had area at least 2π. But T itself has total area exactly 2π so the other component has area 0. $\lightning$ $\square$

Let $\mathrm{Mod}^{\pm}(T) = \left\{ \begin{array}{l} \text{homeomorphisms } T \circlearrowleft \\ \text{fixing the cusp} \end{array} \right\} \Big/ \text{homotopy}$

i.e. the extended mapping class group.

Lemma

$$\mathrm{Mod}^{\pm}(T) \cong GL(2,\mathbb{Z}) \cong \mathrm{Out}(F_2).$$

Proof of lemma

It is well known that

$$\mathrm{Out}(F_2) \cong \mathrm{Aut}(F_2^{ab}) \cong \mathrm{Aut}(\mathbb{Z}^2) \cong GL(2,\mathbb{Z}).$$

Now, any $f \in \mathrm{Mod}^{\pm}(T)$ induces a map on $H_1(X;\mathbb{Z}) \cong \mathbb{Z}^2 \cong F_2^{ab}$. So $\exists\ \mathrm{Mod}^{\pm}(T) \to GL(2,\mathbb{Z})$. This turns out to be an isomorphism. $\square$

So, given a simple closed curve $\gamma \in T$, $T-\{\gamma\}$ is homeomorphic to $T-\{a\}$ by the first lemma.

By the previous lemma, such a homeomorphism induces a $\phi \in \mathrm{Aut}(F_2)$ taking γ to a. So γ is primitive.

Conversely, any primitive element γ maps to a by some $\phi \in \mathrm{Aut}(F_2)$. If ϕ is conjugation, then γ is homotopic to a which is simple. If $\phi \in \mathrm{Out}(F_2)$ then $\exists f \in \mathrm{Mod}^{\pm}(T)$ taking γ to a so γ must be simple. $\square$

Recalling, that the Dehn twists generate Vieta involutions, every Markoff triple can be reached via a triple of lengths of simple closed geodesics on T.

Similarly, since $\mathrm{Mod}^{\pm}(T)$ is generated by Dehn twists, every simple closed curve can be reached by these moves.

Hence, we have established that a Markoff triple (x, y, z) up to reordering corresponds to a triple of lengths of simple closed curves on T up to isometry and vice versa.

APPENDIX A. HURWITZ'S THEOREM

Let $\alpha \in \mathbb{R}/\mathbb{Q}$ and consider its continued fraction expansion

$$\alpha = [a_0; a_1, a_2, a_3, \ldots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

Let $\frac{p_n}{q_n} := [a_0; a_1, \ldots, a_n]$. So

$$\frac{p_0}{q_0} = \frac{a_0}{1}, \qquad \frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}$$

$$\frac{p_2}{q_2} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = a_0 + \frac{a_2}{a_1 a_2 + 1} = \frac{a_0 a_1 a_2 + a_0 + a_2}{a_1 a_2 + 1}$$

In general, $$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}, \quad n \geq 2.$$

Moreover, we have

$$p_n q_{n+1} - p_{n+1} q_n = (-1)^n.$$

Letting $\alpha_n = [a_n; a_{n+1}, a_{n+2}, \cdots]$, we see that

$$\alpha = [0; a_1, a_2, \cdots, a_{n-1}, \alpha_n]$$

$$= \frac{\alpha_n p_{n-1} + p_{n-2}}{\alpha_n q_{n-1} + q_{n-2}}$$

$$\Rightarrow \alpha_n = \frac{p_{n-2} - \alpha q_{n-2}}{\alpha q_{n-1} - p_{n-1}}.$$

Check this.

Define $\beta_n := \dfrac{q_{n-2}}{q_{n-1}} = [0; a_{n-1}, a_{n-2}, \cdots, a_1]$

Therefore,

$$\alpha_{n+1} + \beta_{n+1} = \frac{p_{n-1} - \alpha q_{n-1}}{\alpha q_n - p_n} + \frac{q_{n-1}}{q_n} = \frac{p_{n-1} q_n - p_n q_{n-1}}{q_n(\alpha q_n - p_n)}$$

$$= \frac{(-1)^n}{q_n(\alpha q_n - p_n)}$$

$$\Rightarrow \quad \alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{(\alpha_{n+1}+\beta_{n+1})q_n^2} . \qquad (*)$$

We will argue that for every $n \geq 1$,

$\exists \frac{p}{q} \in \left\{ \frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}} \right\}$ with

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}\, q^2} .$$

For contradiction, assume not. Then, by $(*)$,

$\exists k \geq 1$ with

$$\max \left\{ (\alpha_k+\beta_k), (\alpha_{k+1}+\beta_{k+1}), (\alpha_{k+2}+\beta_{k+2}) \right\} \leq \sqrt{5}.$$

Since $\sqrt{5} < 3$ and $a_m \leq \alpha_m + \beta_m \ \forall m \geq 1$, we get

$$\max \left\{ a_k, a_{k+1}, a_{k+2} \right\} \leq 2 .$$

If $a_i = 2$ for some $i \in \{k, k+1, k+2\}$, then

$$\alpha_i + \beta_i = [a_i; a_{i+1}, a_{i+2}, \ldots] + [0; a_{i-1}, a_{i-2}, \ldots]$$

$$= 2 + [0; a_{i+1}, a_{i+2}, \ldots] + [0; a_{i-1}, a_{i-2}, \ldots]$$

$$\geq 2 + [0; 2, 1] = 2 + \frac{1}{3} > \sqrt{5} \ \lightning.$$

So $a_k = a_{k+1} = a_{k+2} = 1$.

Let $x = \dfrac{1}{\alpha_{k+2}}$ and $y = \beta_{k+1} = \dfrac{q_{k-1}}{q_k} \in \mathbb{Q}$.

So

$$\alpha_{k+1} = 1 + \frac{1}{\alpha_{k+2}} = 1 + x$$

$$\alpha_k = 1 + \frac{1}{\alpha_{k+1}} = 1 + \frac{1}{1+x}$$

$$\beta_{k+2} = \frac{1}{1 + \beta_{k+1}} = \frac{1}{1+y}.$$

$$y = \beta_{k+1} = \frac{1}{1+\beta_k} \Rightarrow \beta_k = \frac{1}{y} - 1.$$

So, we have

$$\max\Big\{ \underset{(1)}{\frac{1}{1+x} + \frac{1}{y}},\ \underset{(2)}{1 + x + y},\ \underset{(3)}{\frac{1}{x} + \frac{1}{1+y}} \Big\} \leq \sqrt{5}.$$

From (1) and (2):

$$\sqrt{5} \geq \frac{1}{1+x} + \frac{1}{y} \geq \frac{1}{\sqrt{5}-y} + \frac{1}{y} = \frac{\sqrt{5}}{y(\sqrt{5}-y)}$$

$$\Rightarrow \quad y(\sqrt{5}-y) \geq 1 \quad \Rightarrow \frac{\sqrt{5}-1}{2} \leq y \leq \frac{\sqrt{5}+1}{2}.$$

(A)

From (2) and (3):

$$\sqrt{5} \geq \frac{1}{x} + \frac{1}{1+y} \geq \frac{1}{\sqrt{5}-y-1} + \frac{1}{1+y} = \frac{\sqrt{5}}{(1+y)(\sqrt{5}-(1+y))}$$

$$\Rightarrow \frac{\sqrt{5}-1}{2} \leq y+1 \leq \frac{\sqrt{5}+1}{2} \Rightarrow \frac{\sqrt{5}-3}{2} \leq y \leq \frac{\sqrt{5}-1}{2}.$$

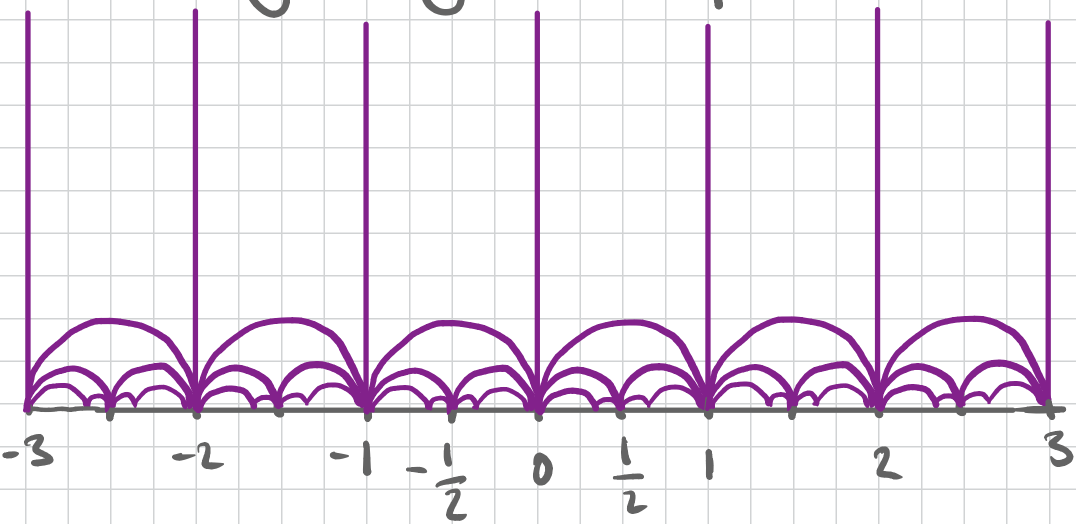
(B)

Then (A) and (B) force $y = \frac{\sqrt{5}-1}{2}$ but this contradicts $y = \beta_{k+1} = \frac{q_{k-1}}{q_k} \in \mathbb{Q}$.

$\square$

APPENDIX B. FAREY NEIGHBOURS

The Farey triangulation of $\mathbb{H}$ is



For $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$ $\left(\infty = \frac{0}{1}\right)$, we connect $\frac{p}{q}, \frac{r}{s}$ by the unique geodesic between them $\iff |ps - qr| = 1$. We say $\frac{p}{q}$ and $\frac{r}{s}$ are Farey adjacent or Farey neighbours.

In the determination of the modular

torus, we calculated that

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ 1 & a_{22} \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ 1 & b_{22} \end{pmatrix}.$$

Moreover, $A(\infty) = \frac{a_{11}}{1} = c \Rightarrow a_{11} = c$

and $B(\infty) = \frac{b_{11}}{1} = a \Rightarrow b_{11} = a$.

So $A = \begin{pmatrix} c & a_{12} \\ 1 & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b & b_{12} \\ 1 & b_{22} \end{pmatrix}$.

So a and ∞, and c and ∞ are Farey neighbours. Now, $PSL(2,\mathbb{Z})$ sends Farey neighbours to Farey neighbours and A sends ∞ and a to b and c while B send ∞ and c to a and b. Hence a and b, and b and c are also Farey neighbours.

We can apply a Möbius transformation (in fact, an element of $PSL(2,\mathbb{Z})$) to map ∞ to ∞, a to -1 and c to 1. Hence, b gets sent to the mutual Farey neighbour of -1 and 1 that is not ∞ and so we must have $b = 0$.

Then $A(\infty) = 1$ and $A(-1) = 0$ and $tr(A) = 3$ forces

$$A(z) = \frac{z+1}{z+2} \to A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and $B(\infty) = -1$ and $B(1) = 0$ and $tr(B) = 3$ forces

$$B(z) = \frac{z-1}{-z+2} \rightsquigarrow B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

as claimed.

APPENDIX C. $PSL(2,\mathbb{Z})$

One can apply Bass-Serre Theory for the action of $PSL(2,\mathbb{Z})$ on the Farey tree (the tree dual to the Farey triangulation) to see that $PSL(2,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ with generators $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $U = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$.

Now we see that the abelianisation is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \cong \mathbb{Z}/6\mathbb{Z}$. Hence,

$[PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})]$ has index 6.

Kurosh's subgroup theorem says that a finite index subgroup H of a free product is the free product of a free group of finite rank with intersections of H with conjugates of the free factors.

Here, the commutator subgroup has trivial intersections with conjugates of the free factors since they are abelian. Hence, $[PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})]$ is a free group.

We use Euler characteristics to determine the rank.

i. a finite group G has $\chi(G) = \frac{1}{|G|}$.

ii. a free product $G_1 * G_2$ has

$$\chi(G_1 * G_2) = \chi(G_1) + \chi(G_2) - 1$$

iii. a finite index subgroup $H \leq G$ has

$$\chi(H) = [G:H]\chi(G)$$

iv. a free group of rank r has

$$\chi(F_r) = 1 - r.$$

We have

$$\chi(PSL(2,\mathbb{Z})) = \chi(\mathbb{Z}/2\mathbb{Z}) + \chi(\mathbb{Z}/3\mathbb{Z}) - 1$$

$$= \frac{1}{2} + \frac{1}{3} - 1 = -\frac{1}{6}.$$

So $\chi([PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})]) = -1$

$$\Rightarrow [PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})] \cong F_2 .$$

Now,

$$[R,U] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$$

and

$$[R,U^{-1}] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix}$$

freely generate.

This is exactly the group we used to build be modular torus.

If $A \in PSL(2,\mathbb{R})$ normalises $PSL(2,\mathbb{Z})$ then it must send fixed points of parabolics in $PSL(2,\mathbb{Z})$ to other fixed points of parabolics. These fixed points are $\mathbb{Q} \cup \{\infty\}$. So $A \in PSL(2,\mathbb{Q})$. By checking the conjugates with $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, we find that A must have integer entries. So $N_{PSL(2,\mathbb{R})}(PSL(2,\mathbb{Z})) = PSL(2,\mathbb{Z})$.

A finite index subgroup of $PSL(2,\mathbb{Z})$ has the same fixed point set for parabolics. A similar argument (using $\begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$)

gives $N_{PSL(2,\mathbb{R})}([PSL(2,\mathbb{Z}), PSL(2,\mathbb{Z})]) = PSL(2,\mathbb{Z})$

again.