EVIDENCE FOR MCMULLEN'S EXPANDER GRAPH CONJECTURE

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1. THE SQUARE TORUS

Surfaces are objects that are intrinsically 2-dimensional. Take, for example, a **sphere** (the surface of a ball) or a **torus** (the surface of a ring doughnut). In fact, all surfaces of this type (closed and orientable) are classified by the number of 'doughnut holes'. We call this number g the **genus** of the surface.

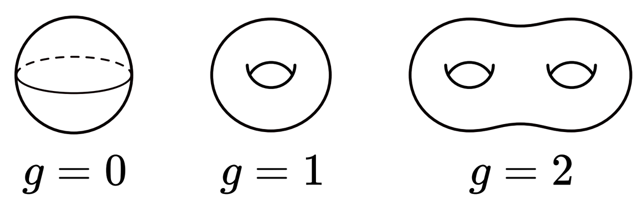

Fig. 1: Closed orientable surfaces of genus 0, 1, and 2

We can build a torus from a square in the following way. See Fig. 2. Firstly, we identify the top and bottom sides by translation to obtain a cylinder. Secondly, we identify the ends of the cylinder (again by translation) to form a torus. We call this the **square torus**.

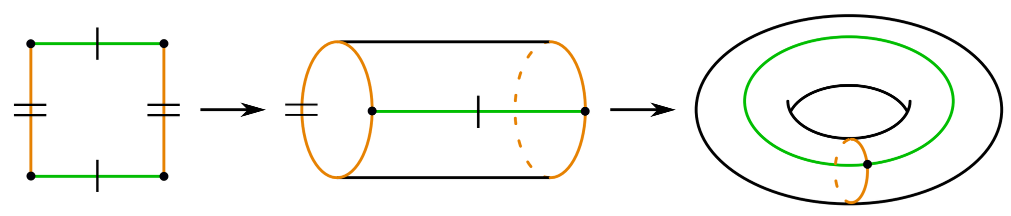

Fig. 2: Gluing the sides of a square to obtain the 'square torus'

2. SQUARE-TILED SURFACES

Generalising the construction of the square torus, a **square-tiled surface** is a surface realised by identifying by translation the sides of a collection of unit squares. See for example the surface shown in Fig. 3. Sides with the same label are identified by translation. Visualising the construction is more challenging here, but it can be checked that the resulting surface has genus 2.

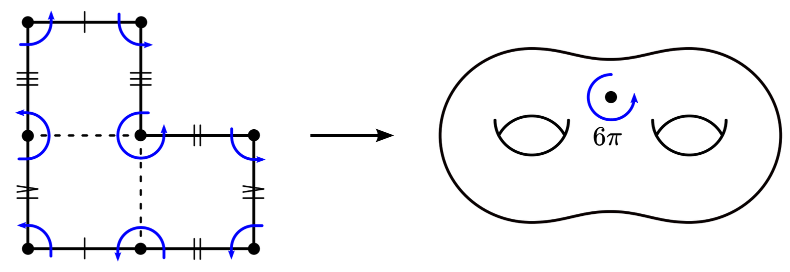

Fig. 3: A genus 2 square-tiled surface

The vertices of the squares are identified to a single point on the surface. Observe that there is 6π angle around this point. We call such a point a **singularity** of excess angle 4π (since $6\pi = 2\pi + 4\pi$).

In general, if a square-tiled surface has n singularities p_i each of excess angle $2k_i\pi$ then we say that the surface lies in the **stratum** $\mathcal{H}(k_1, \ldots, k_n)$. In particular, the surface in Fig. 3 lies in the stratum $\mathcal{H}(2)$.

In practice, we consider square-tiled surfaces 'up to cut and paste' which means that we are allowed to cut along straight lines between vertices and then reglue. See Fig. 4.

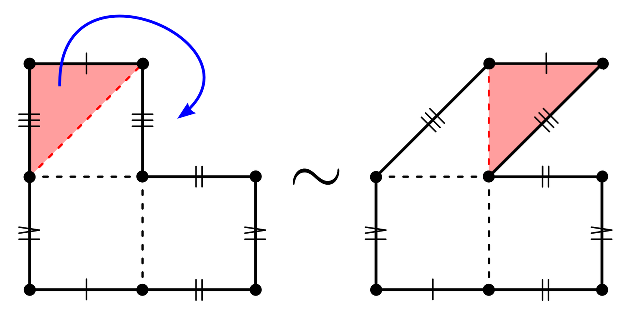

Fig. 4: The 'cut and paste' equivalence on square-tiled surfaces

3. THE ACTION OF SL(2,Z)

Recall that SL$(2,\mathbb{Z})$ is the group of 2×2 integral matrices of determinant 1. An element of SL$(2,\mathbb{Z})$ acts on polygons in the plane by acting on the vectors determining their sides. In particular, we can act on the polygons used to construct a square-tiled surface. It can be checked that pairs of parallel sides of the same length are sent to pairs of parallel sides of the same length and so the resulting polygons (after cutting and pasting) will again give rise to a square-tiled surface.

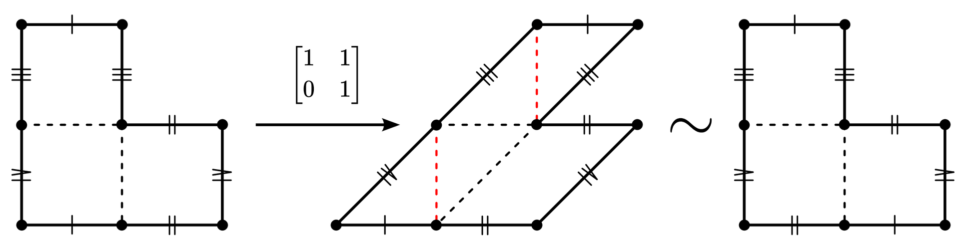

Fig. 5: The action of the element $T = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$ on a square-tiled surface

The group SL$(2,\mathbb{Z})$ is generated by the matrices

$$T = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \text{ and } S = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}.$$

The matrix T acts by shearing the surface horizontally to the right. See Fig. 5. Similarly, the matrix S acts by shearing vertically upwards. The action of SL$(2,\mathbb{Z})$ fixes the number of squares that a square-tiled surface is constructed from and also the stratum that it lies in.

4. EXPANDER GRAPHS

A family $\{\Gamma_n\}_{n\in\mathbb{N}}$ of d-regular graphs of increasing size is said to be a **family of expander graphs** if the Laplacians L_n of Γ_n have a uniform spectral gap (once n is large enough).

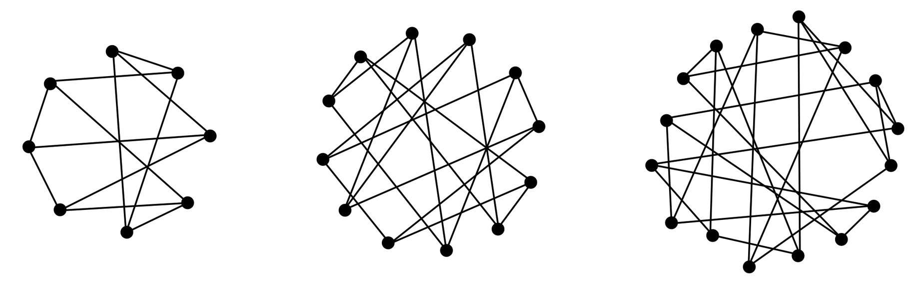

Fig. 6: Ramanujan graphs provide constructions of expander graphs

Expander graphs possess many desirable properties. For example, they are hard to disconnect and have small diameter despite having relatively few edges. These properties make them useful to computer science in the areas of network design, coding theory and pseudo-random number generation.

5. A LONG-STANDING CONJECTURE OF MCMULLEN

The SL$(2,\mathbb{Z})$-orbits of (primitive) n-squared square-tiled surfaces in $\mathcal{H}(2)$ were classified by McMullen and Hubert–Lelièvre. There is a single orbit for each $n=3$ and $n\geq 4$ even, and two orbits called the A and B orbits for $n\geq 5$ odd. The orbit for $n=3$ is shown at the left of Fig. 7. Since the matrices T and S generate SL$(2,\mathbb{Z})$, each orbit can be turned into a 4-valent graph where vertices are the surfaces in the orbit and two surfaces are connected by an edge if one is mapped to the other by T or S. We call these graphs **orbit graphs** and denote them by $\mathcal{G}_n$ in the case of a single orbit, and by $\mathcal{G}_n^A$ or $\mathcal{G}_n^B$ when there are two orbits. The orbit graph $\mathcal{G}_3$ for $n=3$ is shown at the right of Fig. 7.

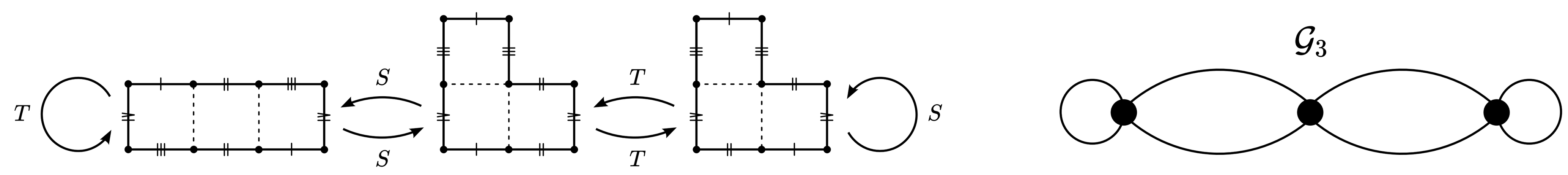

Fig. 7: The SL$(2,\mathbb{Z})$-orbit for $n=3$ and the associated orbit graph $\mathcal{G}_3$

The following conjecture of McMullen has been open for nearly 30 years.

Conjecture. (McMullen) *The orbit graphs of primitive square-tiled surfaces in the stratum $\mathcal{H}(2)$ form a family of expander graphs.*

6. PROPERTIES OF EXPANDER GRAPHS

The **genus**, $g(\Gamma)$, of a graph Γ is the smallest genus of a surface on which it embeds (i.e., on which it can be drawn without edges crossing). For example, planar graphs have genus zero. By work of Gilbert–Hutchinson–Tarjan, if $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a family of expander graphs, then $g(\Gamma_n)\to\infty$ as $n\to\infty$.

The **diameter** of a graph Γ is the largest distance between any pair of vertices in the graph. For example, an n-cycle has diameter $\lfloor \frac{n}{2} \rfloor$. If $\{\Gamma_n\}_{n\in\mathbb{N}}$ is a family of expander graphs, then the diameter of Γ_n is $O(\log(|\Gamma_n|))$; i.e., the diameter is logarithmic in the size of the graph.

7. EVIDENCE FOR THE CONJECTURE

We establish the following indirect evidence for McMullen's conjecture:

Theorem 1. [J–Matheus, '26$^+$] *The orbit graphs of McMullen's conjecture satisfy $g(\mathcal{G}_n), g(\mathcal{G}_n^A), g(\mathcal{G}_n^B)\to\infty$ as $n\to\infty$.*

and

Theorem 2. [J–Matheus, '26] *The orbit graphs $\mathcal{G}_n$, $\mathcal{G}_n^A$, and $\mathcal{G}_n^B$ of McMullen's conjecture have diameter $O(n^2\log(n))$.*

Since, in this situation, the graphs have size $O(n^3)$, we see that this is far from the optimal diameter bound for expander graphs, which would be $O(\log(n))$. However, it is the first non-trivial bound.