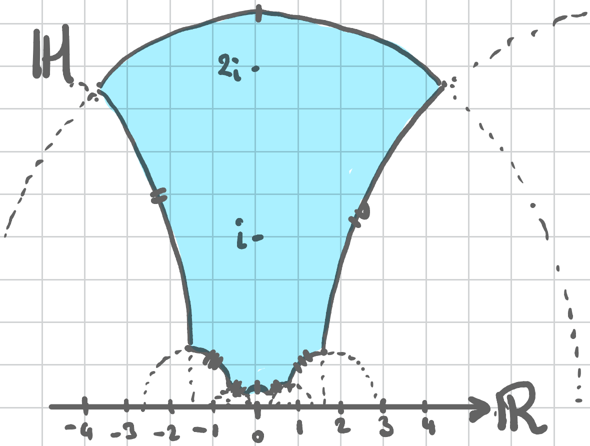
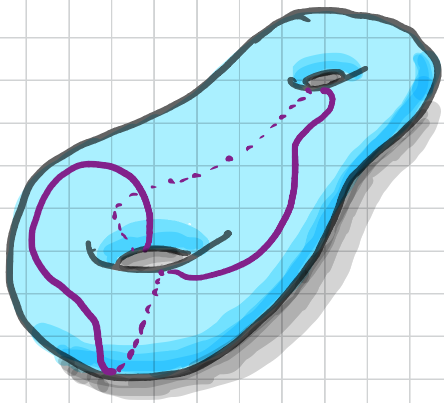
TCC: Hyperbolic surfaces, their length spectra and connections to Markoff triples





$$x^2+y^2+z^2=3xyz$$

- (1,1,1) — (1,1,2) — (1,2,5)
  - (1,5,13)
    - (1,13,34) ...
    - (5,13,194) ...
  - (2,5,29)
    - (5,29,433) ...
    - (2,29,169) ...

Plan for the course:

1. The hyperbolic plane, its isometries and geodesics
2. Hyperbolic surfaces and length spectra
3. Pairs of pants decompositions, Fenchel-Nielsen coordinates and rigidity of the length spectrum
4. Isospectral hyperbolic surfaces
5. Multiplicities in the length spectrum and trace equalities
6. Markoff triples and geodesics in hyperbolic tori
7. A proof of McShane's identity via Markoff triples
8. Counting Markoff triples and simple closed geodesics on hyperbolic tori

Lecture 1: The hyperbolic plane

We will focus on the upper half-plane model.

Let $\mathbb{H} = \mathbb{H}^2 := \{ z \in \mathbb{C} \mid \operatorname{Im}(z) > 0 \}$.

We call this the upper half-plane.

As a subset of $\mathbb{C}$, $\mathbb{H}$ inherits

1. A smooth structure.

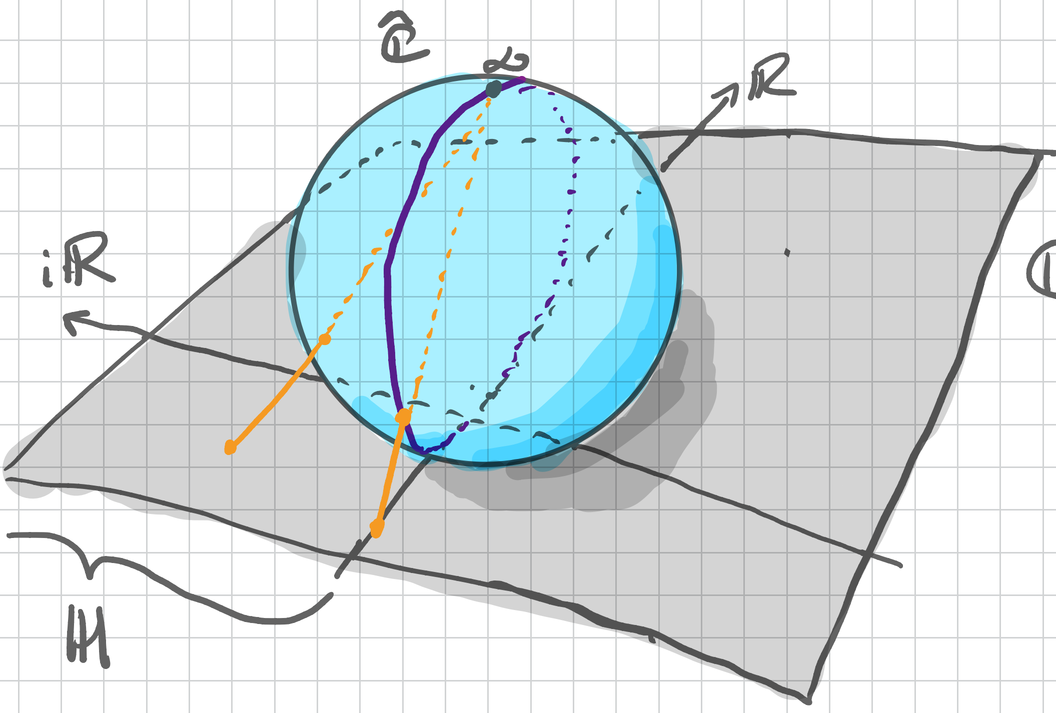
2. A complex structure.

We will consider the automorphisms of these structures of $\mathbb{H}$ and determine the Riemannian metrics on $\mathbb{H}$ (conformal to the Euclidean metric) that are preserved by these automorphisms.

The metric of constant curvature -1 in this family will be the hyperbolic metric on $\mathbb{H}$.

To simplify the proofs, we consider $\mathbb{H}$ as a subset of the Riemann sphere

$$\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}.$$



A Möbius transformation is a map $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which, for $a,b,c,d \in \mathbb{C}$, $ad-bc \neq 0$, is defined by

$$f(z) = \frac{az+b}{cz+d}, \quad z \in \mathbb{C}$$

$$f(w) = \frac{a+bw}{c+dw}, \quad w = \frac{1}{z} \text{ in a neighbourhood of } z = \infty.$$

Exercise

Prove that Möbius transformations form a group under composition generated by

i. $z \mapsto az+b$, $a,b \in \mathbb{C}$, $a \neq 0$, $(a,b) \neq (0,0)$

ii. $z \mapsto \frac{1}{z}$.

__Note__ By simultaneously scaling a,b,c and d, we can assume that $ad-bc=1$.

Prop The group of Möbius transformations of $\hat{\mathbb{C}}$, $\text{Möb}(\mathbb{C})$, is isomorphic to

$$PSL(2,\mathbb{C}) \cong SL(2,\mathbb{C}) / \langle \pm I \rangle$$

Proof: The map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{az+b}{cz+d}$$

is a surjective homomorphism with kernel $\{\pm I\}$. $\square$

Exercise

Prove that $PSL(2,\mathbb{C})$ acts transitively but not freely on $\hat{\mathbb{C}}$ and on pairs of distinct points.

transitive $\Leftrightarrow$ $\forall x, y \in \hat{\mathbb{C}}$, $\exists A \in PSL(2,\mathbb{C})$ s.t. $A \cdot x = y$.

free $\Leftrightarrow$ $\forall A \in PSL(2,\mathbb{C})$, $A \neq I \Rightarrow A \cdot x \neq x \ \forall x$.

Prop $\text{Möb}(\hat{\mathbb{C}})$ acts transitively and freely on triples of points in $\hat{\mathbb{C}}$.

Proof It suffices to send any triple of pts to $(0,1,\infty)$ and to prove that any map fixing $(0,1,\infty)$ is the identity.

i. Consider (z_1,z_2,z_3). Then

$$f(z) = \frac{(z_2-z_3)z - z_1(z_2-z_3)}{(z_2-z_1)z - z_3(z_2-z_1)}$$

sends (z_1,z_2,z_3) to $(0,1,\infty)$.

ii. Now, if $f(z)=\frac{az+b}{cz+d}$ fixes $(0,1,\infty)$ we see that $c=0=b$ and $\frac{a}{d}=1$. So $f(z)=z$. $\square$

Recall that $\hat{\mathbb{C}}$ inherits an inner product on tangent spaces from the Euclidean inner product $\langle\cdot,\cdot\rangle_{\mathbb{R}^2}$ on $\mathbb{C}\cong\mathbb{R}^2$. Hence, we can measure angles between tangent vectors at a point $z\in\hat{\mathbb{C}}$.

(We can only measure angles! We do not get a global metric)

A smooth map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is conformal if it preserves angles between tangent vectors.

That is, $\forall z \in \hat{\mathbb{C}}$, $u, v \in T_z \hat{\mathbb{C}}$,

$$\langle Df_z(u), Df_z(v) \rangle_{T_{f(z)}\hat{\mathbb{C}}} = \lambda^2(z) \langle u, v \rangle_{T_z \hat{\mathbb{C}}}$$

for some smooth $\lambda(z) > 0$.

Let

$$\mathrm{Aut}(\hat{\mathbb{C}}) := \{\text{conformal bijections of } \hat{\mathbb{C}}\}.$$

as a group.

<u>Prop</u>

$$\mathrm{Aut}(\hat{\mathbb{C}}) \cong \text{Möb}(\hat{\mathbb{C}}) \cong PSL(2, \mathbb{C})$$

<u>Proof</u>:

<u>$\supseteq$:</u> Let $f(z) = \frac{az+b}{cz+d} = u(x,y) + iv(x,y)$. Then we have

$$Df_z = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \overset{\substack{\text{Cauchy} \\ \text{Riemann}}}{=} \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} = \begin{pmatrix} \mathrm{Re}(f'(z)) & -\mathrm{Im}(f'(z)) \\ \mathrm{Im}(f'(z)) & \mathrm{Re}(f'(z)) \end{pmatrix}$$

$$= |f'(z)| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for $\theta = \operatorname{Arg}(f'(z))$.

Hence, since $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2,\mathbb{R})$,

$$\langle Df_z(u), Df_z(v)\rangle = |f'(z)|^2 \langle u, v\rangle$$

and $|f'(z)|^2 = \dfrac{|ad-bc|^2}{|cz+d|^4} > 0$.

$\subseteq$: Let $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a conformal automorphism. By composing with Möbius transformations we can assume that $f(0)=0$ and $f(\infty)=\infty$.

We can then show that $g(z) = \dfrac{f(z)}{z}$ is entire and bounded on $\mathbb{C}$. Hence, by Liouville's theorem, $g(z) = c$ for some $c \in \mathbb{C}^*$. Therefore, $f(z) = z \cdot g(z)$ is linear and hence Möbius. $\square$

A circle in $\hat{\mathbb{C}}$ is the locus given by

$|z-a| = r$ for some $r \in \mathbb{R}_{>0}$.

Now

$$|z-a| = r \iff |z-a|^2 = r^2$$

$$\iff (z-a)(\bar{z}-\bar{a}) = r^2$$

$$\iff |z|^2 - \bar{a}z - a\bar{z} + (|a|^2 - r^2) = 0.$$

So a circle has the form

$$A|z|^2 + Bz + \bar{B}\bar{z} + C = 0, \quad A, C \in \mathbb{R}, B \in \mathbb{C},$$

with $A \neq 0$ and $|B|^2 - AC > 0$.

A line in $\hat{\mathbb{C}}$ has the form $\mathrm{Im}(z) = \lambda \cdot \mathrm{Re}(z) + d$

So,

$$\frac{z-\bar{z}}{2i} = \lambda \frac{z+\bar{z}}{2} + d \iff (\lambda i - 1)z + (\lambda i + 1)\bar{z} + 2id = 0$$

$$\iff (\lambda + i)z + (\lambda - i)\bar{z} + 2d = 0.$$

So a line has the form

$$Bz + \bar{B}\bar{z} + C = 0, \quad B \in \mathbb{C}, C \in \mathbb{R}.$$

Prop $\mathrm{Aut}(\hat{\mathbb{C}})$ sends circles and lines to circles and lines.

Proof We have seen that $\mathrm{Aut}(\hat{\mathbb{C}})$ is generated by $z \mapsto az+b$ and $z \mapsto \frac{1}{z}$. It can then be checked that the form $A|z|^2 + Bz + \overline{B}\overline{z} + C = 0$ maps to $A'|z|^2 + B'z + \overline{B'}\overline{z} + C' = 0$ for some other $A', C' \in \mathbb{R}$, $B' \in \mathbb{C}$. So we are done. $\square$

Now, we restrict to $\mathbb{H}$.

Let $\mathrm{Aut}(\mathbb{H}) := \{ f \in \mathrm{Aut}(\hat{\mathbb{C}}) : f(\mathbb{H}) = \mathbb{H} \}$.

Prop $\mathrm{Aut}(\mathbb{H}) \cong PSL(2,\mathbb{R}) \cong SL(2,\mathbb{R}) / \{\pm I\}$.

Proof Let $\partial\mathbb{H} := \mathbb{R} \cup \{\infty\} \subseteq \hat{\mathbb{C}}$. Then an $f \in \mathrm{Aut}(\mathbb{H})$ must satisfy $f(\partial\mathbb{H}) = \partial\mathbb{H}$. So, if $f(z) = \frac{az+b}{cz+d}$, then

$$f(0) = \frac{b}{d} \in \partial\mathbb{H}, \quad f(1) = \frac{a+b}{c+d} \in \partial\mathbb{H}, \quad f(\infty) = \frac{a}{c} \in \partial\mathbb{H}.$$

This implies that $\exists \lambda \in \mathbb{C}\setminus\{0\}$ such that $a' = \lambda a$, $b' = \lambda b$, $c' = \lambda c$ and $d' = \lambda d$ are all real and $f(z) = \dfrac{a'z+b'}{c'z+d'}$, with $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in PSL(2,\mathbb{R})$.

Conversely, note that if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$, then

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{|ad-bc|}{|cz+d|^2} \cdot \operatorname{Im}(z).$$

So $\operatorname{Im}(z) > 0 \Rightarrow \operatorname{Im}\left(\frac{az+b}{cz+d}\right) > 0$. So $\mathbb{H}$ is preserved bijectively. Hence, $\frac{az+b}{cz+d} \in \operatorname{Aut}(\mathbb{H})$.

$\square$

We now focus on finding metrics on $\mathbb{H}$ invariant by $PSL(2,\mathbb{R})$.

Recall that the Euclidean metric on $\mathbb{C} \cong \mathbb{R}^2$ is given by

$$ds^2 = dx^2 + dy^2 = |dz|^2.$$

This gives the standard inner product on tangent spaces.

A Riemannian metric on $\mathbb{H}$ is conformal to the Euclidean metric if

$$ds = \lambda(z)\,|dz|$$

for some smooth $\lambda(z) > 0$.

Prop Up to scale, there is a unique conformal metric on $\mathbb{H}$ invariant by $\mathrm{Aut}(\mathbb{H}) \cong PSL(2,\mathbb{R})$. Namely,

$$ds^2 = \frac{|dz|^2}{\mathrm{Im}(z)^2} = \frac{dx^2+dy^2}{y^2}.$$

Proof:

For $f \in \mathrm{Aut}(\mathbb{H})$ we require

$$\lambda(f(z)) \cdot |f'(z)| \cdot |dz| = \lambda(z)\,|dz|.$$

Consider $f(z) = vz + u$. At $z = i$, we require

$$\lambda(u+vi) \cdot v = \lambda(i).$$

Scaling λ so that $\lambda(i) = 1$, we get the functional equation

$$\lambda(z) = \frac{1}{\mathrm{Im}(z)}.$$

We can check that this is also invariant under $z \mapsto -1/z$, and so we are done. $\square$

Recall that a Riemannian metric of the form $ds^2 = g_1 dx^2 + g_2 dy^2$ has Gaussian curvature given by

$$K = \frac{-1}{\sqrt{g_1 g_2}}\left[\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{g_1}}\frac{\partial}{\partial x}\sqrt{g_2}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{g_2}}\frac{\partial}{\partial y}\sqrt{g_1}\right)\right].$$

Since

$$ds^2 = \frac{|dz|^2}{\mathrm{Im}(z)^2} = \frac{dx^2 + dy^2}{y^2},$$

we have $g_1 = g_2 = \frac{1}{y^2}$, so all $\frac{\partial}{\partial x}$ are 0 and

$$K = \frac{-1}{1/y^2}\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}\left(\frac{1}{y}\right)\right)$$

$$= -y^2 \frac{\partial}{\partial y}\left(-\frac{1}{y}\right)$$

$$= -y^2 \cdot \frac{1}{y^2} = -1.$$

So the metric has constant curvature -1.

Prop In fact, any Riemannian metric on $\mathbb{H}$ invariant by $PSL(2,\mathbb{R})$ must already be conformal.

<u>Proof</u>

The stabiliser in $PSL(2,\mathbb{R})$ of i can be checked to be $SO(2,\mathbb{R})$. In particular, $f = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$ stabilises i. We have $Df_i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. So $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}$ is invariant at i if and only if $Df_i^T g \, Df_i = g \Leftrightarrow g = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{11} \end{pmatrix}$.

That is, $ds^2 = g_{11}(dx^2 + dy^2)$.

Transitivity of the action of $PSL(2,\mathbb{R})$ then implies the same globally.

So $ds^2 = \lambda^2(z)|dz|^2$ and so is indeed conformal. $\square$

The Riemannian metric

$$ds^2 = \frac{|dz|^2}{\mathrm{Im}(z)^2} = \frac{dx^2+dy^2}{y^2}$$

gives a length metric on $\mathbb{H}$ defined by

$$d_{\mathbb{H}}(z,w) = \inf \left\{ \ell(\gamma) \;\middle|\; \begin{array}{l} \gamma:[a,b] \to \mathbb{H} \text{ piecewise } C^1 \\ \gamma(a)=z,\ \gamma(b)=w \end{array} \right\}$$

where

$$\ell(\gamma) = \int_a^b \frac{1}{\mathrm{Im}(\gamma(t))} |\gamma'(t)|\,dt$$

$$= \int_a^b \frac{1}{y(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

if $\gamma(t) = x(t) + i\,y(t)$.

Let $f:\mathbb{H} \to \mathbb{H}$ be a smooth bijection. Then f is an orientation preserving isometry if

$\det(Df_z) > 0 \ \forall z \in \mathbb{H}$, $d_{\mathbb{H}}(f(z), f(w)) = d_{\mathbb{H}}(z,w)$

$\forall z, w \in \mathbb{H}$.

Let $\mathrm{Isom}^+(\mathbb{H})$ be the group of orientation preserving isometries.

By construction, $PSL(2,\mathbb{R}) \subseteq \mathrm{Isom}^+(\mathbb{H})$.

We shall prove the reverse inclusion.

We first consider geodesics.

Recall that geodesics are paths that locally minimise distance.

Prop

Vertical lines in $\mathbb{H}$ are geodesics. Moreover, if $b > a$, then $d_{\mathbb{H}}(ai, bi) = \log \frac{b}{a}$.

Proof: Consider any path $\gamma(t) = x(t) + iy(t)$, $t \in [t_0, t_1]$, joining ai to bi. Then

$$\ell(\gamma) = \int_{t_0}^{t_1} \frac{1}{y(t)} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\geq \int_{t_0}^{t_1} \frac{1}{y(t)} \frac{dy}{dt}\, dt$$

$$= \int_a^b \frac{1}{y}\, dy = \log \frac{b}{a}$$

with equality if and only if $\frac{dx}{dt} = 0$; i.e. $\gamma(t)$ is the vertical line. So the vertical line minimises distance locally (and globally) and we are done. $\square$

Prop Let $z, w \in \mathbb{H}$. If $\mathrm{Re}(z) = \mathrm{Re}(w)$, then the geodesic from z to w is the vertical line from z to w, otherwise it is the arc of the circle with centre on $\partial\mathbb{H}$ joining z to w.

__Proof__

In the first case, applying the isometry $f(u) = u - \mathrm{Re}(z)$ moves z and w to $i\mathbb{R}$ and the result follows from the previous proposition.

Otherwise, let $\mu < \eta$ be the endpoints of the semi-circle with centre on $\partial\mathbb{H}$ joining z to w. Applying the isometry

$$f(u) = \frac{u-\eta}{u-\mu} \qquad \left(\text{note } \begin{vmatrix} 1 & -\eta \\ 1 & -\mu \end{vmatrix} = \eta - \mu > 0\right)$$

sends η to 0 and μ to ∞. Hence, since $PSL(2,\mathbb{R}) \subseteq PSL(2,\mathbb{C})$ sends circles and lines to circles and lines and preserves angles, the semicircle is sent to $i\mathbb{R}$, the unique geodesic between $f(z)$ and $f(w)$. So the original semicircle was the unique geodesic between z and w. $\square$

Prop

Given $z, w \in \mathbb{H}$,

$$\cosh(d_{\mathbb{H}}(z,w)) = 1 + \frac{|z-w|^2}{2\,\mathrm{Im}(z)\,\mathrm{Im}(w)}.$$

Proof Both sides are invariant under $PSL(2,\mathbb{R})$ and the formula holds for $z = ai$ and $w = bi$. $\square$

Exercise: Use this formula to show that hyperbolic circles in $\mathbb{H}$ are also Euclidean circles (with different centres).

Hint First consider circles around i.

Prop $\mathrm{Isom}^+(\mathbb{H}) \subseteq PSL(2,\mathbb{R})$

Proof

Let $T \in \mathrm{Isom}^+(\mathbb{H})$. Suppose $T(i) = z$ and $T(2i) = w$. Consider the map $f_1 \in PSL(2,\mathbb{R})$ sending the geodesic between z and w to $i\mathbb{R}$. Now, if $f_1(z) = ai$ apply $f_2(u) = \frac{u}{a}$.

So $f_2 \circ f_1 \circ T(i) = i$ and, by isometries,

$$d_{\mathbb{H}}(f_2 \circ f_1 \circ T(i), f_2 \circ f_1 \circ T(2i)) = \log 2$$

so $f_2 \circ f_1 \circ T(2i) \in \{2i, \frac{i}{2}\}$.

So, if needed apply $f_3(u) = \frac{-1}{u}$, so that one of $f_2 \circ f_1 \circ T$ and $f_3 \circ f_2 \circ f_1 \circ T$ fixes both i and $2i$. Call this map S.

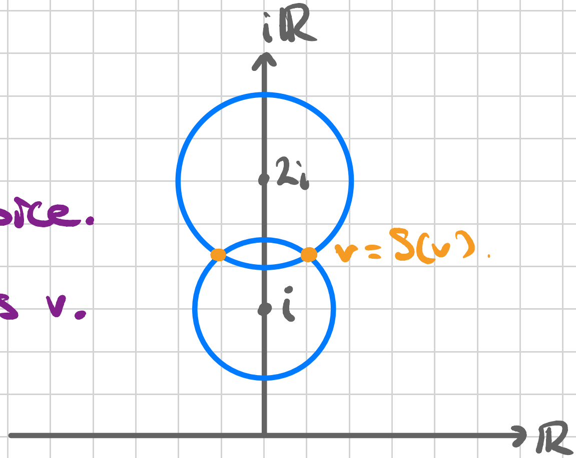
Now consider some $v \in \mathbb{H}$. We have

$$r_1 := d_{\mathbb{H}}(i, v) = d_{\mathbb{H}}(i, S(v))$$

and

$$r_2 := d_{\mathbb{H}}(2i, v) = d_{\mathbb{H}}(2i, S(v)).$$

So v and $S(v)$ both lie on the hyperbolic circles of radius r_1 about i and r_2 about $2i$.



These are Euclidean circle intersecting twice.

One of these points is v.

But, since S is orientation preserving v and $S(v)$ must lie on the same side of $i\mathbb{R}$. So we have $S(v) = v$ and so $S = I \Rightarrow T \in PSL(2, \mathbb{R})$.

$\square$

So, we have achieved the following:

Thm

$$\mathrm{Aut}(\mathbb{H}) \cong \mathrm{Isom}^+(\mathbb{H}) \cong PSL(2, \mathbb{R})$$

We will now classify elements of $PSL(2,\mathbb{R})$ by their fixed points. The key information will be the trace $\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+d$.

Suppose, $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R})$ and

$$z = f(z) = \frac{az+b}{cz+d}.$$

Then

$$cz^2 + (d-a)z - b = 0$$

$$\Rightarrow \quad z = \frac{a-d \pm \sqrt{(a-d)^2 + 4bc}}{2c}$$

$$= \frac{a-d \pm \sqrt{a^2 - 2ad + d^2 + 4bc}}{2c}$$

$$= \frac{a-d \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2c}$$

$$= \frac{a-d \pm \sqrt{\operatorname{tr}(T)^2 - 4}}{2c}.$$

So we see the following.

Classification

(1) If $\mathrm{tr}(T) = \pm 2$, then $T = \mathrm{id}$ or T has a single fixed point in $\partial\mathbb{H}$. We call T parabolic.

(2) If $|\mathrm{tr}(T)| > 2$, then T has two fixed points in $\partial\mathbb{H}$. We call T hyperbolic.

(3) If $|\mathrm{tr}(T)| < 2$, then T has two complex conjugate fixed pts in $\hat{\mathbb{C}}$ and so a single fixed point in $\mathbb{H}$. We call T elliptic.

Each type has a canonical element in its conjugacy class.

We do this by moving the fixed points to ∞, $(0, \infty)$, or i, in each case.

Parabolics:

Let η be the fixed point in $\partial\mathbb{H}$. We conjugate by $f(z) = \dfrac{-1}{z-\eta}$ to obtain a map T fixing ∞. Hence, $T(z) = \dfrac{az+b}{cz+d}$ has $c=0$ and $ad-bc = ad = 1$. So $T(z) = z + \frac{b}{d}$.

Hyperbolics:

Conjugate by $f(z) = \dfrac{z-\eta^+}{z-\eta^-}$, $\eta^\pm$ fixed in $\partial\mathbb{H}$ to get a map T fixing 0 and ∞. Then $T(z) = \dfrac{az+b}{cz+d}$ has the form $T(z) = \lambda z$ for some $\lambda > 0$. So $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}$.

T sends $z = ai$ to $\lambda z = \lambda ai$ hence translates along $i\mathbb{R}$ by $d_{\mathbb{H}}(z, \lambda z) = \log\lambda$.

Observe that

$$2\cosh\left(\frac{\log\lambda}{2}\right) = \operatorname{tr} T.$$

Elliptic:

Let $\eta \in \mathbb{H}$ and $\bar{\eta} \in \hat{\mathbb{C}} \setminus \mathbb{H}$ be the fixed points. Conjugate by $f(z) = \dfrac{z - \mathrm{Re}(\eta)}{\mathrm{Im}(\eta)}$ to obtain a map with fixed points at $\pm i$. We saw earlier that the stabiliser of i is $SO(2, \mathbb{R})$. So

$$T(z) = \frac{\cos\theta \cdot z + \sin\theta}{-\sin\theta \cdot z + \cos\theta}$$

for some θ.

Then $T'(i) = \dfrac{1}{(-\sin\theta\, i + \cos\theta)^2} = e^{2i\theta}$.

So T rotates about i by an angle of 2θ.

Notice, this angle of rotation ψ satisfies

$$2\cos(\psi/2) = \operatorname{tr} T.$$