TCC: Hyperbolic surfaces, their length spectra, and connection to Markoff triples

Lecture 5: Multiplicities in the length spectrum and trace equalities

The objective of today's lecture is to prove the following result.

Theorem: Horowitz – Randol

For any hyperbolic surface X, any $N \geq 1$, there exists a family $(\gamma_i)_{1 \leq i \leq N}$ of distinct closed geodesics such that,

$\forall\, 1 \leq i, j \leq N$, $\ell_X(\gamma_i) = \ell_X(\gamma_j)$.

In other words, the length spectrum of any hyperbolic surface has unbounded multiplicities.

This is specific to hyperbolic surfaces: the length spectrum of generic negatively curved manifolds is simple ($\ell_X(\gamma) \neq \ell_X(\gamma')$ if $\gamma \neq \gamma'$).

We will prove this result in a pair of pants or once-holed torus. Since any hyperbolic surface contains a pair of pants, this is enough to conclude.

We here present an algebraic approach, which will be relevant in the following lectures. A geometric proof can be found in Appendix 3.7 of Buser 92.

I) The free group of two generators

The following will rely on algebra on free groups. Let us introduce the necessary notations.

Definition: Let S be a finite set.

→ A word in S is any expression of the form $\omega = s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}$ where $n \geq 0$, and $\forall\, 1 \leq i \leq n$, $s_i \in S$, $\varepsilon_i \in \{\pm 1\}$.

→ The only word of length $n = 0$ is the empty word.

→ The inverse of $\omega = s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}$ is defined by $\omega^{-1} = s_n^{-\varepsilon_n} \dots s_1^{-\varepsilon_1}$.

→ For two words $\omega = s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}$, $\omega' = s_1'^{\varepsilon_1'} \dots s_{n'}'^{\varepsilon_{n'}'}$, the concatenation $\omega\omega'$ is defined as $\omega\omega' = s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n} s_1'^{\varepsilon_1'} \dots s_{n'}'^{\varepsilon_{n'}'}$.

→ A redundant pair of $\omega = s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}$ is an index $1 \leq i < n$ such that $s_i = s_{i+1}$ and $\varepsilon_i = -\varepsilon_{i+1}$.

If ω has a redundant pair at i, we can reduce it to a new word $\omega' = s_1^{\varepsilon_1} \dots s_{i-1}^{\varepsilon_{i-1}} s_{i+2}^{\varepsilon_{i+2}} \dots s_{n}^{\varepsilon_n}$.

→ A word is called reduced if it has no redundant pair.

We make the following key observation, left as an exercise.

Lemma: Any word can be reduced repeatedly to a unique reduced word.

Example: $S = \{a, b\}$. $aba^{-1}b$ is reduced. $abb^{-1}aa^{-1}a$ reduces to aa.

Definition: Let $k \geq 1$. We define the **free group** with k generators as the set $\mathbb{F}_k = \langle a_1, \dots, a_k \rangle$ of reduced words on $S = \{a_1, \dots, a_k\}$, with:

→ the empty word as the neutral element;

→ for $w \in \mathbb{F}_k$, w^{-1} the inverse of w;

→ for $w, w' \in \mathbb{F}_k$, $w \bullet w'$ is the reduced word obtained from ww'.

Exercise: Prove this defines a group, generated by $\{a_1, \dots, a_n\}$.

In the following we will mostly talk about $\mathbb{F}_2 = \langle a, b \rangle$.

Example: $w = aba^{-1} \in \mathbb{F}_2$, $w^{-1} = ab^{-1}a^{-1}$ and $w^2 = abba^{-1}$.

We notice that, by grouping the exponents, any $w \in \mathbb{F}_k$ can be written uniquely as $a_{j_1}^{m_1} \dots a_{j_n}^{m_n}$ where $\forall 1 \leq i \leq n$, $j_i \in \{1, \dots, k\}$, $m_i \in \mathbb{Z}$, and $\forall 1 \leq i < n$, $j_i \neq j_{i+1}$.

We will need to know what the conjugacy classes are in free groups.

Definition: Two words w, w' are said to be **cyclically equivalent** if $w = s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}$ and $w' = s_k^{\varepsilon_k} \dots s_n^{\varepsilon_n} s_1^{\varepsilon_1} \dots s_{k-1}^{\varepsilon_{k-1}}$ for a $1 \leq k \leq n$.

This is an equivalence relation, and we will denote as $[w]$ the equivalence classes for this relation.

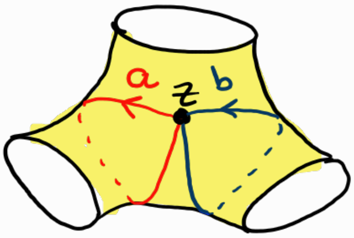
Lemma: $w, w' \in \mathbb{F}_k$ are conjugate $\Leftrightarrow$ w, w' are cyclically equivalent.

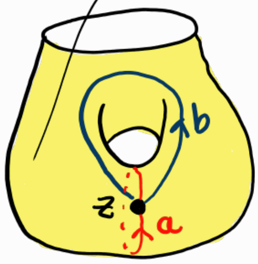
Proof: Exercise.

II) Free groups as fundamental groups.

The motivation for introducing free groups is the following.

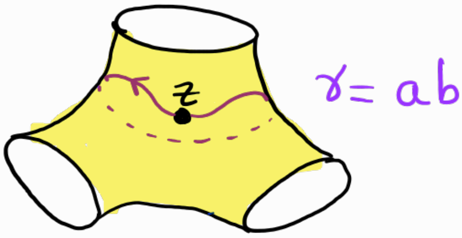
Property: Let X be a topological pair of pants or once-holed torus. For $z \in X$, the fundamental group $\pi_1(X,z)$ is the free group $\mathbb{F}_2 = \langle a,b \rangle$, where a, b are the following closed paths:





This means that any closed path γ with endpoints at z is homotopic, with fixed endpoints, to a reduced word $\omega(a,b) \in \mathbb{F}_2$.

Example:



Let us now equip X with a hyperbolic structure.

As we saw in Lectures 2 & 3, by the uniformisation theorem (adapted to surfaces with a boundary), this means viewing X as a quotient $X = \tilde{X}/\Gamma$ where $\tilde{X}$ is a domain of $\mathbb{H}$ with geodesic boundary, and $\Gamma \leq PSL(2;\mathbb{R})$ is a Fuchsian group with no elliptic elements.

Let us fix $z \in X$ and a lift $\tilde{z}$ of z in $\tilde{X}$.

We saw in Lecture 2 that any $\gamma \in \pi_1(X, z)$ corresponds to a unique lift $\tilde{\gamma}$ from $\tilde{z}$ to $\tilde{\gamma}(1) = T_\gamma \tilde{z} \in \tilde{X}$. This yields the following isomorphism:

$$\rho : \begin{cases} \pi_1(X, z) \longrightarrow \Gamma \\ \gamma \longmapsto T_\gamma . \end{cases}$$

Notation: Let $A, B \in SL(2; \mathbb{R})$ be representatives of $\rho(a), \rho(b) \in PSL(2; \mathbb{R})$.

Lemma: Γ is the free group generated by A, B.

Any closed geodesic $[\gamma]$ on X corresponds to a conjugacy class $[w]$ of Γ, and its length satisfies $\cosh\left(\frac{\ell_X(\gamma)}{2}\right) = \frac{1}{2} \left| \operatorname{tr} \rho(\gamma) \right|$.

Proof: This is simply a reminder of Lecture 2.

In other words,

$$\begin{array}{ccc} \text{closed geodesics} & \leftrightarrow & \text{conjugacy classes of } \Gamma = \langle A, B \rangle \\ \text{lengths} & \leftrightarrow & \text{traces } \operatorname{tr}(w(A,B)). \end{array}$$

Using this viewpoint reduces the question to trace identities in $SL(2; \mathbb{R})$. We will show that there exists words $(w_i)_{1 \leq i \leq N}$ such that $w_i(A,B)$ are non-conjugate and all have the same trace for any $A, B \in SL(2; \mathbb{R})$.

III) Trace identities in $SL(2;\mathbb{R})$

Lemma: For $A, B \in SL(2;\mathbb{R})$,

(i) $\operatorname{tr} A^{-1} = \operatorname{tr} A$

(ii) $\operatorname{tr} A = \operatorname{tr}(BAB^{-1})$

(iii) $\operatorname{tr}(AB) = \operatorname{tr} A \operatorname{tr} B - \operatorname{tr}(AB^{-1})$.

Proof: (i) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$

then $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ so $\operatorname{tr} A^{-1} = d + a = \operatorname{tr} A$.

(ii) Classical property of trace.

(iii) Write $B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $a'd' - b'c' = 1$. Then,

$$\operatorname{tr}(AB) = \operatorname{tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) = aa' + bc' + cb' + dd'$$

$$\operatorname{tr}(AB^{-1}) = ad' - bc' - cb' + da'$$

so $\operatorname{tr}(AB) + \operatorname{tr}(AB^{-1}) = aa' + ad' + da' + dd' = \operatorname{tr} A \operatorname{tr} B$.

Proposition: For any word $w \in \mathbb{F}_2$, there exists a unique polynomial P_w such that, for all $A, B \in SL(2;\mathbb{R})$,

$$\operatorname{tr}(w(A,B)) = P_w(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB).$$

Remark: In a pair of pants, A, B and AB are freely homotopic to the three boundary curves of the pair of pants.

It is a consequence of the uniqueness of the pair of pants with 3 fixed boundary lengths that the length of any closed geodesic γ is a function of $\mathrm{tr}\, A$, $\mathrm{tr}\, B$ and $\mathrm{tr}(AB)$. Here we claim $\cosh\left(\frac{\ell_\gamma}{2}\right)$ is a polynomial function of these three lengths.

<u>Proof</u>: Any word in $\mathbb{F}_2$ is conjugate to a word

→ $\omega = A^m$ or B^m where $m \in \mathbb{Z}$

→ or $\omega = A^{m_1} B^{n_1} \cdots A^{m_k} B^{n_k}$ where $m_i, n_i \in \mathbb{Z} \setminus \{0\}$.

- Let $\omega = A^m$ for a $m \in \mathbb{Z}$. If $m=0$ or 1 the conclusion is immediate.

  Since $\mathrm{tr}(A^m) = \mathrm{tr}(A^{-m})$, WLOG we can assume $m \geq 2$.

  Then $\mathrm{tr}(A^m) = \mathrm{tr}(A^{m-1}.A) = \mathrm{tr}(A^{m-1})\,\mathrm{tr}(A) - \mathrm{tr}(A^{m-1}.A^{-1})$

  $$= \mathrm{tr}(A^{m-1})\,\mathrm{tr}(A) - \mathrm{tr}(A^{m-2}).$$

  This proves that $\mathrm{tr}(A^m) = P_m(\mathrm{tr}\, A)$ where P_m is the polynomial defined by:
  $$\begin{cases} P_0 = 2 \\ P_1 = X \\ \forall m \geq 2, \quad P_m = X P_{m-1} - P_{m-2}. \end{cases}$$

  Remark: $P_m(X) = 2\,T_m(X/2)$ where T_m is the Chebyshev polynomial of the first kind.

- Obviously, $\mathrm{tr}(B^m) = P_m(\mathrm{tr}\, B)$ for the same polynomial P_m.

- For $w = A^{m_1}B^{n_1} \dots A^{m_k}B^{n_k}$, we prove the property by induction on $k \geq 1$.

→ If $k = 1$, i.e. $w = A^m B^n$, notice that

$$tr(A^m B^n) = tr(B^{-n} A^{-m}) = tr(A^{-m} B^{-n}) \text{ so WLOG } n \geq 1.$$

Then $tr(A^m B^n) = tr(A^m B^{n-1})\, tr(B) - tr(A^m B^{n-2})$

so an induction on n reduces the computation to $n \in \{1, 0\}$.

The case $n=0$ has already been covered. If $n=1$,

$$tr(A^m B) = tr(A \cdot A^{m-1} B) = tr(A)\, tr(A^{m-1}B) - tr(A^{m-2}B)$$

and

$$= tr(A^{m+1} B A^{-1}) = tr(A^{m+1}B)\, tr(A^{-1}) - tr(A^{m+1}BA)$$

$$= tr(A)\, tr(A^{m+1}B) - tr(A^{m+2}B)$$

so whether $m \geq 2$ or $m \leq -2$ we can reduce to $m \in \{0, 1, -1\}$.

The case $m=0$ is $tr(B)$, $m=1$ is $tr(AB)$ and for $m=-1$,

$$tr(A^{-1}B) = tr(BA^{-1}) = tr B\, tr A^{-1} - tr(BA) = tr A\, tr B - tr(AB).$$

→ If $k \geq 2$ we observe that

$$tr(w) = tr\big((A^{m_1}B^{n_1} \dots A^{m_{k-1}}B^{n_{k-1}})(A^{m_k}B^{n_k})\big)$$

$$= tr(A^{m_1}B^{n_2} \dots A^{m_{k-1}}B^{n_{k-1}}) \cdot tr(A^{m_k}B^{n_k})$$

$$- tr\big(A^{m_2}B^{n_2} \dots A^{m_{k-1}}B^{n_{k-1}}B^{-n_k}A^{-m_k}\big).$$

We can rewrite this last trace as

$$tr(A^{m_2 - m_k} B^{n_2} \dots B^{n_{k-1} - n_k})$$ to which the induction applies.

Now, for the uniqueness, let $w \in \mathbb{F}_2$ and P, Q be two possible polynomials. Then, for all $A, B \in SL(2;\mathbb{R})$,

$$P(\operatorname{tr}A, \operatorname{tr}B, \operatorname{tr}(AB)) = Q(\operatorname{tr}A, \operatorname{tr}B, \operatorname{tr}(AB)).$$

The image of $SL(2;\mathbb{R})^2$ by $(A,B) \mapsto (\operatorname{tr}A, \operatorname{tr}B, \operatorname{tr}(AB))$ contains an open set of $\mathbb{R}^3$ on which P and Q coincide, so they must be equal.

Corollary: For $w \in \mathbb{F}_2$, let $\Theta(w) = w(a^{-1}, b^{-1})$. Then $P_{\Theta(w)} = P_w$.

Proof: Let $A, B \in SL(2;\mathbb{R})$.

$$\operatorname{tr}(\Theta(w)(A,B)) = \operatorname{tr}(w(A^{-1},B^{-1})) = P_w(\operatorname{tr}A^{-1}, \operatorname{tr}B^{-1}, \operatorname{tr}(A^{-1}B^{-1}))$$

$$= P_w(\operatorname{tr}A, \operatorname{tr}B, \operatorname{tr}(AB)) \quad \text{so} \quad P_{\Theta(w)} = P_w.$$

IV) Examples of high multiplicities

Notation: For $k \geq 1$ let $\mathcal{S}_k = \{\pm\}^k = \{(\varepsilon_1, \dots, \varepsilon_k),\ \varepsilon_i = \pm\}$.

Let $W_+ = ba^{-1}b^{-1}ab$, $W_- = \Theta(W_+)$ and

for $k \geq 0$, $\varepsilon \in \mathcal{S}_k$, $\begin{cases} W_{\varepsilon+}(a,b) = W_\varepsilon(a, W_+(a,b)) \\ W_{\varepsilon-}(a,b) = \Theta(W_{\varepsilon+}(a,b)) \end{cases}$

Example.

For $k=1$, $W_+(a,b) = ba^{-1}b^{-1}ab$ and $W_-(a,b) = b^{-1}aba^{-1}b^{-1}$.

For $k=2$, $W_{++}(a,b) = W_+(a, ba^{-1}b^{-1}ab)$

$$= \underbrace{ba^{-1}b^{-1}ab}_{W_+(a,b)}\, a^{-1}\, \underbrace{b^{-1}a^{-1}bab^{-1}}_{W_+(a,b)^{-1}}\, a\, \underbrace{ba^{-1}b^{-1}ab}_{W_+(a,b)}.$$

Proposition : Let $k \geq 1$. For any $\varepsilon, \sigma \in \mathcal{S}_k$, $P_{W_\varepsilon} = P_{W_\sigma}$ and $W_\varepsilon, W_\varepsilon^{-1}, W_\sigma$ and W_σ^{-1} are non-conjugate in F_2.

In particular, for any hyperbolic pair of pants or once-holed torus, if $\Gamma = \langle A, B \rangle$ as before, then $\{ W_\varepsilon(A,B), \varepsilon \in \mathcal{S}_k \}$ is a family of 2^k distinct closed geodesics of the same lengths.

Proof :

- Let us first prove that $P_{W_\varepsilon} = P_{W_\sigma}$ by induction on k.

  → For $k=1$, by definition $P_{W_-} = P_{\Theta(W_+)} = P_{W_+}$ by the previous lemma.

  → If $k \geq 1$, let $\varepsilon_0 \in \mathcal{S}_{k-1}$. For any $\varepsilon \in \mathcal{S}_{k-1}$, $P_{W_{\varepsilon -}} = P_{\Theta(W_{\varepsilon +})} = P_{W_{\varepsilon +}}$ and

$$\begin{aligned} tr(W_{\varepsilon +}(A,B)) &= tr(W_\varepsilon(A, W_+(A,B))) \\ &= P_\varepsilon(tr A, tr W_+(A,B), tr\, A W_+(A,B)) \quad \text{induction hyp} \\ &= P_{\varepsilon_0}(tr A, tr W_+(A,B), tr(AW_+(A,B))) \\ &= tr(W_{\varepsilon_0 +}(A,B)) \end{aligned}$$

  so $P_{W_{\varepsilon +}} = P_{W_{\varepsilon_0 +}}$.

- Let us now prove the words are non-conjugate, by induction.

  For $k=1$, $[W_+] = [b a^{-1} b^{-1} a b] = [a^{-1} b^{-1} a b^2]$, $[W_+^{-1}] = [a^{-1} b a b^{-2}]$

  whilst $[W_-] = [a b a^{-1} b^{-2}]$ and $[W_-^{-1}] = [a b^{-1} a^{-1} b^2]$

  which we check are all different.

The induction step rests on the following lemma.

Lemma: Let Ω, $\bar{\Omega}$ be two reduced cyclic words in $\mathbb{F}_2$.

We assume:
- $\Omega \neq \bar{\Omega}$ and $\Omega \neq \bar{\Omega}^{-1}$
- Ω and $\bar{\Omega}$ each contain a factor $b^{\pm 2}$.

Then, $\Omega^{\pm 1}(a, W_+(a,b))$, $\Theta(\Omega^{\pm}(a, W_+(a,b)))$, $\bar{\Omega}^{\pm 1}(a, W_+(a,b))$ and $\Theta(\bar{\Omega}^{\pm}(a, W_+(a,b)))$ are 8 distinct cyclic reduced words, and each contain a factor $b^{\pm 2}$.

Proof: Take $\mathcal{W}$ one of the eight words listed, so that $\mathcal{W} = \Omega_0(a^{\varepsilon}, W_+(a^{\varepsilon}, b^{\varepsilon}))$ for $\varepsilon, \delta \in \{\pm 1\}$, $\Omega_0 \in \{\Omega, \bar{\Omega}, \Omega^{-1}, \bar{\Omega}^{-1}\}$.

Since Ω_0 contain a factor $b^{2\delta}$ for $\delta \in \{\pm 1\}$, Θ must contain $W_+(a^{\varepsilon}, b^{\varepsilon})^{2\delta}$.

Note that $W_+(a^{\varepsilon}, b^{\varepsilon})^{2\delta} = b^{\delta\varepsilon} a^{-\varepsilon} b^{-\delta\varepsilon} a^{\varepsilon} b^{2\delta\varepsilon} a^{-\varepsilon} b^{-\delta\varepsilon} a^{\varepsilon} b^{\delta\varepsilon}$.

Hence $\mathcal{W}$ contains a factor $b^{\pm 2}$.

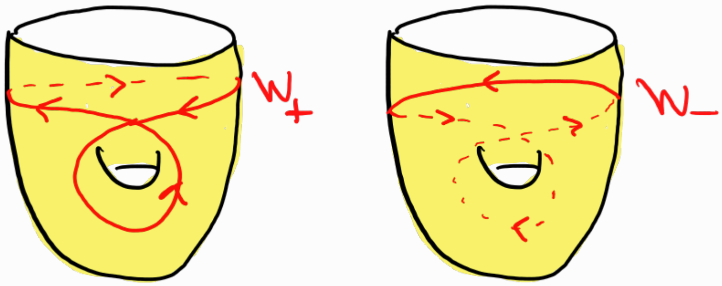
Since $W_+ = ba^{-1}b^{-1}ab$ contains no power $\neq \pm 1$, $\mathcal{W}$ only contains factors $a^{\pm 1}$, $b^{\pm 1}$, $b^{\pm 2}$. Any factor $b^{\pm 2}$ is obtained as above. The following letter determines ε, and the one after δ. Then one can directly read off the word Ω_0 from $\mathcal{W}$.

V) The geometry of rigid pairs

We have proven the existence of arbitrarily large families of closed curves $(\gamma_i)_{i \in \mathcal{I}}$ s.t. $\ell_X(\gamma_i) = \ell_X(\gamma_j)$ for any hyperbolic metric X.

Let us provide more information on their geometry.

<u>Example:</u> For X a once-holed torus, the examples $k=1$ correspond to:



which can be obtained by proving the once-holed torus has an isometry sending w_+ on w_-.

It is a open problem to give a topological description of pairs γ, γ' s.t. $\ell_X(\gamma) = \ell_X(\gamma')$ for each hyperbolic metric X. We prove one fact.

Proposition: Let γ, γ' be two closed curves. We assume γ is simple. Then there exists a hyperbolic metric X for which $\ell_X(\gamma) \neq \ell_X(\gamma')$.

In order to prove this result we shall admit the following consequence of the collar theorem.

Theorem: (Buser 92, theorem 4.1.6)
Any non-simple closed geodesic on a hyperbolic surface has length at least 1.

<u>Proof</u> : (of the Proposition)

We split the proof into 3 cases.

→ γ' not simple

Complete γ into a pair of pants decomposition $(\gamma_1, \dots, \gamma_N)$.

Define $X(x)$ to be the hyperbolic surface with Fenchel-Nielsen coordinates

$\ell_1 = \ell_{X(x)}(\gamma) = x$, $\ell_i = 1$ for $i \neq 1$, $\tau_i = 0$ for all i.

Notice that γ' is not simple so the admitted theorem implies

$$\forall x > 0, \quad \ell_{X(x)}(\gamma') \geq 1.$$

Then for $x \in (0,1)$, $\ell_{X(x)}(\gamma) = x < 1 = \ell_{X(x)}(\gamma')$

so in particular $\ell_{X(x)}(\gamma) \neq \ell_{X(x)}(\gamma')$.

→ γ' simple and γ, γ' intersect

Let $X(x)$ as above.

By the corollary of the collar lemma from Lecture 4,

$\ell_{X(x)}(\gamma') \geq 2 \operatorname{arcsinh}\left(\frac{1}{\sinh\left(\frac{x}{2}\right)}\right)$ since γ' must enter and leave the collar around γ.

Since $\lim_{x \to 0} \operatorname{arcsinh}\left(\frac{1}{\sinh\left(\frac{x}{2}\right)}\right) = +\infty$, for small enough x,

$$\ell_{X(x)}(\gamma) = x < 2 \operatorname{arcsinh}\left(\frac{1}{\sinh\left(\frac{x}{2}\right)}\right) \leq \ell_{X(x)}(\gamma')$$

and in particular $\ell_{X(x)}(\gamma) \neq \ell_{X(x)}(\gamma')$.

→ γ' is simple and disjoint from γ

We complete (γ, γ') into a pair of pants decomposition and pick a metric X by Fenchel-Nielsen coordinates s.t. $\ell_X(\gamma) = 1$ and $\ell_X(\gamma') = 2$. $\square$