A Beam-Tendon System with an Eccentrically Mounted Tendon: Parametric Studies

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It has been recently suggested to incorporate a tendon into a rotorcraft blade to provide a means of controlling the blade’s dynamic properties. While the previous studies demonstrated the functionality of such a system, no systematic evaluation of the effect of the tendon cross-sectional location and the effect of the cross-sectional parameters on the response of the system has been conducted. Therefore, the objective of this paper is to perform a series of numerical parametric studies to gain a better understanding of free vibration of the beam-tendon system with an eccentrically placed tendon. The beam-tendon system is modelled using a set of partial differential equations and numerical free vibration analysis is carried out using a boundary problem solver. The sensitivity analysis is conducted to evaluate the effect of the system’s parameters on the natural frequencies of the system and further parametric studies are carried out to assess the effect of the tendon’s eccentricity in detail. The results demonstrate that the location of the tendon significantly influences natural frequency shifts caused by the applied axial load and it is observed that some natural frequencies can even increase with the increasing axial loading for certain locations of the tendon. The paper concludes with a suggestion of practical applications of a beam-tendon system with an eccentrically placed tendon.

I. Nomenclature

\[
\begin{align*}
A & = \text{Area of the cross-section, m}^2 \\
 b & = \text{Width of the cross-section, m} \\
e_y, e_z & = \text{Coordinates of the tendon location, m} \\
E & = \text{Young’s modulus of elasticity, Pa} \\
f & = \text{Natural frequency, Hz} \\
G & = \text{Shear modulus of elasticity, Pa} \\
h & = \text{Height of the cross-section, m} \\
l_{tip} & = \text{Mass moment of inertia of the tip fixture, kg m}^2 \\
l_y, l_z & = \text{Sectional moments of inertia of the cross-section about y and z axis, m}^4 \\
J & = \text{Torsional constant, m}^4 \\
k & = \text{Number of modes considered in sensitivity study} \\
l & = \text{Number of parameters considered in sensitivity study} \\
L & = \text{Total length of the beam and tendon, m} \\
L_y, L_z & = \text{External excitation forces in y and z directions, N} \\
m & = \text{Mass of the beam per unit length, kg m}^{-1} \\
m_t & = \text{Mass of the tendon per unit length, kg m}^{-1} \\
M_{tip} & = \text{Tip mass, kg} \\
p_j & = j\text{th parameter used in the equations of motion} \\
p_0 & = \text{nominal value of a parameter used in the equations of motion} \\
P & = \text{Applied axial force, N} \\
P_{cr} & = \text{Critical force, N} \\
Q & = \text{External excitation moment about elastic axis (torque), N m} \\
r_c & = \text{Polar radius of gyration about mass axis, m} \\
S & = \text{scaled sensitivity matrix}
\end{align*}
\]

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II. Introduction

It has been recently suggested to incorporate a tendon into a rotorcraft blade to provide a means of controlling the blade’s dynamic properties. While the previous studies demonstrated the functionality of such a system [1,2], no systematic evaluation of the effect of tendon cross-sectional location and the effect of the cross-sectional parameters on the response of the system has been conducted. In this paper, a number of parametric studies concerning vibration behaviour of a beam-tendon system with an eccentrically placed tendon is presented. Dynamics of beams and tendons (taut strings, cables) has been previously extensively studied due to their frequent usage in engineering applications. The beams mostly present the main structural elements while tendons are often utilised as actuators or means of vibration control [3,4]. A number of studies that focus on different aspects of beam and string modelling, application and testing has been published [5–8]. A number of studies also investigated vibration of axially loaded beams [9] and their stability [10,11]. It was often assumed that the axial force acts through the centroid of an open cross-section. It has been also acknowledged that, despite not being extensively studied, the eccentric axial loading can significantly influence the structural stability of thin-walled beams [12]. The equations of motion describing the stability of eccentrically loaded thin-walled beams were derived in [11,13], and further augmented by an effect of warping and axial displacement in [10]. In the previous studies, the effect of the axial loading on the structural stability and vibration has been investigated, but it was not considered how such an axial force is applied. In this paper, the axial load is introduced using a tendon which is attached at the tip of the beam, freely passes through its body parallel to the elastic axis, and is fixed and loaded in the same plane as the beam. Since the motions of the beam and the tendon influence each other, they create a coupled beam-tendon system.

A simple beam-tendon system was previously investigated in [14,15] both numerically and experimentally. It was found that the beam-tendon system exhibits reduction in the beam’s natural frequencies due to the tendon-induced axial force as well as veering between beam-dominated and tendon-dominated modes. These findings were further numerically examined in [2] for rotating pre-twisted beams and for a system with intermittently attached tendon in [16,17]. In all these studies, the tendon was attached to the beam’s tip so that the place of the attachment coincided with the shear centre of the beam. Moreover, in the experimental studies [14,15,17] a cantilever beam with a double symmetry cross-section has been used so the elastic and mass axis were the same.

The beam-tendon system, consisting of a mono-symmetric cross-section beam and an eccentrically placed tendon, which is the main focus of the paper, has been also previously studied in [18]. In this study, the numerical model was proposed and validated using a finite element model and a series of experiments. In these experiments, experimental modal analysis was conducted to observe the effect of the tendon location on the frequency-loading diagrams (the relationship between the natural frequencies and the tendon-induced axial load). The experimentally obtained frequency-loading diagrams from [18] can be seen in Fig. 4. Four positions of the tendon were experimentally investigated - location 1 was far away from the centre of gravity of the cross-section in the direction of the shear centre, location 2 was the position of the shear centre, location 3 corresponded to the centre of gravity of the cross-section, and location 4 was located further from the shear centre in the direction of the centre of gravity. It is clear that depending on the position of the tendon, some of the natural frequencies may increase due to the applied axial load. This observation, which
motivates the present study, is not intuitive as the axial load is usually associated with the loss of stability and decrease in the natural frequencies.

Preliminary parametric study on the effect of the tendon location on the natural frequency of the first three vibration modes (first out-of-plane bending, first in-plane bending and first torsion) was presented in [18] and it is repeated here to introduce the main observations made therein. The parametric studies were performed for three magnitudes of the applied forces: \( P = 200, 400 \) and \( 600 \) N. The location of the tendon varied between \( e_z \in [-h, h] \) and \( e_y \in [2y_0, -y_0] \), where \( e_z \) and \( e_y \) are the co-ordinates of the tendon location, \( h \) in the height of the cross-section and \( y_0 \) is the position of the elastic axis. The resulting variation in the natural frequencies can be seen in Fig. 2. The main observation is that the first lead-lag mode (and all other lead-lag modes) is not influenced as much as the flap and torsion modes. Even for the highest applied force, its natural frequency is shifted by approximately 0.5% while the changes for the other two modes are about 20%. The natural frequency of the first mode (1F) is highest when the tendon is placed in the shear centre and always lower for any other location. On the other hand, the natural frequency of the first torsion mode is higher when the tendon is placed in \( 2y_0 \). It can be also noticed that the flapping and torsion modes are mainly influenced by \( e_y \) while the lead-lag mode is influenced mainly by \( e_z \). All these observation were also confirmed by the experimental data presented in [18]. In addition, it can be noticed that the natural frequency of the first lead-lag mode varies not only with \( e_z \) but also with \( e_y \) for \( e_z \neq 0 \). This is given by the fact that the general location of the tendon leads to the coupling between the two bending directions, and hence the natural frequencies are also influenced. The effect of \( e_z \) on the 1L is however small when compared to the effect of \( e_y \) on flapping modes. In this parametric study, a single case was investigated in detail, but no systematic parametric studies of the effect of the cross-sectional parameters on the modal properties of the system have been conducted, so it is not clear whether the increase in the natural frequencies can be observed for other system’s parameters as well. Such parametric studies are the subject of this paper.

The paper is organised as follows: the theoretical model of the system, given by a set of partial differential equations...
III. Computational free vibration of beam-tendon system

The system under consideration is an isotropic straight thin-walled cantilever beam with a mono-symmetric cross-section loaded by an eccentrically placed tendon as depicted in Fig. 3. The eccentrically placed tendon is attached at the tip of the beam, passes through its whole body parallel to the elastic axis, and is fixed and loaded in the same plane as the beam. Unlike in the previous studies [1,2,15,19], the tendon does not coincide with the elastic or mass axis and can be placed in any point of the cross-section. The tendon is attached using a tip fixture, characterised by its mass and moment of inertia. The tip fixture is assumed to be perfectly rigid and allows the tendon to be placed even outside the beam’s cross-section. The motion of the beam is described by flapping (out-of-plane, vertical bending in the z direction) displacement of the elastic axis, lead-lag (in-plane, horizontal bending in the y direction) displacement of the elastic axis, and torsion of the cross-section about the elastic axis. The out-of-plane bending and torsion motions are mutually coupled through the cross-sectional offset of the elastic and mass axes. The parameters characterising the beam are uniformly distributed, i.e. they do not vary along the span of the beam. It is assumed that the beam meets all the requirement of the Euler-Bernoulli theory and can therefore be modelled by the linear Euler-Bernoulli theory with warping [10,11]. It is further assumed that the static bending deflection caused by an eccentric placement of the tendon is small and can be neglected. The tendon is modelled using the wave equations and it is assumed that it does not change its cross-sectional area under loading [8]. Both the beam and the tendon are fixed (clamped) at one end and coupled with each other through the geometrical and loading boundary conditions at the other end. At the tip,
Fig. 3 The beam-tendon system \cite{18}: it consists of an isotropic thin-walled cantilever beam which features an offset between the mass and elastic axes, and which is subjected to an eccentric tendon-induced axial load. The tendon is connected to the tip of the beam, passes freely through its whole body parallel to the elastic axis and is fixed and loaded in the same plane as the beam.

Their displacements are identical and the tendon-induced axial force contributes to the shear and moment boundary conditions of the beam. Unlike \cite{16, 17}, no other connectivity conditions were enforced since the tendon is free to vibrate inside the beam.

The equations of motion (EoM) describing the coupled beam-tendon system consist of three equations for the motion of the beam, and two equations for the motion of the tendon. The Hamilton’s principle can be used to derive the equations of the axially loaded beam as detailed in \cite{10} as well as the wave equations describing the tendon as shown in \cite{5}. The EoM describing the coupled beam-tendon system can be then written as

\begin{align}
EI_z w'''' + Pw'' - P(y_0 - e_y)\phi'' + m(\ddot{w} - y_0\ddot{\phi}) = L_z, \\
EI_y v'''' + Pv'' - Pe_y\phi'' + m\ddot{v} = L_y, \\
\Gamma \phi'''' - (GJ - P\beta_y e_y - Pr^2_c)\phi'' - Pe_z v'' - P(y_0 - e_y)w'' + m(r^2_c \ddot{\phi} - y_0 \ddot{w}) = Q, \\
- Pw'' + m_t \ddot{w}_t = 0, \\
- Pv'' + m_t \ddot{v}_t = 0.
\end{align}

where $\beta_y$ is a cross-sectional parameter defined \cite{10} as

$$
\beta_y = \frac{1}{I_y} \int_A y(y^2 + z^2) \, dA - 2y_0,
$$

and the notation is provided in the Nomenclature.

The equations of motion are accompanied by a set of boundary conditions (BCs) which ensure the coupling between the beam and the tendon. The BCs at the fixed end of the system for $x = 0$ are

$$w = w' = v = v' = \phi = \phi' = w_t = v_t = 0. \tag{2}$$

The BCs for the free end (for $x = L$) in which the connection between the beam and the tendon is reflected can be written as

\begin{align}
- Pw' + P(y_0 - e_y)\phi' - EI_z w'''' + M_{tip} \ddot{w} + Pw'_t = 0, \tag{3a} \\
EI_z w'' = 0, \tag{3b} \\
- Pv' + Pe_z \phi' - EI_y v'''' + M_{tip} \ddot{v} + Pv'_t = 0, \tag{3c} \\
EI_y v'' = 0 \tag{3d}.
\end{align}
with the BCs, define a boundary value problem. This boundary value problem is then solved by the Matlab bvp4c
solver [20] for the unknown natural frequencies \( \omega \) and corresponding mode shape components \( W(x), V(x), \phi(x), W_i(x), V_i(x) \). This solver is very versatile since it uses a collocation method but may suffer from a decreased numerical performance if an appropriate starting guess is not provided.

Equations (1)–(3) create a system of three fourth-order and two second-order partial differential equations with the corresponding number of boundary conditions. The novelty of these equations and a detailed description of individual terms can be found in [18]. All these equations must be solved simultaneously as there is the coupling between the motions of the beam and the tendon due to asymmetry of the beam’s cross-section and the connection between the tendon and the beam.

In order to evaluate the modal properties (natural frequencies and mode shapes) of the beam-tendon system, the procedure as in [15,17] is used. The excitation forces and moments in Eq. (1) are set to zero and an assumption of the time-invariant mode shape and the time-varying harmonic function of the constant frequency in the following form

\[ w(t, x) = W(x)e^{j\omega t}, \quad v(t, x) = V(x)e^{j\omega t}, \quad \phi(t, x) = \Phi(x)e^{j\omega t}, \]

\[ w_i(t, x) = W_i(x)e^{j\omega t}, \quad v_i(t, x) = V_i(x)e^{j\omega t}. \]

The complete mode shape of the system can then be formally written as \( \Psi_i(x) = [W(x), V(x), \Phi(x), W_i(x), V_i(x)] \). In the rest of the paper, if the mode shape is dominated by the motion of the beam, it will be refer to as a beam-dominated mode. On the other hand, if the mode shape is determined by the tendon activity, the term tendon-dominated mode will be used instead.

As in the previous studies [15,17], substituting the normal mode forms into Eqs. (1), (2) and (3) allows one to eliminate time and rewrite the PDEs into a system of first order ordinary differential equations (ODEs) that, together with the BCs, define a boundary value problem. This boundary value problem is then solved by the Matlab bvp4c solver [20] for the unknown natural frequencies \( \omega \) and corresponding mode shape components \( W(x), V(x), \phi(x), W_i(x), V_i(x) \). This solver is very versatile since it uses a collocation method but may suffer from a decreased numerical performance if an appropriate starting guess is not provided.

IV. Results

In this section, the studies performed using the mathematical model introduced in the previous section are conducted. Firstly, a sensitivity study for the system presented in [18] is performed in section IV.A. Then, a focus is placed on the effect of the eccentrical placement of the tendon in section ??.

A. Sensitivity study

A sensitivity study is a powerful tool for interrogating the underlying physics of the system. The sensitivity study of the beam-tendon system is performed for the parameters of the system which was experimentally characterised in [18]. The parameters of this system, in the rest of the paper termed as case A, are also summarised in Table 1. The sensitivity of a natural frequency \( \omega \) to a parameter \( p \) is found as:

\[ \frac{\delta \omega_i}{\delta p_j} \approx \frac{\omega_i(p_{0j} + \Delta p_j) - \omega_i(p_{0j} - \Delta p_j)}{2\Delta p_j}, \text{ for } i = 1, \ldots, k, \quad j = 1, \ldots, l \]  

(5)

The sensitivities were evaluated in a proximity of nominal values from Table 1 with \( \Delta p \) chosen to be 0.1% of the nominal value. The sensitivities can be organised in the scaled sensitivity matrix as

\[
\mathbf{S} = \begin{pmatrix}
\frac{\partial \omega_1}{\partial p_1} & \frac{\partial \omega_1}{\partial p_2} & \cdots & \frac{\partial \omega_1}{\partial p_k} \\
\frac{\partial \omega_2}{\partial p_1} & \frac{\partial \omega_2}{\partial p_2} & \cdots & \frac{\partial \omega_2}{\partial p_k} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \omega_k}{\partial p_1} & \frac{\partial \omega_k}{\partial p_2} & \cdots & \frac{\partial \omega_k}{\partial p_k}
\end{pmatrix}
\]  

(6)
The results of the sensitivity analysis are shown in Fig. 4 where three applied forces, which are normalised by the critical force, have been considered. It can be seen that the most influential parameter for the change of the natural frequencies of the beam as well as the tendon is the system’s length $L$. The parameter has a prominent influence on all the investigated modes whereby its increase leads to a significant decrease of all natural frequencies. Besides the length of the system $L$, the tendon natural frequencies are only influenced by the tendon’s own mass $m_t$ and the applied tension $P$. No other parameters have any impact on the natural frequencies of the tendon. Therefore, it can be concluded that it does not matter where the tendon is placed or what type of beam it is placed in, its natural frequencies can be tuned independently by considering only its length, mass and tension. The influence of the tip parameters, $M_{\text{tip}}$ and $I_{\text{tip}}$, is very small and for the case of the $I_{\text{tip}}$ it is practically negligible. The flapping stiffness $EI_z$ can be seen to influence all the flapping modes as well as the torsional mode. This is caused by the fact that these modes are coupled bending-torsion modes. In contrast, the lead-lag stiffness $EI_y$ has impact on the lead-lag modes only since there is no coupling involving lead-lag. The torsional parameters, $GJ$ and $r_c^2$, influence $F_2$, $F_3$, and $T_1$ due to the bending-torsional coupling, but have little impact on $F_1$. The same can be said about the offset of the elastic and mass axis $y_0$. The warping stiffness $E\Gamma$ and the cross-sectional parameter $\beta_y$ have a very little impact on the natural frequencies of the system.

It can be noticed that the impact of the system parameters on its natural frequencies varies with the level of loading used in the nominal case. It appears that the natural frequencies of the beam are more sensitive for higher applied load $P$. The natural frequencies are not very sensitive to the parameters $e_y$ and $e_z$ that govern the placement of the tendon when compared to the other structural parameters. However, it will be shown in the next section that the eccentric placement of the tendon can significantly change the qualitative behaviour of the system when larger variation of $e_y$ is examined.

**B. The effect of the eccentric placement of the tendon**

The effect of the eccentric placement of the tendon will be investigated for three different cases of the beam whose parameters are summarised in Table 1. The parameters of the first case, termed as the case A, are exactly the same as in [18] with the exception of the mass of the tendon due to the reason explained in the following paragraph. The length of the cross-section of the other two cases is doubled and halved, respectively, while other independent parameters remain unchanged and the dependent parameters are evaluated accordingly. The parameters are normalised by the parameters from [18] for which the experimental validation was performed. Therefore, all independent and dependent parameters of the case A in Table 1 are equal to 1 (with the exception of the tendon mass) which means that all parameters are equal to those from [18]. For the case B, the cross-sectional length is equal to 2 which means it is twice of that of the case A, i.e. 0.0508 m, while other independent parameters are equal to 1, i.e. they are the same as in [18], and the dependent parameters are evaluated based on the doubled length of the cross-section. The same is true for the case C, but the cross-section length is equal to 0.5, i.e. 0.0127 m and the other parameters are evaluated based on this length.

The parameter that will be extensively used in the rest of the paper is the chord-wise eccentricity of the tendon. This
Table 1 Three beam-tendon cases for which the effect of tendon eccentricity is explored in detail. The parameters are normalised with respect to the parameters from [18].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Case A</th>
<th>Case B</th>
<th>Case C</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>0.0127 m</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>0.0254 m</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$t_w$</td>
<td>0.001 626 m</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$t_f$</td>
<td>0.001 626 m</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$L$</td>
<td>1 m</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$m$</td>
<td>0.259 kg m$^{-1}$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$E$</td>
<td>$5.9818 \times 10^{10}$ Pa</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.3</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$M_{tip}$</td>
<td>0.045 kg</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$I_{tip}$</td>
<td>$1.0805 \times 10^{-7}$ kg m$^2$</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$m_t$</td>
<td>0.011 kg m$^{-1}$</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>$G$</td>
<td>$2.3007 \times 10^{10}$ Pa</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$I_z$</td>
<td>$2.6649 \times 10^{-9}$ m$^4$</td>
<td>1</td>
<td>2.151</td>
</tr>
<tr>
<td>$I_y$</td>
<td>$6.2746 \times 10^{-9}$ m$^4$</td>
<td>1</td>
<td>1.9571</td>
</tr>
<tr>
<td>$J$</td>
<td>$8.8664 \times 10^{-11}$ m$^4$</td>
<td>1</td>
<td>7.0168</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$1.6382 \times 10^{-13}$ m$^6$</td>
<td>1</td>
<td>1.821</td>
</tr>
<tr>
<td>$r_c^2$</td>
<td>$5.6923 \times 10^{-4}$ m$^2$</td>
<td>1</td>
<td>7.3481</td>
</tr>
<tr>
<td>$\beta_y$</td>
<td>0.04586 m</td>
<td>1</td>
<td>4.3606</td>
</tr>
<tr>
<td>$y_0$</td>
<td>$-0.021 863$ m</td>
<td>1</td>
<td>2.15</td>
</tr>
</tbody>
</table>

is marked by $e_y$ in the equations of motion, and in the following it is normalised by the offset of the mass and elastic axis for each case as $\overline{e_y} = e_y/y_0$. For instance, when the tendon coincides with the elastic axis, $\overline{e_y} = 1$, and when the tendon is placed in the centre of gravity of the cross-section, $\overline{e_y} = 0$. The eccentric tendon placement between $\overline{e_y} = -2$ and $\overline{e_y} = 2$ will be explored in detail.

Before examining the effect of the tendon eccentricity, it is convenient to eliminate the vibration modes of the tendon from consideration. The reason for this is the presence of veering between tendon-dominated and beam-dominated modes. If the veering is present, the natural frequency of the beam-dominated modes can be locally increasing, but not due to the eccentricity of the tendon placement, but rather due to the nature of the veering. Therefore, in order to accurately assess the effect of the tendon eccentricity on the natural frequencies of the beam, the mass of the tendon was decreased so that there are no tendon-dominated vibration modes in the considered range of loading and frequencies.

To show that the impact of decreasing the tendon mass on the beam’s frequency loci is minimal, two frequency-loading diagrams are shown in Fig. 5. It can be seen that the frequency loci of the system with significantly lower tendon mass (red dashed lines) follow the frequency loci of the unchanged system (blue solid lines) very well. Most importantly, all the features of the frequency-loading diagrams are retained - the frequency loci have the same slope and the critical force is the same. The veering is however completely removed and therefore it cannot influence the results of the further parametric studies. The slight frequency offset visible in Fig. 5 is given by the lower mass of the system and it is considered unimportant with regards to the present studies. All the following calculations have been carried out for the mass of the tendon equal to 1% of its nominal value as documented in Table 1.

1. The effect of the tendon position on the critical force ($P_{cr}$ vs $e_y$)

It is well known that the beams subjected to axial loading lose their stability when the frequency of the first mode drops to zero as a consequence of the applied force. The applied force causing the loss of stability is termed as the critical force and it is an important characteristic of the system. In the beam-tendon system, the beam can also lose stability due to the tendon-induced axial force. The effect of the tendon eccentricity $\overline{e_y}$ on the critical force for all three beams is shown in Fig. 6. It can be seen that the critical force varies for the cases A and B, but it remains constant for
The effect of the tendon mass on the frequency-loading diagram of the system. The frequency loci slopes and the critical force are not affected by the tendon mass, while the veering regions are removed.

The effect of the tendon eccentricity on the critical force of the system. The case C. The reason for this is the fact that the case C loses its stability on the first lead-lag mode as opposed to the first flapping mode for the cases A and B (the corresponding mode shapes can be seen in Fig. 7). For the cases A and B, the maximum critical force always occur when the tendon is placed in the elastic centre since the stability is lost purely on the first bending mode. For the eccentric placement of the tendon, the bending-torsional coupling is increased and the beam loses stability through a combination of bending and torsion. This fact has been also observed in previous studies [10, 13, 18]. It is important to note the effect of $e_y$ on the magnitude of the critical force. In particular, it can be seen in Fig. 6 when the tendon is placed far from the elastic axis, the critical force is significantly reduced. For the case B, which has the higher critical force for $e_y \in [2, -0.1666]$, the critical force falls below the one of the case A at $-0.1666$ and both of these critical forces are lower than the critical force of the case C close to $e_y = -2$. The variation of the critical force with the tendon location is therefore very significant. When $e_y = 1$, the critical force of the case C is approximately 6 times smaller than the critical force of the case B. The critical force for the case B is lower than that of the case C when close $e_y = -2$. It can be therefore concluded that the position of the tendon as well as the tendon-induced axial force
have a significant influence on stability of the beam-tendon system.

The mode shapes corresponding to the loss of stability can be seen in Fig. 7. As previously mentioned, the case C

losses stability on the first lead-lag mode which does not change with the variation of $\bar{e}_y$. On the other hand, the other two cases lose stability on the first flapping mode whose mode shape is a combination of the beam bending and torsion. When the tendon is placed in the elastic axis, there is little amount of torsion present, but when the tendon is moved further away from the elastic axis, the amount of torsion increases, and the character of the stability loss changes while the critical force decreases.

2. Frequency-loading diagrams ($\omega$ vs. $P$)

The frequency-loading diagrams are one of the main tools for the study of the natural vibration of the beam-tendon systems. Usually, all the natural frequencies are plotted in one graph, such as in Fig. 5 but in this section, each natural frequency is plotted separately. In addition, the natural frequencies are normalised to the frequencies of the unloaded beams. This allows one to compare the frequency loci of different modes with each other and with frequencies of other beams as well. The frequency-loading diagrams for the flapping and torsional modes are shown in Fig. 8. A number of observations can be made from these diagrams:

- The natural frequency of the first flapping mode never increases with the applied axial force for any position of the tendon and any system parameters. However, it can be seen that the frequency loci shapes vary with the position of the tendon. For instance, in Fig. 8(a) starting with the blue frequency locus for $\bar{e}_y = 2$ it can be seen that for decreasing $\bar{e}_y$, the loci initially shift to the right, but then turn the direction of their change to left at $\bar{e}_y = 1$ and continue shifting to the left with decreasing $\bar{e}_y$. Therefore, the same frequency loci, i.e. the same reduction of the first natural frequency by the applied tension, can be achieved for multiple position of the tendon when $\bar{e}_y \in [0.3, 2]$. Similar conclusions can be made about natural frequencies of the first flapping mode of other two cases.

- The second mode is qualitatively different than the first one. It can be clearly seen that for some positions of the tendon, the frequency loci increase, i.e. the applied tendon-induced axial force leads to the increase of the natural frequencies. This may seem counter-intuitive but similar results have been also observed experimentally, and they can be explained by the presence of the additional force components arising from the boundary conditions at the tip of the beam and acting in the transversal directions. The rate of increase of the natural frequencies is different for each case. While for the case A the natural frequency increased by as much as 20% and for the case B the increase is close to 80%, for the case C it is only close to 2%. Moreover, it can be noticed that for the case A the frequency locus corresponding to $\bar{e}_y = 1$ is almost flat, i.e. the frequency is almost unchanged. On the other hand, for the case B the same frequency locus increases, and for case C it decreases. In addition, while for the case A and B the frequency loci change smoothly with the change of the tendon’s position, the case C shows a different trend. The maximal increase is achieved for $\bar{e}_y = 2$ and maximal decrease achieved for $\bar{e}_y = 0$. The trend then reverses and for $\bar{e}_y = -2$ the frequencies are very close to the frequency of the unloaded system. This means that
Fig. 8 Frequency-loading diagrams. The first column is for case A, the second for B, and the third for C. The first row shows the first flapping mode, the second row is for the second flapping mode, the third for the third flapping mode, and the fourth is the torsional mode.

A similar frequency decrease can be achieved for the selected tendon’s position in a range of $\epsilon_y \in [-2, 0]$.

- For the third flapping and first torsional modes, the frequency evolution is similar to the second flapping mode, with a notable exception of the case C. For the case C, the two natural frequencies do not increase at all and, in fact, exhibit a significant decrease. Unlike in the case of the second flapping mode, the highest and lowest frequency loci correspond to $\epsilon_y = 2$ and $\epsilon_y = -2$, respectively.

3. The effect of the tendon position on the natural frequencies ($\omega$ vs. $\epsilon_y$)

To further examine the effect of the tendon position on the natural frequencies, the relation between $\epsilon_y$ and $\omega$ for four selected applied forces can be seen in Fig. 9. The curves in Fig. 9 represent the slices of the frequency-loading diagrams in Fig. 8. It can be easily seen that the cases A and B are qualitatively similar to each other and the case C is very different. As for the frequency-loading diagrams, the following observations can be made:

- For the first flapping mode of the nominal case in Fig. 9(a), it can be seen that for the lowest applied force, the
natural frequency does not change significantly for any position of the tendon. On the other hand, for the higher applied forces, the position of the tendon significantly influences the first natural frequency, and can also lead to the loss of stability (when the frequency drops to zero).

The situation is slightly different for the first flapping mode of the case B in Fig. 9(b). This case looses stability for any considered applied load and the reduction in the frequency is significant.

As previously said, the graph of the first flapping mode of the case C in Fig. 9(c) is very different. In this graph, no frequency drops to zero is found, i.e. the stability is not lost. This is given by the fact that the stability is lost on the first lead-lag mode. Consequently, the reduction in the natural frequency is not so significant as for the two previous cases.

• The second flapping mode is seen in the second row of Fig. 9. The cases A and B are again very similar, but the case B has a lower range due to the critical force. It is important to notice that there appears to be a single point in

**Fig. 9** The effect of the tendon eccentricity on the natural frequencies of the system for four applied forces. The first column is for case A, the second for B, and the third for C. The first row shows the first flapping mode, the second row is for the second flapping mode, the third for the third flapping mode, and the fourth is the torsional mode.
which all the curves cross each other. If such a point indeed existed and the tendon was placed in it, it would be possible to load the beam by the axial force without changing the natural frequency of the second flapping mode. However, it will be shown in Fig. 10 that the curves do not intersect each other in a single point.

A similar behaviour can be seen for the second flapping mode of the case C (Fig. 9(f)) as well. However, the curves are qualitatively very different - the part of the curves above the nominal frequency is very small, so for most of the tendon positions, the frequency will be reduced. The reduction is not monotonic, but it reaches its minimum at $\bar{e}_y = -0.25$ and then reverses back and moves closer to the nominal frequency again. It must be emphasised, however, that there is only a small difference in the frequency (less than 4% as opposed to up to 100% for the cases A and B).

- Similarly to the frequency-loading diagrams, the third flapping mode and the first torsional mode are similar in nature. For the cases A and B both natural frequencies can be increased or decreased based on the position of the tendon, and the range of the stable tendon positions is reduced for the case B. For the case C, both natural frequencies always decrease.

4. Is there a position of the tendon for which the applied axial force does not change the natural frequencies?

Lastly, to see if there is a position of the tendon for which the tendon-induced axial force does not change the natural frequencies, the position of the tendon $\bar{e}_y$ for which the natural frequencies are not changed, for given applied load $P/P_{cr}$, was found and it is shown in Fig. 10. From the curves shown in Fig. 10 it can be seen that there is no position of the tendon for which any of these frequencies stays constant for multiple magnitudes of the applied force. Moreover, each of these frequencies require a different loading and tendon position to stay unchanged. This means that it is not possible to tune the beam-tendon system in such a way that one of the frequencies is not influenced by the tendon.

It is also interesting to realise that the curves shown in Fig. 10 define the boundary between the decrease and increase of the natural frequencies. If, for a given applied force, the position of the tendon is under the curve shown, the frequency will be reduced by the axial force and vice versa. Interestingly, for each studied case this boundary between increase and decrease is very different. For the case A this boundary is located close to the elastic axis, for the case B it is in between the elastic axis and mass axis, and for the case C, it is far away from the elastic axis and even the cross-section.

Since the findings presented in this section vary for each case, it seems that no general rule of thumb about the frequency-loading evolution can be given. It seems that for each set of system parameters, the analysis must be carried out separately.
5. A note on the lead-lag modes

All previous results and findings have been made for flapping and torsion modes only. The lead-lag modes have not been discussed. The reason for this is that it has been already observed in [18], and reiterated in introduction, that there is a negligible influence of $e_y$ on the lead-lag modes. The effect of $e_z$ was observed in [18] and it was concluded that the actual effect of this parameter on the lead-lag natural frequencies is very small. However, it should be noted that when $e_z \neq 0$ a coupling between flapping, torsion and lead-lag motion is exhibited, but as said, the effects of this coupling is very small.

V. Discussion

In this paper, the theoretical model of the beam axially loaded by an eccentrically placed tendon, which was previously experimentally characterised in [18], has been introduced and used to study the effect of the tendon’s eccentricity on the natural frequencies of the system. In this section, the main findings, and a possible application of the beam-tendon system are discussed.

The effect of the tendon location was examined in detail and the main findings can be summarised as follows:

• The location of the tendon has significant impact on the beam-dominated modes, while the tendon-dominated modes remain the same.
• Due to the additional coupling introduced by an eccentrically located tendon, veering between the beam-dominated and tendon-dominated modes is complex and bending-bending-torsion coupling can be observed.
• Each natural frequency is affected differently by the tendon’s location. For the investigated systems, it can be concluded that (i) the natural frequency of the first mode always decreases regardless of the tendon’s location, (ii) the natural frequencies of other flapping and torsional modes may increase or decrease depending on the tendon’s position, but there is no position for which these frequencies remain unchanged for any magnitude of the applied load, and (iii) the lead-lag modes are influenced by the eccentric placement of the tendon only when the tendon is not placed on the axis of symmetry.
• The cross-sectional location of the tendon has significant impact on the stability of the system. The critical force is at its maximum when the tendon is placed in the shear centre and the stability is lost due to pure bending of the beam in this case. The critical force decreases with the distance from the shear centre and the loss of stability due to bending-torsion is observed.

The motivation behind this research was the application of an active blade-tendon system in rotorcraft as suggested in [2, 21]. It was shown that the blade-tendon system with the tendon placed in the elastic axis can be used as a means of resonance avoidance [1, 22]. The active tendon concept should ultimately introduce control capabilities by which the natural frequencies of rotorcraft blade could be adjusted and, in turn, this should enable rotorcraft to operate over a wider range of rotor cases or rotor speeds, thereby increasing their efficiency and decreasing their fuel burn, air pollution and noise emissions in the future. As seen in this paper, the sectional location of the tendon can significantly influence the modal properties of the underlying beam/blade as well as its stability. Therefore, the future development of the active tendon concept should take this into account and examine the possibility that the resonance avoidance can be achieved not only by the applied force, but also by adjusting the position of the tendon within the cross-section.

VI. Conclusion

In this paper, the beam-tendon system with an eccentrically placed tendon was studied. The effect of the tendon location was investigated in detail and it was observed that (i) it has a significant effect on the natural frequencies of the beam-dominated modes while it does not influence the tendon-dominated ones, (ii) the natural frequency of the first system mode decreases regardless of the location of the tendon, while the natural frequencies of other modes can increase or decrease depending on the specifics of the tendon’s location and (iii) the location of the tendon has a significant effect on the structural stability and the critical force reaches its maximum when the tendon is placed in the shear centre.

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References


