TCC: Hyperbolic surfaces, their length spectra and connections to Markov triples

See Farb-Margalit "A Primer on Mapping Class Groups" Chapter 10.

Lecture 3: Pairs of pants, Fenchel-Nielsen coordinates and length spectrum rigidity

Recall that a **hyperbolic surface** is a topological surface equipped with charts to $\mathbb{H}$ whose transition functions are **isometries**.

Equivalently, a **hyperbolic surface** is a quotient, $\mathbb{H}/\Gamma$, of $\mathbb{H}$ by a **Fuchsian group** (i.e. discrete subgroup of $PSL(2,\mathbb{R})$) acting **freely**.

The surface is equipped with a metric pulled back from $\mathbb{H}$.

Given a homotopy class $[\alpha]$ of an essential

closed curve α on X, there exists a unique geodesic $\gamma \in [\alpha]$.

I. LENGTH SPECTRA

Define the **length of $[\alpha]$ in X** to be

$$\ell_X([\alpha]) = \ell_X(\alpha) = \text{length of the unique geodesic } \gamma \in [\alpha]$$

The **length spectrum of X** is the multiset

$$\{\ell_X([\alpha]) \mid [\alpha] \text{ homotopy class of essential closed curve}\}.$$

(essential: not homotopic to a point or a cusp)

The **simple length spectrum** restricts to homotopy classes of simple (i.e. non-self-intersecting) essential closed curves.

Note: We only record the lengths of curves, we do not record which curves have those lengths.

The marked (simple) length spectrum of X is a function

$$f: \left\{ \begin{array}{l} \text{homotopy classes} \\ \text{of essential (simple)} \\ \text{closed curves} \end{array} \right\} \longrightarrow \mathbb{R}$$

$$[\alpha] \longmapsto \ell_X(\alpha).$$

The goal for today is to prove the following:

Theorem (9g-9 Theorem)

Let X_1 and X_2 be closed hyperbolic surfaces of genus $g \geq 2$. There exists a set $\{[\alpha_i]\}_{i=1}^{9g-9}$ of homotopy classes of essential simple closed curves such that, if we have $\ell_{X_1}([\alpha_i]) = \ell_{X_2}([\alpha_i])$, $\forall\ 1 \leq i \leq 9g-9$, then X_1 and X_2 are isometric.

Corollary

Closed hyperbolic surfaces are marked (simple) length spectrum rigid.

II. MARKED HYPERBOLIC SURFACES

Given a curve α, how do we know what α looks like on two different surfaces X_1 and X_2?

Markings solve this by fixing a reference topological surface.

Let S_g be a fixed topological surface of genus $g \geq 2$.

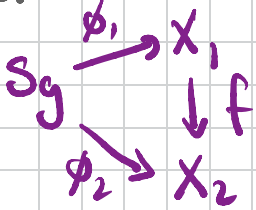
Let X be a hyperbolic surface of genus g.

A **marking** of X is a homeomorphism

$$\phi : S_g \to X.$$

We call X, or (X, ϕ), a **marked hyperbolic surface**.

Two marked hyperbolic surfaces X_1 and X_2 are **homotopic** if $\exists$ an isometry $f: X_1 \to X_2$ such that



commutes up to homotopy.

Now it makes sense to talk about lengths of curves.

Let α be an essential closed curve on S_g, then

$$\ell_X([\alpha]) := \text{length of the geodesic representative of } [\phi(\alpha)] \text{ in } X.$$

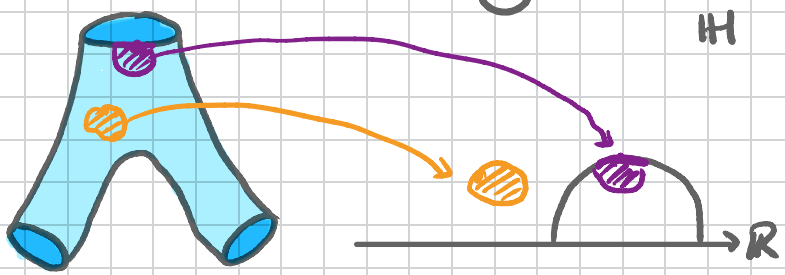
Changing α or X up to homotopy does not change this length function. So we actually have a map well-defined on

$$\{\text{marked hyperbolic surfaces}\}/_{\text{homotopy}} =: \text{Teich}(S_g).$$

III PAIRS OF PANTS

A **pair of pants** is a surface homeomorphic to a sphere with three boundary components.

Boundary is "totally geodesic"



We shall see that a hyperbolic metric on a pair of pants is determined by the lengths of the boundary curves.

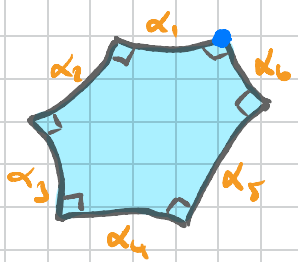
We first consider hyperbolic right-angled hexagons.

A **marked hexagon** is a hexagon with a distinguished vertex. Two marked right-angled geodesic hexagons in IH are equivalent if there is a distinguished vertex preserving

isometry of $\mathbb{H}$ taking one to the other.

Proposition

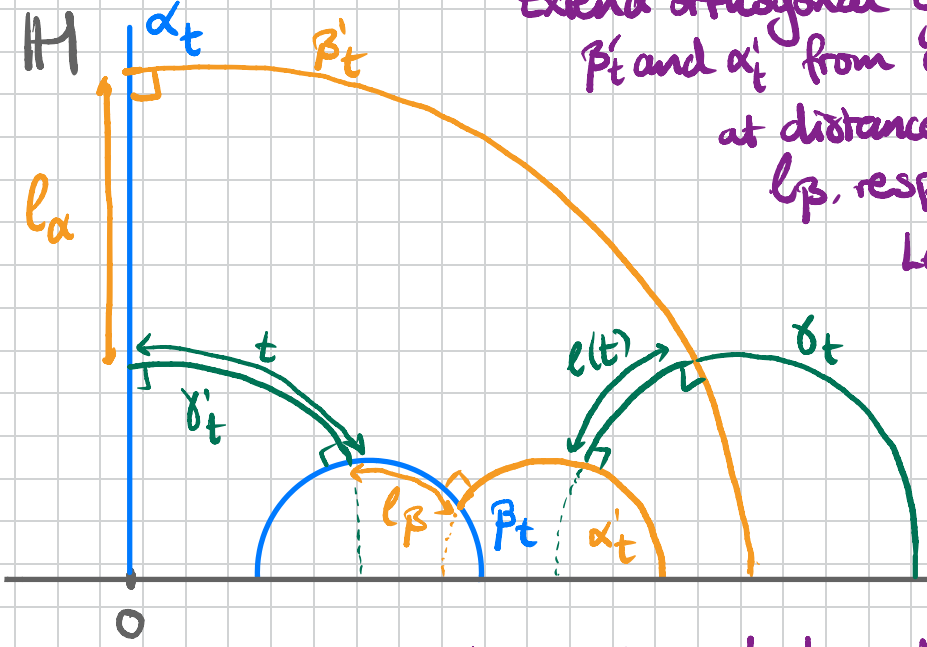
The map $\left\{\begin{matrix}\text{marked hyperbolic} \\ \text{right-angled hexagons}\end{matrix}\right\}/\sim \to \mathbb{R}_+^3$ sending a hexagon to the lengths of every other side starting at the marked vertex and going anti-clockwise is a bijection.



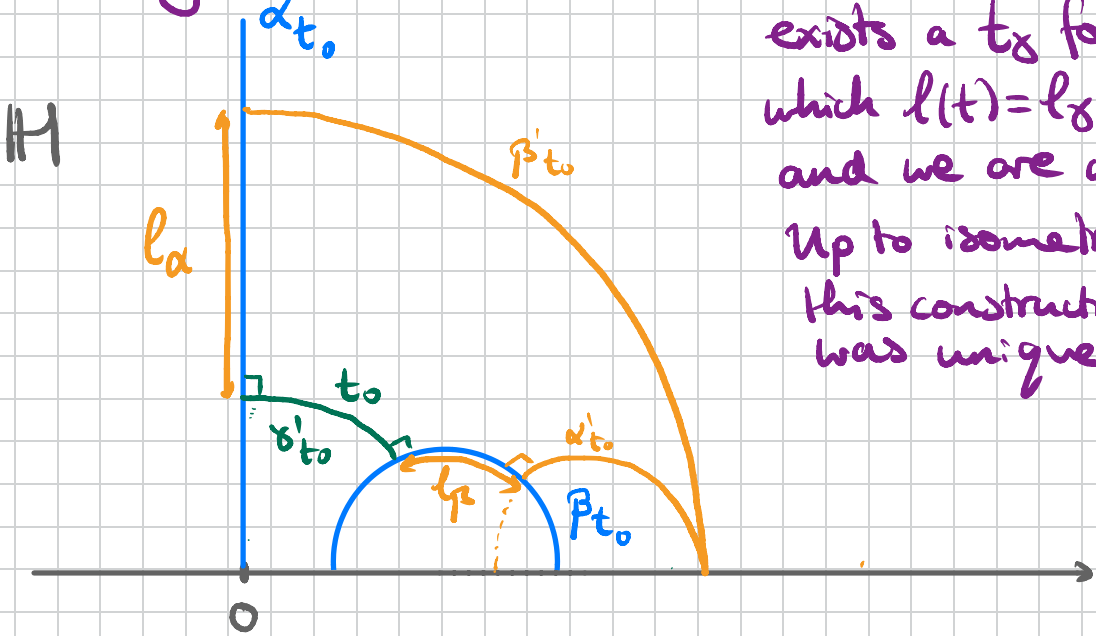
$\mapsto (\ell(\alpha_1), \ell(\alpha_3), \ell(\alpha_5))$.

Proof: We will define a two-sided inverse from $\mathbb{R}_+^3$, by building a hexagon from a given triple of lengths $(\ell_\alpha, \ell_\beta, \ell_\gamma) \in \mathbb{R}_+^3$.

For any $t > 0$, let α_t and β_t be a pair of geodesics in $\mathbb{H}$ at distance t and let γ_t' be the unique geodesic between them. It meets both at right angles.

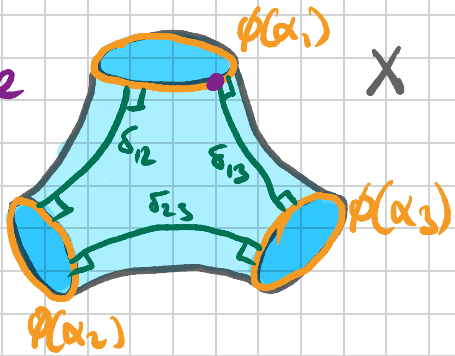


There is a value $t_0 > 0$ for which α'_t and β'_t share an endpoint. Sending t from t_0 to infinity sends $l(t)$ from 0 to infinity. So there exists a t_γ for which $l(t) = l_\gamma$ and we are done. Up to isometry, this construction was unique. □



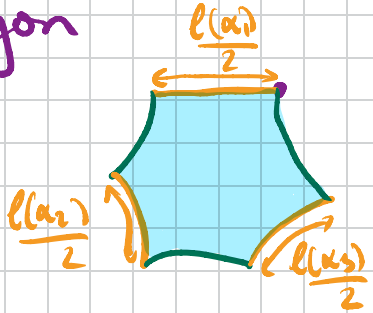
Proposition

Let P be a pair of pants with boundary components α_1, α_2 and α_3. Let X be a marked hyperbolic pair of pants. Then X is determined (up to homotopy) by the triple $(\ell_X(\alpha_1), \ell_X(\alpha_2), \ell_X(\alpha_3))$. That is, $\text{Teich}(P) \cong \mathbb{R}_+^3$.



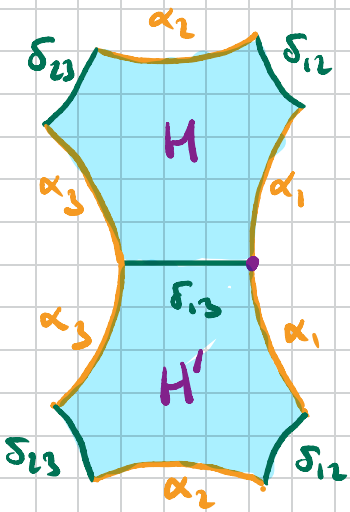
X

Proof: Let δ_{ij} be the unique geodesic arc between $\phi(\alpha_i)$ and $\phi(\alpha_j)$. By the first variation principle, the δ_{ij} meet the boundary at right-angles. Hence, $X - \cup\delta_{ij}$ is a union of two hyperbolic right-angled hexagons H_1 and H_2. Since the lengths of the δ_{ij} are equal on both, H_1 and H_2 are isometric by the previous proposition. The marked

hexagon 

is then determined up to isometry by $(\ell_X(\alpha_1), \ell_X(\alpha_2), \ell_X(\alpha_3))$.

Conversely, given these lengths, we build the hyperbolic right-angled hexagon H with triple $\left(\frac{\ell_X(\alpha_1)}{2}, \frac{\ell_X(\alpha_2)}{2}, \frac{\ell_X(\alpha_3)}{2}\right)$ labelled as shown. Reflect across δ_{13} to get another hexagon H'. Gluing along δ_{ij} gives a hyperbolic pair of pants X with a marking $\phi: P \to X$ given by the labels.



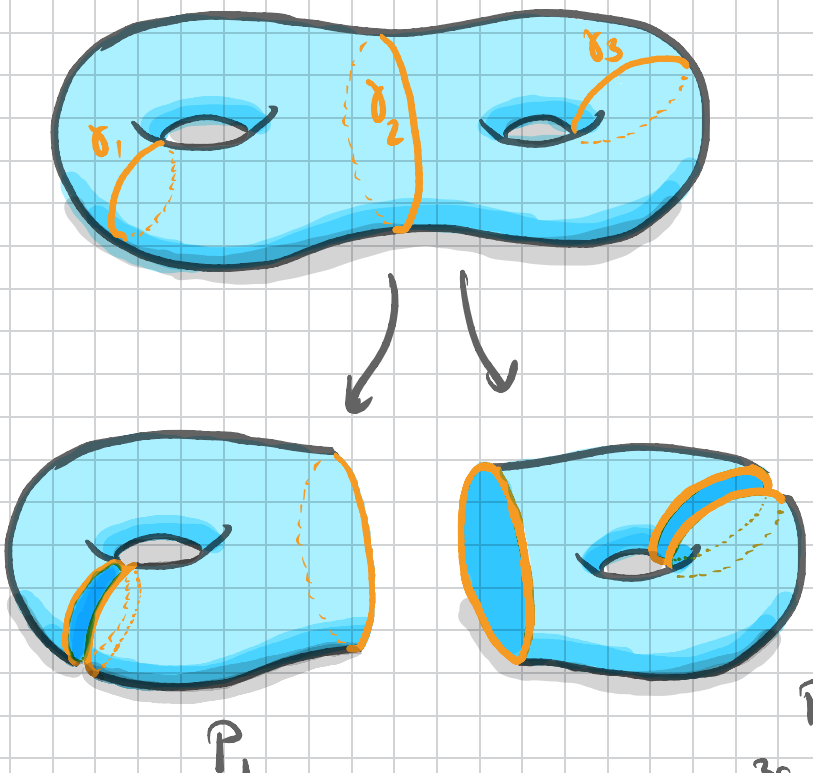
Everything is well-defined up to homotopy. □

We will now use pairs of pants to understand hyperbolic surfaces of genus $g \geq 2$.

IV. FENCHEL-NIELSEN COORDINATES

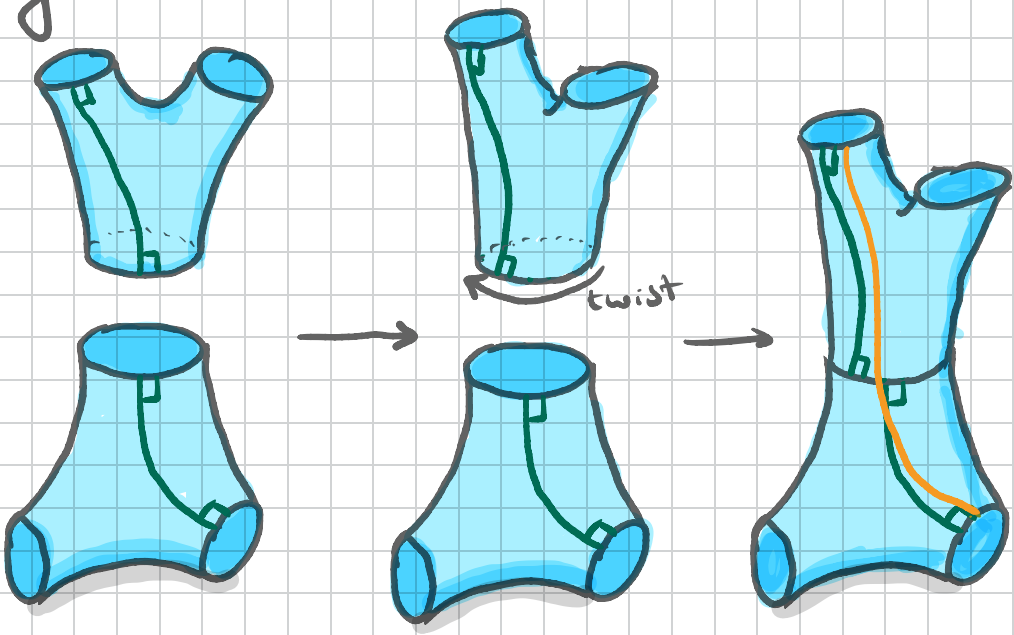
Let S_g be a closed surface of genus $g \geq 2$.

Given $3g-3$ simple closed curves γ_i in S_g (and pairwise non-homotopic) that are pairwise disjoint, $S_g - \cup \gamma_i$ will be a disjoint union of $2g-2$ pairs of pants.



We call the set of curves $\{\gamma_i\}_{i=1}^{3g-3}$ a pants decomposition.

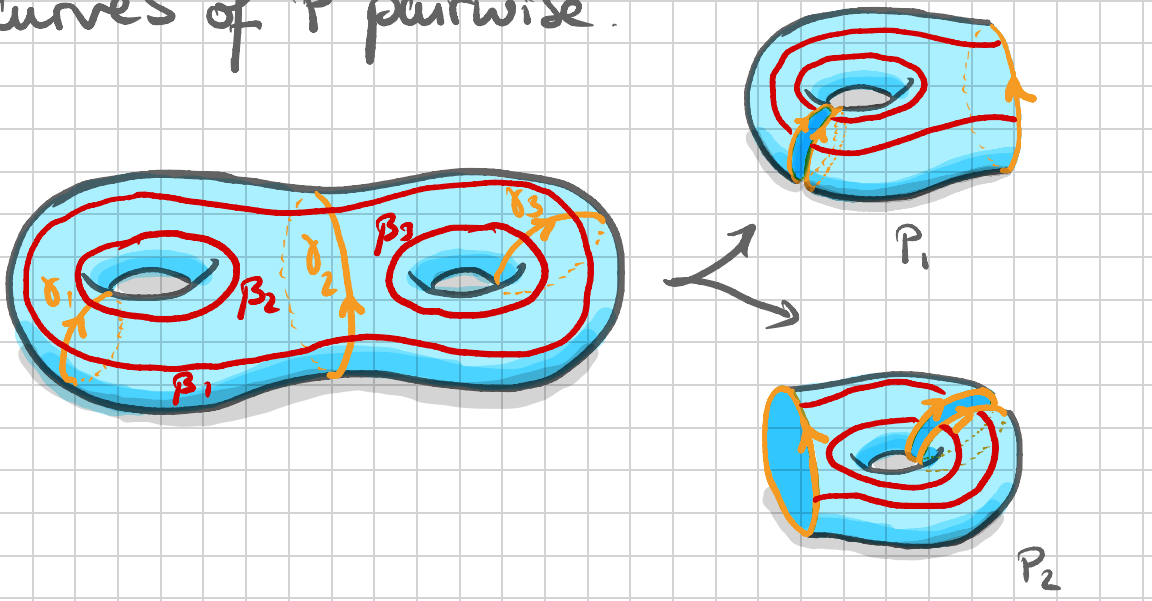
The hyperbolic metrics on the pairs of pants are uniquely determined by the lengths of the γ_i. The hyperbolic metric on the original surface X can be recovered if we know how to identify the boundary curves. The only choice we can make is a choice of twist when we reglue.



Twisting affects geodesic arcs.

Consider the following data:

- a pants decomposition $\{\gamma_1, \ldots, \gamma_{3g-3}\}$ of oriented simple closed curves.
- a set of seams $\{\beta_1, \ldots, \beta_n\}$; that is, a collection of disjoint simple closed curves in S_g so that for any pair of pants P in the decomposition, $P \cap (\cup \beta_j)$ is a union of three disjoint arcs connecting the boundary curves of P pairwise.



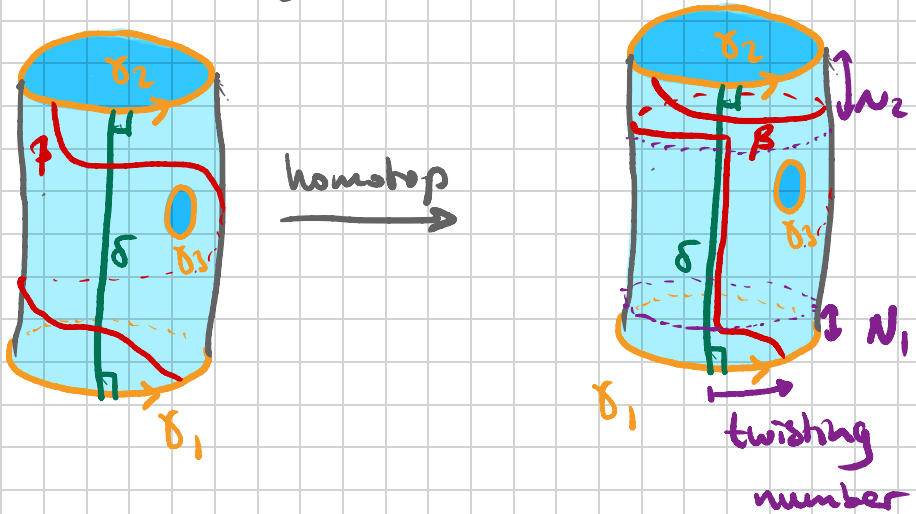
We use the seams to measure twists.

So, suppose β is a geodesic arc between the boundary curves γ_1 and γ_2 on some pair of pants P. Then:

i. Choose neighbourhoods N_1 of γ_1 and N_2 of γ_2;

ii. Homotope β so that it agrees with the unique geodesic δ between γ_1 and γ_2 outside of $N_1 \cup N_2$;

iii. The twisting number of β at γ_1 is the signed amount of winding that takes place travelling along β inside N_1.



Let (X, ϕ) be a marked hyperbolic surface. The **length parameters** of X are defined to be $\ell_X(\gamma_i)$ for γ_i in the pants decomposition. On each side of $\phi(\gamma_i)$ there is a pair of pants and some $\phi(\beta_j)$ gives rise to a geodesic arc in each. Let t_L and t_R be the twisting numbers for each of these arcs on the left and right sides of $\phi(\gamma_i)$, respectively.

The **i^{th} twist parameter** of X is defined to be

$$\theta_i(X) = 2\pi \frac{t_L - t_R}{\ell_X(\gamma_i)}.$$

Theorem

The hyperbolic surface X is determined up to homotopy by

$$(\ell_X(\gamma_1), \ldots, \ell_X(\gamma_{3g-3}), \theta_1(X), \ldots, \theta_{3g-3}(X)).$$

Proof Let $(l_1, \dots, l_{3g-3}, \theta_1, \dots, \theta_{3g-3}) \in \mathbb{R}_+^{3g-3} \times \mathbb{R}^{3g-3}$. We will show that there is a unique X up to homotopy with these parameters.

1. Let $P_{i,j,k}$ be the pair of pants with boundaries $\gamma_i, \gamma_j, \gamma_k$. There is a unique hyperbolic pair of pants $X_{i,j,k}$ whose boundary components have lengths l_i, l_j and l_k and a homeomorphism $P_{i,j,k} \to X_{i,j,k}$ respecting labels.

2. For each pair of boundaries on $X_{i,j,k}$, draw the unique geodesic arc between them. Adjust each such seam by twisting by $\left(\frac{\theta_m}{2\pi}\right) l_m$ on the left of γ_m for each $m \in \{i,j,k\}$. Identify seams of $P_{i,j,k}$ with those of $X_{i,j,k}$.

3. These homeomorphisms extend to $\phi: S_g \to \bigsqcup X_{i,j,k}/\sim$ identifying boundaries by matching seams. Every choice was unique up to homotopy. □

These $6g-6$ numbers are called the Fenchel-Nielsen coordinates of X.

V. THE 9g-9 THEOREM

We are now ready to prove the following:

Theorem (9g-9 Theorem)

Let X be a marked hyperbolic surface, then there exist simple closed curves $\{\delta_1, \dots, \delta_{9g-9}\}$ so that the tuple

$$(\ell_X(\delta_1), \dots, \ell_X(\delta_{9g-9}))$$

determines X up to homotopy.

We will use the following proposition as a black box for now.

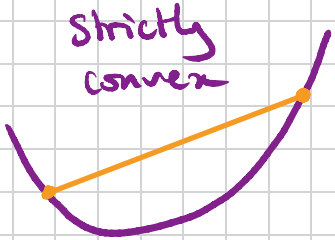
Proposition

Let X be a marked hyperbolic surface and let X_s for $s \in \mathbb{R}$ be the family of surfaces obtained by changing the twist parameter of X by s around some pants curve γ. Let b be

a homotopy class of simple closed curves on S_g that intersects γ. Then the function

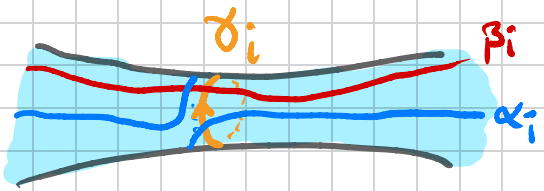
$$s \mapsto \ell_{X_s}(b)$$

is strictly convex.



Proof of the $9g-9$ Theorem

Let $\{\gamma_1, \dots, \gamma_{3g-3}\}$ be a pants decomposition and choose simple closed curves $\{\beta_1, \dots, \beta_{3g-3}\}$ so that β_i intersects $\gamma_j \Leftrightarrow i = j$. Let α_i be the curves obtained from β_i by twisting once around γ_i.



Choose a set of seams for the $\{\gamma_i\}$ to define Fenchel-Nielsen coordinates. We now already have the $\ell_X(\gamma_i)$. We need to use

the $\ell_X(\delta_i)$, $\ell_X(\alpha_i)$ and $\ell_X(\beta_i)$ to determine the twist parameters $\theta_i(X)$.

Let X_t be the hyperbolic surface with the same length parameters as X but with twist parameters $\underline{t} = (t_1, \ldots, t_{3g-3})$. We can assume $X = X_{\underline{0}}$. We will show that if $t_i \neq 0$ for some i then either $\ell_X(\alpha_i) \neq \ell_{X_t}(\alpha_i)$ or $\ell_X(\beta_i) \neq \ell_{X_t}(\beta_i)$.

Consider the functions

$$A(t) = \ell_{X_t}(\alpha_1) \quad \text{and} \quad B(t) = \ell_{X_t}(\beta_1).$$

Since α_1 and β_1 only intersect δ_1, these are just functions of $s := t_1$. By the above proposition, $A(s)$ and $B(s)$ are strictly convex, hence so is $(A+B)(s)$. Also, by definition, $A(s+2\pi) = B(s)$.

Assume $A(s) = A(0)$ for some $s \neq 0$. Suppose $s > 0$, for concreteness. Since $A(s) = A(0)$,

it follows from strict convexity that $A(t) < A(0)$ for $t \in (0, s)$ and $A(t)$ is strictly increasing for $t > s$.

If $s < 2\pi$, then $s < 2\pi < 2\pi + s$, so $A(2\pi) < A(2\pi + s) \Rightarrow B(0) \neq B(s)$.

If $s > 2\pi$, then $0 < 2\pi < s < 2\pi + s$ so $A(2\pi) < A(0) = A(s) < A(2\pi + s)$.

Finally, if $s = 2\pi$, we cannot have $A(2\pi) = A(2\pi + s)$, for then $A(t)$ would take the same value at 0, 2π, and 4π, breaking strict convexity.

So, in all cases, $B(0) \neq B(s)$ or $A(0) \neq A(\quad)$.

This works for any t_i and so if $\ell_X(\alpha_i) = \ell_{X_t}(\alpha_i)$ and $\ell_X(\beta_i) = \ell_{X_t}(\beta_i)$ for all i, then we must have $t = 0$ and so $X_t = X$.

□

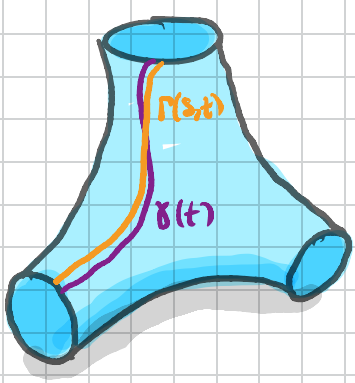
APPENDIX A. FIRST VARIATION PRINCIPLE

Let $\gamma : [0,1] \to M$ be a curve between two boundary components of Riemannian manifold M with metric g. The length of γ is

$$L(\gamma) = \int_0^1 |\dot{\gamma}(t)|\, dt$$

$$= \int_0^1 \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))}\, dt.$$

A variation of γ is a family of small perturbations of γ, $\Gamma(s,t)$, $s \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, with $\Gamma(0,t) = \gamma(t)$.



Let $V(t) = \left.\frac{d\Gamma}{ds}\right|_{s=0}$ be the variation vector field, i.e. the direction of the perturbation.

$\dot{\gamma}(t) = \left.\frac{d\Gamma}{dt}\right|_{s=0}$ recovers the tangent vector to γ.

The first variation principle says that if γ minimises L, then

$$\delta L := \frac{d}{ds}\Big|_{s=0} L(\Gamma(s,t)) = 0 .$$

We have

$$\delta L = \frac{d}{ds}\Big|_{s=0} L(\Gamma(s,t))$$

$$= \frac{d}{ds}\Big|_{s=0} \int_0^1 \sqrt{g(\Gamma_t, \Gamma_t)}\, dt$$

$$= \int_0^1 \frac{d}{ds}\Big|_{s=0} \sqrt{g(\Gamma_t, \Gamma_t)}\, dt$$

Chain rule

$$= \int_0^1 \left(\frac{1}{2\cdot\sqrt{g(\Gamma_t,\Gamma_t)}} \cdot \frac{d}{ds} g(\Gamma_t, \Gamma_t) \right)\Big|_{s=0} dt$$

Lie derivative ↔ covariant derivative

$$= \int_0^1 \frac{1}{2\cdot|\dot\gamma(t)|} \cdot 2\, g(\nabla_S \Gamma_t, \Gamma_t)\, dt$$

$$= \int_0^1 \frac{g(\nabla_S \Gamma_t, \Gamma_t)}{|\dot\gamma(t)|}\, dt = \int_0^1 \frac{g(\nabla_t \Gamma_S, \Gamma_t)}{|\dot\gamma(t)|}\, dt$$

swapped order.

$$= \int_0^1 \frac{g(\nabla_t V(t), \dot{\gamma}(t))}{|\dot{\gamma}(t)|} dt$$

$$= \int_0^1 \frac{1}{|\dot{\gamma}(t)|} \left(\frac{d}{dt} g(V(t), \dot{\gamma}(t)) - g(V(t), \nabla_t \dot{\gamma}(t)) \right) dt$$

$$= \left[\frac{g(V(t), \dot{\gamma}(t))}{|\dot{\gamma}(t)|} \right]_0^1 - \int_0^1 \frac{g(V(t), \nabla_t \dot{\gamma}(t))}{|\dot{\gamma}(t)|} dt$$

assume arc length parameterisation (ie. $|\dot{\gamma}(t)| = 1$)

$$= \left[g(V(t), \dot{\gamma}(t)) \right]_0^1 - \int_0^1 g(V(t), \nabla_t \dot{\gamma}(t)) dt$$

Since $V(t)$ is arbitrary, this can only be zero for all possible $V(t)$ if both terms are independently zero.

The latter forces $\nabla_t \dot{\gamma}(t) = 0$, which is the geodesic equation. The former forces $\dot{\gamma}(t)$ to be orthogonal to any variation at the boundary so $\gamma(t)$ is a geodesic arc orthogonal to the boundary.

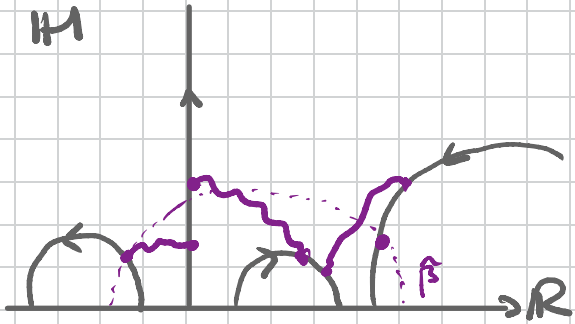
APPENDIX B. DEPENDENCE ON SEAMS

The definition of $\Theta_i(x)$ made use of only one seam $\phi(\beta_j)$ passing through $\phi(\gamma_i)$.

We must check that $\Theta_i(x)$ is independent of this choice. This can be done by considering the universal cover of the neighbourhood N_i around γ_i. This is handled in Figure 10.8 of Chapter 10 of Farb-Margalit.

APPENDIX C. CONVEXITY

Given some curve β in X, its image in X_S can be lifted to a broken path between lifts of γ (the twisting curve) in $\mathbb{H}$. Each lift is displaced by 2 along the direction of the lift of γ.



This is called an earthquake map.

One shows that minimising the length of this broken path gives a strictly convex function of the displacement s.
The details can be found at the end of Chapter 10 of Fathi-Margalit.