Second order approximations in model updating

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Abstract
The local quality of model parameterisation is one of the key ingredients of the methods used in the experimentally-based model updating of the mechanical systems. Traditional measures of the parameterisation quality are based on the sensitivity of the primary observed structural responses to the changes in model parameters. This paper explores the use of second order information in model updating. While the use of these data is reported in the general literature on inverse problems and nonlinear regression, the use of this information is relatively unexplored in the field of model updating. It can be shown that there exists a direct link between regularisation approaches used in model updating and the second order derivatives originating from the extended part of the Taylor series approximation of the model data quadratic distance measure. In this sense, the second order derivatives act as regularising information in the model updating problem. The goal of the paper is to evaluate the use of second order derivatives in model updating. The theoretical concepts will be applied to the real model updating case study of a three story aluminium frame structure.

1 Introduction

The purpose behind this work is to provide initial investigation into algorithmic aspects and qualitative effects stemming from the use of second order sensitivity information, i.e. second order derivatives, in the context of model updating. This topic is not widely covered within the model updating community, while it can be speculated that the use of the second order approximations can have some useful features for further consideration. Two potential uses, excluding full scale second order algorithm deployment, is the use of this mechanism within or with standard sensitivity based methods (e.g. during initial updating stages), or, alternatively, to use the second order information for preliminary (or intermediate) qualitative problem assessment, particularly with respect to parameterisation extent and nonlinear aspects.

The use of second order derivatives in the area of model updating was demonstrated by Ojalvo and Pilon [1]. Another work published in closely related field of general inverse problems was presented by Hettlich and Rundell [2]. Both publications are relevant to currently presented work. The methods introduced in [1] and [2] will be discussed later in section 3. The problem of model updating and its link to the algorithms including second order derivatives is presented by Titurus and Friswell [3]. The main focus of paper [3] is on regularisation and, in this context, the use of second order information can be seen as a form of problem regularisation. Similarly, this approach is also adopted in nonlinear regression community as demonstrated by Seber and Wild [4]. The methods exploiting the use of second order derivatives use the formulas for the first and second order eigenvalue and eigenvector derivatives summarised in [5], [6], [7] and [8].

This paper is divided as follows: Section 1 provides brief summary of previous uses of the second order information. Section 2 introduces standard terminology, notation and some assumptions applied in model updating. Section 3 represents the central part of the theoretical presentation delivered by this paper and
section 4 demonstrates the use of second order derivatives in the updating process. The final summary and conclusions are presented in section 5.

2 Model updating: basic problem statement

Let us assume that a number of structural responses are selected to constitute a representative basis describing behaviour of the structural system. The structural responses used in this work are modal properties corresponding to the undamped or lightly damped system. A generic notation will be adopted here for these, where \( z_i \) represents \( i \)-th response. The main assumption in model updating is that it is possible to measure the corresponding experimental values on the existing structure. This measured response is denoted here as \( z_m \).

Secondly, the model providing \( z_i \) is parameterised by a set of carefully selected parameters \( p_j \) (i.e. parameters that are thought to be, directly or indirectly, sources of discrepancy between the model and the measured observations). The number of response quantities is assumed to be \( N_z \) and the number of parameters is \( N_p \). A number of scenarios is generated by specific relationships between \( N_z \) and \( N_p \). These thoughts are deferred for later consideration applied in the context of the case study, or whenever they are needed to clarify the dimensional structure of the problem. Further, responses are assumed to be assembled in the response vector \( z \in \mathbb{R}^N_z \), and parameters are stored in the parameter vector \( p \in \mathbb{R}^N_p \), where \( \mathbb{R}^N_p \) usually represents physically feasible parameters, e.g. \( p_j > 0 \). The model generating instances of the vector \( z \) is denoted \( \mathcal{Z}(p) \) and its inverse is \( \mathcal{Z}^{-1}(z) \).

The ability to accumulate measured responses in the vector \( m \) gives rise to the consideration of deterministic model updating as the problem of minimising the “distance” between \( z = \mathcal{Z}(p) \) and \( z_m \)

\[
\min \text{dist}(\mathcal{Z}(p), z_m) \quad \text{for} \quad p \in \mathcal{D}_p \subseteq \mathbb{R}^{N_p}.
\]

where \( \text{dist}(a, b) \) is some measure of difference between the two vectors. One suitable form to implement this measure is via the generalised vector norm, \( ||a||_W^2 = a^T W a \), in the following form

\[
\min ||\delta z(p)||_{W_z} = \min (\mathcal{Z}(p) - z_m)^T W_z (\mathcal{Z}(p) - z_m), \quad p \in \mathcal{D}_p
\]

where the weighting matrix \( W_z = \text{diag}(w_{z,j}) \) will be assumed such that \( w_{z,j} = 1 \) and \( \delta z(p) = \mathcal{Z}(p) - z_m \).

Problem (2) is a nonlinear least-squares type problem with the cost function \( J_{\text{LS}} = ||\delta z(p)||_{W_z}^2 \). The result of the operation suggested by equation (2) is one possible implementation of the inverse problem of finding the parameter vector, where \( p_m \approx \mathcal{Z}^{-1}(z_m) \). The practical ways that can be employed to address this task are summarised in the following section.

3 Local approximations in iterative model updating

3.1 Quantities used in iterative model updating

To build a more general framework for further algorithmic consideration and exploitation of the structure of the problem (2), it is useful to consider the gradient vector and the Hessian matrix of the scalar function \( J_{\text{LS}}(p) \). A more detailed description of this representation is provided in [3]. The gradient vector \( g \) of \( J_{\text{LS}} \) is

\[
g(p) = \nabla J_{\text{LS}}(p) = \left[ \frac{\partial J_{\text{LS}}(p)}{\partial p_i} \right] = 2 S^T (z - z_m) \in \mathbb{R}^{N_p},
\]

(3)
and the Hessian matrix $H_J$ of $J_{LS}$ is

$$H_J(p) = \nabla^2 J_{LS}(p) = \left[ \frac{\partial^2 J_{LS}(p)}{\partial p_i \partial p_j} \right] = 2 \sum_{i=1}^{N} \mathbf{H}_i(z_i - z_{mi}) + S^T S \in \mathbb{R}^{N_x \times N_p},$$

where $S$ is Jacobian matrix of $Z(p)$ and $H_k$ is Hessian matrix of response $z_k$. The matrices $S$ and $H_k$ are defined as

$$S(p) = \nabla z = \left[ \frac{\partial z_i(p)}{\partial p_j} \right] = \left[ s_{ij} \right] \in \mathbb{R}^{N_x \times N_p}, \quad H_k(p) = \nabla^2 z_k = \left[ \frac{\partial^2 z_k(p)}{\partial p_i \partial p_j} \right] = \left[ h_{k,lm} \right] \in \mathbb{R}^{N_x \times N_p}.$$

In the current work, finite element models are used, where the structural mass, damping and stiffness matrices are parameterised, i.e. in general $M=M(p)$, $D=D(p)$ and $K=K(p)$. The systems are assumed to have symmetric matrices, low or zero damping ($D \approx 0$), and a simple structure (i.e. diagonalizable). In this context, the structural responses are taken to be $N_z$ undamped natural frequencies $\omega_k$.

### 3.2 Local approximations for parameter stepping in model updating

The two useful forms that can serve as a starting point for demonstration of the parameter stepping strategies, as well as for demonstration of the links that exist between them, are based on a truncated Taylor series expansion to the second order at the current parameter point $p_k$.

The first form that was presented in the context of model updating in [3] is based on the local approximation of the scalar cost function $J_{LS}(p)$. The use of the Taylor series expansion in this case leads to the quadratic approximation $J_{NR}(\delta p_k)$ around $p_k$, in terms of the quantity $\delta p_k = p - p_k$

$$J_{NR}(\delta p_k) = J_{LS,k} + g_{i,k}^T \delta p_k + \frac{1}{2} \delta p_k^T H_{J,k} \delta p_k$$

where $J_{LS,k} = J_{LS}(p_k)$, $g_{i,k} = g(p_k)$, and $H_{J,k} = H_k(p_k)$.

The second useful form exploits the second order Taylor expansion and is based on the selected vector of the model responses $z = Z(p)$. The application of the Taylor series expansion for the individual response $z_i$ around the parameter point $p_k = \{ p_{k,i} \}$ leads to the following relationship

$$z_{Q,k,i} = z_k + \sum_{r=1}^{N} s_{i,r}^k \delta p_{k,r} + \frac{1}{2} \sum_{r=1}^{N} \sum_{s=1}^{N} h_{i,r,s} \delta p_{k,r} \delta p_{k,s}$$

where $S_k = \{ s_{i,r}^k \}$, $\delta p_{k,r} = p_{r} - p_{k,r}$ and $H_{i,j} = \{ h_{i,j} \}$. Equation (7) can be written in a more compact “multi-response” form, which is

$$z_{Q,k} = z_k + S_k \delta p_k + \frac{1}{2} \left[ \delta p_k^T H_k \delta p_k \right]_{i=1}^{N_x}$$

where $z_k = Z(p_k)$, $S_k = S(p_k)$, $\delta p_k = p - p_k$, $H_k = H_k(p_k)$ and $[\cdot]^T_{i=1}^{N_x}$ is a column sequence of $N$ blocks (scalars or vectors), each labeled by numeric index $i$.

Both equations (6) and (8) represent local quadratic approximations around parameter point $p_k$, however, equation (8) is an example of a multi-response approximation scenario. Further, it is obvious from (8) that the simpler and more frequently used linear local approximation $z_{L,k} = z_k + S_k \delta p_k$ is related to the quadratic local approximation and both represent different degrees of approximation of the response model $Z$. 
\[ Z(p_k + \delta p_k) \approx z_{Q,k} = \begin{bmatrix} S_k \end{bmatrix}^T \delta p_k = \begin{bmatrix} S_k \end{bmatrix}^T \begin{bmatrix} \delta p_k \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \sum_{j=1}^{N_z} \delta p_k^T H_{k,j} \delta p_k \end{bmatrix}^{N_z} \]

Both approximation approaches, (6) and (8), are well known in the area of nonlinear regression [4]. The problem of model updating as introduced in section 2 can be interpreted as equivalent to some of these problems. However, it is important to take notice of some specifics of model updating in structural dynamics such as: standard problem size, in terms of \( N_z \) and particularly \( N_p \); the cost per single parameter point evaluation \( Z(p_k) \); qualitative properties of employed responses and their derivatives; and the inverse, i.e. often ill-posed nature, of model updating problems.

### 3.3 Specific parameter-stepping strategies in model updating

The first approximation form, equation (6), is the basis of the Newton methods. The local step in this instance is based on the condition \( \frac{\partial J_{NR}}{\partial \delta p_k} = 0 \) and the definitions introduced in equations (3) and (4). The condition for the parameter step is

\[ \begin{align*}
S_k \delta p_k & = -S_k^T (z_k - z_w) \\
\end{align*} \]  

which provides the Gauss-Newton parameter step \( \delta p_{GN,k} \). It is interesting to note that problem (11) can be also formulated as a linear least squares problem, and can be directly related to the original definition (2). In this case, the linear model approximation \( z_{L,k} = z_k + S_k \delta p_k \approx Z(p) \) is used in the reformulated cost function

\[ \min J_{GN,k} = \min \| z_{L,k} - z_w \|^2 = \min \| S_k \delta p_k - \delta z_k \|_i^2 \]

where \( \delta z_k = z_m - z_k \). This strategy, however, tends to be unstable in the model updating context. Additional stabilisation is introduced through problem regularisation [3]. In this case we only assume one instance of this approach used in combination with problem (12). The step length will be affected by the augmented part of the overall cost function

\[ \min J_{GNReg,k} = \min ((1-\alpha)\| S_k \delta p_k - \delta z_k \|_i^2 + \alpha \| \delta p_k \|^2) \]

where \( \alpha \in [0,1] \) is the regularisation parameter. Further, equation (13) can be expressed in the form of its associated linear equation

\[ \begin{bmatrix} (1-\alpha)^{1/2} S_k \\ \alpha^{1/2} I \end{bmatrix} \delta p_k = \begin{bmatrix} (1-\alpha)^{1/2} \delta z_k \\ 0 \end{bmatrix} \]

The resulting parameter step in this case is denoted as \( \delta p_{GNReg,k}(\alpha) \). The degree of participation of the augmented part depends on the magnitude of \( \alpha \). Too low a value may not ensure the required stability during parameter stepping, while too large a value may cause a significant increase in the number of iterations required for the solution to converge. In parallel to equations (13) and (14) it is possible to derive a similar associated linear equation for problem (10), which is

\[ \begin{bmatrix} S_k \\ D_{H,k}^T U_{H,k}^T \end{bmatrix} \delta p_k = \begin{bmatrix} \delta z_k \\ 0 \end{bmatrix} \]
where \( \sum H_{i,k}(z_i - z_{w,i}) = U_{H,k} D_{H,k} U_{H,k}^T \) is a symmetric decomposition of part of the Hessian matrix \( H_{j,k} \).

While equation (10) uses second order information in the separate Hessian matrices \( H_{i} \), its origin is in the scalar quadratic form (6), whose differentiation leads to the linear equations. On the other hand, the use of the vector form \( z_{Q,k} \), equation (8), gives a direct second order approximation of the model \( Z(p) \), whose application in the context of equation (2), and analogous to equation (12), does not produce a quadratic system. The new system in this case is quartic in \( \delta p_i \) and its differentiation results in a cubic equation in \( \delta p_i \). The application of this process begins with \( z_{Q,k} \) applied to equation (2)

\[
\min J_{QQ,k} = \min \| z_{Q,k} - z_w \|^2 = \min \| S_k \delta p_k + \frac{1}{2} \left[ \delta p_k^T H_{H,k} \delta p_k \right]_{i=1}^{N_x} - \delta z_k \|^2.
\]

Alternatively, a direct matrix equation equivalent to this optimisation formulation can be obtained from equation (8)

\[
z_w \approx z_{Q,k} = z_k + S_k \delta p_k + \frac{1}{2} \left[ \delta p_k^T H_{H,k} \delta p_k \right]_{i=1}^{N_x} \delta p_k.
\]

This formal rearrangement was assumed by Hettlich and Rundell, [2], in the context of inverse potential problems. This approach was also applied by Ojalvo and Pilon, [1], to construct their set of resulting equations. Casting approximate equality in (17) into a matrix equation format provides

\[
\left( S_k + \frac{1}{2} \left[ \delta p_k^T H_{H,k} \right]_{i=1}^{N_x} \right) \delta p_k = \delta z_k.
\]

A formal approach proposed in [1] introduces a two-level algorithm to address the nonlinear (quadratic) nature of this “primary” equation. A similar approach was developed in [2]. Both approaches exploit the combination of equation (18) with equation (11) to handle this nonlinearity. Moreover, Hettlich and Rundell combined the baseline scheme with an additional regularisation term of the type used in equation (13) and an iteration stopping criterion. The common features of this process are shown here

\[
T \delta p_k = \delta z_k, \quad T \in \mathbb{R}^{N_x \times N_x},
\]

\[
T = T(S_k, H_{H,k}, \delta p_{GN,k}) = S_k + \frac{1}{2} \left[ \delta p_{GN,k}^T H_{H,k} \right]_{i=1}^{N_x}, \quad i = 1, \ldots, N_z.
\]

While paper [1] discusses the use of pseudo-inverse form \( \delta p_{QQ,k} = (T^T)^{-1} T^T \delta z_k \), any alternative and numerically more efficient process can be used to achieve \( \delta p_{QQ,k} = T^T \delta z_k \), where \((\cdot)^{-1}\) denotes the matrix pseudo-inverse operator. Finally, the solution process proposed by Hettlich and Rundell, [2], leads to the equation \( \delta p_{QReg,k} = (T^T, \beta^\dagger \beta^\dagger I) \right)^T [\delta z_k^T, 0]^T \), where \( \beta_i > 0 \) is the regularisation parameter and instead of the parameter step \( \delta p_{GN,k} \) they propose the use of a regularised parameter step \( \delta p_{QReg,k}(\beta_i) \), again with \( \beta_i > 0 \).

A number of parameter stepping strategies applicable in the area of model updating have been discussed in this section. The goal of this exposition is to demonstrate the links between existing approaches with particular focus on the methods using second order response derivatives. In the context of model updating the approach summarised in equation (19) was performed by Ojalvo and Pilon, [1]. The case study section of this paper will focus on the method based on equation (10).

### 3.4 First and second order derivatives of eigenvalues

The complete updating scheme in equation (10) requires the computation of the response derivatives, \( z_{i,j} = \partial z_i / \partial p_j \) and \( z_{i,k} = \partial z_i / \partial p_j \delta p_k \). Although a range of responses could be used, such as the FRF or
the mode shapes, here we will demonstrate the approach using natural frequencies, so that \( z_i = \omega_i^2 \).

Adhikari and Friswell, [5], summarized the methods to calculate the first and second order derivatives of eigenvalues and eigenvectors for the general case (asymmetric, non-conservative second order systems). In the following, the systems are assumed to be undamped (i.e. conservative) and modelled with symmetric structural matrices. Furthermore the systems are assumed to have distinct eigenvalues (apart from, possibly, repeated zero eigenvalues representing rigid body modes). Friswell, [6], gave more detail of the issues in updating systems with repeated eigenvalues.

The eigenvalues, \( \lambda_i = -\omega_i^2 \), and eigenvectors, \( \phi_i \), are given by

\[
[\lambda_i M(p) + K(p)]\phi_i = 0.
\]  

(20)

We will assume the eigenvectors are mass normalized so that \( \phi_i^T M \phi_i = 1 \). The first derivative of the \( i \)th eigenvalue, [7], is then

\[
\frac{\partial \lambda_i}{\partial p_j} = -\phi_i^T \left[ \lambda_i \frac{\partial M}{\partial p_j} + \frac{\partial \lambda_i}{\partial p_j} M \phi_i + \frac{\partial \lambda_i}{\partial p_j} \phi_i + \frac{\partial^2 M}{\partial p_j \partial p_j} \right] \phi_i.
\]  

(21)

Note that the response derivative is actually

\[
\frac{\partial z_i}{\partial p_j} = \frac{\partial \omega_i}{\partial p_j} = \frac{1}{2\omega_i} \frac{\partial \lambda_i}{\partial p_j}.
\]  

(22)

The second derivative of the \( i \)th eigenvalue is

\[
\frac{\partial^2 \lambda_i}{\partial p_j \partial p_k} = -\phi_i^T \left[ \lambda_i \frac{\partial^2 M}{\partial p_j \partial p_k} + \frac{\partial \lambda_i}{\partial p_j} \frac{\partial M}{\partial p_k} + \frac{\partial \lambda_i}{\partial p_k} \frac{\partial M}{\partial p_j} + \frac{\partial^2 M}{\partial p_j \partial p_k} \right] \phi_i
\]  

(23)

\[ -\phi_i^T \left[ \frac{\partial \lambda_i}{\partial p_j} M + \lambda_i \frac{\partial M}{\partial p_j} \right] \phi_i - \phi_i^T \left[ \frac{\partial \lambda_i}{\partial p_k} M + \lambda_i \frac{\partial M}{\partial p_k} \right] \phi_i \]

and the second derivative of the response (i.e. the natural frequency) may be calculated as

\[
\frac{\partial^2 z_i}{\partial p_j \partial p_k} = \frac{\partial^2 \omega_i}{\partial p_j \partial p_k} = -\frac{1}{2\omega_i} \frac{\partial \lambda_i}{\partial p_j} \frac{\partial \lambda_i}{\partial p_k} - \frac{1}{\omega_i} \frac{\partial \lambda_i}{\partial p_j} \frac{\partial \lambda_i}{\partial p_k}.
\]  

(24)

One slight difficulty is that the first derivative of the \( i \)th eigenvector is required to calculate the second derivative of the \( i \)th eigenvalue. The eigenvector derivatives may be calculated as a sum of contributions from all of the modes [7], or using the approach given by Nelson [8]. Here we use the latter approach, which for systems with a large number of degrees of freedom is computationally more efficient. The eigenvector derivative is written as

\[
\frac{\partial \phi_i}{\partial p_j} = v_i + c_i \phi_i
\]  

(25)

for unknown vector \( v_i \) and scalar \( c_i \). Differentiating the eigenvalue equation, and substituting the expression for the eigenvector derivative gives the following equation for \( v_i \):

\[
[\lambda_i M + K] v_i = -\frac{\partial \lambda_i}{\partial p_j} M \phi_i - \left[ \lambda_i \frac{\partial M}{\partial p_j} + \frac{\partial K}{\partial p_j} \right] \phi_i.
\]  

(26)

Note that the matrix \([\lambda_i M + K]\) is rank deficient and so this equation cannot be solved directly. Nelson [8] obtained a well-conditioned solution by setting the element of \( v_i \) corresponding to the largest element
of $\phi_i$ to zero. This may be accomplished efficiently by replacing the corresponding row and column of $[\lambda_i \mathbf{M} + \mathbf{K}]$ to the identity matrix, and the corresponding element of the right side of equation (26) to zero. The constant $c_i$ is then obtained by differentiating the mass normalization equation, to give

$$c_i = -\phi_i^T \mathbf{M} \phi_i - 2 \phi_i^T \mathbf{M} \mathbf{v}_i.$$  (27)

## 4 Case studies

### 4.1 Discrete 3 dof system

A simple 3 degree-of-freedom (dof) spring-mass system is chosen to provide fully controlled environment for numeric experiment. The response Hessian matrices $\mathbf{H}_{i,j}$ are calculated according to the theory presented in section 3.4. Both, initial “baseline” system state and “experimental” data are determined from the same model through suitable selection of model parameter values. In this section, a comparative study is presented between the Newton-Rhapson (N-R) scheme, equation (10), and regularised Gauss-Newton (G-N) scheme, equation (14). System description is shown in Figure 1.

![Figure 1: Discrete 3 dof spring-mass system.](image)

Reference parameters of the model are selected as follows:

\[
[m_1,m_2,m_3]=[1.00,2.00,1.05] \text{ kg and } [k_1,k_2,k_3,k_4,k_5]=[1.0,1.5,1.5,1.0,3.0] \times 10^3 \text{ N/m}. \quad (28)
\]

Further, “baseline” and “experimental” parameters are defined as follows:

\[
k_{1,\text{base}}=k_{1,\text{ref}} \times 0.15, \quad k_{4,\text{base}}=k_{4,\text{ref}} \times 0.05 \quad \text{and} \quad k_{1,\text{exp}}=k_{1,\text{ref}} \times 2.00, \quad k_{4,\text{exp}}=k_{4,\text{ref}} \times 5.00. \quad (29)
\]

Updating problem is further specified so that $N_p=2$ and $N_z=3$, where $\mathbf{p}=[k_1,k_5]^T$ and $\mathbf{z}=[\omega_1,\omega_2,\omega_3]^T$. Finally, regularised G-N updating scheme, equation (14), is evaluated for $\alpha \in [10^{-8},10^{-3}]$. During iterations, parameter stepping is completed when: (i) the parameter step convergence threshold is reached, (ii) the maximum number of iterations is exceeded, or (iii) the algorithm fails due to non-physical parameter updates. The results of this study are shown in Figure 2.
In Figure 2, the cases where the updating process failed are marked with 0 iterations, while maximum number of iterations was set to 200. The red dashed line represents constant number of N-R iterations, which is independent of the regularisation parameter $\alpha$. The regularised G-N scheme requires the use of the parameter $\alpha$, in this case, to restrict the extent of parameter steps during the iterations. Clearly, the use of very low parameter values, including $\alpha=0$, leads to the failure of the updating algorithm (i.e. excessive initial parameter steps). The use of relatively high parameter values, $\alpha>10^{-4}$, causes excessive number of updating steps required to achieve the convergence. Finally, significant part of the parameter interval, centred around $\alpha\approx10^{-6}$, allows convergence rates comparable with that of the N-R scheme. However, the N-R scheme can be seen, in this case, as robust alternative to the regularised G-N scheme as it does not require specification of the unknown regularisation parameter $\alpha$.

4.2 Aluminium frame structure

To evaluate the application of the N-R scheme, equation (10), a more complex configuration is selected in this section. A real case study is performed on the aluminium frame structure shown in Figure 1(a). The structure was built from Meroform M12 system using the standard aluminium beams and nodes.

Figure 3: Experimental setup and FE model of the aluminium frame.
The structure was supported in free-free conditions and it was instrumented with 33 single-axis B&K accelerometers. The data acquisition and Experimental Modal Analysis (EMA) data processing was performed with the multi-channel LMS measurement system. The structure was repetitively excited with the impact hammer to allow identification of 33 FRFs, followed by the mode shape and natural frequency extraction. The experiment parameters (frequency range, sampling, sensor density etc.) were adjusted to allow complete identification of the first eight “global” (flexible) mode shapes. The results of the identification process are summarised in Figure 4. Figure shows first 8 mode shapes and their associated (undamped) natural frequencies.

![Figure 4: Experimental mode shapes and natural frequencies.](image)

The finite element (FE) model of the aluminium frame structure is shown in Figure 3(b). The model consists of 168 elements (Euler-Bernoulli beams, point masses and rigid links) with the total of 708 dofs. The material properties are those of the standard aluminium and the model is unconstrained. The baseline FE model is compared with experimental data in Figure 5 and Figure 6. In Figure 5, the analytical FRFs are compared with identified natural frequencies (vertical red lines). This initial comparison indicates that the baseline FE model is stiffer than the real structure. The second comparison uses the Modal Assurance Criterion (MAC) between the sets of experimental and baseline model modes. A very good correlation is achieved for the first eight modes. These modes represent the global modes located in a distinctive mode cluster as shown in Figure 4 and Figure 5. The analytical and experimental modes located in the next mode cluster, Figure 5, can not be matched with confidence due to their “local” nature, Figure 6.

![Figure 5: Frequency domain comparison between measured and baseline model data.](image)
Reference FE model

Experimental modes

Figure 6: MAC between experimental data and baseline FE model.

The model parameterisation is chosen to suit the purposes of the current presentation. The three parameters are selected for element subsets: (i) and (ii) the central parts of aluminium members (blue elements in Figure 3(b)) are parameterised with the second moments of inertia, $I_{\text{Z,mid}}$ and $I_{\text{Y,mid}}$; and (iii) the joint parts of aluminium members (red elements in Figure 3(b)) are parameterised with the Young’s modulus $E_{\text{joint}}$. This parameter arrangement allows to take into considerations some uncertain (stiffness) modelling aspects concentrated particularly in the joint areas.

Figure 7: The progression of the N-R model updating scheme.
In model updating, the first eight “nonzero” natural frequencies $\omega_i$ were used to define $z_m$. Therefore, the dimensionality of the problem is $N_p = 3$ and $N_q = 8$. The FE model was implemented in the in-house built Matlab FE code. The core calculations were based on Matlab’s function “eigs” for modal analysis. Figure 7 documents the updating process when the N-R scheme with the above parameter/response combination is used. The quantities shown are: (a) the response residuals (top subplot), (b) the parameter changes (middle subplot), and (c) the diagonal MAC coefficients (lower subplot).

The convergence is achieved in approximately 10 iterations and significant improvement in frequency differences can be observed, with the residuals reduced from original 10−15% to −1% and less. Further, even relatively insensitive MAC criterion indicates very good and in the cases of the modes 4 and 6 improving shape correlation. Despite not using additional regularisation terms, the parameters $I_{z,mid}$ and $I_{y,mid}$ converge to comparable final values. Finally, as expected, the major parameter change is represented by the global joint stiffness reduction, described by the parameter $E_{joint}$ ($\approx 30\%$ with respect to its baseline value). To put these results into context, the N-R scheme is compared with the regularised G-N scheme in Figure 8.

![Figure 8: Comparison between the N-R and regularised G-N model updating schemes.](image)

Four test cases are shown in Figure 8. The top subplot shows the results of the N-R scheme (see also Figure 7) and the G-N scheme ($\alpha=0$). The middle subplot shows the iteration process for the regularised G-N scheme with $\alpha=1$, representing the case with too much regularisation. The lower subplot uses the same scheme as the middle subplot with $\alpha=10^{-2}$, representing the case with improved convergence rate. The regularisation term in this case is $\|p_i-p_0\|^2$, where $p_0$ is the vector of baseline parameter values (see [3]).

It is not common that the direct use of the G-N scheme survives initial iterations due to excessive, often non-physical, estimated parameter variations. In this case, the G-N scheme provides stable results that agree with those provided by the N-R scheme and others. In this respect, this method can be seen as the
best updating approach, however, this is likely to be the case only due to current problem specifics related to clearly defined areas of modelling uncertainties (i.e. local joint stiffness representation). The success of the G-N scheme indicates nearly linear nature of the response variations required. This, however, provides more insight into the reasons behind the behaviour of the N-R and regularised G-N schemes. Near linearity of the updating problem allows fast deployment of the G-N scheme, while the added term in the regularised G-N scheme slows down the convergence rate, depending on the magnitude of the parameter $\alpha$. The introduction of the second order information in the N-R scheme acts as more “physical” regularising term, capturing some aspects of the local curvature and allowing faster convergence with respect to the employed regularised first order schemes.

5 Conclusion

The paper presents the theory describing the use of second order derivatives in the model updating. Theoretical section of the paper links different second order approaches and puts them into context of the regularised model updating methods. The use of the Newton-Rhapson model updating scheme is demonstrated in two case studies. The numeric experiment presented in the first case study demonstrates that this scheme can be seen as a more robust alternative to the regularised Gauss-Newton scheme, not requiring specification of the unknown regularisation parameter. The second case study demonstrates the use of second order updating scheme on the case of real frame structure with significant response residuals. In this case, it is found that near linearity of the problem allows the direct use of the Gauss-Newton scheme. Newton-Rhapson scheme provides competitive updating rate without the need to specify unknown regularisation parameter. The investigation of the influence of the second order derivatives on the updating processes can therefore improve our understanding of the function and issues associated with standard regularisation approaches.

References