TCC: Hyperbolic surfaces, their length spectra, and connection to Markoff triples

Lecture 4: Isospectral hyperbolic surfaces

Recall:

For $[\alpha]$ a homotopy class of essential closed curves on X, denote as $\ell_X([\alpha])$ the length of the unique closed geodesic in $[\alpha]$. The map $[\alpha] \longmapsto \ell_X([\alpha])$ is the marked length spectrum: it is marked because we record which curve has which length. In lecture 3 we proved the following.

> **Theorem:** There exists a set of $9g-9$ homotopy classes of simple closed curves $\{[\alpha_i]\}_{1\leq i\leq 9g-9}$ such that, for any marked hyperbolic surfaces X_1, X_2 of genus g, if
> $\forall i \in \{1, \dots, 9g-9\}$, $\quad \ell_{X_1}([\alpha_i]) = \ell_{X_2}([\alpha_i])$,
> then X_1 and X_2 are isometric.

In other words the marked length spectrum is rigid: it entirely determines the hyperbolic metric. Even so, we only need a fixed number of simple closed curves to determine it.

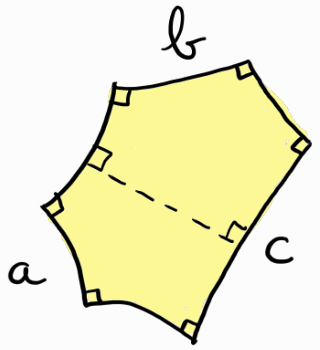
The objective of today's lecture is to prove that, to the contrary, the (unmarked) length spectrum is not rigid.

Theorem: There exists two hyperbolic surfaces X_1, X_2 such that:

- X_1 and X_2 are not isometric;
- X_1 and X_2 have the same length spectrum, i.e. there exists a bijective map $\Psi : \{\text{closed geodesics on } X_1\} \to \{\text{closed geodesics on } X_2\}$ such that $\forall \gamma,\ \ell_{X_1}(\gamma) = \ell_{X_2}(\Psi(\gamma))$.

I. Right-angled pentagons

We saw last lecture that for any $a, b, c > 0$ there exists a unique right-angled hexagon with non-consecutive sides of lengths a, b, c:



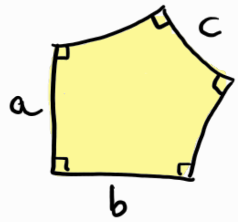
Cutting such a hexagon along the common perpendicular of two opposite sides splits it into two right-angled pentagons.

We shall use the following trigonometric formula.

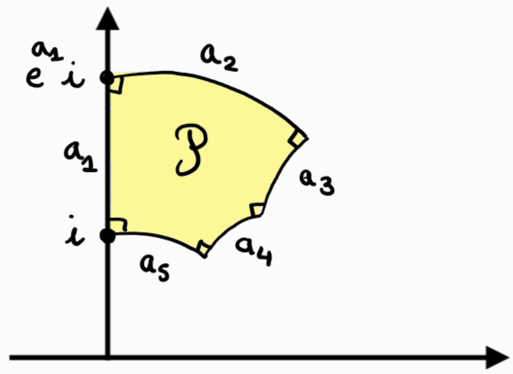
Proposition:

The side-lengths of a right-angled pentagon must satisfy $\cosh(c) = \sinh(a)\sinh(b)$.

In particular, $\sinh(a)\sinh(b) > 1$.



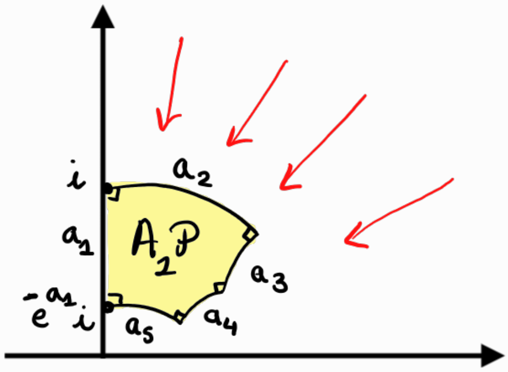
Proof: Let us label the side lengths $a_1, \dots, a_5$. We apply a isometry of $\mathbb{H}$ so that the pentagon sits as follows:

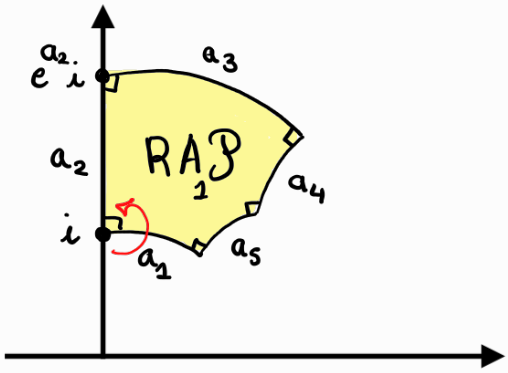


We now apply the two consecutive isometries:

$$A_1 = \begin{pmatrix} e^{-a_1/2} & 0 \\ 0 & e^{a_1/2} \end{pmatrix} \qquad \text{and} \qquad R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$z \mapsto e^{-a_1} z$ rotation of angle $\frac{\pi}{2}$ around i





Notice how this has simply rotated the pentagon. For this reason, if for each $i \in \{1, \dots, 5\}$ we write $A_i = \begin{pmatrix} e^{-a_i/2} & 0 \\ 0 & e^{a_i/2} \end{pmatrix}$ then

$$\circledast \quad RA_5RA_4RA_3RA_2RA_1 = I_2.$$

We will deduce the formula from this matrix relation.

Write $$B_i = RA_iR = \begin{pmatrix} -\sinh \frac{a_i}{2} & \cosh \frac{a_i}{2} \\ -\cosh \frac{a_i}{2} & \sinh \frac{a_i}{2} \end{pmatrix}.$$

Then $\circledast$ can be rewritten as $M := B_5 A_4 B_3 = A_1^{-1} R^{-1} A_2^{-1}$.

We observe that $A_i^{-1} = \begin{pmatrix} e^{a_i/2} & 0 \\ 0 & e^{-a_i/2} \end{pmatrix}$ and $R^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

Identifying the matrix coefficients yields the four equations:

$$\begin{cases} m_{11} = \dfrac{e^{\frac{a_1+a_2}{2}}}{\sqrt{2}} = -e^{\frac{a_4}{2}} \cosh\frac{a_3}{2} \cosh\frac{a_5}{2} + e^{-\frac{a_4}{2}} \sinh\frac{a_3}{2} \sinh\frac{a_5}{2} \\ m_{21} = \dfrac{e^{\frac{a_2-a_1}{2}}}{\sqrt{2}} = -e^{\frac{a_4}{2}} \cosh\frac{a_3}{2} \sinh\frac{a_5}{2} + e^{-\frac{a_4}{2}} \sinh\frac{a_3}{2} \cosh\frac{a_5}{2} \\ m_{12} = -\dfrac{e^{\frac{a_1-a_2}{2}}}{\sqrt{2}} = e^{\frac{a_4}{2}} \sinh\frac{a_3}{2} \cosh\frac{a_5}{2} - e^{-\frac{a_4}{2}} \cosh\frac{a_3}{2} \sinh\frac{a_5}{2} \\ m_{22} = \dfrac{e^{\frac{-a_1-a_2}{2}}}{\sqrt{2}} = e^{\frac{a_4}{2}} \sinh\frac{a_3}{2} \sinh\frac{a_5}{2} - e^{-\frac{a_4}{2}} \cosh\frac{a_3}{2} \cosh\frac{a_5}{2} . \end{cases}$$

Then, using $\sinh(2x) = 2\cosh(x)\sinh(x)$ and $\cosh(2x) = \cosh^2(x) + \sinh^2(x)$,

$$\begin{aligned} e^{a_2} &= 2\, m_{11} m_{21} \\ &= e^{a_4} \cosh^2\frac{a_3}{2} \sinh a_5 + e^{-a_4} \sinh^2\frac{a_3}{2} \sinh a_5 - \sinh a_3 \cosh a_5 \end{aligned}$$

and

$$\begin{aligned} -e^{-a_2} &= 2\, m_{12} m_{22} \\ &= e^{a_4} \sinh^2\frac{a_3}{2} \sinh a_5 + e^{-a_4} \cosh^2\frac{a_3}{2} \sinh a_5 - \sinh a_3 \cosh a_5 . \end{aligned}$$

We conclude writing

$$\begin{aligned} \cosh(a_2) &= \frac{e^{a_2} + e^{-a_2}}{2} \\ &= \frac{1}{2}\Big[e^{a_4} \underbrace{\Big(\cosh^2\frac{a_3}{2} - \sinh^2\frac{a_3}{2}\Big)}_{1} + e^{-a_4} \underbrace{\Big(\sinh^2\frac{a_3}{2} - \cosh^2\frac{a_3}{2}\Big)}_{-1} \Big] \sinh a_5 \end{aligned}$$

which is the claim: $\cosh(a_2) = \sinh a_4 \sinh a_5$. $\square$

II) The collar theorem

The collar theorem is a fundamental theorem of hyperbolic geometry which allows to describe collars around simple closed geodesics. See Chapter 4 in Buser 92.

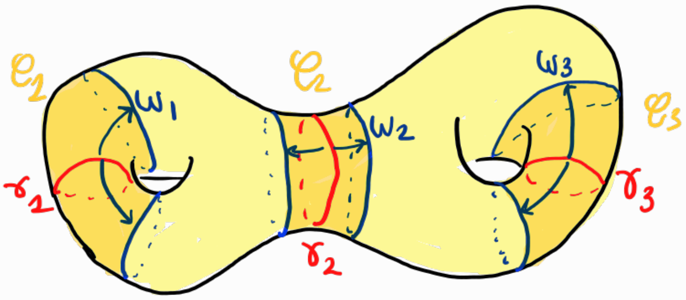
Theorem: Let X be a compact hyperbolic surface of genus g.
Let $\gamma_1, \dots, \gamma_m$ be pairwise disjoint simple closed geodesics on X.
Then,

(i) $m \leq 3g-3$ and we can complete $(\gamma_i)_{1 \leq i \leq m}$ into a pair-of-pants decomposition of X;

(ii) if $w(\ell) := \operatorname{arcsinh}\left(\frac{1}{\sinh(\ell)}\right)$, then the collars

$$\mathcal{C}_i = \left\{ z \in X : d(z, \gamma_i) \leq w\left(\frac{\ell_X(\gamma_i)}{2}\right) \right\}$$

are disjoint and isometric to cylinders.



Proof: (i) $(\gamma_i)_{1 \leq i \leq m}$ are simple closed geodesics with no intersections. As a consequence, cutting X along them yields a family of surfaces $(X_j)_{1 \leq j \leq n}$ with boundary components identified with $(\gamma_i)_{1 \leq i \leq m}$.
Since $(\gamma_i)_{1 \leq i \leq m}$ are distinct geodesics, they are not contractible nor can they be homotopic to one another. Hence for all $1 \leq j \leq n$,

X_i is not a disk or a cylinder, which implies $\chi(X_i) < 0$.

We can hence cut each X_i into pairs of pants. We obtain a completed family $(\gamma_i)_{1\leq i\leq m'}$ cutting X into pairs of pants $(P_j)_{1\leq j\leq n'}$.

By additivity of the Euler characteristic,

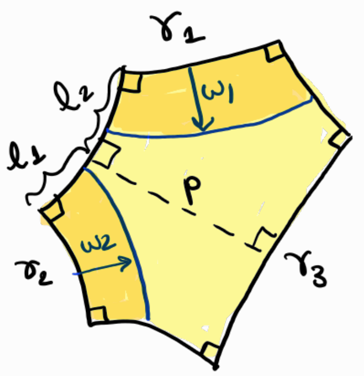
$$\chi(X) = 2-2g = \sum_{j=1}^{n'} \underbrace{\chi(P_i)}_{=-1} = -n' \quad \text{so} \quad n' = 2g-2.$$

Each pair of pants has 3 boundary components, i.e. $6g-6$ boundary components in total. The geodesics $(\gamma_i)_{1\leq i\leq m'}$ are obtained by pairing up these components so $m' = \frac{6g-6}{2} = 3g-3$.

(ii) Let us cut X along the $(\gamma_i)_{1\leq i\leq 3g-3}$ and each pair of pants into right-angled hexagons. This reduces the proof to the following lemma.

Lemma: In a right-angled hexagon of non-consecutive sides $\gamma_1, \gamma_2, \gamma_3$, the half-collars $\mathcal{C}_i = \{z : d(z,\gamma_i) < \omega(\ell(\gamma_i))\}$ are disjoint.

Proof: We draw the common perpendicular p of γ_3 and its opposite side γ_3'. In this process, γ_3' is split into two pieces of respective lengths l_1, l_2. By trigonometry of right-angled pentagons, for $i = 1,2$,



$$\sinh(l_i)\sinh(\ell(\gamma_i)) > 1.$$

For any point z on p, $d(z,\gamma_i) \geq l_i > \operatorname{arcsinh}\left(\frac{1}{\sinh(\ell(\gamma_i))}\right) = \omega(\ell(\gamma_i))$

so p separates both half-collars $\mathcal{C}_1$ and $\mathcal{C}_2$. $\square$

In this lecture we will mostly use the following consequence.

Corollary:

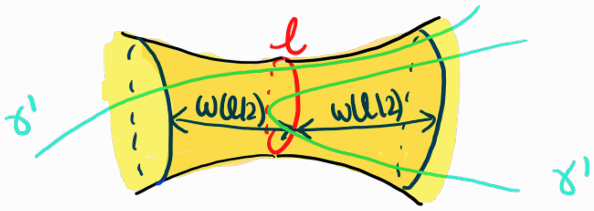
All simple closed geodesics of lengths $< 2\,\mathrm{arcsinh}(1)$ are disjoint.

Remark: $2\,\mathrm{arcsinh}(1) \approx 1.762$.

Proof: Let γ be a simple closed geodesic of length $\ell < 2\,\mathrm{arcsinh}(1)$.

By the collar theorem, the neighbourhood of width $w(\ell/2)$ around γ is a cylinder. Any other closed curve γ' intersecting γ must enter the cylinder, reach γ, and exit, and hence has length

$\geqslant 2w(\ell/2) > 2\,\mathrm{arcsinh}(1)$.



$\square$

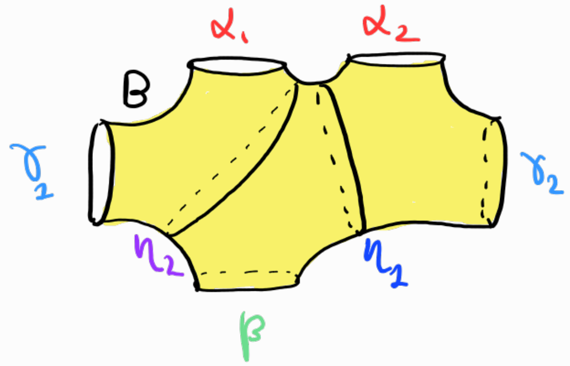
III. Construction of isospectral, non-isometric, hyperbolic surfaces

Construction:

We shall build two surfaces from copies of one elementary block B constructed by gluing three hyperbolic pairs of pants chosen so that

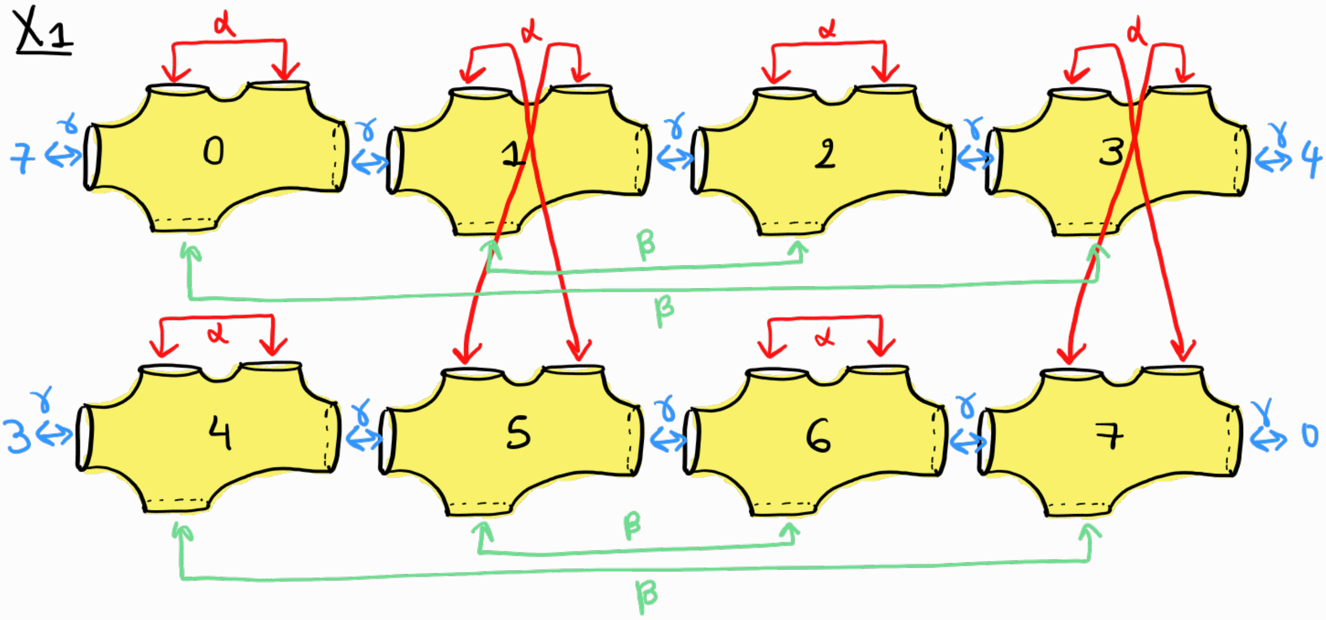
$$\ell(\alpha_1) = \ell(\alpha_2) < \ell(\beta) < \ell(\gamma_1) = \ell(\gamma_2) < \ell(\eta_1) < \ell(\eta_2) \leqslant 1.$$

with no twists, according to this scheme:

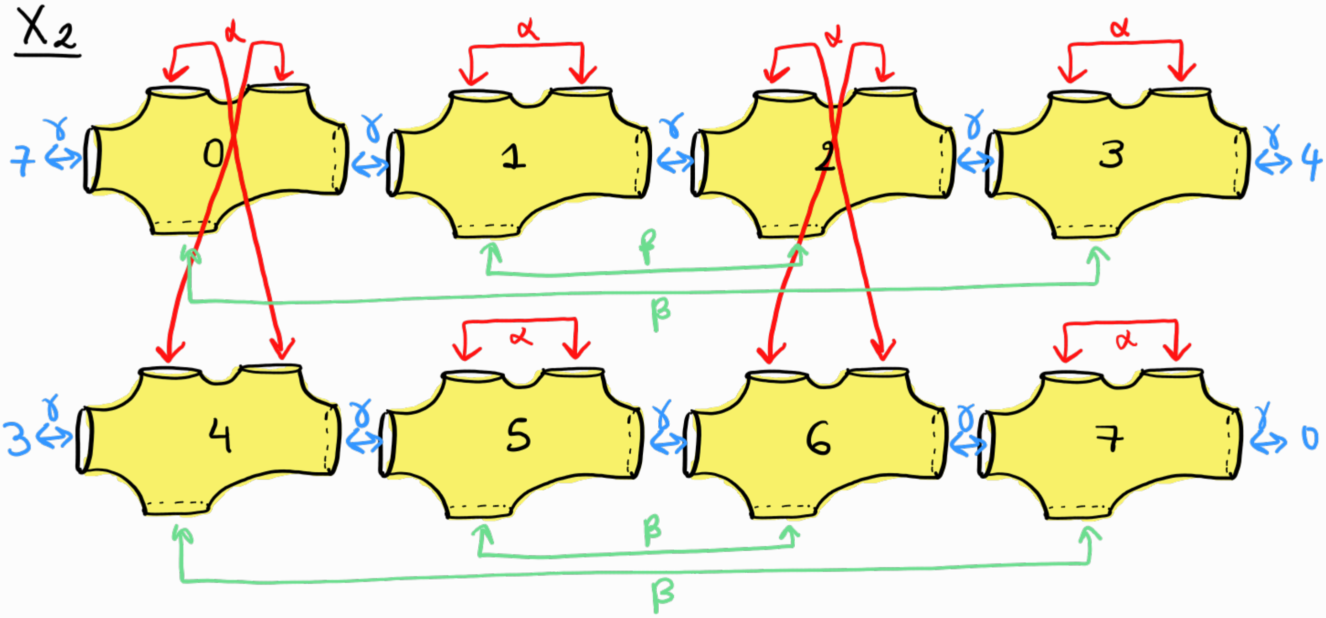


We then define X_1 and X_2 by gluing 8 copies of B, numbered $(B_i)_{0 \le i \le 7}$, according to two different schemes:

X_1



X_2



Note that each pair (α_1, α_2), resp. (γ_1, γ_2), is glued onto a new simple closed curve we denote α, resp. γ. We prove the following.

Theorem:

(i) X_1 and X_2 are not isometric.

(ii) There exists a 1-to-1 map $\{\text{closed geods of } X_1\} \to \{\text{closed geod of } X_2\}$, $\delta \longmapsto \delta^*$

such that $\forall \delta$, $\ell_{X_1}(\delta) = \ell_{X_2}(\delta^*)$.

Proof of (i): Let us assume by contradiction that there exists an isometry $\varphi : X_1 \to X_2$.

- Since φ is an isometry, for any simple closed geodesic δ on X_1, $\varphi(\delta)$ is a simple closed geodesic of the same length on X_2.
- Let δ be a simple closed geodesic of length ≤ 1 on X_1 or X_2. By the consequence of the collar theorem, δ must either be a copy of $\alpha, \beta, \gamma, \eta_1, \eta_2$, or disjoint from all of them, which is impossible since these curves cut X_1, X_2 into pairs of pants, which contain no essential simple closed curves.
- Since $\ell(\alpha) < \ell(\beta) < \ell(\gamma) < \ell(\eta_1) < \ell(\eta_2)$, φ sends copies of α on copies of α etc, which we use to deduce that φ sends each block B_i of X_1 on a block $B_{\phi(i)}$ of X_2 for a 1-to-1 map $\phi : \{0, \dots, 7\} \circlearrowleft$.
- Because of the identifications of the copies of γ, there exists $k \in \{0, \dots, 7\}$ such that $\phi(i) = i + k \pmod 8$ for all $i \in \{0, \dots, 7\}$.
- Because of the identifications of the copies of α, $k \in \{1, 3, 5, 7\}$.
- Because of the identifications of the copies of β, $k \in \{0, 4\}$.

We have reached a contradiction. $\square$

Proof of (ii):

For $i,j \in \{0,\dots,7\}$ let φ_{ij} denote the identification sending the bock B_i of X_1 on the block B_j of X_2.

Let δ be a closed geodesic on X_1.

→ If δ is included in a block B_i, then we define

$$\delta^* = \varphi_{ii}(\delta).$$

→ Otherwise we pick the origin of δ to be an inside point of a block B_{i_1}, and decompose δ as a concatenation $\delta_1 \dots \delta_N$ where the cuts are exactly the intersections of δ with α, β, γ.

→ Let
$$\begin{cases} i_\alpha = \#\text{ intersections of } \delta \text{ with } \alpha \\ i_\beta = \#\text{ intersections of } \delta \text{ with } \beta \\ i_\gamma = \#\text{ intersections of } \delta \text{ with } \gamma. \end{cases}$$

and define the shift $s_1 = \begin{cases} 0 \text{ if } & i_\alpha \text{ is even} \\ 1 \text{ if } & i_\alpha \text{ is odd and } i_\beta \text{ even} \\ 2 \text{ if } & i_\alpha \text{ and } i_\beta \text{ are odd.} \end{cases}$

We define $\delta_1^* = \varphi_{i_1, i_1^*}(\delta_1)$ where $i_1^* = i_1 + s_1$.

→ At the end of δ_1, it reaches a curve α, β, γ and does a crossing leading to a block B_{i_2}, called a $\alpha/\beta/\gamma$ jump.

On X_2, δ_1^* does a jump of the same type and reaches a block $B_{i_2^*}$ on M_2.

We define $\delta_2^* = \varphi_{i_2 i_2^*}(\delta_2)$.

→ We continue this process until we have covered all the intersections, and define $\delta^* = \delta_1^* \dots \delta_N^*$.

→ Let us check that δ is closed $\Rightarrow$ δ^* is closed.

At each step k, we have a shift $s_k = i_k^* - i_k$

δ^* is closed $\Leftrightarrow$ $s_N = s_1$.

- Case 1: i_α is even.

  We observe that if $s_k \in \{0, 4\}$, then:

  → for a α-jump, $s_{k+1} = s_k + 4 \pmod 8$;

  → for a β or γ-jump, $s_{k+1} = s_k$.

  Since $s_1 = 0$ and i_α is even, $s_N = 0$.

- Case 2: i_α is odd and i_β is even.

  We observe that if $s_k \in \{\pm 1\}$ then:

  → for a α or γ-jump, $s_{k+1} = s_k$;

  → for a β-jump, $s_{k+1} = -s_k \pmod 8$.

  Since $s_1 = 1$ and i_β is even, $s_N = s_1$.

- Case 3: i_α and i_β are odd.

  We observe that if $s_k \in \{\pm 2\}$ then:

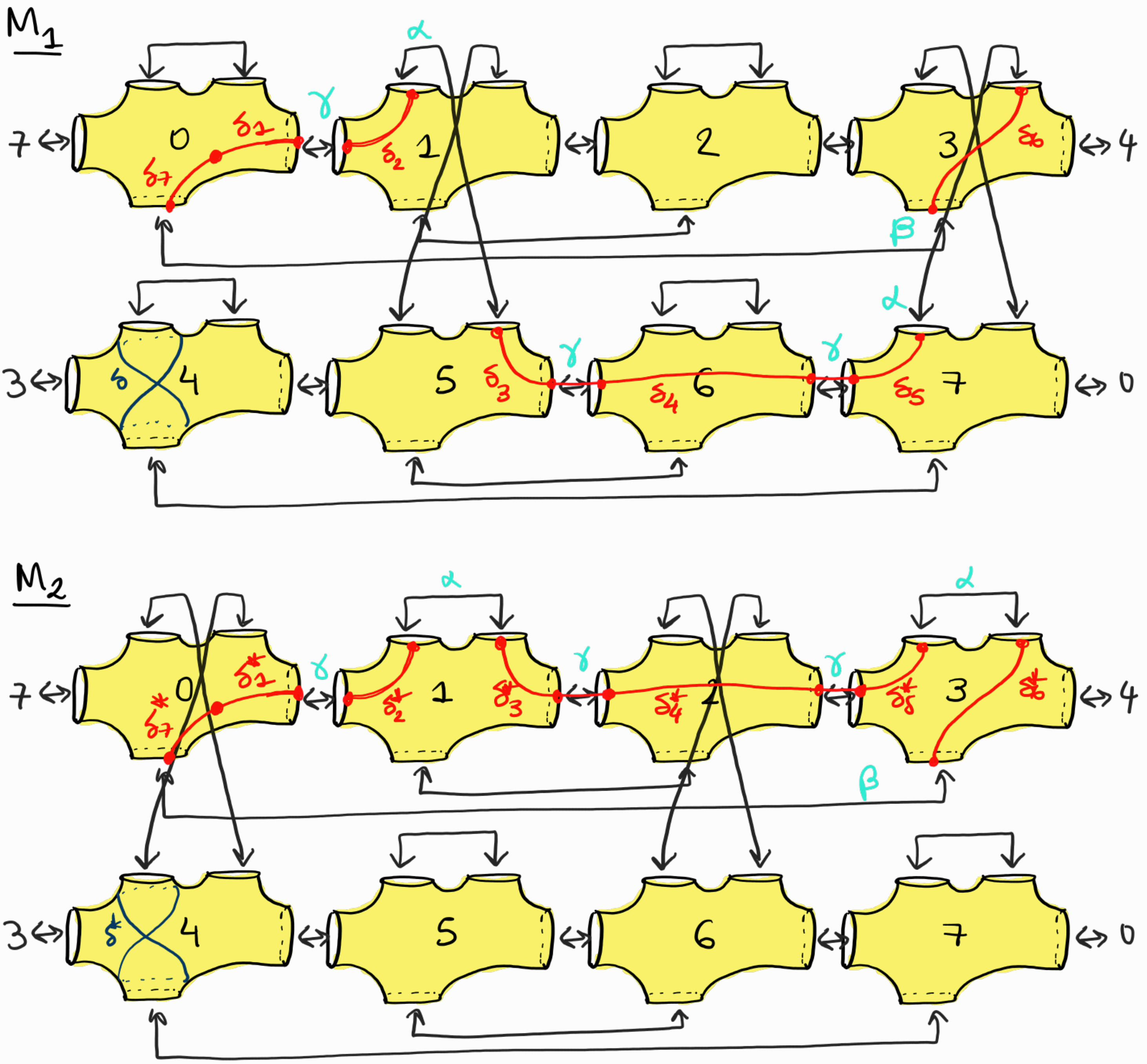
  → for a α or β-jump, $s_{k+1} = -s_k \pmod 8$;

  → for a γ-jump, $s_{k+1} = s_k$.

  Since $s_1 = 2$ and $i_\alpha + i_\beta$ is even, $s_N = s_1$.

We have hence defined a map $\delta \mapsto \delta^*$ sending the closed geodesics of X_1 on those of X_2.

We can proceed the same way to create the inverse $\delta^* \mapsto \delta$ and conclude the map is 1-to-1. $\square$



IV. Link with the spectrum of the Laplace-Beltrami operator

We conclude this lecture by mentioning the relationship of this question to an important problem of spectral geometry. We will not discuss this further in these lectures.

The Laplace-Beltrami operator $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ on $\mathbb{H}$ is invariant by isometries and defines an unbounded operator on $L^2(X)$ for $X = \mathbb{H}/\Gamma$.

If X is compact then this operators has discrete spectrum

$$0 = \lambda_0(X) < \lambda_1(X) \leq \dots \leq \lambda_n(X) \xrightarrow[n\to\infty]{} \infty.$$

These fundamental values correspond to the sounds the surface would make if played as a drum.

In 1966, Kac asked a celebrated question: "Can one hear the shape of a drum?". In other words, are isospectral surfaces (surfaces with the same $(\lambda_n)_{n\geq 0}$) isometric?

The answer is no and the surfaces we constructed today are a counter-example because, for any compact hyperbolic surfaces X_1, X_2,

X_1 and X_2 are isospectral $\Leftrightarrow$ X_1 and X_2 have the same unmarked length spectrum.