TCC: Hyperbolic surfaces, their length spectra, and connection to Markoff triples

Lecture 2: Hyperbolic surfaces and their length spectra

Our objectives today:

- → Define and describe hyperbolic surfaces.
- → Talk about their closed geodesics.

(I) Hyperbolic surfaces

Definition: A hyperbolic structure on a topological surface X is a maximal collection of coordinate charts, i.e. open sets $U_i \subset X$ and maps $\Phi_i : U_i \to \mathbb{H}$ such that:

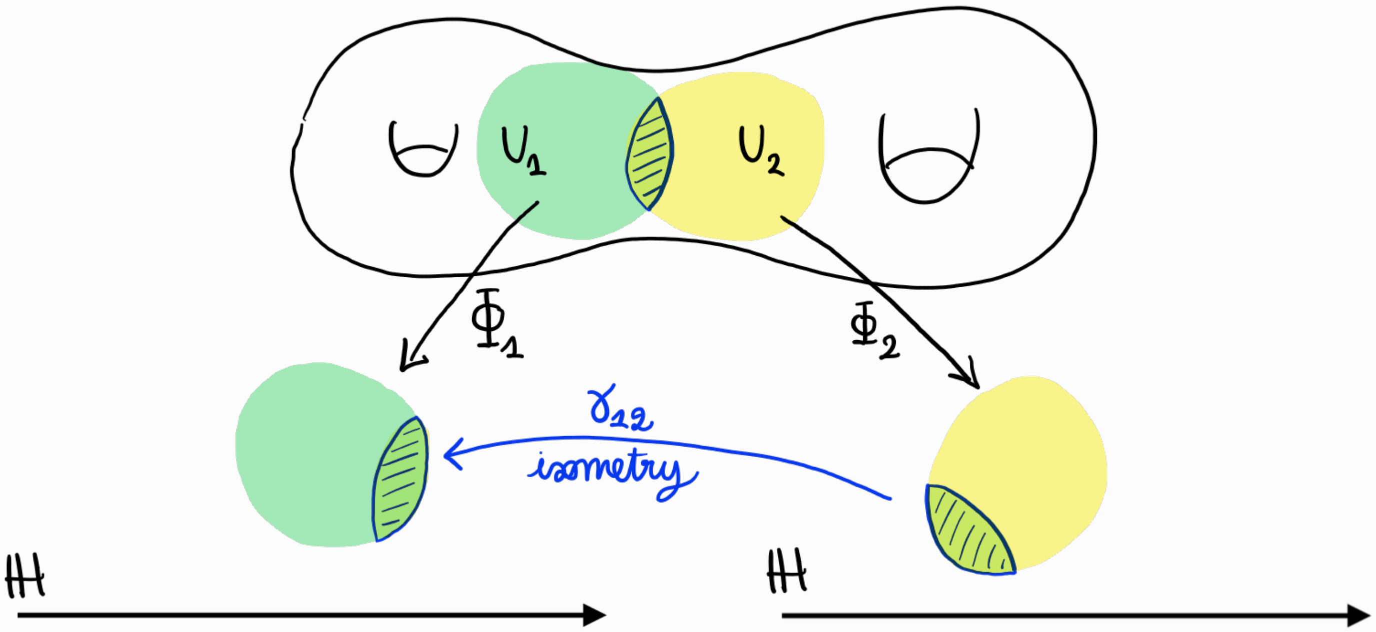
(i) $\Phi_i : U_i \to \Phi_i(U_i)$ is a homeomorphism for all i;

(ii) the sets $(U_i)_i$ cover X;

(iii) the "overlap maps" are isometries, i.e. for all i, j,

$$\gamma_{ij} : \Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \longrightarrow \Phi_i(U_i \cap U_j)$$

coincides with an element of $\mathrm{Isom}^+(\mathbb{H})$.



This is the usual definition of manifold except we add the fact that the overlap maps are isometries of $\mathbb{H}$. ◻

A hyperbolic structure on X equips it with a Riemannian metric of constant curvature -1, and an associated concept of length of a curve.

Ⅱ Groups acting on $\mathbb{H}$ and quotients

Ⓟ Use groups acting on $\mathbb{H}$ to define hyperbolic surfaces as quotients of $\mathbb{H}$.

ⓐ Action of $\Gamma \leq PSL(2;\mathbb{R})$ on $\mathbb{H}$

Recall: The group of positive isometries of $\mathbb{H}$ is $PSL(2;\mathbb{R})$.
To $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we associate the isometry $z \mapsto \frac{az+b}{cz+d}$ of $\mathbb{H}$.

As a consequence, any subgroup $\Gamma \leq PSL(2;\mathbb{R})$ defines an action on $\mathbb{H}$ by isometries: $\forall T \in \Gamma, \forall z \in \mathbb{H}, \quad T \cdot z = Tz = \frac{az+b}{cz+d}$ if $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We wish to try and equip the quotient space $X = \mathbb{H}/\Gamma$ with a hyperbolic structure. Here the topology of X is the quotient topology, which guarantees the projection map $\pi : \begin{cases} \mathbb{H} \to X \\ z \mapsto \Gamma z \end{cases}$ is continuous.

ⓑ Free actions

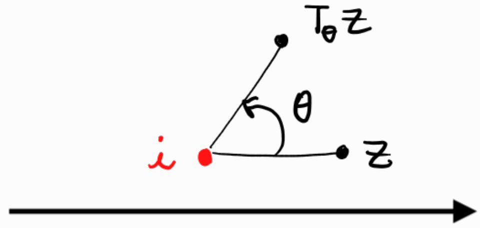
Definition: An action of a group Γ on a space Y is free if, $\forall T \in \Gamma \setminus \{e\}, \forall z \in Y, \quad Tz \neq z$ (ie. T has no fixed points).

Recall: Let $T \in PSL(2;\mathbb{R})$ distinct from id.
- → If $|tr(T)| > 2$ (T is hyperbolic) then T has two fixed points in $\partial\mathbb{H}$.
- → If $|tr(T)| = 2$ (T is parabolic) then T has one fixed point in $\partial\mathbb{H}$.
- → If $|tr(T)| < 2$ (T is elliptic) then T has one fixed point in $\mathbb{H}$.

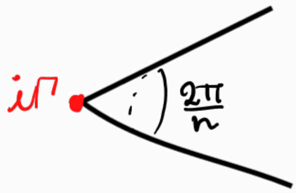
As a consequence, the action of Γ on $\mathbb{H}$ is free iff Γ contains no elliptic elements (as we don't mind fixed points at the boundary $\partial\mathbb{H}$).

Let us see why elliptic elements are undesirable here. We saw that elliptic elements are rotations and all conjugate to a rotation around i of angle θ,

$$T_\theta = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix}$$



→ If $\theta = \frac{2\pi}{n}$ for $n \in \mathbb{Z}_{>0}$, then the quotient $X = \mathbb{H}/\Gamma$ is a cone with an angular point at $i\Gamma$.
↳ X is not a manifold (but an orbifold).



→ Otherwise, the orbit of any z is dense in the circle of center i going through z.

(c) Properly discontinuous actions

You can convince yourself that the case of an irrational rotation above is not a great setting to work on the quotient $X = \mathbb{H}/\Gamma$.
The natural (very general) hypothesis to avoid such issues is the following.

Definition: An action of a group Γ on a topological space Y is *properly discontinuous* if, for all compact set K in Y,
$$\#\{T \in \Gamma : K \cap TK \neq \emptyset\} < \infty.$$

Exercise: Prove that, for any hyperbolic or parabolic element $T \in PSL(2;\mathbb{R})$, the cyclic group $\langle T \rangle$ generated by T acts properly discontinuously on $\mathbb{H}$.
Hint: Prove this notion is invariant by conjugation and use the normal form of hyperbolic / parabolic elements.

Exercise: Find a cyclic subgroup of $PSL(2;\mathbb{R})$ which does not act properly discontinuously on $\mathbb{H}$.

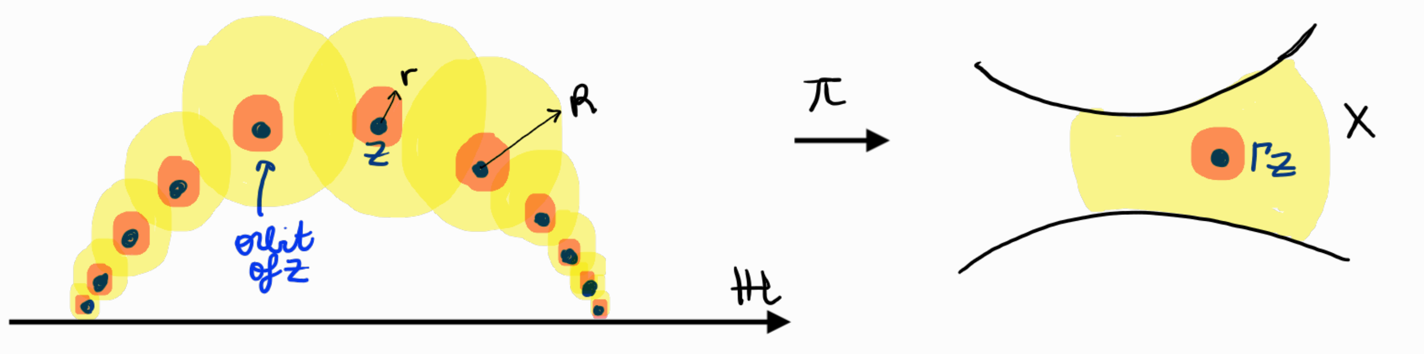
This hypotheses allows us to achieve our goal.

Proposition: Let $\Gamma \leq PSL(2;\mathbb{R})$. We assume that Γ acts freely and properly discontinuously on $\mathbb{H}$. Then we can define a hyperbolic structure on $X = \mathbb{H}/\Gamma$.

Proof: Recall that X comes with the quotient topology, so we "just" need to define charts $\Phi : U \to \mathbb{H}$ on a cover of X.

Let $\Gamma z \in X$ be a coset with representative $z \in \mathbb{H}$.
Let $R>0$. The ball $\overline{B_R(z)}$ of radius R centered at z in $\mathbb{H}$ is compact.
Since the action is properly discontinuous, the number of $T \in \Gamma$ such that $B_R(z)$ intersects $T B_R(z)$ is finite.

Since T is an isometry, $TB_R(z) = B_R(Tz)$.
Note that since the action is free, for all $T \neq id$, $Tz \neq z$.
We can hence find $r \in (0,R)$ such that $\forall T \in \Gamma \setminus \{id\}$, $B_r(z) \cap TB_r(z) = \phi$.



Then the restriction of the projection $\pi : \mathbb{H} \to X$ to $B_r(z)$ is an homeomorphism onto an open neighbourhood U of Γz in X. We take its inverse $\Phi : U \to B_r(z)$ to be a chart.

Exercise: (*) Check the overlap maps are isometries. $\square$

The definition of properly discontinuous is not the simplest. In this specific setting there is a very useful characterisation, which requires to view $PSL(2;\mathbb{R})$ as a topological group.

Definition: A **topological group** is a group equipped with a topology for which the group multiplication and inverse are continuous.

We equip $SL(2;\mathbb{R})$ with a topology by viewing it as a subset of $\mathbb{R}^4$, given by the matrix coefficients. The matrix multiplication and inverse are continuous in the coefficients.

The topological group $PSL(2;\mathbb{R}) = SL(2;\mathbb{R}) / \{\pm Id\}$ is obtained by quotient.

Exercise: Prove that if $T_n \to T$ in $PSL(2;\mathbb{R})$ then $\forall z \in \mathbb{H}$, $T_n z \to Tz$.

Definition: A topological group is **discrete** if its neutral element is an isolated point.

Exercise: Prove that a topological group is discrete iff all of its elements are isolated.

Discrete subgroups of $PSL(2;\mathbb{R})$ are so important they have their own name.

Definition: A **Fuchsian group** is a discrete group of $PSL(2;\mathbb{R})$.

Theorem: Let $\Gamma \leq PSL(2;\mathbb{R})$. Then Γ is discrete iff Γ acts properly discontinuously on $\mathbb{H}$.

Proof:

($\Leftarrow$) Let us assume that Γ acts properly discontinuously on $\mathbb{H}$.
Let $K = \overline{B_1(i)}$ be the ball of center i and radius 1 in $\mathbb{H}$.
This is a compact set. Since Γ acts properly discontinuously on $\mathbb{H}$,
$A = \{T \in \Gamma : K \cap TK \neq \emptyset\}$ is finite.

Let $(T_n)_{n\geq 0}$ be a sequence of elements in Γ converging to id.
Then $T_n i \longrightarrow \mathrm{id}\, i = i$ as $n \to \infty$.
As a consequence, there exists $n_0 \geq 0$ such that, for all $n \geq n_0$,
$T_n i \in K$. In particular $T_n i \in K \cap T_n K$ so $K \cap T_n K \neq \emptyset$, i.e. $T_n \in A$.

We have proven that, for all $n \geq n_0$, T_n lies in the finite set A.
Since $T_n \to \mathrm{id}$, there exists n_1 such that $\forall n \geq n_1$, $T_n = \mathrm{id}$.
Hence any sequence $(T_n)_{n\geq 0}$ in Γ converging to id is stationnary,
which is exactly the claim. $\square$

($\Rightarrow$) This side uses the following lemma.

Lemma: Let $K \subset \mathbb{H}$ be compact. Then $\{T \in PSL(2;\mathbb{R}) : Ti \in K\}$ is compact.

Proof (of Lemma):

By definition of the topology on $PSL(2;\mathbb{R})$ it is sufficient to prove this set, which we denote by E, is closed and bounded.

→ Closed: Let $(T_n)_{n\geq 0}$ be a sequence s.t. $T_n \to T$ in $PSL(2;\mathbb{R})$ and $\forall n \geq 0$, $T_n \in E$. Then, $\forall n$, $T_n i \in K$, and $T_n i \to Ti$. Since K is closed, $Ti \in K$ which means $T \in E$. $\diamond$

→ Bounded: K is compact hence bounded, so there exists $M>0$ such that $\forall z \in K$, $|z| \leq M$ and $\operatorname{Im} z > 1/M$.

Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in E$. By definition, $Ti \in K$ so

$$|Ti| = \left|\frac{ai+b}{ci+d}\right| \leq M \text{ and } \operatorname{Im}(Ti) = \frac{1}{|ci+d|^2} > 1/M.$$

Then $c^2+d^2 = |ci+d|^2 < M$,

and $a^2+b^2 = |ai+b|^2 \leq M^2|ci+d|^2 < M^3$

so a,b,c,d are bounded and hence T is bounded. $\diamond$

We now prove the $(\Rightarrow)$ of the proposition.

Let us assume that Γ is discrete.

Let $K \subset \mathbb{H}$ be compact. Take $R>0$ such that $K \subset \overline{B_R(i)}$.

Let $T \in \Gamma$ such that $K \cap TK \neq \emptyset$. Then in particular $\overline{B_R(i)} \cap T\overline{B_R(i)} \neq \emptyset$.

So there exists $z \in \mathbb{H}$ such that $d(z,i) < R$ and $d(z,Ti) < R$.

Hence $d(i,Ti) \leq 2R$, which means $Ti \in \overline{B_{2R}(i)}$.

Let $E = \{T \in PSL(2,\mathbb{R}) : Ti \in \overline{B_{2R}(i)}\}$. $\overline{B_{2R}(i)}$ is compact, so by the previous lemma, E is compact.

Then $E \cap \Gamma$ is discrete and compact hence finite.

The discussion above proves that $\{T \in \Gamma : K \cap TK \neq \emptyset\} \subset E \cap \Gamma$, so it is finite, which is what we needed to prove. $\square$

ⓓ The uniformisation theorem

We have now introduced a large family of hyperbolic surfaces, constructed by quotient. Strikingly, the beautiful Uniformisation theorem, which we will not have time to prove, states that these are the only examples.

Theorem: Let X be a complete hyperbolic surface. Then there exists a Fuchsian group Γ, with no elliptic elements, so that $X = \mathbb{H}/\Gamma$.

This statement is proven in Caroline Series's lecture notes "Hyperbolic geometry".

III) Fundamental domains

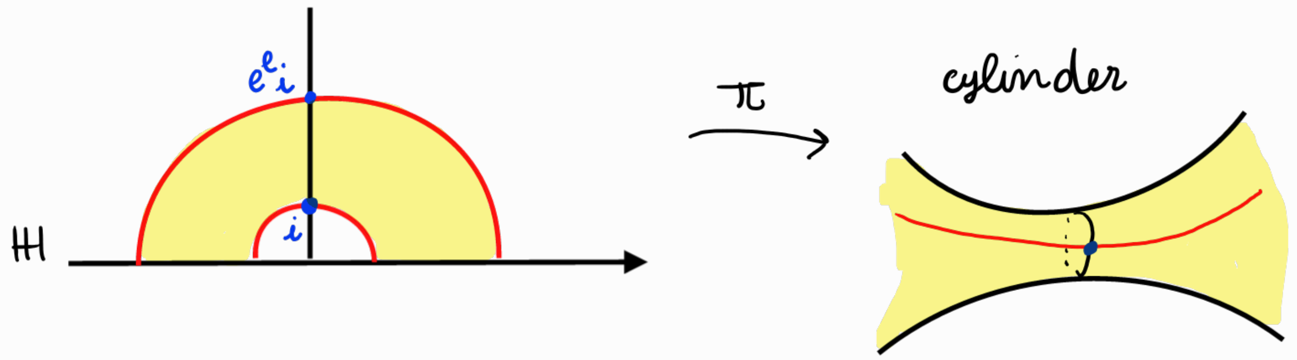
Fundamental domains are a very useful way to describe quotients.

Definition: Let Γ be a Fuchsian group. A fundamental domain for Γ is a subset $R \subset \mathbb{H}$ s.t:

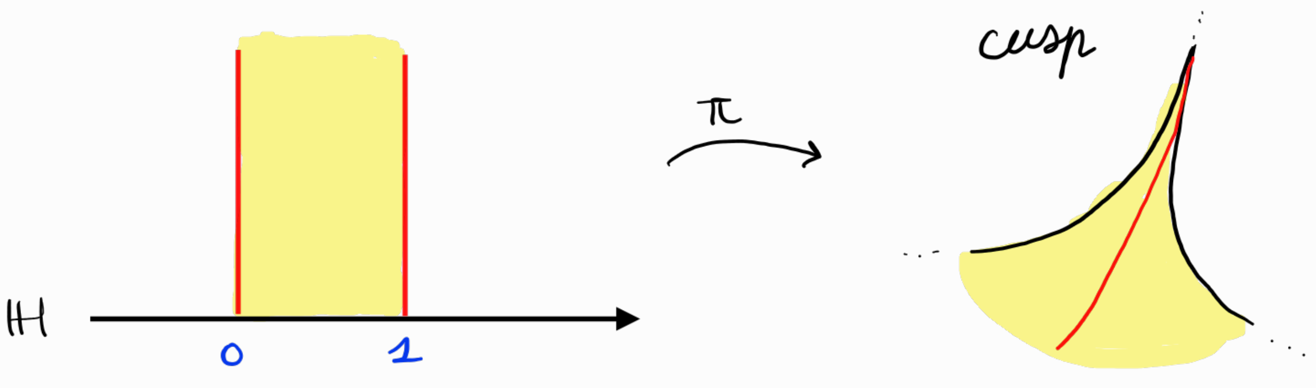
(i) $\forall T \in \Gamma \setminus \{id\}$, $R \cap TR = \phi$

(ii) $\forall z \in \mathbb{H}$, $\exists T \in \Gamma$: $Tz \in \overline{R}$.

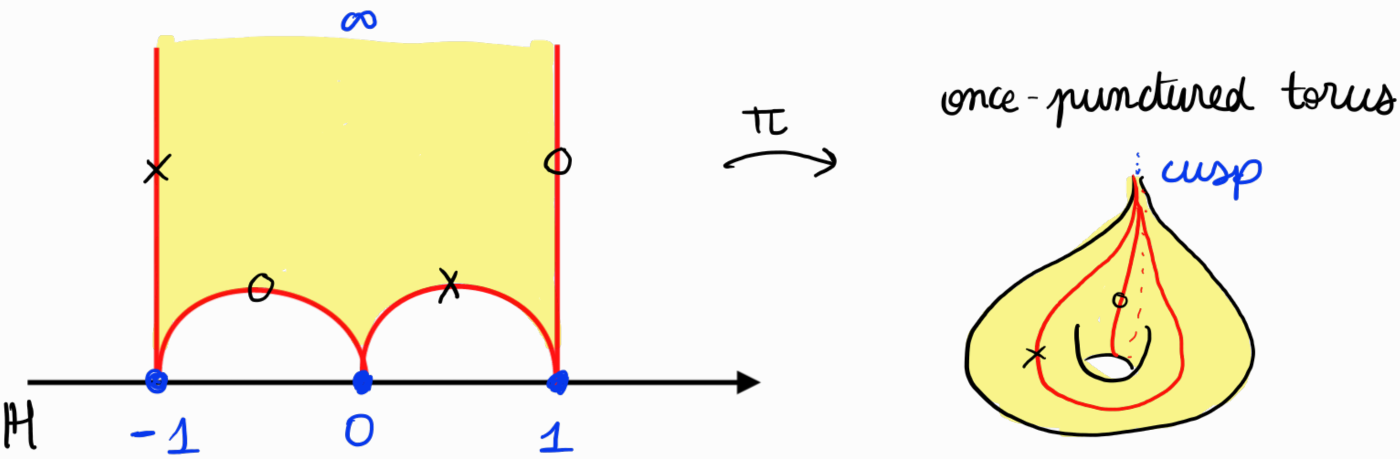
Exercise: Prove that $\{z \in \mathbb{H} : 1 < |z| < e^{\ell}\}$ is a fundamental domain for cyclic group generated by $\begin{pmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{pmatrix}$.



Exercise: Prove that $\{z \in \mathbb{H} : 0 < \operatorname{Re} z < 1\}$ is a fundamental domain for $\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$.



Little spoiler: We will see later that $\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \right\rangle$ is a Fuchsian group, and that the following is a fundamental domain:



There exists simple constructions of fundamental domains:

→ Dirichlet domain of center z_0:

$$R_{z_0} = \{ z \in \mathbb{H} : \forall T \in \Gamma \setminus \{id\},\ d(z, z_0) < d(z, Tz_0) \}.$$

→ Ford domain when Γ has parabolic elements (e.g. example above).

For these constructions, the fundamental domains are "nice":

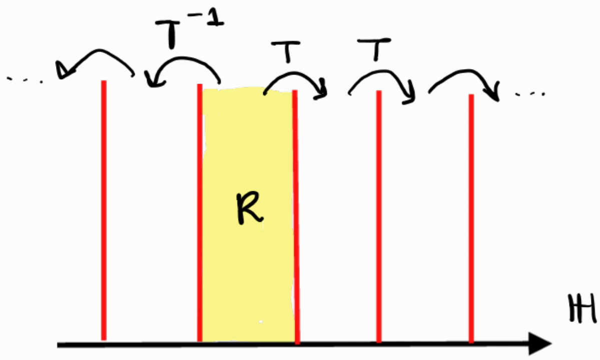
→ R is a convex polygon;

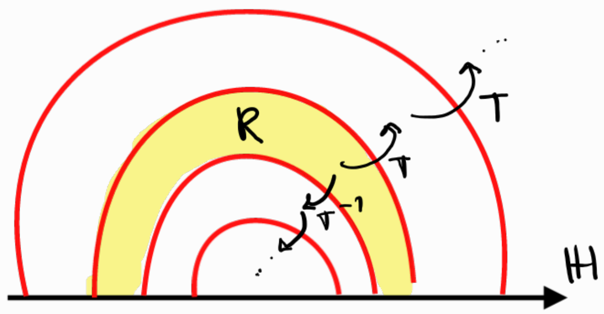
→ R is locally finite (any compact K meets finitely many TR, $T \in \Gamma$);

→ R has finitely many edges iff Γ is finitely generated.

In these cases the copies $(TR)_{T \in \Gamma}$ form a tiling of $\mathbb{H}$.

The sides of R are paired by some elements of Γ.



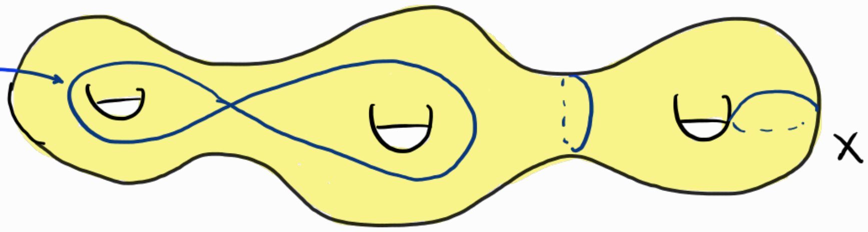


IV Closed geodesics

Let us now study closed geodesics on hyperbolic surfaces.

Let us fix a hyperbolic surface $X = \mathbb{H}/\Gamma$ and $\pi : \mathbb{H} \to X$

we shall see that X has a countable family of closed geodesics



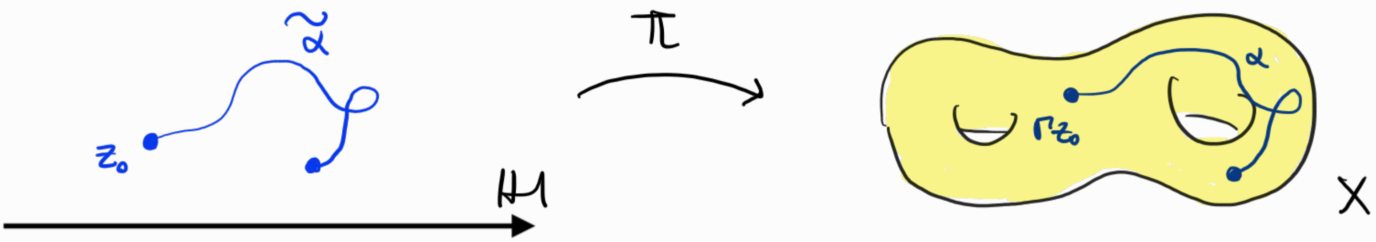
(a) Paths and lifts

Definition: Let Y be a topological space ($\mathbb{H}$ or X).
A path on Y is a continuous map $\alpha : [0,1] \to Y$.

(We will also occasionally have paths $\mathbb{R} \to Y$).

In order to study paths on $\mathbb{H}$ it will be essential to use the following fundamental lifting property.

Theorem: Let $\alpha : [0,1] \to X$ be a path on X.
Let $z_0 \in \mathbb{H}$ be a representative of $\pi^{-1}(\alpha(0)) = \Gamma z_0$.
Then there exists a unique path $\tilde{\alpha} : [0,1] \to \mathbb{H}$ such that $\tilde{\alpha}(0) = z_0$ and $\forall t \in [0,1]$, $\pi(\tilde{\alpha}(t)) = \alpha(t)$.



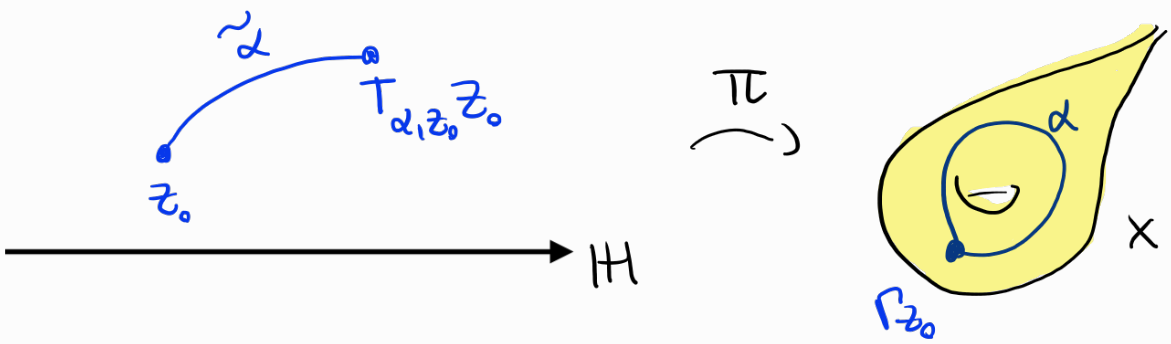
(b) Closed paths and free homotopy

Definition: A path α is closed if $\alpha(0) = \alpha(1)$. We call this point its basepoint.

Interestingly, closed paths on X will not lift to closed paths on $\mathbb{H}$.
However the endpoints of a lift will project to the same point in X.
which motivates the following definition.

Definition: Let α be a closed path on X, and $z_0 \in \pi^{-1}(\alpha(0))$.
Let $\tilde{\alpha}$ be the lift of α starting at z_0.
We denote as T_{α,z_0} the element of Γ sending $z_0 = \tilde{\alpha}(0)$ to $\tilde{\alpha}(1)$.

Remark: The existence of T_{α,z_0} is given by the fact that $\alpha(1) = \alpha(0)$.
The uniqueness follows from the assumption that the action of Γ is free.



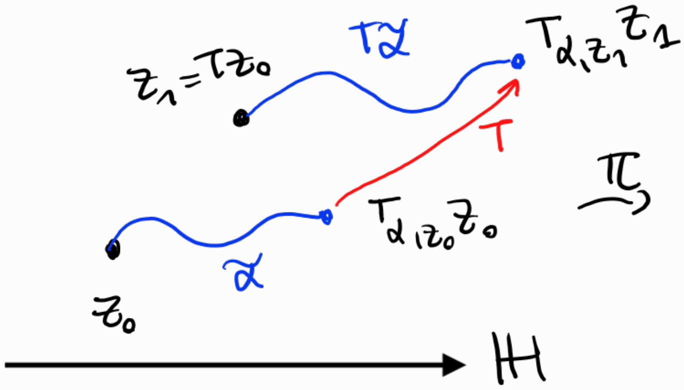
Example: Let $T \in \Gamma$ and $z_0 \in \mathbb{H}$. If γ is the geodesic from z_0 to Tz_0 in $\mathbb{H}$, then its projection $\pi \circ \gamma$ is a closed path based at $\pi(z_0) = \Gamma z_0$, for which $T_{\alpha,z_0} = T$.

We here see that group elements of Γ and closed paths dialogue, which will be a key feature of these lectures.

Let us study the dependency of T_{α,z_0} on α and z_0.
First notice that any other choice of starting point for the lift must be a Tz_0 for a $T \in \Gamma$.

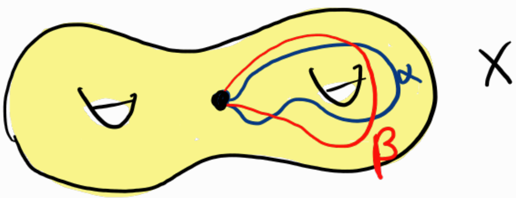
Lemma: For any $T \in \Gamma$, $\quad T_{\alpha, Tz_0} = T\, T_{\alpha, z_0} T^{-1}$.

Proof: Let $z_1 = Tz_0$. $T\tilde{\alpha}$ is the lift of α starting at z_1, so by definition $T_{\alpha,z_1} z_1 = T\tilde{\alpha}(1) = T\, T_{\alpha,z_0} z_0$. Hence $T_{\alpha,z_1} T z_0 = T\, T_{\alpha,z_0} z_0$ which leads to the result since Γ acts freely on $\mathbb{H}$.



Let us now study the dependency of T_{α,z_0} on α.

Definition: α, β are homotopic with fixed endpoints if there exists $h : [0,1] \times [0,1] \to X$ continuous such that, for all $t \in [0,1]$, $h(0,t) = \alpha(t)$ and $h(1,t) = \beta(t)$, and $s \in [0,1] \mapsto h(s,0)$ and $h(s,1)$ are constant.



two paths α, β on X homotopic with fixed endpoints

Lemma: Let α, β be two closed paths on X with the same basepoints. Then,

α and β are homotopic with fixed endpoints

$\Leftrightarrow \forall z_0 \in \pi^{-1}(\alpha(0)),\ T_{\alpha,z_0} = T_{\beta,z_0}$.

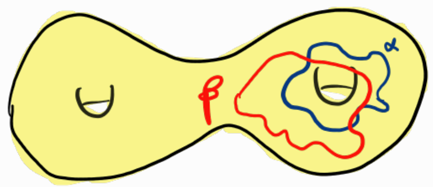
Proof: Exercise. Hint: lift the homotopy h.

We will actually care mostly about a coarser relation.

Definition: Two closed paths α, β are *freely homotopic* if there exists a continuous map $h : [0,1] \times [0,1] \longrightarrow Y$ such that:

$$\forall t \in [0,1], \quad h(0,t) = \alpha(t) \quad \text{and} \quad h(1,t) = \beta(t)$$

and $\forall s \in [0,1]$, $h(s,0) = h(s,1)$.



two freely homotopic closed paths α, β.

Exercise: Prove this is an equivalence relation.

We denote as $[\alpha]$ its equivalence classes.

Definition: For a closed path α on X we let

$$\mathcal{C}_\alpha = \left\{ T_{\alpha, z_0}, \ z_0 \in \pi^{-1}(\alpha(0)) \right\}.$$

By the previous observations, this is a conjugacy class.

Lemma: Let α, β be closed paths on X. Then,

α and β are freely homotopic

$\Leftrightarrow$ $\mathcal{C}_\alpha = \mathcal{C}_\beta$.

Proof: Exercise.

Conclusion: We have the identification

$$\{\text{free homotopy classes on } X\} \leftrightarrow \{\text{conjugacy classes of } \Gamma\}$$
$$[\alpha] \mapsto \mathcal{C}_\alpha$$

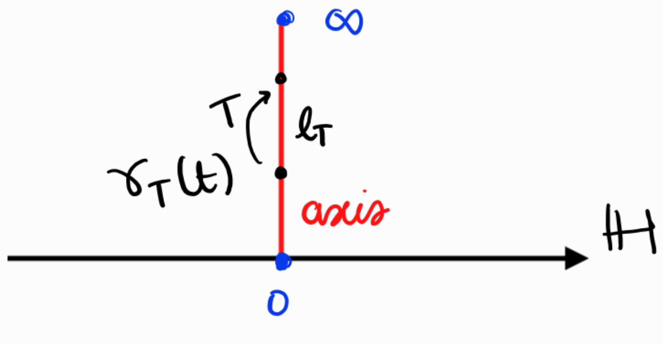
(c) Closed geodesics

Let us now finally fulfil the second goal of this session and talk about closed geodesics. These will correspond to hyperbolic elements of Γ.

Definition: Let $T \in PSL(2;\mathbb{R})$ be hyperbolic. We call the geodesic γ_T connecting its two fixed points at infinity its axis, parametrised by unit speed.

Example: For $T = \begin{pmatrix} e^{\ell/2} & 0 \\ 0 & e^{-\ell/2} \end{pmatrix}$ the axis is

$$\gamma_T : \begin{cases} \mathbb{R} \longrightarrow \mathbb{H} \\ t \mapsto ie^t \end{cases}$$



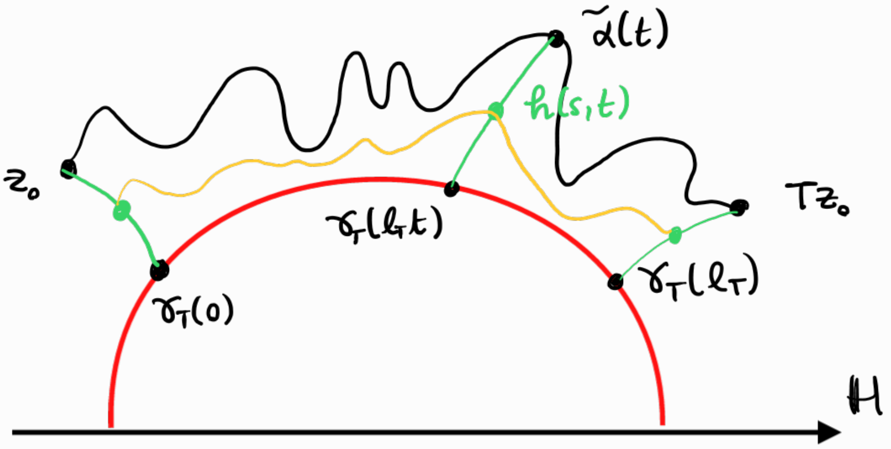
Lemma: Let $T \in PSL(2;\mathbb{R})$ be hyperbolic. Then T acts on its axis by translation of size $\ell_T = 2\,\mathrm{arccosh}\left(\frac{1}{2}|\mathrm{Tr}\,T|\right)$, ie $\forall t,\ T\gamma_T(t) = \gamma_T(t+\ell_T)$.

Proof: Exercise, using the normal form of hyperbolic elements.

Let us prove the following key result.

Theorem: Let α be a closed path on X, and $T \in \mathcal{C}_\alpha$.
Assume T is hyperbolic. Then there exists a unique length-minimiser in $[\alpha]$, which is exactly $t \in [0,1] \mapsto \pi(\gamma_T(\ell_T t))$.
Its length ℓ satisfies
$$\cosh\left(\frac{\ell}{2}\right) = \frac{1}{2}|\mathrm{Tr}\, T|.$$

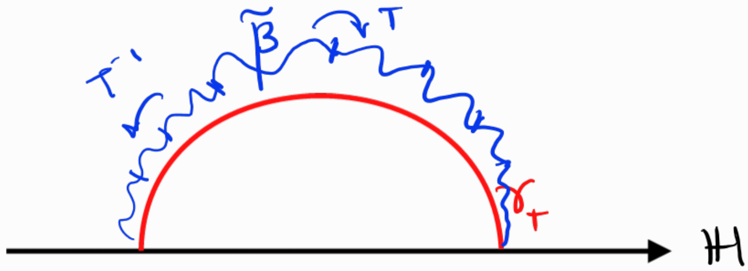
Proof: Let us show that α is freely homotopic to $t \in [0,1] \mapsto \pi(\gamma_T(\ell_T t))$.
Since $T \in \mathcal{C}_\alpha$, $T = T_{\alpha, z_0}$ for a $z_0 \in \pi^{-1}(\alpha(0))$.
Let $\tilde{\alpha}$ be the lift of α starting at z_0, which ends at Tz_0.



For $t \in [0,1]$ we parametrize the geodesic from $\tilde{\alpha}(t)$ to $\gamma_T(\ell_T t)$ by $h(\cdot, t) : [0,1] \to \mathbb{H}$. Then, $h : [0,1] \times [0,1] \to \mathbb{H}$ is continuous, and $h(0,t) = \tilde{\alpha}(t)$, $h(1,t) = \gamma_T(\ell_T t)$.

Furthermore, $Th(s,0) = h(s,1)$ for all s since the isometry T sends z_0 on $Tz_0 = \tilde{\alpha}(1)$ and $\gamma_T(0)$ on $\gamma_T(\ell_T)$.
Hence $\pi \circ h$ is a free homotopy on X.

We have proven that the axis of T belongs in $[\alpha]$.
Let us now justify it is the only length-minimizer.
Let β be a length-minimiser and $\tilde{\beta}$ be its lift starting at z_0.
We extend it to an infinite lift by letting $\tilde{\beta}(t) = T^k \tilde{\beta}(u)$ if $t = k + u$, $k \in \mathbb{Z}$, $u \in [0,1)$.



By length-minimizing, $\tilde{\beta}$ is a geodesic on $\mathbb{H}$.
For all $t \in \mathbb{R}$, $t = k + u$,

$$d(\tilde{\beta}(t), \gamma_T(t\ell_T)) = d(T^k \tilde{\beta}(u), T^k \gamma_T(\ell_T u)) = d(\tilde{\beta}(u), \tilde{\gamma}_T(\ell_T u))$$

is bounded. This implies that $\tilde{\beta}$ and γ_T coincide up to parametrization, which allows to conclude.

<u>Conclusion:</u> $\{$hyperbolic conjugacy classes$\} \leftrightarrow \{$closed geodesics$\}$.

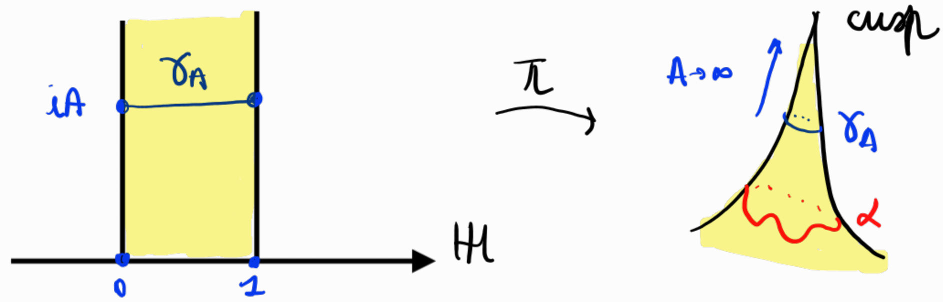
(d) Other conjugacy classes

What happens when T is not hyperbolic?

<u>Exercise:</u> Show that if α is a closed path and $\mathcal{C}_\alpha = \{id\}$ then α is contractible.

<u>Exercise:</u> Let α be a closed path such that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathcal{C}_\alpha$. Let $A > 0$.

1. Show that α is freely homotopic to $\gamma_A : t \in [0,1] \mapsto t + iA$.
2. Show that $\lim_{A \to \infty} \ell(\gamma_A) = 0$.



α is freely homotopic to neighbourhood of a cusp