DIAMETER BOUNDS FOR $\mathrm{SL}(2,\mathbb{Z})$-ORBITS OF ORIGAMIS IN $\mathcal{H}(2)$ AND THE PRYM LOCI IN $\mathcal{H}(4)$ AND $\mathcal{H}(6)$

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ABSTRACT. Using algorithms implicit in the classification of $\mathrm{SL}(2,\mathbb{Z})$-orbits of primitive origamis in the stratum $\mathcal{H}(2)$ due to Hubert-Lelièvre and McMullen, we give diameter bounds on the resulting orbit graphs. Since the machinery of McMullen from $\mathcal{H}(2)$ is generalised and reused in Lanneau and Nguyen's classification of the orbits of Prym eigenforms in $\mathcal{H}(4)$ and $\mathcal{H}(6)$, we are also able to obtain diameter bounds for the orbit graphs in this setting as well. In each stratum, we obtain diameter bounds of the form $O(N^{2/3}\log N)$, where N is the size of the orbit graph.

1. INTRODUCTION

A square-tiled surface is an orientable connected surface obtained from a finite cover of the unit square torus $\mathbb{T}^2 := \mathbb{C}/(\mathbb{Z}\oplus i\mathbb{Z})$ which is possibly branched only at $0\in\mathbb{T}^2$. In the literature, square-tiled surfaces are also called origamis (for short).

Origamis play a special role in the study of the natural $\mathrm{SL}(2,\mathbb{R})$ action on moduli spaces of Abelian differentials. For example, since they correspond to integral points of such moduli spaces, one can compute Masur–Veech volumes of these spaces by counting origamis (cf. [Zo] and [EsOk]). Moreover, their $SL(2,\mathbb{R})$ orbits are closed subvarieties of the moduli spaces known as *arithmetic Teichmüller curves*. We refer the reader to the recent book [AtMa] and survey [Fil] for further explanations about moduli spaces of Abelian differentials and its connections to several topics in Mathematics (including Dynamics and Hodge Theory).

As it turns out, origamis can be organised into $\mathrm{SL}(2,\mathbb{Z})$ orbits leading to Schreier graphs sitting on the corresponding arithmetic Teichmüller curves. In this context, McMullen conjectured that the family of such Schreier graphs associated to primtive origamis of genus 2 with a single conical singularity is expander. In a previous paper [JM], we gave indirect evidence towards McMullen's conjecture by showing that these graphs are non-planar. In the present paper, we give another indirect evidence by establishing the following bound on the diameter of these graphs:

Theorem 1.1. *If X is a primitive origami in the stratum $\mathcal{H}(2)$ of the moduli space of translation surfaces of genus two with a single conical singularity, then the graph $\mathcal{G}(X)$ associated to its $SL(2,\mathbb{Z})$-orbit has a diameter $O(|V|^{2/3}\log|V|) = O(n^2\log n)$, where $|V|$ is the number of vertices of $\mathcal{G}(X)$ and n is the number of unit squares tiling X (i.e., the degree of the branched covering map $X\to\mathbb{T}^2$ defining X).*

This article is organised along the following lines. In Section 2, we quickly review some basic material about moduli spaces of translation surfaces, $\mathrm{SL}(2,\mathbb{Z})$ orbits of primitive origamis, and the works of Hubert–Lelièvre and McMullen on origamis in the minimal stratum of translation surfaces of genus two. In Section 3, we employ Hubert–Lelièvre's method to establish a weaker version of Theorem 1.1, namely, a bound of

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$O(n^{5/2})$ on the diameter of the graph associated to the $\mathrm{SL}(2,\mathbb{Z})$ orbit of an origami in $\mathcal{H}(2)$ tiled by a prime number n of unit squares. In Section 4, we prove Theorem 1.1 using McMullen's results on butterfly moves. Finally, we take advantage of the results of Lanneau and Nguyen extending the technology of butterfly moves to origamis in the Prym loci of the minimal strata of moduli spaces of translation surfaces in genus 3 and 4 (resp.) in order to establish in Sections 5 and 6 (resp.) the analogs of Theorem 1.1 in these settings.

Remark 1.2. *The diameter bounds in this paper are far from optimal: for instance, a positive answer to McMullen's conjecture would imply a diameter bound of the form $O(\log n)$. For this reason, we included short subsections at the end of Sections 3 and 4 where we indicate potential sources of improvements (partly supported by numerical experiments) in the basic arguments.*

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2. Preliminaries

Here, we remind the reader of the necessary background on origamis and their $\mathrm{SL}(2,\mathbb{Z})$-orbits as well as the language required to discuss the works of Hubert-Lelièvre and McMullen.

2.1. **Translation surfaces and their moduli spaces.**

A compact Riemann surface M of genus $g \geq 1$ equipped with a non-trivial Abelian differential ω is called a translation surface. This nomenclature is justified by the fact that the local primitives of ω away from the set Σ of its zeroes yields a collection of charts on $M \setminus \Sigma$ whose changes of coordinates are given by translations of the complex plane $\mathbb{C}$. By the Riemann-Hurwitz theorem, ω has $2g-2$ zeroes counted with multiplicities, i.e., if $\sigma = |\Sigma|$ is the cardinality of Σ and $k_1, \ldots, k_\sigma$ are the vanishing orders of ω at the elements of Σ, then $\sum\limits_{j=1}^{\sigma} k_j = 2g-2$.

By gathering together translation surfaces $X = (M, \omega)$ with a prescribed list $\kappa = (k_1, \ldots, k_\sigma)$ of orders of zeroes of ω, one obtains a stratum $\mathcal{H}(\kappa)$ of the moduli space of Abelian differentials of genus g. This is a complex orbifold of dimension $2g+\sigma-1$ carrying a natural $SL(2,\mathbb{R})$ action (consisting of post-composing the local primitives of ω with the usual action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2 = \mathbb{C}$). For further information and references about these objects, the reader is encouraged to consult Athreya–Masur's book [AtMa].

2.2. **Origamis and their $\mathbf{SL(2,\mathbb{Z})}$-orbits.**

An origami (or square-tiled surface) is a translation surface $(M, \pi^*(dz))$ obtained from a finite cover $\pi: M \to \mathbb{T}^2 = \mathbb{C}/(\mathbb{Z}+i\mathbb{Z})$ possibly branched only at $0 \in \mathbb{T}^2$. Alternatively, an origami is constructed from a pair (h, v) of permutations acting transitively on n symbols by taking unit squares $sq(i)$, $i = 1, \ldots, n$, and gluing the rightmost vertical side of $sq(i)$ to the leftmost vertical side of $sq(h(i))$ and the topmost horizontal side of $sq(i)$ to the bottommost horizontal side of $sq(v(i))$. An origami is said to be primitive if it is not a cover of another origami different from itself or the unit-square torus. Equivalently, an origami is primitive if the group $\langle h, v\rangle$ is primitive as a permutation group.

The group $SL(2,\mathbb{Z})$ leaves invariant the set of primitive origamis. In fact, $SL(2,\mathbb{Z})$ is generated by the matrices

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and one can check that their actions on pairs of permutations are $T(h, v) = (h, vh^{-1})$ and $S(h, v) = (hv^{-1}, v)$ (see, for example, [JM, Figure 2.2]). In particular, the $SL(2,\mathbb{Z})$

orbits of origamis are coded by graphs whose vertices are the elements of these orbits and edges corresponding to elements deduced one from the other by applying T or S.

Remark 2.1. *Later, we will also use the matrix* $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *satisfying* $R = ST^{-1}S$. *Note that* R *rotates the origami by* $\frac{\pi}{2}$ *anti-clockwise.*

The origamis $\pi : (M, \omega) \to (\mathbb{T}^2, dz)$ studied in this paper come equipped with an involution ι of M taking ω to $-\omega$. In this case, one can count the number l_0 of fixed points of ι over $0 \in \mathbb{T}^2$ and the numbers l_1, l_2, l_3 of fixed points of ι over the other three 2-torsion points of $\mathbb{T}^2$. As it turns out, the pair $(l_0, [l_1, l_2, l_3])$, where $[l_1, l_2, l_3]$ is an unordered triple, is an invariant of the $SL(2, \mathbb{Z})$ orbit of (M, ω) called its HLK-invariant.

2.3. **Surface parameters.** The HLK-invariant was originally used by Hubert and Lelièvre [HL] to distinguish between two $SL(2, \mathbb{Z})$ orbits (called A and B) of primitive origamis in $\mathcal{H}(2)$ tiled by a prime number of unit squares. In general, by putting together the works [HL], [McM] and [LR], if X is a primitive origami in $\mathcal{H}(2)$ tiled by $n \geq 3$ squares, then

- X falls into a single $SL(2, \mathbb{Z})$ orbit whenever $n = 3$ or $n \geq 4$ is even, and
- X falls into one of two possible $SL(2, \mathbb{Z})$ orbits (called A and B) whenever $n \geq 5$ is odd: the A orbit has cardinality $\frac{3}{16}(n-1)n^2 \prod\limits_{p|n, p \text{ prime}} (1 - p^{-2})$ and the B orbit has cardinality $\frac{3}{16}(n-3)n^2 \prod\limits_{p|n, p \text{ prime}} (1 - p^{-2})$.

In the language of HLK-invariants, the orbit for even $n \geq 4$ corresponds to the HLK-invariant $(1, [2, 2, 0])$, the A orbit and the case of $n = 3$ corresponds to the HLK-invariant $(0, [3, 1, 1])$, and the B orbit corresponds to the HLK-invariant $(2, [1, 1, 1])$.

Moreover, one can exhibit explicit representatives of the A and B orbits by using the surface parameters introduced by Hubert–Lelièvre. More concretely, an origami X in $\mathcal{H}(2)$ tiled by n unit squares and decomposing into two cylinders in the horizontal direction is characterised by the widths w_1, w_2 and heights h_1, h_2 of these cylinders, and the twists t_1, t_2 (i.e., the relative positions of the conical singularity in the boundaries of these cylinders): cf. Figure 2.1 below. Note that these parameters satisfy $h_1 w_1 + h_2 w_2 = n$ (because the total area of X is n), and, if $n \geq 5$ is odd, one gets a representative of the A orbit, resp. B orbit, by setting $t_1 = 0 = t_2$, $h_1 = 1 = h_2$, $w_1 = 1$, $w_2 = n - 1$, resp. $t_1 = 0 = t_2$, $h_1 = 2$, $h_2 = 1$, $w_1 = 1$, $w_2 = n - 2$. For a primitive origami, we require $\gcd(h_1, h_2) = 1$.

Note that in all cases the size of the orbit is $O(n^3)$.

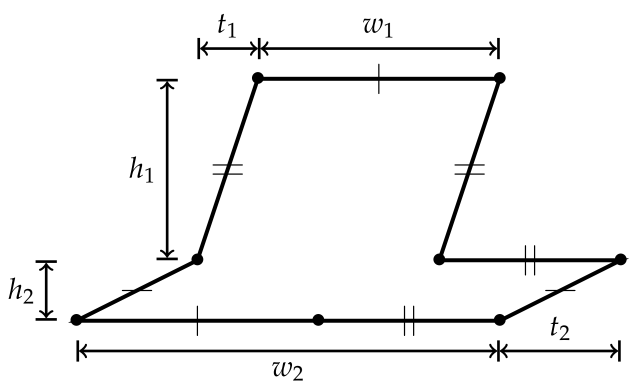

FIGURE 2.1. Two-cylinder surface parameters in $\mathcal{H}(2)$.

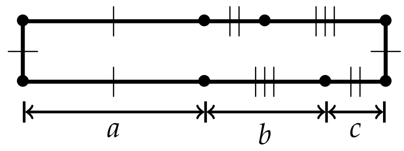

FIGURE 2.2. A one-cylinder cusp representative in $\mathcal{H}(2)$.

The *cusp* of an origami, X, is its orbit under the action of the horizontal shear T and the *cusp width* is the size of this orbit (i.e., the minimal i for which $T^i(X) = X$).

If X is a two-cylinder surface with parameters $(w_1, h_1, t_1, w_2, h_2, t_2)$ then it has cusp width

$$\operatorname{lcm}\left(\frac{w_1}{\gcd(w_1, h_1)}, \frac{w_2}{\gcd(w_2, h_2)}\right).$$

The cusp representative of X is the unique surface in the cusp with $0 \le t_i < \gcd(w_i, h_i)$ (cf. [HL, Lemma 3.1]).

Every cusp of a one-cylinder origami has a representative, as shown in Figure 2.2, with saddle connections of lengths $a, b, c > 0$ such that $n = a + b + c$ and $\gcd(a, b, c) = 1$. For $n \ge 4$, the cusp width is n (for $n = 3$ the surface with $(a, b, c) = (1, 1, 1)$ has cusp width 1). Note that the cusp representative has two cylinders in the vertical direction so that applying R (rotation by $\pi/2$) reaches a two-cylinder surface.

2.4. **Prototypes and butterfly moves.** A Teichmüller curve is an algebraic and isometric immersion of a finite-volume hyperbolic Riemann surface into the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g. McMullen [McM] classified all of the Teichmüller curves in genus two. The main source of such Teichmüller curves are the so-called Weierstrass curves W_D, parameterised by integers (called *discriminants*) $D \ge 5$ with $D \equiv 0$ or 1 modulo 4. These curves consist of those Riemann surfaces $M \in \mathcal{M}_2$ whose Jacobians admit real multiplication by the quadratic order $\mathcal{O}_D := \mathbb{Z}[x]/\langle x^2 + bx + c\rangle, b, c \in \mathbb{Z}$ with $D = b^2 - c$,and for which there exists a holomorphic one-form ω on M such that $(M, \omega) \in \mathcal{H}(2)$ and $\mathcal{O}_D \cdot \omega \subset \mathbb{C} \cdot \omega$ (such forms are said to be *eigenforms* for real multiplication by $\mathcal{O}_D$).

A translation surface $X = (M, \omega) \in \mathcal{H}(2)$ projects to W_{n^2} if and only if X is an n-squared origami. The classification of the $\mathrm{SL}(2, \mathbb{Z})$-orbits of primitive origamis in $\mathcal{H}(2)$ is equivalent to the fact that W_{n^2} is connected for $n = 3$ and $n \ge 4$ even, and has two connected components for $n \ge 5$ odd.

McMullen [McM, Section 3] defines a 4-tuple (a, b, c, e) to be a *prototype of discriminant* D if we have

- $D = e^2 + 4bc$;
- $0 < b, c$;
- $0 \le a < \gcd(b, c)$;
- $c + e < b$; and
- $\gcd(a, b, c, e) = 1$.

We collect these in the set $\mathcal{P}_D$. A prototype is said to be *reduced* if it has the form $(0, b, 1, e)$. The set of reduced prototypes is denoted by $\mathcal{S}_D$.

A prototype (a, b, c, e) corresponds to a prototypical splitting of an eigenform (M, ω) as a connected sum $(M, \omega) = (E_1, \omega_1) \underset{I}{\#} (E_2, \omega_2)$ as shown in Figure 2.3. Here, $\lambda = \frac{e+\sqrt{D}}{2}$.

Given a prototypical splitting of M as $E_1 \underset{I}{\#} E_2$, a *butterfly move* B_q, for some $q \in \{1, 2, \dots\} \cup \{\infty\}$, changes the splitting to $F_1 \underset{J}{\#} F_2$ with $J = [(b, 0) + q(a, c)]$ for finite q or $J = [(a, c)]$ for $q = \infty$. See Figure 2.3. However, this is only possible for certain admissible q.

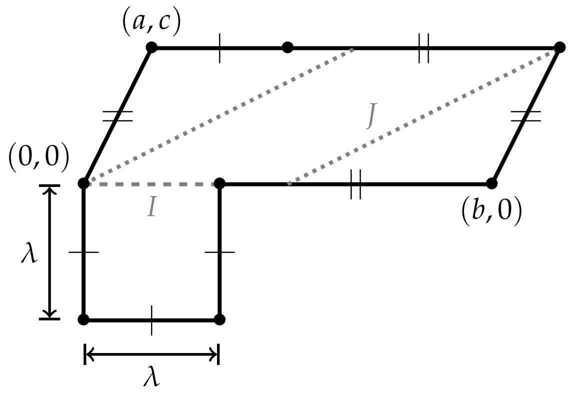

FIGURE 2.3. The surface associated to the prototypical splitting (a,b,c,e). The splitting realises the surface M as $E_1 \underset{I}{\#} E_2$. A butterfly move changes the splitting to $F_1 \underset{J}{\#} F_2$ with $J = [(b,0) + q(a,c)]$.

For a given prototype, the set of admissible q values is

$$\{q \mid q \in \mathbb{N}, (e+2qc)^2 < D\} \cup \{\infty\}.$$

Observe that $q = 1$ is always admissible since

$$(e+2c)^2 = e^2 + 4ec + 4c^2 = e^2 + 4(e+c)c < e^2 + 4bc = D.$$

Admissibility is also equivalent to $[(b,0) + q(a,c)]$ (or $[(a,c)]$ for $q = \infty$) not intersecting I.

McMullen's classification of $\mathrm{SL}(2,\mathbb{Z})$-orbits of origamis essentially relies on the fact that all prototypes can be connected by a sequence of butterfly moves.

Given a q value that is admissible for (a,b,c,e), McMullen [McM, Section 7] determined that the butterfly move B_q has the following effects:

- if q is finite, then $B_q(a,b,c,e) = (a',b',c',e')$ with $c' = \gcd(qc, b+qa)$ and $e' = -e-2qc$, with b' determined from the condition that $D = (e')^2 + 4b'c'$. McMullen does not determine a formula for a'.
- if $q = \infty$, then $B_q(a,b,c,e) = (a',b',c',e')$ with $c' = \gcd(a,c)$ and $e' = -e-2c$, with b' determined from these values as above. Once again, no formula for a' is given.

Since we need to understand a' in order to bound the number of butterfly moves. We will unpack further the action of the butterfly move on the prototype (a,b,c,e).

Firstly, recall that the butterfly move B_q for finite q sends a splitting $M = E_1 \underset{I}{\#} E_2$ with homology

$$H_1(M,\mathbb{Z}) = H_1(E_1,\mathbb{Z}) \oplus H_1(E_2,\mathbb{Z}) = (\mathbb{Z}(\lambda,0) \oplus \mathbb{Z}(0,\lambda)) \oplus (\mathbb{Z}(a,c) \oplus \mathbb{Z}(b,0))$$

to a splitting $M = F_1 \underset{J}{\#} F_2$ with homology

$$H_1(M,\mathbb{Z}) = H_1(F_1,\mathbb{Z}) \oplus H_1(F_2,\mathbb{Z}) = \left(\mathbb{Z}(\lambda',0) \oplus \mathbb{Z}(0,\lambda')\right) \oplus \left(\mathbb{Z}(a',c') \oplus \mathbb{Z}(b',0)\right)$$

by first taking the basis of $H_1(F_1,\mathbb{Z})$ to be

$$L_1 = \begin{pmatrix} b+qa & \lambda+a \\ qc & c \end{pmatrix}$$

and the basis of $H_1(F_2,\mathbb{Z})$ to be

$$L_2 = \begin{pmatrix} \lambda & b+qa \\ 0 & \lambda+qc \end{pmatrix}.$$

The change of basis matrix for $H_1(F_1,\mathbb{Z})$ is then $\lambda' L_1^{-1}$ which sends L_2 to

$$N = \lambda' L_1^{-1} L_2 = \begin{pmatrix} c & -a-e-qc \\ -qc & b+qa \end{pmatrix}.$$

We can then change the basis by a Euclidean algorithm like operation in order to obtain a basis of the form

$$\begin{pmatrix} a^* & b' \\ c' & 0 \end{pmatrix}$$

which can then by modified (by basis changes and the action of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$) to a basis of the form

$$\begin{pmatrix} a' & b' \\ c' & 0 \end{pmatrix}.$$

So a^* is some linear combination of c and $-a-e-qc$ determined by the Euclidean algorithm to obtain $c' = \gcd(b+qa, qc)$ and then $a' = a^* \mod \gcd(b',c')$.

If $q = \infty$, then we have

$$L_1 = \begin{pmatrix} a & \lambda-b \\ c & 0 \end{pmatrix}, L_2 = \begin{pmatrix} \lambda & a \\ 0 & \lambda+c \end{pmatrix}, \text{ and } N = \begin{pmatrix} 0 & b-e-c \\ -c & a \end{pmatrix}.$$

Note that, if $a=0$, then we have $a'=0$, $b'=b-e-c$ and $c'=c$. Otherwise, we again apply the Euclidean algorithm procedure to get a basis

$$N \sim \begin{pmatrix} a^* & b' \\ c' & 0 \end{pmatrix} \sim \begin{pmatrix} a' & b' \\ c' & 0 \end{pmatrix}$$

with a^* some multiple of $b-e-c$ determined by the Euclidean algorithm procedure and $a' = a^* \mod \gcd(b',c')$.

2.4.1. *Applying butterfly moves to square-tiled surfaces.* Here, we describe how to apply a butterfly move to a two-cylinder square-tiled surface.

Consider the origami

$$O = ((1,2)(3,4)(5,6)(7,8)(9,10,11,12,13,14),(1,3,5,7,9)(2,4,6,8,14,13,12,11,10))$$

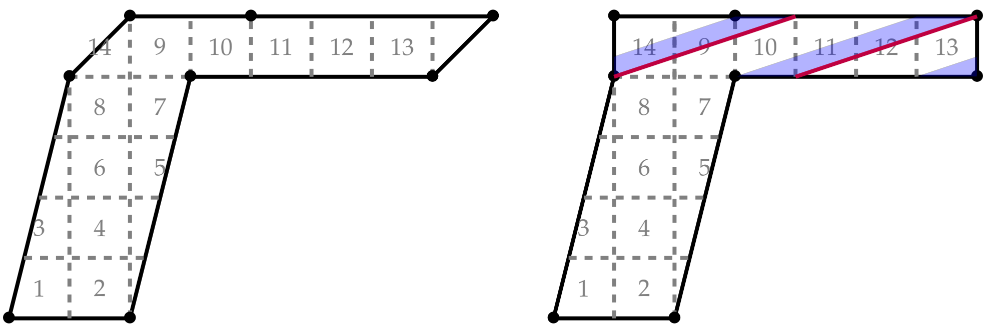

FIGURE 2.4. The origami O and its cusp representative. The direction corresponding to the butterfly move B_2 is also shown, with the resulting torus F_1 shaded in blue.

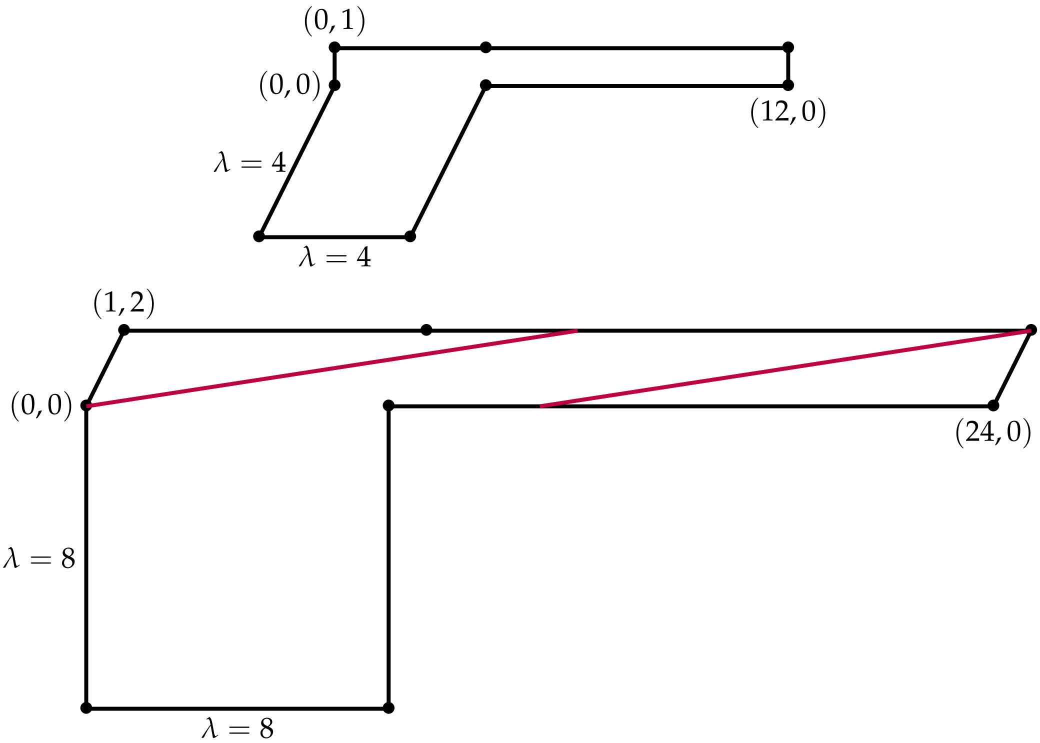

FIGURE 2.5. Realising the prototype corresponding to O. The direction corresponding to the butterfly move B_2 is also shown.

shown on the left of Figure 2.4. We must determine the corresponding prototype. First, we find the cusp representative (i.e., the origami with $0 \le t_i < \gcd(w_i, h_i)$). We achieve this by applying T^{-1} to get the origami

$$((1,2)(3,4)(5,6)(7,8)(9,10,11,12,13,14),(1,3,5,7,9,2,4,6,8,14))$$

shown on the right of Figure 2.4. To realise the surface in the form of Figure 2.3, we must make the bottom cylinder square. So we apply the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. The resulting surface is shown in the top of Figure 2.5. We now also need to undo the twist in the bottom cylinder. We would need to apply $\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$ to do this. However, this will not preserve the integer values of the vertices of the top cylinder. So we must first scale the surface by $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. We must also then normalise the twist of the top cylinder to satisfy the requirements on a. The resulting surface is shown in Figure 2.5. At this point, we must check that $e^2 + 4bc = D = n^2$. Here, this is true. However, it is possible that the surface requires further scaling. We then see that the prototype of the surface is $(1, 24, 2, 2)$ with admissible q values $\{1, 2, \infty\}$.

If we want to apply the butterfly move B_2, say, then we want to make the direction $(b + 2a, 2c)$ horizontal. In the cusp representative of the original origami, this was the direction $(w_2 + 2t_2, 2h_2) = (6, 2)$. To make this horizontal, we apply T^{-2} followed by S^{-1} and obtain the origami

$$((1,2)(3,4,5,6,7,8,9,10,11,12,13,14),(1,3,10,5,12,7,14,9,2,4,11,6,13,8))$$

whose cusp representative

$$((1,2)(3,4,5,6,7,8,9,10,11,12,13,14),(1,3,14,13,12,11,10,9,8,7,6,5,2,4))$$

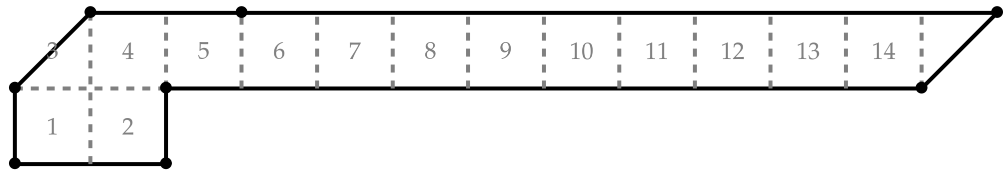

FIGURE 2.6. The cusp representative corresponding to the prototype $(1, 12, 2, -10)$.

is shown in Figure 2.6. It can then be checked that (after applying $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$) the corresponding prototype is $(1, 12, 2, -10)$ which agrees with $B_2(1, 24, 2, 2)$.

So the full butterfly move B_2 is realised by T^{-1} to reach the cusp representative followed by $S^{-1} \circ T^{-2}$ in order to change to the cusp of the image prototype.

3. BOUNDS VIA THE ALGORITHM OF HUBERT-LELIÈVRE

We use the language of Subsection 2.3.

3.1. **From two cylinders cusps to one cylinder cusps.** Let X be a primitive square-tiled surface in $\mathcal{H}(2)$ tiled by a prime number n of unit squares. Suppose that X decomposes into two horizontal cylinders with heights h_i, widths w_i, and twists t_i, $i = 1, 2$. Recall that $w_1 h_1 + w_2 h_2 = n$ and that, by applying T^k with $0 \leq k \leq w_1 w_2 \leq n^2$ to X, we can assume that $0 \leq t_i < \gcd(w_i, h_i) \leq \sqrt{n}$ for $i = 1, 2$.

Denote by $h_{total} = h_1 + h_2$ the total height of X. The values of the twists allow us to distinguish four cases:

- Case (I): $t_1, t_2 \neq 0$
- Case (II): $t_1 = 0$, $t_2 \neq 0$
- Case (III): $t_1 \neq 0$, $t_2 = 0$
- Case (IV): $t_1 = t_2 = 0$

We want use the inductive procedure in [HL, §5.2] to reduce h_{total}. For this sake, let us recall the algorithm introduced by Hubert and Lelièvre.

3.1.1. *Reduction of Case* (I). We apply R to X. In this way, we get a one-cylinder surface or a two-cylinder surface with total height $h'_{total} \leq t_1 + t_2 < h_1 + h_2$. Note that $h'_{total} \leq 2\sqrt{n}$.

3.1.2. *Reduction of Case* (II). We apply R to X. If the resulting surface has two cylinders, its total height is $h'_{total} \leq t_2 < h_2$. Note that $h'_{total} \leq \sqrt{n}$.

3.1.3. *Reduction of Case* (III). We apply R to X in order to obtain a surface with two cylinders: the top cylinder has zero twist and the bottom cylinder has height $\leq t_1 < h_1$. By applying T^l to $R(X)$ with an adequate choice of $0 \leq l \leq n^2$ in order to minimize the twist parameters in the cusp of $R(X)$, we get a surface with $t'_1 = t'_2 = 0$ (Case (IV)), or $t'_1 = 0$ and $0 \neq t'_2$ (Case (II)).

3.1.4. *Reduction of case* (IV). By [HL, Lemma 5.3], we have that X has a single cylinder in the direction (w_1, h_2). This direction can be made horizontal with a Euclidean algorithm procedure that takes $O(\max\{w_1, h_2\}) = O(n)$ steps.

3.1.5. *Two-cylinder counts:* In summary, given a square-tiled surface X with two cylinders and total height $h_{total} = h_1 + h_2$, after taking at most n^2 steps, we can assume that its twist parameters are $0 \leq t_i < \gcd(w_i, h_i) \leq \sqrt{n}$. At this point, we have the following possibilities for reaching a one-cylinder surface:

- **From a Case (I) surface of height > 2:** Each surface along the way must have been a Case I-III surface. At each step $O(n^2)$ moves are used to rotate and normalise the twists, there are at most $O(n^{\frac{1}{2}})$ steps and so there are a total of $O(n^{\frac{5}{2}})$ moves to reach a one-cylinder surface.
- **From a Case (II) surface of height > 2:** As in the previous case, this requires $O(n^{\frac{5}{2}})$ steps.
- **From a surface of height 2:** We use at most $O(n^{\frac{5}{2}})$ moves to reach a surface of height 2. Following this, we use the $O(n^2)$ steps of [HL, Lemma 5.2]. This is a total of $O(n^{\frac{5}{2}} + n^2) = O(n^{\frac{5}{2}})$ moves.
- **From a Case (IV) surface:** It takes at most $O(n^{\frac{5}{2}})$ steps to reach a Case (IV) surface. As discussed above, applying [HL, Lemma 5.3] requires an additional $O(n)$ steps. The total is $O(n^{\frac{5}{2}})$.

So any two-cylinder surface can be connected to a one-cylinder surface in at most $O(n^{\frac{5}{2}})$ moves.

3.2. **Connecting one cylinder cusps among themselves.** Let X be a square-tiled surface tiled by a prime number n of unit squares. Assume that X decomposes into a single horizontal cylinder. In this case, the surfaces in the cusp determined by X are determined by the lengths (a, b, c) of the saddle-connections in the bottom of the cylinder (so that $a + b + c = n$) and a twist parameter.

3.2.1. *From (a, b, c) surfaces to $(1, *, *)$ surfaces.* By taking at most n steps, we can assume that our (a, b, c) surface X has top saddle-connections of lengths b, a and c. As it is explained in [HL, §5.3.1], $R(X)$ is a $(\delta, k\delta, \gamma)$ surface in the direction $(1 + t, d)$, where $d = \gcd(a, b)$, $0 \leq t < (a + b)/d$ is some twist parameter, $\delta = \gcd(1 + t, d)$, and $\gcd(\gamma, \delta) = 1$. Hence, by applying a Euclidean algorithm procedure to make the $((1 + t)/\delta, d/\delta)$ direction horizontal, we see that any (a, b, c) surface X can be joined to a $(\delta, k\delta, \gamma)$ surface with δ dividing $\gcd(a, b) = 1$ and $\gcd(\delta, \gamma) = 1$ in $O(n)$ steps. Thus, by using this procedure once more (with a, b replaced by γ, δ), we derive that any (a, b, c) surface X can be joined to a $(1, *, *)$ surface in $O(n)$ steps.

3.2.2. *From $(1, b, c)$ surfaces with b, c odd to $(1, 1, n - 2)$ surfaces.* Consider the L-shaped surface Y with arms of widths one and lengths b, c. Since it is a $(1, b, c)$ surface in the direction $(1, 1)$, it suffices to connect Y to a $(1, 1, n - 2)$ surface. For this sake, we observe that $R \circ T^2(Y)$ is a surface with two cylinders of heights one. By applying T^l for some $0 \leq l \leq n^2$, we have that $Z = T^l \circ R \circ T^2(Y)$ has two cylinders of heights one and both twist parameters equal to zero. As it turns out, the $((n - b)/2, 1)$ direction in Z is a $(1, 1, n - 2)$ surface, so that Z can be joined to a $(1, 1, n - 2)$ surface in $O(n)$ steps. So the original $(1, b, c)$ surface is connected to a $(1, 1, n - 2)$ surface in $O(n^2)$ steps.

3.2.3. *From $(1, b, c)$ surfaces with b, c even to $(1, 2, n - 3)$ surfaces.* Let X be a $(1, b, c)$ surface with $b = 2b'$ and $c = 2c'$.

Assume first that $b \neq c$, say $b' < c'$. The surface X decomposes into two cylinders in the direction $(c' - b', 1)$: the resulting surface Y has a top cylinder with $h_1 = b$, $w_1 = 2$, and a bottom cylinder with $h_2 = 1$, $w_2 = 2 + \ell$. As it is explained in [HL, §5.3.3], the surface Y gives rise to a $(d, 2d, *)$ surface, $d = \gcd(\ell, b')$, in the direction (ℓ, b'). Hence, by taking at most $O(n)$ steps, X can be joined to a $(d, 2d, *)$ surface. Finally, since any $(d, 2d, *)$ surface gives rise to a $(1, 2, *)$ surface in the direction $(d, 1)$, we conclude that X can be joined to a $(1, 2, *)$ surface in $O(n)$ steps.

Assume now that $b = c$. As it is explained in [HL, §5.3.3], by taking at most n steps in the cusp of X, we get a $(1, b, c)$ surface whose $(b', 1)$ leads to a $(2, 2, *)$ surface Y. In the $(2, 1)$ direction of Y, we see a two cylinders surface with $h_1 = 2$, $w_1 = 1$, $h_2 = 1$. By choosing an adequate $0 \leq l \leq n^2$ and by acting with T^l, we can set the twist parameters to zero, so that we obtain a $(1, 2, n-3)$ surface by looking in the direction $(1, 1)$. In particular, X can be joined to a $(1, 2, n-3)$ surface in $O(n^2)$ steps.

3.3. **Resulting diameter bounds.** We see that all of the one-cylinder surfaces are connected to the appropriate target surface in $O(n^2)$ moves. Any two-cylinder surface is connected to a one-cylinder surface in $O(n^{\frac{5}{2}})$ moves. Hence, the diameter bound is $O(n^{\frac{5}{2}}) = O(|V|^{\frac{5}{6}})$.

3.4. **Potential improvements.** The computer suggests that a two-cylinder surface can be sent to a one-cylinder surface after only a small number (maybe $O(1)$ or $O(\log n)$) of applications of the steps discussed in Subsection 3.1. Each step requires at most $O(n^2)$ steps to move inside the cusps. So this would suggest an improvement from $O(n^{\frac{5}{2}})$ to $O(n^2)$ or $O(n^2 \log n)$ giving the diameter bound $O(|V|^{\frac{2}{3}})$ or $O(|V|^{\frac{2}{3}} \log |V|)$. Furthermore, it is likely possible to improve the $O(n^2)$ bound for moving in the cusp to some bound of the form $O(n^c)$ with $c < 2$, since $O(n^2)$ cusp width requires small height.

4. BOUNDS VIA THE ALGORITHM OF MCMULLEN

We use the language of Subsection 2.4.

4.1. **Bounding the number of butterfly moves.** Here we will argue that McMullen's algorithm requires $O(\log n)$ butterfly moves to reach a reduced prototype and then a further $O(n)$ butterfly moves to reach any target reduced prototype.

4.1.1. *Connecting to reduced prototypes.* Consider a prototype (a, b, c, e). We will argue that $O(\log n)$ repeated applications of B_1 will reach a prototype of the form $(0, b, 1, e)$.

First, observe that after applying B_1 we obtain the prototype (a', b', c', e') with $c' = \gcd(b + a, c)$. Now either $c' < c$ or $c' = c$. In the latter case, letting $b + a = mc$ we have the basis matrix

$$N \sim \begin{pmatrix} -c & -a-e-c \\ c & b+a \end{pmatrix} \sim \begin{pmatrix} -c & mc-a-e-c \\ c & 0 \end{pmatrix} = \begin{pmatrix} -c & b-e-c \\ c & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & b-e-c \\ c & 0 \end{pmatrix},$$

since $-c \mod \gcd(c, b-e-c) = 0$. So we have either arrived at a prototype (a', b', c', e') with $c' < c$, or at a prototype of the form $(0, b', c', e') = (0, b-e-c, c, -e-2c)$.

Consider the latter prototype. We apply B_1 again to obtain a prototype of the form (a'', b'', c'', e''). Again, we either have $c'' = \gcd(b', c') < c' = c$ or we have $c'' = c' = c$. In the latter case, by a similar argument to the above, we will have that $(a'', b'', c'', e'') = (0, b'-e'-c', c', -e'-2c')$ and we notice that, since $\gcd(b', c') = c'$, $\gcd(b'-e'-c', c') = \gcd(e', c') = \gcd(0, b', c', e') = 1$. The latter equality coming from the defining property of a prototype. So another application of B_1 will arrive at a prototype of the form $(0, b''', 1, e''')$.

To summarise, if $\gcd(b + a, c) < c$, then applying B_1 once will give a prototype (a', b', c', e') with $c' < c$. Otherwise, if $\gcd(b + a, c) = c$, applying B_1 again will give a prototype of the form (a'', b'', c'', e'') with $c'' < c$ unless $\gcd(b-e-c, c) = c$ in which case applying B_1 a third time will give rise to a prototype of the form $(0, b''', 1, e''')$.

So we can always iterate B_1 in order to arrive at a prototype with $c = 1$. Note that if $c' < c$, then $c' = \gcd(b, c) \leq c/2$, so it will take at most $O(\log c)$ applications. Now, $n^2 = D = e^2 + 4bc$. Hence, $O(\log c) = O(\log n)$.

4.1.2. *Connecting reduced prototypes to the target prototype.* The reduced prototypes of discriminant D

$$\{(0, b, 1, e) \,:\, e^2 + 4b = D\}$$

correspond to the set

$$\mathcal{S}_D := \{e \equiv D \mod 2 \,:\, e^2 < D \text{ and } (e+2)^2 < D\}.$$

As such, there are $O(\sqrt{D}) = O(n)$ reduced prototypes. McMullen equips $\mathcal{S}_D$ with the equivalence relation produced by butterfly moves that preserve being reduced. That is, $e \sim e'$ if there exists a sequence $e = e_0, e_1, \ldots, e_k = e'$, so that $e_i \in \mathcal{S}_D$ for all $0 \leq i \leq k$ and for all $0 \leq i \leq k-1$ there exists some $q > 0$ such that

$$(0, b_{i+1}, 1, e_{i+1}) = B_q(0, b_i, 1, e_i) \text{ or } (0, b_i, 1, e_i) = B_q(0, b_{i+1}, 1, e_{i+1})$$

With this relation in mind, McMullen proved the following theorem.

Theorem 4.1 ([McM, Theorem 10.1])**.** *Assume $D \geq 5$ and $D \neq 9, 49, 73, 121$ or 169. Then S_D has exactly two components when $D \equiv 1 \mod 8$, and otherwise just one.*

The two components for $D \equiv 1 \mod 8$ correspond to the orbits distinguished by the spin-invariant (Hubert-Lelièvre's A- and B-orbits).

So we see that, in the general case, each reduced prototype can be connected to its target prototype by a path of undirected butterfly moves between reduced prototypes. Note that even though butterfly moves are directed the underlying SL(2, $\mathbb{Z}$) moves can be performed in the reverse order. There are $O(n)$ prototypes in each component of $\mathcal{S}_D$ and so we require at most $O(n)$ butterfly moves to do this.

4.2. **The resulting diameter bound.** First, note that every one-cylinder surface can be connected to a two-cylinder surface in $O(n)$ moves (the cusp representative of Figure 2.2 has two cylinders in the vertical direction).

The arguments above show that all non-reduced prototypes can be connected to the target prototype in $O(\log n)$ butterfly moves.

As above, each butterfly move is achieved by travelling inside the two-cylinder cusp to reach the cusp representative - this takes of $O(n^2)$ moves - followed by a Euclidean algorithm operation to make the direction $(w_2 + qt_2, qh_2)$ (or (t_2, h_2) for $q = \infty$) horizontal - this takes $O(\max\{w_2 + qt_2, qh_2\})$ steps. We can connect to a reduced prototype with $O(\log n)$ applications of B_1. Since $O(\max\{w_2 + t_2, h_2\}) = O(n)$, each application of B_1 is $O(n^2)$ and so this whole process takes $O(n^2 \log n)$ steps.

Now we must connect the reduced prototypes to the target.

Lemma 4.2. *A primitive two-cylinder origami in $\mathcal{H}(2)$ corresponding to a reduced prototype $(0, b, 1, e)$ has surface parameter $h_2 = 1$ and cusp width equal to $w_2 = O(n)$.*

Proof. Consider Figure 4.1. The prototype $(0, b, 1, e)$ has $\lambda = \frac{e+\sqrt{D}}{2} = \frac{e+n}{2}$. The surface therefore has area $A := \lambda^2 + b = \left(\frac{e+n}{2}\right)^2 + b = \frac{en+n^2}{2} = n\lambda$. Recall that the prototype is obtained from the origami by scaling and shearing. Suppose that the origami was vertically scaled by l. Then the horizontal scaling factor is $\frac{\lambda}{l}$, and we have

$$\lambda = lh_1,\ \lambda = \frac{\lambda}{l}w_1,\ 1 = lh_2,\ b = \frac{\lambda}{l}w_2,$$

from which we obtain that $1 = w_1 h_2$. Hence, the origami has parameters $w_1 = 1$, $h_1 = \lambda$, $w_2 = \frac{b}{\lambda}$ and $h_2 = 1$. Hence, it lies in a cusp of width

$$\operatorname{lcm}\left(\frac{w_1}{\gcd(w_1, h_1)}, \frac{w_2}{\gcd(w_2, h_2)}\right) = \operatorname{lcm}\left(\frac{1}{\gcd(1, \lambda)}, \frac{w_2}{\gcd(w_2, 1)}\right) = w_2 = O(n).$$

$\square$

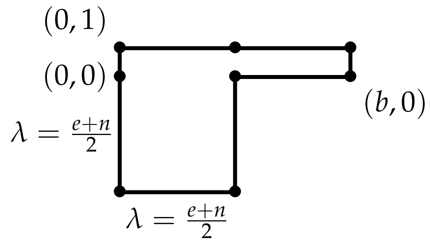

FIGURE 4.1. The surface corresponding to prototype $(0, b, 1, e)$.

Hence, the cusp representative with $t_2 = 0$ can be arranged in $O(n)$ steps and the Euclidean algorithm part of the butterfly move only costs $O(\max\{w_2, q\}) = O(n)$ steps (recall that $q = O(n)$ for admissible q). So each butterfly move between reduced prototypes costs $O(n)$ steps and we make $O(n)$ such butterfly moves. Hence, connecting reduced prototypes to the target costs $O(n^2)$ steps.

Therefore, we obtain the diameter bound $O(n^2 \log n) = O(|V|^{\frac{2}{3}} \log |V|)$.

4.3. **Potential improvements.** Currently, we apply a butterfly move by travelling to the cusp representative before performing the Euclidean algorithm operation. Is it possible that there exists a cheap (i.e., still $O(n)$) Euclidean algorithm operation that one can perform at any origami within the cusp? If so, we could improve the $O(n^2)$ cost of a butterfly move in this setting to $O(n)$, and connecting to a reduced prototype would only cost $O(n \log n)$.

Reduced prototypes have the form $(0, b, 1, e)$. If $e = 0$ (which can only happen if $D \equiv 0 \mod 4$), then we are the prototype $(0, \frac{D}{4}, 1, 0)$. If $e > 0$, then apply B_1 again to obtain the prototype $(0, b - e - 1, 1, -e - 2)$ with $-e - 2 < 0$. So we can assume that $e < 0$.

For $D \equiv 0 \mod 4$, so that e is even, set the target prototype to be $(0, \frac{D}{4}, 1, 0)$. If $\gcd(b, -\frac{e}{2}) = 1$, then applying $B_{-\frac{e}{2}}$ will reach the target prototype. We also see that $q = -\frac{e}{2}$ is admissible since $e < 0$ and even, and

$$(e + 2qc)^2 = (e + 2(-\frac{e}{2}))^2 = 0 < D.$$

If $\gcd(b, -\frac{e}{2}) \neq 1$, then the computer suggests that we can find an admissible q with $-e - 2q < 0$ and $\gcd(b, q) = 1 = \gcd(\frac{e}{2} + q, b - eq - q^2) = \gcd(\frac{e}{2} + q, b - \frac{e}{2}q)$. In such a case, the composition of butterfly moves $B_{\frac{e}{2}+q} \circ B_q$ will send $(0, b, 1, e)$ to $(0, \frac{D}{4}, 1, 0)$.

Can we prove that such a q value always exists?

If $D \equiv 1 \mod 4$ then set the target prototype to be $(0, \frac{(D-1)}{4}, 1, \pm 1)$, depending on the spin invariant if $D \equiv 1 \mod 8$. The same argument as above but with $q = \frac{-e \pm 1}{2}$ should also work. Again, there is a computationally supported conjecture that $\gcd(b, \frac{-e \pm 1}{2}) = 1$ can be achieved after possibly applying another butterfly move. We also see that $q = \frac{-e \pm 1}{2}$ is admissible since

$$(e + 2qc)^2 = (e - e \pm 1)^2 = 1 < D.$$

If the existence of such q can be proved then the reduced prototypes can be connected in $O(1)$ butterfly moves that each cost $O(n)$.

If all of the above improvements are achieved, then we would obtain the bound $O(n \log n) = O(|V|^{\frac{1}{3}} \log |V|)$.

5. Prym loci in $\mathcal{H}(4)$

The Prym locus $\Omega\mathcal{E}_D(4)$ is the subset of $\mathcal{H}(4)$ consisting of those (M, ω) for which M admits a holomorphic involution ι with 4 fixed points taking ω to $-\omega$, and admitting real multiplication by $\mathcal{O}_D$ with $\mathcal{O}_D \cdot \omega \subset \mathbb{C} \cdot \omega$. Lanneau-Nguyen [LN14] classify the $\mathrm{GL}^+(2, \mathbb{R})$ connected components of $\Omega\mathcal{E}_D(4)$ and we direct the reader to their paper for more details on Prym loci. They prove that, for $D \geq 17$, $\Omega\mathcal{E}_D(4)$ is non-empty if and only if $D \equiv 0, 1, 4 \bmod 8$.

Lanneau-Nguyen, as a consequence of their determination of the connected components of $\Omega\mathcal{E}_D(4)$, give the following classification of the SL$(2, \mathbb{Z})$-orbits of Prym origamis in $\mathcal{H}(4)$.

Theorem 5.1 ([LN14, Proposition B.1 and Corollary B.2])**.** *Fix $n \geq 5$ and let X be a primitive origami that is a Prym eigenform in $\mathcal{H}(4)$. Then*

- *if n is odd, there is a single $SL(2, \mathbb{Z})$-orbit of such origamis and the eigenforms have discriminant $D = n^2$.*
- *if $n \equiv 0 \mod 4$ or $n = 6$, there is a single $SL(2, \mathbb{Z})$-orbit of such origamis and the eigenforms have discriminant $D = n^2$.*
- *if $n \equiv 2 \mod 4$, $n \geq 10$, there are two $SL(2, \mathbb{Z})$-orbits of such origamis: one containing eigenforms of discriminant $D = n^2$ and one containing eigenforms of discriminant $D = \frac{n^2}{4}$.*

In the language of HLK-invariants, when n is odd the origamis have HLK-invariant $(0, [1, 1, 1])$; when $n \equiv 0 \mod 4$ or $n = 6$ the origamis have HLK-invariant $(1, [2, 0, 0])$; and when $n \equiv 2 \mod 4$, $n \geq 10$, then the origamis of discriminant n^2 have HLK-invariant $(1, [2, 0, 0])$ while the origamis of discriminant $\frac{n^2}{4}$ have HLK-invariant $(3, [0, 0, 0])$.

5.1. **Surface parameters.** In the Prym locus in $\mathcal{H}(4)$, an origami can have one, two, or three cylinders. The cusp representatives for one-cylinder and two-cylinder origamis are shown in Figure 5.1. Here, (a, b, c) is a triple of positive integers with $2a + 2b + c = n$ in the one-cylinder case, $2a + 2b + 2c = n$ in the two-cylinder case, and $\gcd(a, b, c) = 1$ in both cases. It can be seen that two-cylinder origamis do not exist if n is odd, and one-cylinder origamis do not exist for even n with HLK-invariant $(3, [0, 0, 0])$. One-cylinder cusps have width equal to n, while two-cylinder cusps have width equal to $\frac{n}{2}$.

An origami with three cylinders can have one of the three forms shown in Figure 5.2. These correspond, respectively, to the prototypes $A+$, $A-$, and B discussed in the following subsection. The cusp width is given by the same formula used for two-cylinder origamis in $\mathcal{H}(2)$. Namely, the cusp width is

$$\mathrm{lcm}\left(\frac{w_1}{\gcd(w_1, h_1)}, \frac{w_2}{\gcd(w_2, h_2)}\right).$$

Moreover, we see that the orbits will have size $O(n^3)$.

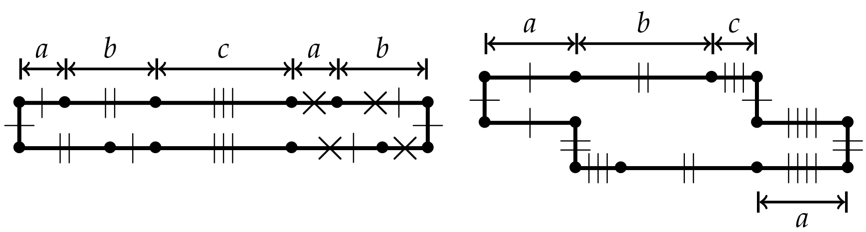

FIGURE 5.1. One-cylinder and two-cylinder cusp representatives in the Prym locus of $\mathcal{H}(4)$.

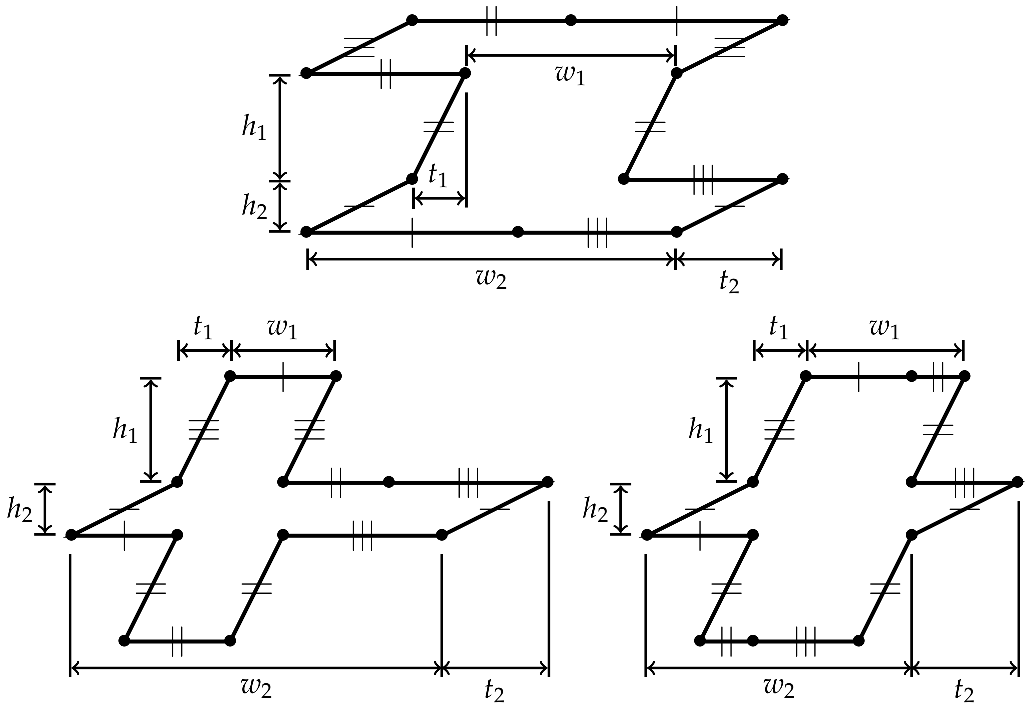

FIGURE 5.2. Three-cylinder surface parameters in $\mathcal{H}(4)$ corresponding to the $A+$, $A-$ and B prototypes.

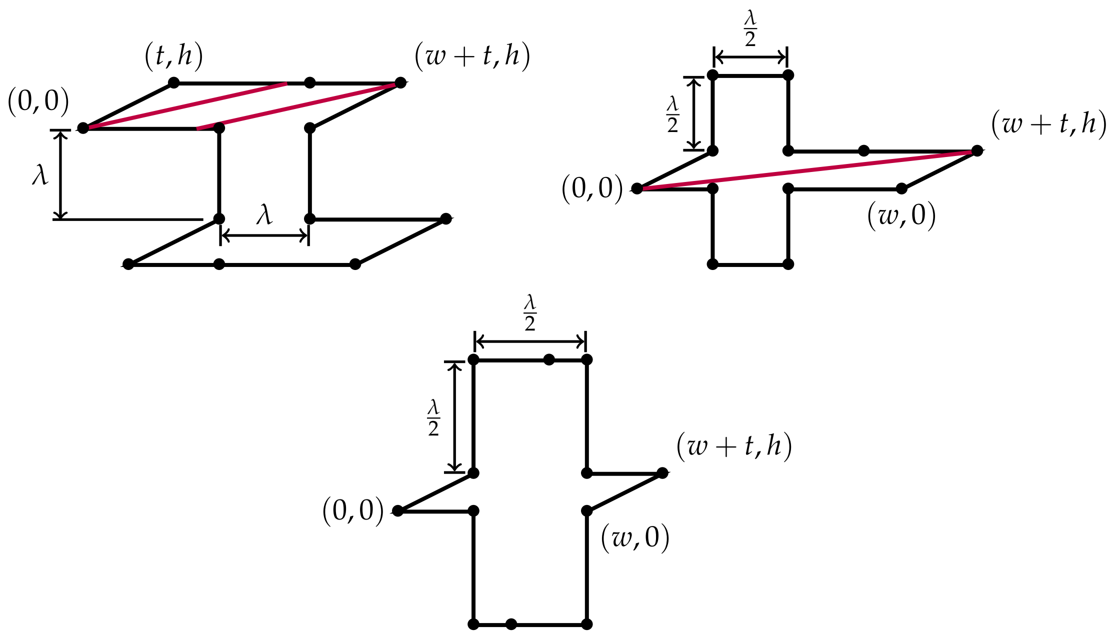

FIGURE 5.3. Three-cylinder prototypes of type $A+$ (top left), $A-$ (top right) and B (bottom). Directions corresponding to butterfly moves ($q = 2$ and $q = 1$, resp.) are shown in the top prototypes.

5.2. **Prototypes and butterfly moves.** Similar to the work of McMullen discussed above, Lanneau-Nguyen define prototypical splittings of discriminant D. Here, there are three prototypical splittings: type $A+$, type $A-$, and type B. See Figure 5.3. Prototypes of type

$A\pm$ are parameterised by the set

$$\mathcal{Q}_D := \{(w,h,t,e,\varepsilon) \in \mathbb{Z}^4 \times \{+,-\} : w > 0, h > 0, e + 2h < w,$$
$$0 \le t < \gcd(w,h), \gcd(w,h,t,e) = 1, D = e^2 + 8wh\}.$$

As before, $\lambda := \frac{e+\sqrt{D}}{2}$.

In this setting, butterfly moves exist for such prototypes and send a prototype of type $A+$ to a prototype of type $A-$, and vice versa. A value of q is admissible if $q = \infty$ or $(e+4qh)^2 < D$. The quadruples that forget the type $(\pm)$ are parameterised by the set

$$\mathcal{P}_D := \{(w,h,t,e) \in \mathbb{Z}^4 : (w,h,t,e,\pm) \in \mathcal{Q}_D\}.$$

In this setting, a reduced prototype is a prototype of the form $(w,1,0,e)$. Such prototypes are parameterised by the set

$$\mathcal{S}_D := \{e \in \mathbb{Z} : e^2 \equiv D \mod 8, e^2, (e+4)^2 < D\}.$$

Lanneau-Nguyen prove the following result describing the affect of a butterfly move.

Proposition 5.2 ([LN14, Propositions 7.5 and 7.6])**.** *Let $(w,h,t,e,\pm) \in \mathcal{Q}_D$, then, for any admissible q, the butterfly move sends $(w,h,t,e,\pm)$ to $(w',h',t',e',\mp)$ with*

$$e' = -e - 4qh$$
$$h' = \gcd(qh, w+qt)$$

if $q < \infty$, or

$$e' = -e - 4qh$$
$$h' = \gcd(h,t)$$

if $q = \infty$. In each case, w' is determined by the condition that $D = e^2 + 8wh = (e')^2 + 8w'h'$.

Similar to McMullen's proof in genus two, this is proved by considering the basis changes for homology associated to the matrix

$$T = \begin{pmatrix} -2qh & 0 & h & -e-t-2qh \\ 0 & -2qh & -qh & w+qt \\ 2w+2qt & 2e+2t+4qh & e+2qh & 0 \\ 2qh & 2h & 0 & e+2qh \end{pmatrix}$$

for $q < \infty$, or

$$T = \begin{pmatrix} -2h & 0 & 0 & -e+w-2h \\ 0 & -2h & -h & t \\ 2t & 2e-2w+4h & e+2h & 0 \\ 2qh & 0 & 0 & e+2h \end{pmatrix}$$

for $q = \infty$. The key minors in each case are the upper right minors

$$\begin{pmatrix} h & -e-t-2qh \\ -qh & w+qt \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & -e+w-2h \\ -h & t \end{pmatrix}$$

which resemble those considered in genus two.

Consider the origami $((1,2,3,4,5,6)(8,9,10,11,12,13),(6,8,7))$ shown in Figure 5.4. It can be checked that this origami corresponds to the prototype $(6,1,0,-11,+) \in \mathcal{Q}_{169}$. In particular, $q = 2$ is admissible. The direction $(w_2 + qt_2, qh_2) = (6,2)$ can be seen in

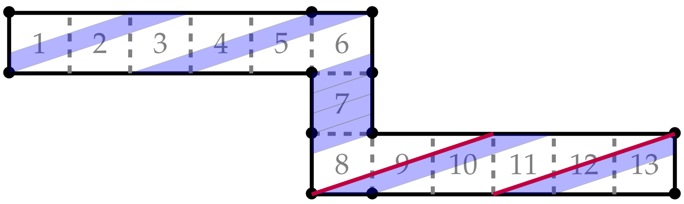

FIGURE 5.4. An origami corresponding to the prototype $(6, 1, 0, -11, +) \in \mathcal{Q}_{169}$. The direction corresponding to the butterfly move B_2 and the resulting simple cylinder are shown in purple and blue, respectively.

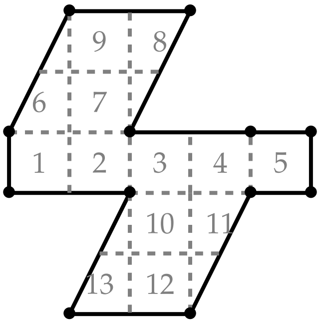

FIGURE 5.5. An origami corresponding to the prototype $(10, 2, 1, 3, -) \in \mathcal{Q}_{169}$.

Figure 5.4. Carrying out the butterfly move, we perform $S^{-1} \circ T^{-2}$ (followed by T to reach the cusp representative). The resulting origami

$$((1, 2, 3, 4, 5)(6, 7)(8, 9)(10, 11)(12, 13), (1, 6, 8, 2, 7, 9)(3, 13, 11, 4, 12, 10))$$

is shown in Figure 5.5 and it can be checked that it does indeed correspond to the prototype $B_2(6, 1, 0, -11, +) = (10, 2, 1, 3, -)$.

As in the genus two case above, we see that the butterfly move costs $O(n^2)$ to move to the cusp representative followed by a Euclidean algorithm operation costing $O(\max\{w_2 + qt_2, qh_2\})$, for finite q, or $O(\max\{t_2, h_2\})$ for $q = \infty$.

5.3. **Bounding the number of butterfly moves.** The argument is similar to that given in Subsection 4.1.

5.3.1. *Connecting to reduced prototypes.* Given a prototype $(w, h, t, e) \in \mathcal{P}_D$ we can apply the butterfly move B_1 $O(\log n)$ times to achieve a reduced prototype $(w', 1, 0, e')$. Indeed, the key minor we are reducing is the matrix

$$\begin{pmatrix} -h & -t - e - 2h \\ h & w + t \end{pmatrix}.$$

If $\gcd(w + t, h) < h$, then we have $h' \leq \frac{h}{2}$. Otherwise, if $\gcd(w + t, h) = h$, we obtain

$$\begin{pmatrix} -h & -t - e - 2h \\ h & w + t \end{pmatrix} \sim \begin{pmatrix} 0 & w - e - 2h \\ h & 0 \end{pmatrix}$$

corresponding to the prototype $(w', h', t', e') = (w - e - 2h, h, 0, -e - 4h)$.

Applying B_1 again, if $\gcd(w' + t', h') = \gcd(w', h') < h'$, we obtain (w'', h'', t'', e'') with $h'' \leq \frac{h'}{2}$, or we have $\gcd(w', h') = h'$. In the latter case, similar to the argument in

Subsection 4.1, we obtain

$$\begin{pmatrix} -h' & -e'-2h' \\ h' & w' \end{pmatrix} \sim \begin{pmatrix} 0 & w'-e'-2h' \\ h' & 0 \end{pmatrix}$$

so that, since $\gcd(w''+t'', h'') = \gcd(w'-e'-2h', h') = \gcd(e', h') = 1$, a further application of B_1 achieves $(w''', 1, 0, e''')$.

5.3.2. *Connecting reduced prototypes to the target prototype.* Similar to the case in $\mathcal{H}(2)$, the reduced prototypes in $\mathcal{P}_D$ are parameterised by the set

$$\mathcal{S}_D := \{e \in \mathbb{Z} : e^2 \equiv D \mod 8, \text{ and } e^2, (e+4)^2 < D\}.$$

So, again, we have $|\mathcal{S}_D| = O(\sqrt{D}) = O(n)$.

In this setting, Lanneau and Nguyen prove the following.

Theorem 5.3 (Follows from [LN14, Theorems 8.2 and 8.6])**.** *Once $D \equiv 0, 1, 4 \mod 8$ is large enough, the set $\mathcal{S}_D$ is non-empty and either*

- $D \equiv 4 \mod 16$ *and $\mathcal{S}_D$ has two connected components $\{e \equiv 2 \mod 8\}$ and $\{e \equiv -2 \mod 8\}$; or*
- *$\mathcal{S}_D$ is connected.*

However, again once D is large enough, $\mathcal{P}_D$ is connected.

If we are in the case where $\mathcal{S}_D$ is connected, then we can reach any target reduced prototype in $O(n)$ butterfly moves. In the case that $\mathcal{S}_D$ has two components, Lanneau-Nguyen prove that the two components can be connected as follows:

- if $D = 4 + 16k$, k odd, then one can connect the two components via the following path in $\mathcal{P}_D$:
$$(2k-4, 1, 0, -6) \xrightarrow{B_2} (k, 2, 0, -2) \xrightarrow{B_\infty} (k-2, 2, 0, -6) \xrightarrow{B_1} (2k, 1, 0, -2).$$
- if $D = 4 + 32k$, k odd, then one can connect the two components via the following path in $\mathcal{P}_D$:
$$(4k, 1, 0, 2) \xrightarrow{B_2} (2k-6, 2, 1, -10) \xrightarrow{B_2} (2k-2, 2, 1, -6) \xrightarrow{B_1} (4k, 1, 0, -2).$$
- if $D = 4 + 32k$, k even, then one can connect the two components via the following path in $\mathcal{P}_D$:
$$(4k-4, 1, 0, -6) \xrightarrow{B_4} (k-3, 4, 0, -10) \xrightarrow{B_\infty} (k-1, 4, 0, -6) \xrightarrow{B_1} (4k-12, 1, 0, -10).$$

5.4. **The resulting diameter bound.** We must now bound the number of SL(2, $\mathbb{Z}$)-steps required to connect to some target origami.

Firstly, note that the cusp representatives of one-cylinder and two-cylinder origamis have a three-cylinder decomposition of type $A\pm$ in the vertical direction. So we can arrive at a surface corresponding to such a prototype in $O(n)$ steps.

If we have a three-cylinder origami corresponding to a prototype of type B, then the proof of [LN14, Proposition 4.7] gives us that the cusp representative of this origami has a three-cylinder decomposition of type $A\pm$ in one of the following directions:

- $(t_1 + t_2 - w_2, h_1 + h_2)$;
- $(2t_1 + t_2, 2h_1 + h_2)$;
- $(w_1 + t_1 + t_2, h_1 + h_2)$; or
- $(2t_1 + t_2 + y - w_1 - w_2, 2h_1 + h_2)$, for some integer $1 \leq y < w_1$.

In each case, this direction can be made horizontal in $O(n)$ steps, after reaching the cusp representative in $O(n^2)$ steps.

So, any origami can be taken to a three-cylinder origami corresponding to a prototype of type $A\pm$ in $O(n^2)$ steps.

We now make use of the following result of Lanneau-Nguyen and its proof.

Theorem 5.4 ([LN14, Theorem 9.2]). *Let $D > 16$ be an even discriminant with $D \equiv 0,4 \bmod 8$. If $D \notin \{48, 68, 100\}$, then $\mathcal{Q}_D$ has only one component.*

Since, we are interested in asymptotics, we can ignore the exceptional discriminants here.

5.4.1. $D \equiv 1 \bmod 8$. In this case, Lanneau-Nguyen prove that $\mathcal{Q}_D$ has two components. Indeed, for each $(w, h, t, e) \in \mathcal{P}_D$, they prove that $(w, h, t, e, +)$ and $(w, h, t, e, -)$ lie in different $\mathrm{GL}^+(2, \mathbb{R})$-orbits [LN14, Theorem 6.1].

For example, it can be shown that if $\sqrt{D} \equiv 1 \bmod 4$ then the prototype $(\frac{D-1}{8}, 1, 0, -1, -)$ corresponds to an origami with $\sqrt{D}$ squares (i.e., n is odd). Whereas, $(\frac{D-1}{8}, 1, 0, -1, +)$ corresponds to an origami with $2\sqrt{D}$ squares (i.e., $n \equiv 2 \bmod 4$). Conversely, when $\sqrt{D} \equiv 3 \bmod 4$, $(\frac{D-1}{8}, 1, 0, -1, +)$ corresponds to an origami with $\sqrt{D}$ squares while $(\frac{D-1}{8}, 1, 0, -1, -)$ corresponds to an origami with $2\sqrt{D}$ squares.

In this setting, take any $(w, h, t, e, \varepsilon) \in \mathcal{Q}_D$ and apply the $O(\log n)$ applications of B_1 to reach a prototype of the form $(w', 1, 0, e', \varepsilon')$ with $e' \in \mathcal{S}_D$. Each butterfly move requires $O(n^2)$ steps within the cusp before $O(\max\{w_2 + t_2, h_2\}) = O(n)$ steps to perform the Euclidean algorithm. So this process costs $O(n^2 \log n)$.

Now, by Theorem 5.3, we can apply $O(n)$ butterfly moves to connect the reduced prototype to the target reduced prototype, and we are done. Similar to Lemma 4.2, we have the following.

Lemma 5.5. *An origami corresponding to a reduced prototype $(w, 1, 0, e, \epsilon) \in \mathcal{Q}_D$ has $h_2 \in \{1, 2\}$ and cusp width $k \in \{\frac{w_2}{2}, w_2, 2w_2\}$. So, $k = O(n)$.*

Proof. Suppose that $D = n^2$ and $\varepsilon = +$. The prototype has area $\lambda^2 + 2w = n\lambda$. Suppose that in achieving the prototype the origami is scaled by l vertically and $\frac{\lambda}{l}$ horizontally. We then have

$$\lambda = lh_1,\ \lambda = \frac{\lambda}{l}w_1,\ 1 = lh_2,\ w = \frac{\lambda}{l}w_2,$$

from which we obtain that $1 = w_1 h_2$. Hence the origami has surface parameters $w_1 = 1$, $h_1 = \lambda$, $h_2 = 1$, $w_2 = \frac{w}{\lambda}$ and cusp width

$$k = \operatorname{lcm}\left(\frac{w_1}{\gcd(w_1, h_1)}, \frac{w_2}{\gcd(w_2, h_2)}\right) = \operatorname{lcm}\left(\frac{1}{\gcd(1, \lambda)}, \frac{w_2}{\gcd(w_2, 1)}\right) = w_2.$$

When $D = \frac{n^2}{4}$ and $\varepsilon = +$, the prototype has area $\lambda^2 + 2w = n\frac{\lambda}{2}$. As above, suppose that in achieving the prototype the origami is scaled by l vertically and $\frac{\lambda}{2l}$ horizontally. We then have

$$\lambda = lh_1,\ \lambda = \frac{\lambda}{2l}w_1,\ 1 = lh_2,\ w = \frac{\lambda}{2l}w_2,$$

from which we obtain that $2 = w_1 h_2$. Hence,

$$(w_1, h_1, w_2, h_2) \in \left\{\left(2, \lambda, \frac{2w}{\lambda}, 1\right), \left(1, 2\lambda, \frac{w}{\lambda}, 2\right)\right\}.$$

However, since $\gcd(h_1, h_2) = 1$ for primitive origamis, we must have

$$(w_1, h_1, w_2, h_2) = \left(2, \lambda, \frac{2w}{\lambda}, 1\right)$$

and so the origami has cusp width

$$k = \operatorname{lcm}\left(\frac{w_1}{\gcd(w_1, h_1)}, \frac{w_2}{\gcd(w_2, h_2)}\right) = \operatorname{lcm}\left(\frac{2}{\gcd(2, \lambda)}, \frac{w_2}{\gcd(w_2, 1)}\right) \in \{w_2, 2w_2\}.$$

When $D = n^2$ and $\varepsilon = -$, the prototype has area $2(\frac{\lambda}{2})^2 + w = n\frac{\lambda}{2}$. So, if the vertical scaling is l and horizontal scaling is $\frac{\lambda}{2l}$, then

$$\frac{\lambda}{2} = lh_1, \ \frac{\lambda}{2} = \frac{\lambda}{2l}w_1, \ 1 = lh_2, \ w = \frac{\lambda}{2l}w_2,$$

giving $1 = w_1h_2$, again. Hence, $w_1 = 1$, $h_1 = \frac{\lambda}{2}$, $w_2 = \frac{2w}{\lambda}$, $h_2 = 1$, and the origami has cusp width

$$k = \operatorname{lcm}\left(\frac{w_1}{\gcd(w_1,h_1)}, \frac{w_2}{\gcd(w_2,h_2)}\right) = \operatorname{lcm}\left(\frac{1}{\gcd(1,\frac{\lambda}{2})}, \frac{w_2}{\gcd(w_2,1)}\right) = w_2.$$

Finally, when $D = \frac{n^2}{4}$ and $\varepsilon = -$, the prototype has area $2(\frac{\lambda}{2})^2 + w = n\frac{\lambda}{4}$. So, if the vertical scaling is l and horizontal scaling is $\frac{\lambda}{4l}$, then

$$\frac{\lambda}{2} = lh_1, \ \frac{\lambda}{2} = \frac{\lambda}{4l}w_1, \ 1 = lh_2, \ w = \frac{\lambda}{4l}w_2,$$

giving $2 = w_1h_2$, again. Hence,

$$(w_1,h_1,w_2,h_2) \in \left\{\left(1,\lambda,\frac{2w}{\lambda},2\right), \left(2,\frac{\lambda}{2},\frac{4w}{\lambda},1\right)\right\}.$$

So, the cusp width is

$$k = \operatorname{lcm}\left(\frac{w_1}{\gcd(w_1,h_1)}, \frac{w_2}{\gcd(w_2,h_2)}\right) \in \left\{\frac{w_2}{2}, w_2, 2w_2\right\}.$$

□

Therefore, these $O(n)$ butterfly moves between reduced prototypes each require $O(n + \max\{w_2, qh_2\}) = O(n)$ steps in SL(2, $\mathbb{Z}$), costing $O(n^2)$ in total.

So we obtain a diameter bound of $O(n^2 \log n)$.

5.4.2. $D \equiv 0 \bmod 8$. By Theorem 5.4, we know that $\mathcal{Q}_D$ is connected. The proof of connectivity argues that there exists an $e \in \mathcal{S}_D$ that is sent to itself by a sequence of butterfly moves of odd length. This sends $(w,h,t,e,+)$ to $(w,h,t,e,-)$ and the connectivity of $\mathcal{S}_D$ from Theorem 5.3 completes the proof.

The required paths are:

- $B_2(2k-1,1,0,-4) = (2k-1,1,0,-4)$ if $D = 8+16k$, $k \geq 1$;
- if $D = 32k$, then
$$(4k-2,1,0,-4) \xrightarrow{B_2} (2k-1,2,0,-4) \xrightarrow{B_\infty} (2k-1,2,0,-4) \xrightarrow{B_1} (4k-2,1,0,-4);$$
- if $D = 16+32k$, $k > 1$, then
$$(4k-6,1,0,-8) \xrightarrow{B_2} (2k+1,2,0,0) \xrightarrow{B_\infty} (2k-3,2,0,-8) \xrightarrow{B_2} (4k-6,1,0,-8).$$

As above, we use $O(\log n)$ applications of B_1 to send (w,h,t,e,ε) to $(w',1,0,e',\varepsilon')$. This requires $O(n^2 \log n)$ steps. Now, if we need to change ε' in order to reach the target prototype, we use $O(n)$ butterfly moves to connect to one of the reduced prototypes in the list above. Each move requires $O(n)$ steps. Next, we apply one of the odd length sequences above. Each sequence requires at most $O(n^2)$ steps (since q is bounded or equal to ∞ in each case). Finally, we apply $O(n)$ butterfly moves to connect to the target reduced prototype, each costing $O(n)$ steps.

Hence, we again obtain a diameter bound of $O(n^2 \log n)$.

5.5. $D \equiv 4 \bmod 8$. The proof of Theorem 5.4 deals with this case using the path

$$B_1\left(\frac{D-4}{8}, 1, 0, -2\right) = \left(\frac{D-4}{8}, 1, 0, -2\right).$$

This requires $O(n^2)$ steps.

If $D \not\equiv 4 \bmod 16$, then $\mathcal{S}_D$ is connected and a similar argument to those above gives a diameter bound of $O(n^2 \log n)$.

Otherwise, $\mathcal{S}_D$ is disconnected. In this case, we may be required to use one of the paths given below Theorem 5.3 in order to change component. Each such path requires at most $O(n^2)$ steps. All other costing is the same as before. Hence, we once again obtain a diameter bound of $O(n^2 \log n)$.

6. Prym loci in $\mathcal{H}(6)$

Similar to the previous section, the Prym locus $\Omega\mathcal{E}_D(6)$ is the subset of $\mathcal{H}(6)$ consisting of those (M, ω) for which M admits a holomorphic involution ι with 2 fixed points taking ω to $-\omega$, and admitting real multiplication by $\mathcal{O}_D$ with $\mathcal{O}_D \cdot \omega \subset \mathbb{C} \cdot \omega$. Lanneau-Nguyen [LN20] classify the $\mathrm{GL}^+(2, \mathbb{R})$ connected components of $\Omega\mathcal{E}_D(6)$ and we again direct the reader to their paper for more details on Prym loci. They prove that, for $D \equiv 0, 1 \bmod 4$ with $D \neq 4, 9$, $\Omega\mathcal{E}_D(6)$ is non-empty and connected. Moreoever, they show that $\Omega\mathcal{E}_4(6)$ and $\Omega\mathcal{E}_9(6)$ are both empty.

The relevant result for origamis is the following.

Theorem 6.1 ([LN20, Theorem 1.2]). *Fix an even $n \geq 8$ and let X be a primitive origami that is a Prym eigenform in $\mathcal{H}(6)$. Then there is a single $SL(2, \mathbb{Z})$-orbit of such origamis and the eigenforms have discriminant $D = \frac{n^2}{4}$.*

If n is odd, then there are no such origamis.

Here, the origamis have HLK-invariant $(1, [0, 0, 0])$. Moreover, since our discriminant is a square, we can ignore the case of $D \equiv 5 \bmod 8$.

6.1. **Surface parameters.** In the Prym locus in $\mathcal{H}(6)$, an origami can have two, or four cylinders (since there can only be one fixed point that is not the zero of the differential and a fixed cylinder gives rise to two fixed points). The possible cusp representatives for two-cylinder origamis are shown in Figure 6.1. Here, (a, b, c) is a triple of positive integers with $2a + 4b + 2c = n$ in the first case, $2a + 4b + 4c = n$ in the second, and $\gcd(a, b, c) = 1$ in both cases. Two-cylinder cusps have width equal to $\frac{n}{2}$.

Origamis with four cylinders have one of the two forms shown in Figure 6.2. As above, the cusp width of such an origami is

$$\mathrm{lcm}\left(\frac{w_1}{\gcd(w_1, h_1)}, \frac{w_2}{\gcd(w_2, h_2)}\right).$$

Moreover, we see that the orbits will have size $O(n^3)$.

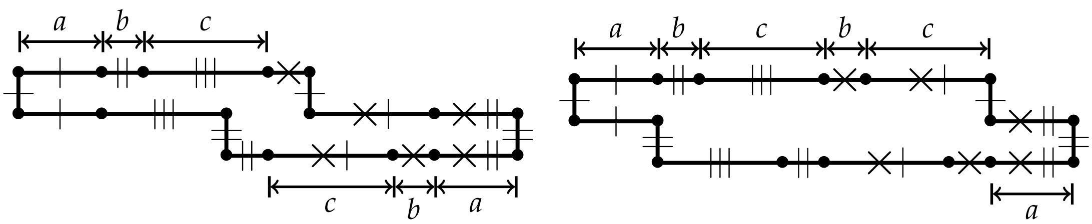

FIGURE 6.1. Two-cylinder cusp representatives in the Prym locus of $\mathcal{H}(6)$.

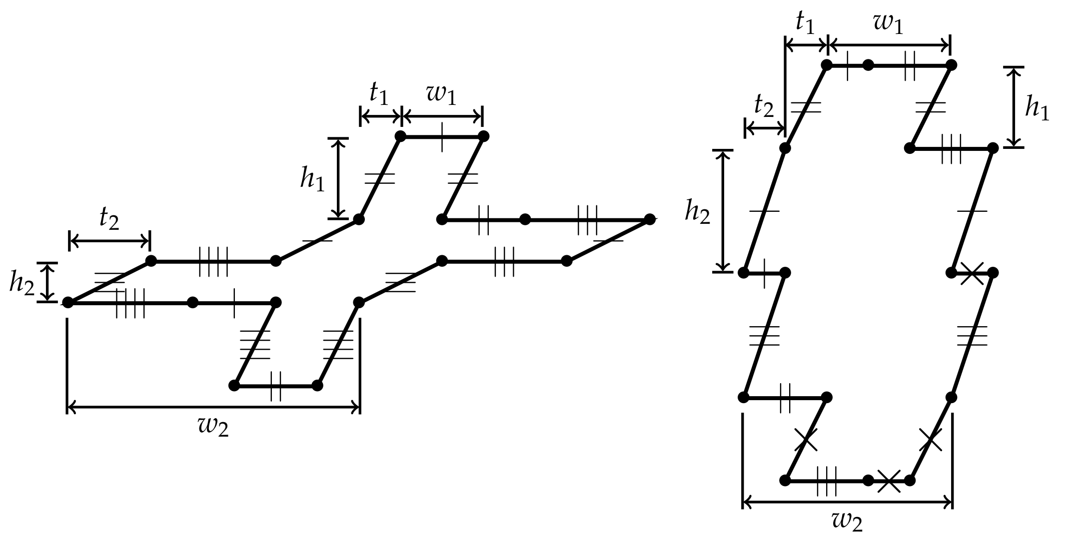

FIGURE 6.2. Four-cylinder surface parameters in $\mathcal{H}(6)$ corresponding to prototypes of type A and B, respectively.

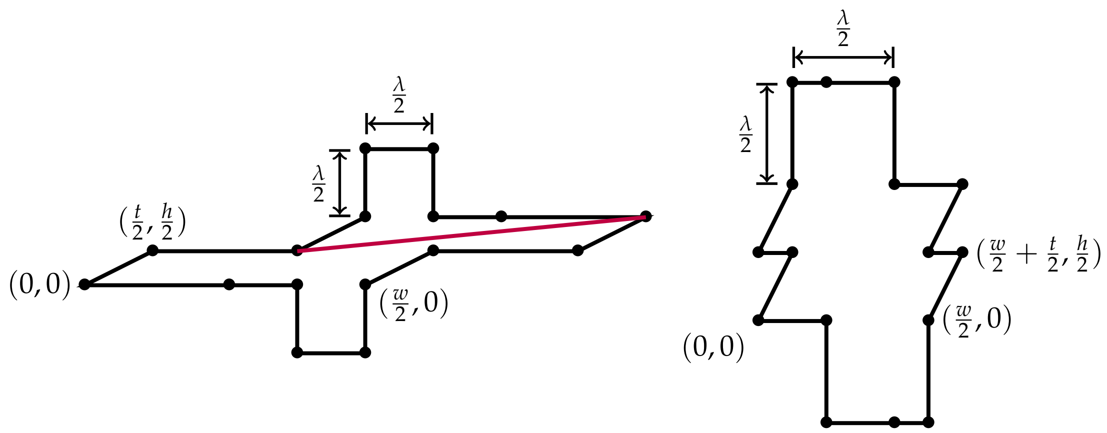

FIGURE 6.3. Four cylinder prototypes of type A and B, respectively. The direction corresponding to the butterfly move with $q = 1$ is shown.

6.2. **Prototypes and butterfly moves.** Similar to the previous two sections, Lanneau-Nguyen define protoypes of discriminant D in this setting. They define prototypes of type A and type B as shown in Figure 6.3. Both prototypes are parameterised by the set

$$\mathcal{P}_D := \Big\{ (w,h,t,e) \in \mathbb{Z}^4 \,:\, w > 0, h > 0, 0 \le t < \gcd(w,h)\,, \\ \gcd(w,h,t,e) = 1, D = e^2 + 4wh, 0 < \lambda := \frac{e + \sqrt{D}}{2} < w, \lambda \neq \frac{w}{2} \Big\}$$

with those of type A being specifically parameterised by the set

$$\mathcal{P}_D^A := \Big\{ (w,h,t,e) \in \mathcal{P}_D \,:\, \lambda < \frac{w}{2} \Big\}$$

and those of type B parameterised by

$$\mathcal{P}_D^B := \Big\{ (w,h,t,e) \in \mathcal{P}_D \,:\, \frac{w}{2} < \lambda < w \Big\}.$$

In this setting we still have reduced prototypes $(w, 1, 0, e)$ parameterised by

$$\mathcal{S}_D^1 := \{e \in \mathbb{Z} : e^2 \equiv D \mod 8, e^2, (e+4)^2 < D\}.$$

For $D \equiv 1 \mod 8$, we also have almost-reduced prototypes of the form $(w, 2, 0, e)$, with w even, parameterised by the set

$$\mathcal{S}_D^2 := \{e \in \mathbb{Z} : e^2 \equiv D \mod 16, e^2, (e+8)^2 < D\}.$$

Butterfly moves can be defined on the prototypes of type A and map such prototypes among themselves. Here, a value of q is admissible if $q = \infty$, or if $(e + 4qh)^2 < D$. We have the following result.

Proposition 6.2 ([LN20, Propositions 2.6 and 2.7]). *Let $(w, h, t, e) \in \mathcal{P}_D^A$ and let q be admissible. We have $B_q(w, h, t, e) = (w', h', t', e')$ with*

$$\begin{aligned} e' &= -e - 4qh \\ h' &= \gcd(w + qt, qh) \end{aligned}$$

for finite q, and

$$\begin{aligned} e' &= -e - 4h \\ h' &= \gcd(t, h) \end{aligned}$$

for $q = \infty$. In each case, w' is determined by $D = e^2 + 4wh = (e')^2 + 4w'h'$.

Similar to the above, the proof relies on the reduction of the matrices

$$\begin{pmatrix} h & -t - 2e - 4qh \\ -qh & w + qt \end{pmatrix}$$

for $q < \infty$, or

$$\begin{pmatrix} 0 & w - 2e - 4h \\ -h & t \end{pmatrix}$$

for $q = \infty$, which appear as minors of a larger matrix acting on homology.

Consider the origami $((2,3,4,5,6,7)(8,9,10,11,12,13), (1,12,2)(3,13,4))$ shown in Figure 6.4. It can be checked that this origami corresponds to the prototype $(6, 1, 0, -5) \in \mathcal{P}_{49}^A$. In particular, $q = 2$ is admissible. The direction $(w_2 + qt_2, qh_2) = (6, 2)$ can be seen in Figure 6.4. Carrying out the butterfly move, we perform $S^{-1} \circ T^{-2}$ (followed by T^{-4} to reach the cusp representative). The resulting origami

$$((1,2)(3,4,5,6,7)(8,9,10,11,12)(13,14), (1,9,3)(2,10,4)(5,13,11)(6,14,12))$$

is shown in Figure 6.5 and it can be checked that it does indeed correspond to the prototype $B_2(6, 1, 0, -5) = (5, 2, 0, -3)$.

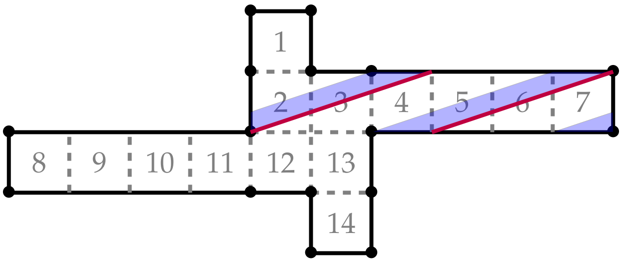

FIGURE 6.4. An origami corresponding to the prototype $(6, 1, 0, -5) \in \mathcal{P}_{49}^A$. The direction corresponding to the butterfly move B_2 and one of the resulting simple cylinders are shown in purple and blue, respectively.

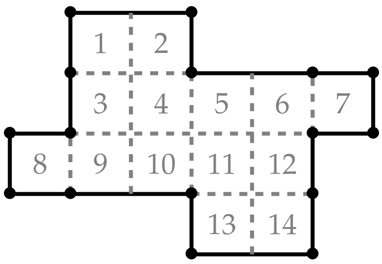

FIGURE 6.5. An origami corresponding to the prototype $(5, 2, 0, -3) \in \mathcal{P}_{49}$.

As in the genus two case above, we see that the butterfly move costs $O(n^2)$ to move to the cusp representative followed by a Euclidean algorithm operation costing $O(\max\{w_2 + qt_2, qh_2\})$, for finite q, or $O(\max\{t_2, h_2\})$ for $q = \infty$.

6.3. **Bounding the number of butterfly moves.** Again, the argument is similar to that given in Subsection 4.1.

6.3.1. *Connecting to (almost-)reduced protoypes.* Given a prototype $(w, h, t, e) \in \mathcal{P}_D^A$ we can apply the butterfly move B_1 $O(\log n)$ times to achieve a reduced prototype $(w', 1, 0, e')$ or an almost-reduced prototype $(w', 2, 0, e')$ when $D \equiv 1 \bmod 8$. Indeed, the key minor we are reducing is the matrix

$$\begin{pmatrix} -h & -t - 2e - 4h \\ h & w + t \end{pmatrix}.$$

If $\gcd(w + t, h) < h$, then we have $h' \leq \frac{h}{2}$. Otherwise, if $\gcd(w + t, h) = h$, we obtain

$$\begin{pmatrix} -h & -t - 2e - 4h \\ h & w + t \end{pmatrix} \sim \begin{pmatrix} 0 & w - 2e - 4h \\ h & 0 \end{pmatrix}$$

corresponding to the prototype $(w', h', t', e') = (w - 2e - 4h, h, 0, -e - 4h)$.

Applying B_1 again, if $\gcd(w' + t', h') = \gcd(w', h') < h'$, we obtain (w'', h'', t'', e'') with $h'' \leq \frac{h'}{2}$, or we have $\gcd(w', h') = h'$. In the latter case, similar to the argument in Subsection 4.1, we obtain

$$\begin{pmatrix} -h' & -2e' - 4h' \\ h' & w' \end{pmatrix} \sim \begin{pmatrix} 0 & w' - 2e' - 4h' \\ h' & 0 \end{pmatrix}$$

so that, since $\gcd(w'' + t'', h'') = \gcd(w' - 2e' - 4h', h') = \gcd(2e', h') \in \{1, 2\}$, a further application of B_1 achieves $(w''', 1, 0, e''')$ or $(w''', 2, 0, e''')$.

In the latter case, if D is even, then e''' is even and so we must have $\gcd(w''', 2) = 1$. Hence, a further application of B_1 arrives at a reduced prototype. If $D \equiv 1 \bmod 8$, then w''' is either even or odd. In the former case, $(w''', 2, 0, e''')$ is almost-reduced and we cannot reach a reduced prototype. In the latter case, $\gcd(w''', 2) = 1$ and we achieve a reduced prototype after a further application of B_1.

Therefore, we achieve a reduced prototype or, for $D \equiv 1 \bmod 8$, an almost-reduced prototype in $O(\log n)$ applications of B_1.

6.3.2. *Connecting (almost-)reduced prototypes.* Lanneau-Nguyen proved the following results (we have again ignored the exceptional discriminants since we are interested in asymptotics).

Theorem 6.3 ([LN20, Theorem A.2])**.** *For D a large enough discriminant, S_D^1 is non-empty and has*

- *three components*, $\{e \in S_D^1 : e \equiv 0$ *or* $4 \bmod 8\}, \{e \in S_D^1 : e \equiv 2 \bmod 8\}$, *and* $\{e \in S_D^1 : e \equiv -2 \bmod 8\}$, *if* $D \equiv 4 \bmod 8$.
- *two components*,
  - $\{e \in S_D^1 : e \equiv 1$ *or* $3 \bmod 8\}$ *and* $\{e \in S_D^1 : e \equiv -1$ *or* $-3 \bmod 8\}$, *if* $D \equiv 1 \bmod 8$,
  - $\{e \in S_D^1 : e \equiv 0$ *or* $4 \bmod 8\}$ *and* $\{e \in S_D^1 : e \equiv 2$ *or* $-2 \bmod 8\}$, *if* $D \equiv 0 \bmod 8$,
- *only one component, otherwise.*

Moreover, if $D \equiv 1 \bmod 8$, *then* S_D^2 *is non-empty and connected.*

Theorem 6.4 ([LN20, Theorem 3.4]). *For* D *a large enough discriminant,* $\mathcal{P}_D^A$ *is non-empty and has*

- *one component if* $D \equiv 5 \bmod 8$,
- *two components,* $\{(w,h,t,e) \in \mathcal{P}_D^A : e \equiv 0 \bmod 4\}$ *and* $\{(w,h,t,e) \in \mathcal{P}_D^A : e \equiv 2 \bmod 4\}$ *if* $D \equiv 0, 4 \bmod 8$,
- *two components,* $\mathcal{P}_D^{A_i} := \{(w,h,t,e) \in \mathcal{P}_D^A : e \in S_D^i\}$*, for* $i = 1, 2$*, if* $D \equiv 1 \bmod 8$.

In all cases, $|S_D^i| = O(n)$.

If $D \equiv 0 \bmod 8$, we can connect to a target prototype in one of the two components of S_D^1 in $O(n)$ butterfly moves. If $D \equiv 4 \bmod 8$, then the proof of Theorem 6.4 demonstrates that the components $\{e \in S_D^1 : e \equiv 2 \bmod 8\}$, and $\{e \in S_D^1 : e \equiv -2 \bmod 8\}$ are connected by a paths in $\mathcal{P}_D^A$:

- if $D = 12 + 16k, k \geq 2$

$$(4k+2, 1, 0, -2) \xrightarrow{B_2} (2k-3, 2, 0, -6) \xrightarrow{B_\infty} (2k+1, 2, 0, -5) \xrightarrow{B_1} (4k-6, 1, 0, -6)$$

- if $D = 4 + 32k, k \geq 4$

$$(8k, 1, 0, 2) \xrightarrow{B_2} (4k-12, 2, 1, -10) \xrightarrow{B_2} (4k-4, 2, 1, -6) \xrightarrow{B_1} (8k, 1, 0, -2)$$

- if $D = 20 + 32k, k \geq 3$

$$(8k+4, 1, 0, 2) \xrightarrow{B_2} (4k-10, 2, 1, -10) \xrightarrow{B_2} (2k-1, 4, 0, -6) \xrightarrow{B_1} (8k-20, 1, 0, -10).$$

So, we can connect to a reduced prototype in one of the components of $\mathcal{P}_D^A$ listed in Theorem 6.3 in $O(n)$ butterfly moves between reduced prototypes and possible one of the paths in the list above. Finally, if $D \equiv 1 \bmod 8$ then the proof of Theorem 6.4 demonstrates that the two components of S_D^1 can be connected by a path in $\mathcal{P}_D^{A_1}$:

- if $D = 1 + 16k, k \geq 3$

$$(4k-6, 1, 0, -5) \xrightarrow{B_2} (2k-1, 2, 0, -3) \xrightarrow{B_\infty} (2k-3, 2, 0, -5) \xrightarrow{B_1} (4k-2, 1, 0, -3)$$

- if $D = 9 + 16k, k \geq 3$

$$(4k-10, 1, 0, -7) \xrightarrow{B_2} (2k+1, 2, 0, -1) \xrightarrow{B_\infty} (2k-5, 2, 0, -7) \xrightarrow{B_1} (4k+2, 1, 0, -1).$$

So reduced prototypes can be connected to a target reduced prototype using $O(n)$ butterfly moves between reduced prototypes and possible one of the two paths above, and the almost-reduced prototypes can be connected to a target almost-reduced prototype in $O(n)$ butterfly moves between almost-reduced prototypes.

6.4. **The resulting diameter bound.** We must now bound the number of SL$(2,\mathbb{Z})$-steps required to connect to the target origamis.

Firstly, observe that the cusp representative of a two-cylinder origami has four-cylinder direction corresponding to a prototype of type A in its vertical direction. Hence, we can reach such a four-cylinder origami in $O(n)$ steps.

If we have a four-cylinder origami corresponding to a prototype of type B, then the proof of [LN20, Proposition 2.4] gives us that the cusp representative of this origami has a four-cylinder decomposition of type A in one of the following directions:

- (t_1+t_2, h_1+h_2);
- $(t_1+t_2-w_1, h_1+h_2)$; or
- $(t_1+t_2-w_2, h_1+h_2)$.

In each case, this direction can be made horizontal in $O(n)$ steps, after reaching the cusp representative in $O(n^2)$ steps.

So, any origami can be taken to a four-cylinder origami corresponding to a prototype of type A in $O(n^2)$ steps.

Similar to Lemmas 4.2 and 5.5, we will make use of the following lemma when we connect (almost-)reduced prototypes to a target (almost-)reduced prototype.

Lemma 6.5. *An origami corresponding to a reduced prototype $(w,1,0,e)$ in $\mathcal{P}_D$ has $h_2=1$ and cusp width $w_2=O(n)$. An origami corresponding to an almost-reduced prototype $(w,2,0,e)$ in $\mathcal{P}_D$ has $h_2\in\{1,2\}$ and cusp width $k\in\{\frac{w_2}{2}, w_2, 2w_2\}$. So, $k=O(n)$.*

Proof. Let $(w,1,0,e)$ be a reduced prototype. The prototype has area $2(\frac{\lambda}{2})^2+\frac{w}{2}=n\frac{\lambda}{4}$. Suppose that in achieving the prototype the origami is scaled by l vertically and $\frac{\lambda}{4l}$ horizontally. We then have

$$\frac{\lambda}{2}=lh_1,\ \frac{\lambda}{2}=\frac{\lambda}{4l}w_1,\ \frac{1}{2}=lh_2,\ \frac{w}{2}=\frac{\lambda}{4l}w_2,$$

giving $1=w_1h_2$. So the origami has surface parameters $w_1=1$, $h_1=\lambda$, $w_2=\frac{w}{\lambda}$, $h_2=1$ and cusp width

$$k=\operatorname{lcm}\left(\frac{w_1}{\gcd(w_1,h_1)},\frac{w_2}{\gcd(w_2,h_2)}\right)=\operatorname{lcm}\left(\frac{1}{\gcd(1,\lambda)},\frac{w_2}{\gcd(w_2,1)}\right)=w_2.$$

Let $(w,2,0,e)$ be an almost-reduced prototype. The prototype has area $2(\frac{\lambda}{2})^2+w=n\frac{\lambda}{4}$. Suppose that in achieving the prototype the origami is scaled by l vertically and $\frac{\lambda}{4l}$ horizontally. We then have

$$\frac{\lambda}{2}=lh_1,\ \frac{\lambda}{2}=\frac{\lambda}{4l}w_1,\ 1=lh_2,\ \frac{w}{2}=\frac{\lambda}{4l}w_2,$$

giving $2=w_1h_2$. So the origami has surface parameters

$$(w_1,h_1,w_2,h_2)\in\left\{\left(1,\lambda,\frac{w}{\lambda},2\right),\left(2,\frac{\lambda}{2},\frac{2w}{\lambda},1\right)\right\},$$

and cusp width

$$k=\operatorname{lcm}\left(\frac{w_1}{\gcd(w_1,h_1)},\frac{w_2}{\gcd(w_2,h_2)}\right)\in\left\{\frac{w_2}{2},w_2,2w_2\right\}.$$

□

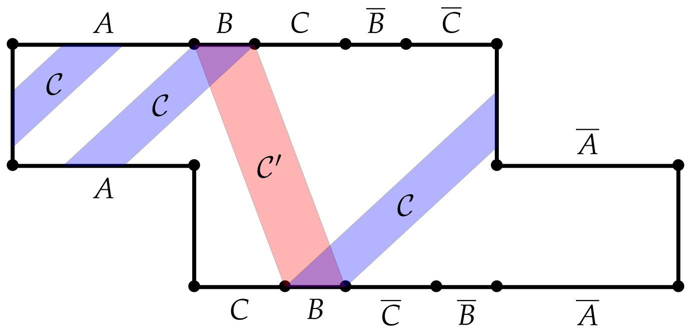

FIGURE 6.6. An origami with cylinders giving rise to prototypes in different components of $\mathcal{P}_D^A$. Cylinder $\mathcal{C}$ is shown in blue while cylinder $\mathcal{C}'$ is shown in red.

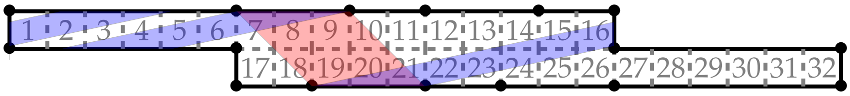

FIGURE 6.7. An origami with cylinder $\mathcal{C}$, shown in blue, giving rise to prototype $(63, 1, 0, 2)$, and cylinder $\mathcal{C}'$, shown in red, giving rise to prototype $(30, 2, 1, -4)$.

6.4.1. $D \equiv 0, 4 \bmod 8$. In this case, we can reach a reduced prototype in each component of $\mathcal{P}_D^A$ using $O(n^2 \log n)$ steps (applying B_1 $O(\log n)$ times followed by $O(n)$ butterfly moves between redcued prototypes which, by Lemma 6.5, each cost $O(n)$ steps, and possibly one of the paths listed below Theorem 6.4 which each cost $O(n^2)$).

To connect the two components of $\mathcal{P}_D^A$, Lanneau-Nguyen (see [LN20, Theorem 7.1]) construct an origami (with height 2) as shown in Figure 6.6 and demonstrate that the cylinders $\mathcal{C}$ and $\mathcal{C}'$ give rise to prototypes in differing components of $\mathcal{P}_D^A$.

For example, when $n = 32$, the origami

$$((1, 2, 3, \ldots, 16)(17, 18, 19, \ldots, 32), \\ (7, 19, 9, 21, 11, 18, 8, 20, 10, 17)(12, 24, 14, 26, 16, 23, 14, 25, 15, 22))$$

shown in Figure 6.7 has a cylinder $\mathcal{C}$ in direction $(14, 3)$ giving rise (after an application of $T^2 \circ S^{-1} \circ T^{-1} \circ S^{-1} \circ T^{-4}$) to the origami

$$((10, 11, \ldots, 16)(17, 18, \ldots, 23), (1, 22, 10, 9, \ldots, 2)(11, 32, 31, \ldots, 23))$$

on the left of Figure 6.8 corresponding to the prototype $(63, 1, 0, 2)$, while the cylinder $\mathcal{C}'$ in direction $(-2, 2)$ (after applying S) gives rise to the origami

$$((1, 2)(3, 4)(5, 6)(7, 8, \ldots, 16)(17, 18, \ldots, 26)(27, 28)(29, 30)(31, 32), \\ (1, 24, 8, 6, 4, 2, 23, 7, 5, 3)(9, 32, 30, 28, 26, 10, 31, 29, 27, 25))$$

on the right of Figure 6.8 corresponding to the prototype $(30, 2, 1, -4)$.

Letting l_A, l_B and l_C denote the lengths of the sides A, B and C, respectively, we see that cylinder $\mathcal{C}$ has direction $(l_A + 2l_B + l_C, 3)$ and so can be made horizontal in $O(\max\{l_A + 2l_B + l_C, 3\}) = O(n)$ steps. Similarly, the cylinder $\mathcal{C}'$ has direction $(-l_C, 2)$ and so can also be made horizontal in $O(n)$ steps. Hence, transitioning between the components of $\mathcal{P}_D^A$ costs only $O(n)$ steps in $\mathrm{SL}(2, \mathbb{Z})$.

Therefore, we obtain the diameter bound $O(n^2 \log n)$.

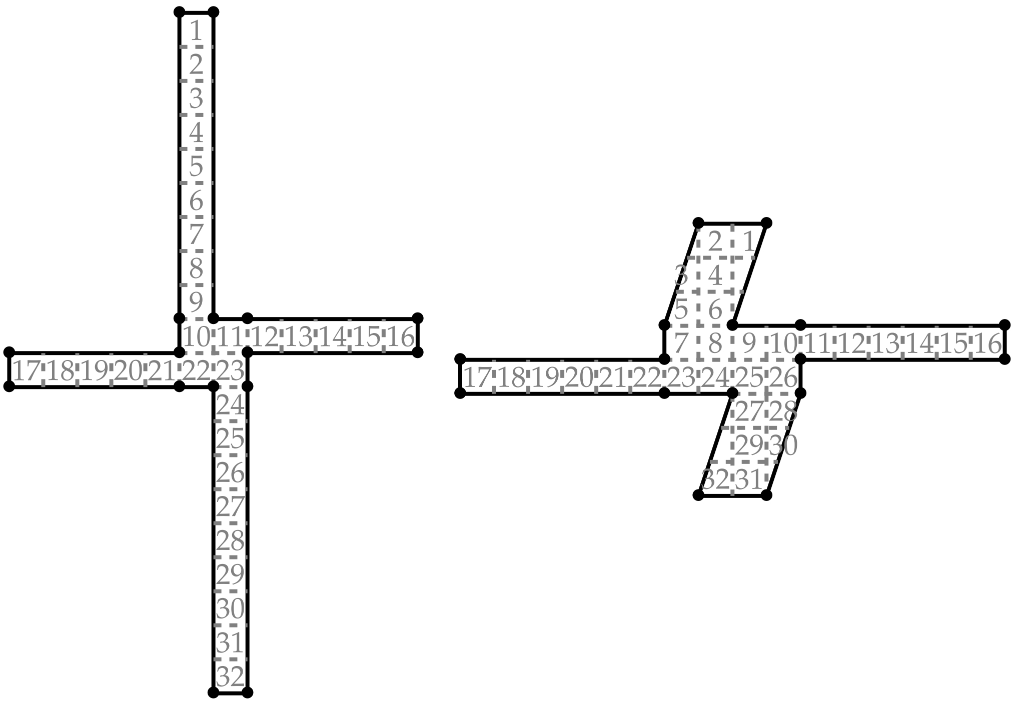

FIGURE 6.8. The origami on the left corresponds to prototype $(63, 1, 0, 2)$. The origami on the right corresponds to prototype $(30, 2, 1, -4)$.

6.4.2. $D \equiv 1 \bmod 8$. As above, each component of $\mathcal{P}_D^A$ has diameter $O(n^2 \log n)$. We must bound the $\mathrm{SL}(2, \mathbb{Z})$ moves required to connect the two components. Here, Lanneau-Nguyen connect the two components of $\mathcal{P}_D^A$ by proving the existence of a prototype in $\mathcal{P}_D^{A_2}$ that contains a simple cylinder in a non-horizontal direction which gives rise to a prototype in $\mathcal{P}_D^{A_1}$. To bound the steps in $\mathrm{SL}(2, \mathbb{Z})$ required to do this, we must determine the direction of this cylinder inside the corresponding origami.

Lanneau and Nguyen prove the following.

Proposition 6.6 (See [LN20, Section 8]). *For any $D \equiv 1 \bmod 8$, with D large enough, there exists $(w, 2, 0, e) \in \mathcal{S}_D^2$ such that there is a simple cylinder in the direction*

$$\left(\frac{w}{2} + \left\lfloor \frac{\lambda}{2} \right\rfloor \frac{\lambda}{2}, j + 1 + \frac{\lambda}{2}\right),$$

for some $j \in \mathbb{N}$, for which the associated prototype lies in $\mathcal{P}_D^{A_1}$.

Now, by Lemma 6.5, we have

$$(w_1, h_1, w_2, h_2) \in \left\{ \left(1, \lambda, \frac{w}{\lambda}, 2\right), \left(2, \frac{\lambda}{2}, \frac{2w}{\lambda}, 1\right) \right\},$$

where in the first case the prototype is obtained from the origami by scaling horizontally by $\frac{\lambda}{2}$ and vertically by $\frac{1}{2}$, and in the second case by scaling horizontally by $\frac{\lambda}{4}$. Hence, on the origami, the direction of the cylinder becomes

$$\left(\frac{w}{\lambda} + \left\lfloor \frac{\lambda}{2} \right\rfloor, 2j + 2 + \lambda\right) = \left(w_2 + \left\lfloor \frac{h_1}{2} \right\rfloor, 2j + 2 + h_1\right),$$

in the first case, or

$$\left(\frac{2w}{\lambda} + 2\left\lfloor \frac{\lambda}{2} \right\rfloor, j + 1 + \frac{\lambda}{2}\right) = (w_2 + 2h_1, j + 1 + h_1),$$

in the second case.

Moreover, [LN20, Lemma 8.3] guarantees that

$$\frac{\lambda}{2}+j+1<\frac{w}{\lambda}+\left\lfloor\frac{\lambda}{2}\right\rfloor \Rightarrow j<w_2=O(n).$$

Hence, it requires only $O(n)$ steps in $\mathrm{SL}(2,\mathbb{Z})$ to make the cylinder direction horizontal in the origami.

For example, the origami

$$((2,3,4,5)(6,7,8,9)(10,11,12,13)(14,15,16,17),\\ (1,16,12,6,2)(3,18,17,13,7)(4,8)(5,9)(10,14)(11,15)),$$

shown on the left of Figure 6.9, corresponds to the prototype $(w,h,t,e)=(4,2,0,-7)\in\mathcal{P}_{81}^{A_2}$. Here, $(w_1,h_1,w_2,h_2)=(1,1,4,2)$. In this case, it can be checked that setting $j=2$ works in the construction of Proposition 6.6. That is, the origami contains a cylinder in the direction $\left(w_2+\left\lfloor\frac{h_1}{2}\right\rfloor,2j+2+h_1\right)=(4,7)$ giving rise to an origami corresponding to a prototype in $\mathcal{P}_{81}^{A_1}$. The cylinder is shown in blue in Figure 6.9. We can make this horizontal by applying $S^{-3}\circ T^{-1}\circ S^{-1}$ (followed by T^{-1} to achieve the cusp representative) to get the origami

$$((3,4,5,6,7,8,9)(10,11,12,13,14,15,16),(1,15,3,2)(4,18,17,16)),$$

shown on the left of Figure 6.10, corresponding to the prototype $(14,1,0,-5)\in\mathcal{P}_{81}^{A_1}$.

Similarly, for an example of the latter possibility for (w_1,h_1,w_2,h_2), we can take the origami

$$((1,2)(3,4,5,6,7,8,9)(10,11,12,13,14,15,16)(17,18),\\ (1,13,3)(2,14,4)(5,17,15)(6,18,16)),$$

shown on the right of Figure 6.9 and corresponding to the prototype $(7,2,0,-5)$, for which $j=2$ again works. Here, we have $(w_1,h_1,w_2,h_2)=(2,1,7,1)$. The cylinder in direction $(w_2+2h_1,j+1+h_1)=(9,4)$ can be made horizontal using $S^{-4}\circ T^{-2}$ (followed by T^{-1} to reach the cusp representative) achieving the origami

$$((7,8,9)(10,11,12),(1,11,7,6,\ldots,2)(8,18,17,\ldots,12)),$$

shown on the right of Figure 6.10 corresponding to the prototype $(18,1,0,3)$.

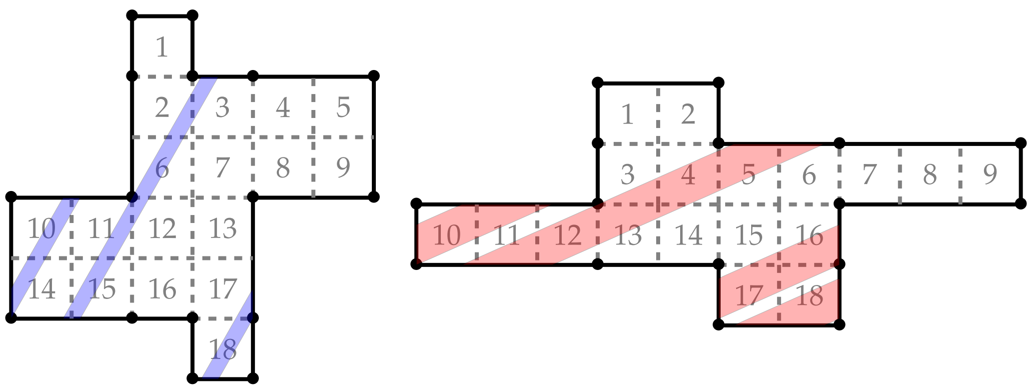

FIGURE 6.9. On the left, an origami corresponding to the prototype $(4,2,0-7)$. A cylinder in the direction $(4,7)$ is shown in blue. On the right, an origami corresponding to the prototype $(7,2,0,-5)$. A cylinder in the direction $(9,4)$ is shown in red.

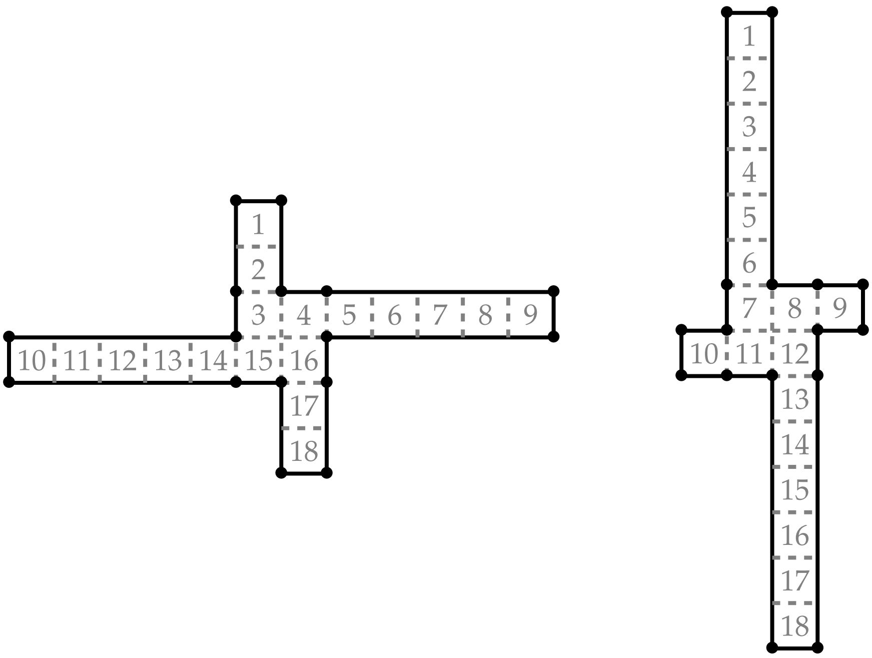

FIGURE 6.10. On the left, an origami corresponding to the prototype $(14, 1, 0, -5)$. On the right, an origami corresponding to prototype $(18, 1, 0, 3)$.

Therefore, each component of $\mathcal{P}_D^A$ has diameter at most $O(n^2 \log n)$ and the two components can be connected in $O(n)$ steps. Hence, we obtain an overall diameter bound of $O(n^2 \log n)$.

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