THE NON-PLANARITY OF SL(2,Z)-ORBITS OF SQUARE-TILED SURFACES

Luke Jeffreys[1] and Carlos Matheus[2]

[1]Heilbronn Research Fellow, School of Mathematics, University of Bristol, UK
[2]Centre de Mathématiques Laurent Schwartz, École Polytechnique, France

Heilbronn Institute for Mathematical Research
University of BRISTOL
School of Mathematics

1. THE SQUARE TORUS

Surfaces are objects that are intrinsically 2-dimensional. Take, for example, a **sphere** (the surface of a ball) or a **torus** (the surface of a ring doughnut). In fact, all surfaces of this type (closed and orientable) are classified by the number of 'doughnut holes'. We call this number g the **genus** of the surface.

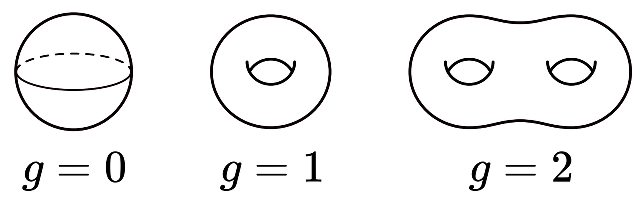

Fig. 1: Closed orientable surfaces of genus 0, 1, and 2

We can build a torus from a square in the following way. Firstly, we identify the top and bottom sides by translation to obtain a cylinder. Secondly, we identify the ends of the cylinder (again by translation) to form a torus. We call this the **square torus**.

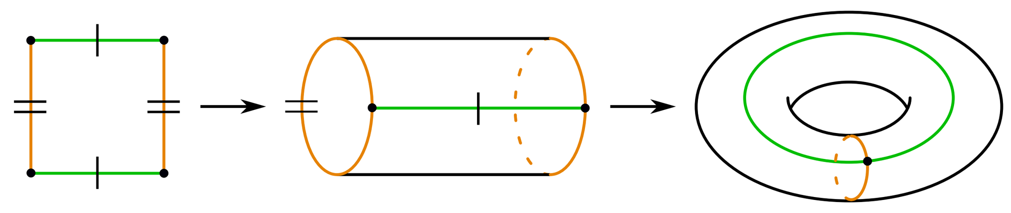

Fig. 2: Gluing the sides of a square to obtain the 'square torus'

2. SQUARE-TILED SURFACES

Generalising the construction of the square torus, a **square-tiled surface** is a surface realised by identifying by translation the sides of a collection of unit squares. See for example the surface shown in Figure 4. Sides with the same label are identified by translation. Visualising the construction is more challenging here, but it can be checked that the resulting surface has genus 2.

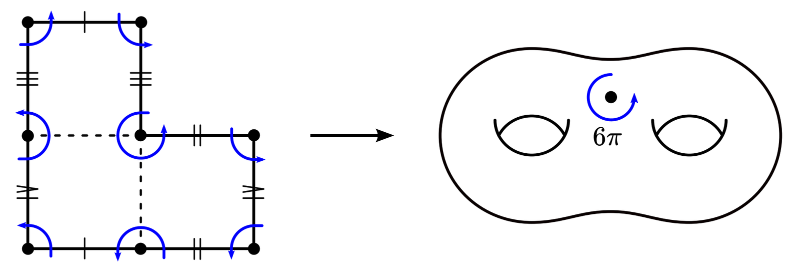

Fig. 4: A genus 2 square-tiled surface

The vertices of the squares are identified to a single point on the surface. Observe that there is 6π angle around this point. We call such a point a **singularity** of excess angle 4π (since $6\pi = 2\pi + 4\pi$).

In general, if a square-tiled surface has n singularities p_i each of excess angle $2k_i\pi$ then we say that the surface lies in the **stratum** $\mathcal{H}(k_1, \ldots, k_n)$. In particular, the surface in Figure 4 lies in the stratum $\mathcal{H}(2)$.

In practice, we consider square-tiled surfaces 'up to cut and paste' which means that we are allowed to cut along straight lines between vertices and then reglue. See Figure 5.

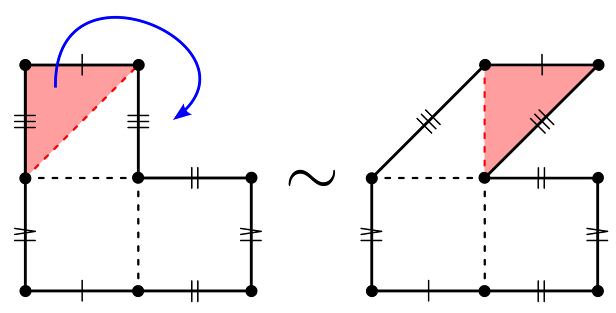

Fig. 5: The 'cut and paste' equivalence on square-tiled surfaces

3. THE ACTION OF SL(2,Z)

Recall that SL$(2,\mathbb{Z})$ is the group of 2×2 integral matrices of determinant 1. An element of SL$(2,\mathbb{Z})$ acts on polygons in the plane by acting on the vectors determining their sides. In particular, we can act on the polygons used to construct a square-tiled surface. It can be checked that pairs of parallel sides of the same length are sent to pairs of parallel sides of the same length and so the resulting polygons (after cutting and pasting) will again give rise to a square-tiled surface.

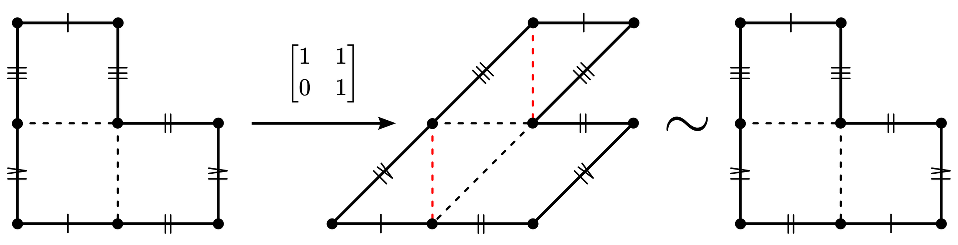

Fig. 3: The action of the element $T = \begin{bmatrix}1&1\\0&1\end{bmatrix}$ on a square-tiled surface

The group SL$(2,\mathbb{Z})$ is generated by the matrices

$$T = \begin{bmatrix}1&1\\0&1\end{bmatrix} \text{ and } S = \begin{bmatrix}1&0\\1&1\end{bmatrix}.$$

The matrix T acts by shearing the surface horizontally to the right. See Figure 3. Similarly, the matrix S acts by shearing vertically upwards. The action of SL$(2,\mathbb{Z})$ fixes the number of squares that a square-tiled surface is constructed from and also the stratum that it lies in.

4. EXPANDER GRAPHS

A family $\{\Gamma_n\}_{n\in\mathbb{N}}$ of d-regular graphs of increasing size is said to be a **family of expander graphs** if the Laplacians Δ_n of Γ_n have a uniform spectral gap.

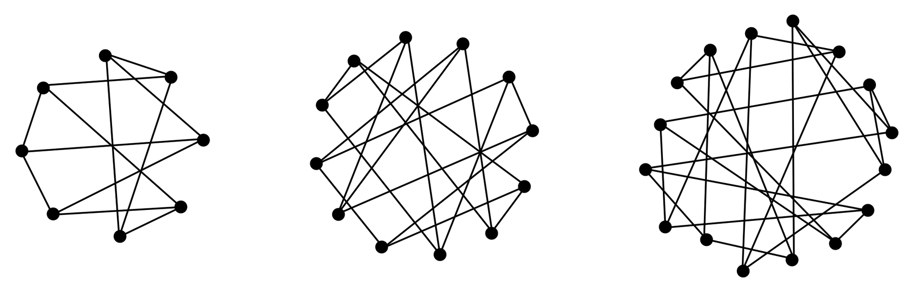

Fig. 6: Ramanujan graphs provide constructions of expander graphs

Expander graphs possess many desirable properties. For example, they are hard to disconnect and have small diameter despite having relatively few edges. These properties make them useful to computer science in the areas of network design, coding theory and pseudo-random number generation.

5. A LONG-STANDING CONJECTURE OF MCMULLEN

The SL$(2,\mathbb{Z})$-orbits of (primitive) n-squared square-tiled surfaces in $\mathcal{H}(2)$ were classified by McMullen and Hubert-Lelièvre. There is a single orbit for each $n=3$ and $n \geq 4$ even, and two orbits called the A and B orbits for $n \geq 5$ odd. The orbit for $n=3$ is shown at the left of Figure 7. Since the matrices T and S generate SL$(2,\mathbb{Z})$, each orbit can be turned into a 4-valent graph with vertices the surfaces in the orbit and two surfaces connected by an edge if one is mapped to the other by T or S. We call these graphs **orbit graphs** and denote them by $\mathcal{G}_n$ in the case of a single orbit, and by $\mathcal{G}_n^A$ or $\mathcal{G}_n^B$ when there are two orbits. The orbit graph $\mathcal{G}_3$ for $n=3$ is shown at the right of Figure 7.

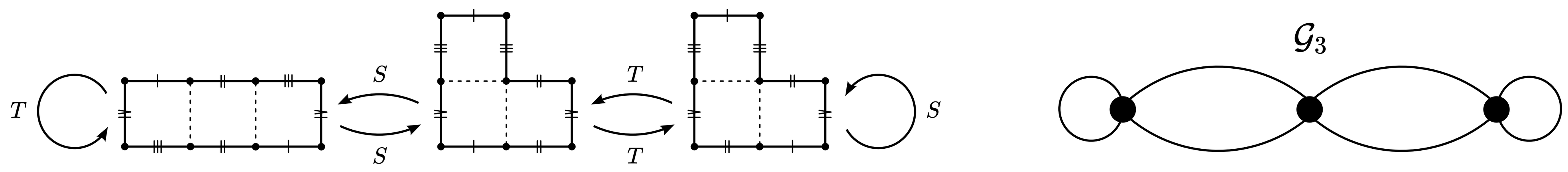

Fig. 7: The SL$(2,\mathbb{Z})$-orbit for $n=3$ and the associated orbit graph $\mathcal{G}_3$

The following conjecture of McMullen has been open for nearly 20 years.

Conjecture. (McMullen) *The orbit graphs of primitive square-tiled surfaces in the stratum $\mathcal{H}(2)$ form a family of expander graphs.*

6. EXPANDER GRAPHS ARE NON-PLANAR

By work of Lipton-Tarjan, a family of expander graphs must be eventually non-planar.

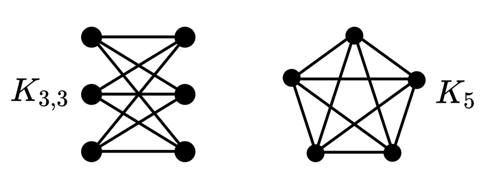

Fig. 8: The complete bipartite graph $K_{3,3}$ and the complete graph K_5 are non-planar

Wagner's and Kuratowski's Theorems state that a graph is non-planar if and only if it can be reduced using specific combinatorial moves to either the complete bipartite graph $K_{3,3}$ or the complete graph K_5.

7. SL(2,Z)-ORBITS ARE NON-PLANAR

In joint work with Carlos Matheus, we prove the following theorem.

Theorem. [J-Matheus, '21] *The orbits graphs $\mathcal{G}_n$, $\mathcal{G}_n^A$, and $\mathcal{G}_n^B$ are all non-planar with the exception of $\mathcal{G}_3$ and $\mathcal{G}_5^B$.*

In particular, we give indirect evidence for the conjecture of McMullen.

To prove the non-planarity of the graphs we reduce each orbit graph to a $K_{3,3}$ using the combinatorial moves that allow us to apply the results of Wagner and Kuratowski.