Observation of a new interference phenomenon in internal conical diffraction

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Abstract: Conical diffraction is observed in biaxial materials when a beam of light is directed along one of the two optic axis directions. When the beam is directed close to but not along an optic axis, a rich interference pattern is observed beyond the material. We observe some of the previously predicted low intensity interference patterns, representing a qualitatively new optical phenomenon in biaxial materials.

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References and links
1. Introduction

Conical diffraction occurs in biaxial materials when light is incident along an optic axis and spreads out as a hollow skewed cone inside the material. The phenomenon was first predicted by Hamilton in 1882 [1]. It was experimentally confirmed later that year by Lloyd using the biaxial material aragonite. Hamilton’s consideration was that of a refractive phenomenon and led to the term ‘conical refraction’. It was not until the development of the theory by Belsky and Khapalyuk [2], which considered diffraction in a paraxial wave-theory model, that the nomenclature was switched to ‘conical diffraction’. This model was later reformulated by Berry [3] and remains a highly accurate description of the phenomenon [4].

The profile of the conically diffracted beam depends on the profile of the incident beam [3,5]. For a Guassian incident beam, the resultant profile changes as it propagates in space away from the focal image plane (FIP), the location where the double-ring structure takes its sharpest appearance. The inner and outer rings begin to broaden and separate, with the inner ring converging upon itself and the outer ring fading away [4]. Interference fringes—the Warnick-Arnold rings—become apparent as intensity oscillations in the inner ring [6, 7]. Further still from the FIP the Raman spot or axial spike, a high intensity region at the centre of the beam, dominates [8].

Many papers have been published reflecting the interest conical diffraction presents, finding applications such as the generation of Bessel beams [9, 10], optical vortices [11], and optical traps [12]. A feature of the beam predicted to occur by Berry and Jeffrey [13, 14] is the presence of dark spots caused by interference between geometrical rays when the incident light is directed close to, but not along an optic axis. These features for misaligned beams are predicted to be very faint and are restricted to a small region of the beam profile, and so up to now have not been observed experimentally. We present observations of these features representing experimental confirmation for the prediction of this qualitatively new optical phenomenon in biaxial materials.

2. Theory

This paper will use the theory set out by Berry and Jeffrey [13] as a basis for the experiments presented in subsequent sections. A biaxial material has three principal refractive indices \( n_1 < n_2 < n_3 \). A beam aligned along one of the optic axis directions in such a material undergoes conical diffraction as shown in Fig. 1 whereby the beam spreads out as a hollow cone with semi-angle

\[
A = \frac{1}{2} \arctan \left[ n_2^2 \sqrt{\left( n_1^2 - n_2^2 \right) \left( n_2^2 - n_3^2 \right)} \right].
\]  
(1)

For many biaxial materials, \( A \) is of the order of \( 10^{-2} \) rad and therefore we may use a paraxial approximation to calculate the radius of the emergent hollow cylinder from such a material of length \( l \):

\[
R_0 \approx Al
\]  
(2)

If the incident beam has \( 1/e \) intensity radius \( w \), we can define the dimensionless radial parameter \( \rho_0 \) as the ratio of the radii of the conically diffracted beam and the incident beam, as well as the related dimensionless radial parameter \( \rho \), which represents radial position in units of \( w \):

\[
\rho_0 = \frac{Al}{w} = \frac{R_0}{w}, \quad \rho = \frac{R}{w}.
\]  
(3)
Furthermore, defining the beam propagation direction to be along the $z$-axis with the narrowest beam waist occurring at $z = 0$, we can define the dimensionless longitudinal position as

$$
\zeta \equiv \frac{l + (z - l)n_2}{n_2k_0w^2}.
$$

(4)

The focal image plane (FIP) occurs at the longitudinal position in the beam for which the conically diffracted ring profile is sharpest, corresponding to $\zeta = 0$.

Consider a light beam with a Gaussian profile incident along an arbitrary direction of a biaxial material. The degree to which the beam is misaligned with the optic axis is represented by the dimensionless parameter $u$, where $u = 0$ means the beam is aligned with the optic axis. Furthermore, position in the plane transverse to the beam propagation direction is now defined to be $\rho = \{\xi, \eta\}$ where the beam is misaligned with the optic axis in the $\xi$ direction. The transverse electric displacement field profile emerging from the material is shown by Berry and Jeffrey [13] to have the following form:

$$
D \propto \begin{pmatrix} B_0 + B_1 \cos \tilde{\phi} & B_1 \sin \tilde{\phi} \\ B_1 \sin \tilde{\phi} & B_0 - B_1 \cos \tilde{\phi} \end{pmatrix},
$$

(5)

$$
B_0(\rho, \tilde{\zeta}, \rho_0) = \int_0^\infty d\tilde{k} \tilde{k} \exp\left(-\frac{1}{2}i\tilde{\zeta} \tilde{k}^2\right) J_0(\tilde{k}\rho_0) \cos(\tilde{k}\rho_0),
$$

(6)

$$
B_1(\rho, \tilde{\zeta}, \rho_0) = \int_0^\infty d\tilde{k} \tilde{k} \exp\left(-\frac{1}{2}i\tilde{\zeta} \tilde{k}^2\right) J_1(\tilde{k}\rho_0) \sin(\tilde{k}\rho_0),
$$

(7)

where $J_v(x)$ is the $v$th order Bessel function of the first kind. The radial and longitudinal parameters have also been complexified:

$$
\tilde{\rho} = \{\xi + iu, \eta\}, \quad \tilde{\zeta} = \zeta - i,
$$

(8)

and the trigonometric expressions are given by

$$
\cos \tilde{\phi} = \frac{\tilde{\zeta} + iu}{\tilde{\rho}}, \quad \sin \tilde{\phi} = \frac{\eta}{\tilde{\rho}},
$$

(9)

where

$$
\tilde{\rho} = \sqrt{(\xi + iu)^2 + \eta^2}.
$$

(10)

The equations (6) and (7) can be combined to give the sum and difference integrals:

$$
A_+ \equiv B_0 + B_1, \quad A_- \equiv B_0 - B_1.
$$

(11)
Using the Bessel function approximation
\[ J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\nu \pi}{2} \right) \]
in Eqs. (6) and (7) give the following expressions for \( A_+ \) and \( A_- \):

\[ A_\pm \approx \sqrt{\frac{2}{\pi \tilde{\rho}}} \int_0^\infty d\tilde{k} \sqrt{\tilde{k}} \exp \left\{ -\frac{1}{2} i \tilde{\zeta} \tilde{k}^2 \right\} \cos \left\{ \tilde{k} (\tilde{\rho} \mp \rho_0) - \frac{\pi}{4} \right\} \].

Expanding the cosines into the component exponentials allows us to use the method of stationary phase in the integrals above, assuming the exponentials are fast-varying:

\[ A_\pm \approx -\frac{\mu_\pm}{\zeta} \left( \frac{\rho_0 \mp \tilde{\rho}}{\tilde{\rho}} \right)^{1/2} \exp \left\{ \frac{i}{2 \zeta} (\tilde{\rho} \mp \rho_0)^2 \right\}, \quad \begin{cases} \mu_+ = \text{sgn} \left[ \text{Im} \left( \frac{\rho_0 - \tilde{\rho}}{\sqrt{1 + i \tilde{\rho}}} \right) \right] \\ \mu_- = i \end{cases} \]

The values \( A_+ \) and \( A_- \) now represent separate geometrical rays, and due to the complexification introduced in Eq. (8) these rays interfere. It can be shown [13] that this interference is only significant close to the branch points \( \rho_b = \{ 0, \pm u \} \) where

\[ |\cos \tilde{\phi}|^2 + |\sin \tilde{\phi}|^2 \approx 2u^2/|\tilde{\rho}|^2 \]

Further requirements for geometrical interference are the conditions

\[ \exp \left\{ -2i \rho_0 \frac{\tilde{\rho}}{\zeta} \right\} = -i, \quad \text{Im} \left( \frac{\tilde{\rho}}{\zeta} \right) = 0. \]

The result is that geometrical interference is confined to an arc of a circle in the \( \rho \)-plane whose equation is given by

\[ (\rho - \rho_c) \cdot (\rho - \rho_c) = R_c^2, \]

where

\[ \rho_c = \left\{ -\frac{u (\zeta^2 - 1)}{2 \zeta}, 0 \right\}, \quad R_c = \frac{u (\zeta^2 + 1)}{2 \zeta}. \]

This circle manifests as a region with exponentially lower intensity than elsewhere, the exception being the region between the two branch points \( \rho_b = \{ 0, \pm u \} \) which lie on this circle. A further equality of phase requirement in Eq. (16) gives dark spots on this circle where

\[ \text{Re} \left( \frac{\tilde{\rho}}{\zeta} \right) = \left( n + \frac{1}{4} \right) \pi, \quad n = \ldots -1, 0, 1, \ldots \]

These dark spots, and the low intensity arc on which they lie, are the focus of our paper.

3. Experimental method

Due to the small size and low intensity nature of the features predicted to occur, an experimental method was required which maximised the resolution and intensity contrast of the images of the conically diffracted beam. To achieve this, a very narrow incident beam waist was generated using a biconvex lens of focal length \( f_1 = 30 \text{ mm} \), denoted \( l_1 \). A Helium Neon (HeNe) laser beam with a peak emission at 632.8 nm was expanded to have a \( 1/e \) intensity radius of 950 \( \mu \text{m} \) and collimated before entering the lens \( l_1 \). The resultant focussed spot had a Gaussian
profile with a 1/e intensity radius of $w = 6.4 \ \mu m$ at its narrowest point, corresponding to $z = 0$. Using Eq. (3) gives $\rho_0 = 56.2$. In order to confirm this, a 20.63 mm long slab of KGd(WO$_4$)$_2$ was placed in the beam with the entrance face at $z = -2 \ \text{cm}$. For this material at 632.8 nm, $n_1 = 2.011$, $n_2 = 2.041$, and $n_3 = 2.095$ \cite{15}. The optic axis was aligned with the incident beam and a colour charge-coupled device (CCD) was placed at $z = 1.03 \ \text{cm}$, corresponding to the focal image plane (FIP). The radius of the conically diffracted ring profile was determined to be $R_0 = 360 \pm 2.3 \ \mu m$. Note that this differs from the predicted value of 409 $\mu m$, a discrepancy which is discussed further in Ref. \cite{16}.

A 10x objective was then used to image the focal image plane onto the CCD, giving a magnification of 3.97. The ring profile was examined by taking intensity profiles along radii at many angles and averaging. This averaged profile was then compared directly to theoretical intensity plots generated using $I = D \cdot D$ from Eq. (5) at various values of $\rho_0$ as shown in Fig. 2. The value which best matched the experimental average was found to be $\rho_0 = 56$, which is in excellent agreement with the value of 56.2 predicted by Eq. (3) using $w = 6.4 \ \mu m$.

![Intensity profile](image)

**Fig. 2.** Averaged radial intensity profile (blue dots) taken from an experimental image (inset) compared to theoretical intensity plots for $\rho_0 = 36$ (solid black line), $\rho_0 = 46$ (dashed black line), and $\rho_0 = 56$ (solid red line).

The objective was replaced with an achromatic biconvex lens $l_2$ of focal length $f_2 = 150 \ \text{mm}$ for all subsequent experiments. This experimental apparatus is shown in Fig. 3. Lens $l_1$ and the biaxial crystal were fixed together on a translation stage with a separation of 3 cm, so that the narrowest beam radius coincided with the crystal’s entrance face at $z = 0 \ \text{cm}$. The relative placement of lenses $l_1$ and $l_2$ allowed the selection of $\zeta$ values.

The biaxial crystal was gradually reoriented so the incident beam was no longer aligned to an optic axis ($u \neq 0$). In order to maximise the visible range of the interference fringes, several images were taken at each location, each with a different CCD exposure level. These images were then combined in order to produce a high dynamic range (HDR) image which allowed very weak features to be observed while not saturating the highest intensity regions. Up to three orders of magnitude are visible in the experimental images Figs. 4, 5, and 6.
4. Results

To examine the low intensity features of the high-dynamic range images generated using the experimental method outlined above, a logarithmic intensity plot was created for each image and compared to theoretical logarithmic intensity plots generated using \( I = \bar{D} \cdot D \) from Eq. (5). In Fig. 4 many fringes are visible along the low intensity arc of the circle described by Eq. (17). These fringes are remnants of the Warnick-Arnold rings [6], features which are visible as symmetric annuli in conically diffracted beams with \( u = 0 \). The location of the low intensity arc of the circle also agrees well with theory.

In Fig. 5 it is possible to see the faint remnants of the Warnick-Arnold rings on the theoretical plot (a) and on the upper half of the experimental image (b). They are visible as weak intensity oscillations and their positions are indicated on the upper half of the theoretical plot using white arcs. The clearer rapid intensity oscillations are as a result of the interference described in Eq. (19), an effect only observed in beams with \( u \neq 0 \). These are clearly visible in both the theoretical plot (a) and the lower half of the experimental image (b). The asymmetry between the upper and lower halves of the experimental image is due to the high sensitivity of the interference pattern to the alignment of the crystal. Although the beam was of course misaligned from the optic axis in the \( \xi \) direction to achieve \( u \neq 0 \), varying the crystal orientation slightly in
the $\eta$ direction was observed to cause the interference fringes to appear and disappear rapidly.

Fig. 5. Comparison of (a) theoretical, and (b) experimental logarithmic intensity plots with $\rho_0 = 56.2, u = 1.8,$ and $\zeta = 31$. The white arcs in the upper part of the images show the remnants of the Warnick-Arnold rings, visible as faint intensity maxima at the locations shown. The white dots in the lower part of the images are the predicted locations of intensity minima from Eq. (19) for $0 \leq n \leq 15$, which occur along an arc of the circle given by Eq. (17).

To examine the interference fringes further, in particular those close to the branch points $\rho_b = \{0, \pm u\}$, the image was magnified by a factor of 13 for $u = 1.8$ and $\zeta = 42$. The results are shown in Fig. 6. The low-intensity fringes are visible in the experimental image (b), and are marked in the lower half of the image by white dots which are solutions of Eq. (19).

Fig. 6. Comparison of (a) theoretical, and (b) experimental logarithmic intensity plots with $\rho_0 = 56.2, u = 1.8,$ and $\zeta = 42$. The white dots in the lower part of the images are the predicted locations of intensity minima from Eq. (19) for $0 \leq n \leq 4$, which occur along an arc of the circle in Eq. (17).

Although the intensity minima in the experimental image are not as clearly defined as those in the theoretical plot due to the large intensity contrast required and saturation of the exper-
5. Conclusion

We have experimentally confirmed the presence of low-intensity interference patterns present when a Gaussian light beam enters a biaxial crystal in a direction close to, but not along an optic axis. Although these patterns exhibited high sensitivity on the orientation of the crystal, it was possible to obtain a stable image with a sufficiently high dynamic range to allow the rich structures to be seen clearly. We anticipate further work on these low intensity features, for example using different incident beam profiles such as top-hat beams.

Although it has been well over a century since conical diffraction was first observed, new features relating to the phenomenon are still being discovered. The interference patterns presented here decorate the transition between double refraction and conical diffraction, a transition which Lloyd [17] eloquently described: “This phenomenon was exceedingly striking. It looked like a small ring of gold viewed upon a dark background; and the sudden and almost magical change of the appearance, from two luminous points to a perfect luminous ring, contributed not a little to enhance the interest.”

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