The detection of damage in joints of mechanical structures

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Abstract
This paper presents the theory of generic elements along with the concept of subspace angles applied in damage location in joints. Structural joints are often considered to be failure-critical regions due to their relatively high uncertainty during mathematical model generation and subsequent simulations and design considerations. Two concepts investigated in this paper are the use of joint subspaces for unique damage location and the combined use of different modal type responses in the subspace angle approach. Both concepts are jointly presented in the concluding example.

1 Introduction

The topic of damage detection is comprehensively reviewed in [1]. Since the time of publication of this review paper a wide range of new approaches have been studied. Common features of recent research in this field are, as dictated by potentially successful industrial application, the attempts to deal with experimental and modeling uncertainty, as well as with the influence of an ever changing environment. To mention only a few of the recent papers addressing these issues, methods based on statistical process control [2], pattern recognition [3], Independent Component Analysis, Principal Component Analysis and Neural Networks [4], [5] have been studied.

A common feature of these approaches is that they try to address uncertainty related problems, often by using tools developed in statistics. One such approach is subset selection, originally introduced in regression [6], which is the basis of damage location algorithms suggested in [7]. This concept was further studied in [16] where its performance was tested on a laboratory scale experimental structure. As this approach is based on the use of a parameterized model, the choice of appropriate parameters determines the spatial and damage mode resolution, as well as its performance. The impetus for the introduction of the concept of generic elements into this area is an expected enrichment of the parametric description with respect to possible joint damage scenarios.

Section 1 is a general and brief introduction to recently studied damage detection techniques, primarily statistically based ones. Section 2 provides comprehensive information about joint stiffness parameterization by the generic element concept. Two groups of parameters are identified here, with a new comprehensive description of the parameter group related to eigenvector transformation. The following section, introduces the concept of a joint subspace to damage location based on subset selection. Moreover, it deals with theoretical aspects of mixing different types of modal responses in the modal residual vector and sensitivity matrix for improved resolution of damage location. The last section of the paper, section 4, presents a study of these techniques on an example, using the results of a simulated experiment performed on a 3-storey aluminium frame structure.
2 Generic elements

2.1 Basics of generic element theory

The concept of generic elements was introduced in [8] and was presented as a potential tool for model updating research. Since then a few studies have been performed in this area, such as [9], [10], [11]. This concept was investigated as a part of PhD research in [12] for model updating and in [13] for damage location. Applications in damage detection were presented at a conference in [14] and in journal format in [15] and [16].

The generic element concept tries to provide an effective, physically based tool for finite element modification/parameterization. The concept is based on manipulating the spectral properties either of a single element or an arbitrary substructure. It is possible to manipulate spectral properties of these blocks either on the level of separate structural matrices, i.e. stiffness and/or mass, or on the level of complete dynamic substructures, i.e. generalized eigenvalue problem on the substructure level. In this paper, we study the parameterization of the stiffness matrix of joint substructures, which appears to be a natural option for the case of damage detection due to its expected manifestation in the form of local stiffness reduction without significant mass modification.

Under these conditions, the generic element concept is introduced from the eigenvalue problem for the joint substructure stiffness matrix,

\[
\mathbf{K}_J \varphi_j = \lambda_j \varphi_j, \quad \mathbf{\Phi}^T \mathbf{K}_J \mathbf{\Phi} = \begin{bmatrix} 0 & 0 \\ 0 & \Lambda_N \end{bmatrix}
\]

\[
\mathbf{\Phi}^T \mathbf{\Phi} = \mathbf{\Phi} \mathbf{\Phi}^T = \mathbf{I}
\]

where the eigenvector matrix is assumed to be organized as follows

\[
\mathbf{\Phi} = [\varphi_1, \ldots, \varphi_{n_Z}, \varphi_{n_Z+1}, \ldots, \varphi_{n_0}] = [\mathbf{\Phi}_Z, \mathbf{\Phi}_N] \in \mathbb{R}^{n_Z \times n_0}, \quad n_Z \leq 6
\]

leading to the eigenvalue decomposition of the joint substructure stiffness matrix as

\[
\mathbf{K}_J = \mathbf{\Phi}_N \Lambda_N \mathbf{\Phi}_N^T = \sum_{j=n_Z+1}^{n_0} \lambda_j \varphi_j \varphi_j^T \in \mathbb{R}^{n_0 \times n_0}
\]

Subscripts \(Z\) and \(N\) in Eq. (1), (2) and (3) represent quantities corresponding to zero and non-zero eigenvalues, respectively. Superscript \(J\) denotes joint quantities. The joint stiffness matrix \(\mathbf{K}_J\) is symmetric, positive semi-definite, with no more then 6 zero eigenvalues, \(n_Z\) is total number of degrees of freedom of joint substructure and \(n_Z\) is number of zero eigenvalues. According to Eq. (1) matrix \(\mathbf{\Phi}\) is orthogonal. If we consider nominal joint stiffness matrix \(\mathbf{K}_J^0\), then a family of elements or the generic element can be obtained by a linear transformation between the nominal and new sets of eigenvectors corresponding to non-zero eigenvalues. A family of elements is thus generated by linear transformation, as it is suggested in [8], as

\[
\mathbf{\Phi}_N = \mathbf{\Phi}_{N0} \mathbf{R}
\]

\[
\mathbf{R}^T \mathbf{R} = \mathbf{I}, \quad \mathbf{R} \in \mathbb{R}^{n_Z \times n_0}, \quad n_N = n_D - n_Z
\]

where 0 represent original or nominal values of generic element quantities and \(\mathbf{R}\) is an orthogonal transformation matrix. Substituting Eq. (4) into Eq. (3) leads to the stiffness matrix of the generic element, expressed in terms of nominal eigenvectors, as

\[
\mathbf{K}_J = \mathbf{\Phi}_{0N} \mathbf{R} \Lambda_N \mathbf{R}^T \mathbf{\Phi}_{0N}^T
\]
There are $n_N$ free parameters due to $\Lambda_N$ and $n_N (n_N - 1)/2$ free parameters due to $\mathbf{R}$, leading to $n_N (n_N + 1)/2$ free parameters for a complete (minimal) parameterization of Eq. (5). In the case of a joint substructure with only 5 nodes with 6 structural degrees of freedom per node, the complete parameterization of Eq. (5) leads to 24 eigenvalue related parameters and 276 parameters related to the eigenvector transformation, providing 300 parameters in total. On one side, this allows for great flexibility in possible element modifications, however, on the other side the number of parameters renders the problem of damage location to be computationally intractable even for very simple examples. Thus, there is a strong motivation for a considerable reduction of number of parameters, while keeping certain aspects of the approach’s flexibility.

2.2 Generic elements and their parameters

2.2.1 Eigenvalue related parameters

Assuming that eigenvectors are not modified in the generic element amounts to setting $\mathbf{R}$ to the unit matrix. This assumption leads to a very simplified form of generic element parameterization with only joint stiffness matrix eigenvalues taken as parameters (where the stiffness matrix is a linear function in these parameters)

$$K^J = \Phi^T_0 N \Lambda_N \Phi_0 N = \sum_{j=n_N+1}^{n_N} \lambda_j \Phi_{0,j} \Phi_{0,j}^T$$

(6)

According to Eq. (6) the non-zero eigenvalues represent the influence of the eigenvectors to the overall stiffness of the joint substructure. Modifying these values translates into a reduction (or increase) of the joint stiffness in terms of reduced (or increased) participation of corresponding eigenvector(s) in the given stiffness matrix with respect to the reference state. Sensible selection of subsets from this set of parameters allows reduced generic element parameterization. Thus, if the following equation is valid for nominal/datum joint substructure

$$K^J_0 = \sum_{j=n_N+1}^{n_N} \lambda_{0,j} \Phi_{0,j} \Phi_{0,j}^T$$

(7)

then the partly parameterized joint stiffness matrix with fixed eigenvectors is

$$K^J = K^J_F + K^J_P \left( \lambda_j \right) = \sum_{i \in U} \lambda_{0,i} \Phi_{0,i} \Phi_{0,i}^T + \sum_{j \in V} \lambda_j \Phi_{0,j} \Phi_{0,j}^T$$

$$W = \{n_z + 1, \ldots, n_D\}, V \subset W, U \subset W, U \cap V = \emptyset, U \cup V = W$$

(8)

where $K^J_F$ and $K^J_P$ are fixed and parameterized parts of joint stiffness matrix, respectively. $W$ is a set of all available indices, $V$ is a subset of set $W$ with indices of eigenvalues selected for actual joint stiffness matrix parameterization and finally subset $U$ is the complement of subset $V$ in set $W$.

2.2.2 Eigenvector related parameters

The situation with parameters related to eigenvectors is more complicated. To retain physical meaning of $K^J$, eigenvector matrix $\Phi$ has to remain orthogonal for all members of the element family, i.e. any realization of the generic element. Thus, the transformation matrix $\mathbf{R}$ also has to be orthogonal, and the linear transformation in (4) is in fact a more specialized orthogonal transformation. Orthogonal transformations are transformations that retain angles and lengths when applied on sets of vectors in vector spaces, e.g. the sub-basis of vector space. The transformation represented by Eq. (4) thus represents...
A generalized rotation (for cases where \( \det(R) = +1 \)) of a vector subspace spanned by eigenvectors corresponding to the non-zero eigenvalues of Eq. (1), in the vector space spanned by all eigenvectors.

Now, two complementary approaches can be used to cope with parameterization of this part of joint stiffness matrix. The first, suggested in [8], is to divide the eigenvectors into smaller groups based on some physical reasoning such as, for instance, retaining symmetrical and anti-symmetrical groups of stiffness matrix eigenvectors, reducing modifications to a few eigenvectors corresponding to the lower eigenvalues, etc. In our case, a selection of a subset of eigenvectors could be performed, not necessarily those corresponding to lower eigenvalues, providing us with a low dimensional vector subspace with potential to cover particular, failure-critical damage scenarios described either by detailed joint models or a separate experimental study. Thus, if we assume two independent groups of eigenvectors, transformation (4) can be modified as follows

\[
\begin{bmatrix}
\Phi_{N,1}, \Phi_{N,2}
\end{bmatrix} = \begin{bmatrix}
\Phi_{N,0,1}, \Phi_{N,0,2}
\end{bmatrix} \begin{bmatrix}
R_1 & 0 \\
0 & R_2
\end{bmatrix}
\]

where square matrices \( R_j, j \in \{1, 2\} \) are again orthogonal with appropriate sizes. If, for instance, vectors in the second group are to be kept unchanged, then \( R_2 \) is set to the unit matrix. These smaller matrices are supposed to be easier to handle than the whole transformation matrix \( R \) in terms of the following discussion, as well as in terms of overall number of parameters.

The second approach, applied either to the complete matrix \( R \) or to the reduced matrices \( R_j \), is based on direct parameterization of these transformation matrices, so that they remain orthogonal as required by (4). In the case of a matrix of size 1, matrix \( R_j \) is by definition constant and equal to one. Cases of matrices of sizes 2 and 3 can be handled with orthogonal matrices representing rotations in the plane for size 2 and consecutive rotations in three spatial planes, e.g. Euler angles, for size 3. Actually, separation into two independent sets of symmetric and anti-symmetric eigenvectors and transformation matrix \( R_j^{(2x2)} \) were used in [8] when looking for transformation between the BCIZ1 and more advanced DKT formulations of a triangular plate element. Parameterized matrices for these cases are summarized in Eq. (10).

\[
\begin{align*}
R_j^{(1x1)} &= 1 = \text{const} \\
R_j^{(2x2)} &= \begin{bmatrix}
\cos \alpha_1 & \sin \alpha_1 \\
-\sin \alpha_1 & \cos \alpha_1
\end{bmatrix} = f(\alpha_1) \\
R_j^{(3x3)} &= \begin{bmatrix}
R_j^{(3x3)}(\alpha_1) & R_j^{(3x3)}(\alpha_2) & R_j^{(3x3)}(\alpha_3)
\end{bmatrix} = f(\alpha_1, \alpha_2, \alpha_3)
\end{align*}
\]

where \( R_j^{(3x3)}(\alpha_k), k \in \{1, 2, 3\} \) represent consecutive rotations in three spatial planes.

Higher dimensional parameterizations can be obtained by Cayley’s transformation, Eq. (11). This transformation is one-to-one and onto, from the set of skew-symmetric matrices to the set of proper orthogonal matrices, i.e. \( \det(R) = +1 \), with no eigenvalues at –1.

\[
\begin{align*}
R_j &= (I - Q)(I + Q)^{-1} = (I + Q)^{-1}(I - Q) \\
Q &= (I - R_j)(I + R_j)^{-1} = (I + R_j)^{-1}(I - R_j)
\end{align*}
\]

where \( Q \) is a skew-symmetric matrix containing \( n_{Nj}(n_{Nj}-1)/2 \) free parameters. This parameterization has been shown to be equivalent to the higher dimensional analog to parameterization of \( R_j^{(3x3)} \) by...
classical Rodrigues parameters, [17]. Classical Rodrigues parameters in 3 dimensional space provide information about the principal rotation, i.e. unique axis and angle for rotation from arbitrary reference to arbitrary final position. These parameters have singular orientation at a principal rotation angle \( \pm \pi \) and this restriction is also present for higher dimensional generalizations provided by Cayley’s transformation. This is a minimal set of parameters of matrix \( R_j \) with \( n_{Nj}(n_{Nj}-1)/2 \) members. Another minimal parameter description in 3 dimensions is the modified Rodrigues parameters with extended singular orientation to \( \pm 2\pi \). A common feature of all minimal parameter set descriptions is the existence of a singular orientation, [17]. An alternative way to avoid the presence of this singular orientation is the use of a one-parameter redundant description, i.e. \( n_{Nj}(n_{Nj}-1)/2 + 1 \), which in case of three dimensional space is the 4-parameter quaternion description, also called Euler parameters. This concept can be again extended to multi-dimensional cases. This subject is comprehensively covered in [17].

An alternative approach to the complete parameterization is based on the formula in Eq. (5) which can be rearranged to the following form

\[
K^J = \Phi_{0N}^T T \Phi_{0N},
\]

\[
T = RA_N R^T, T = T^T, T \in \mathbb{R}^{n_s \times n_s}
\]

(12)

The symmetric matrix \( T \) then consists of \( n_N(n_N+1)/2 \) free parameters, the same number as in the previous approach. While the joint stiffness matrix is a non-linear function of the parameters in \( Q \) due to Eq. (11), the same matrix can be expressed as a linear function of parameters in matrix \( T \), leading to a very simple formula for the joint stiffness matrix sensitivity with respect to parameters in \( T \). However, this simplification leads to a more complicated interpretation of the parameters in \( T \) as each element of \( T \) is a complicated function of both sets of parameters from previous analysis.

This section introduces the concept of joint stiffness parameters corresponding to the eigenvector transformation equation, Eq. (4), for general multidimensional cases that can easily occur in the present approach to damage location. A complementary approach to parameterization is indicated by Eq. (12). However, the application of this particular subset of generic element parameters is deferred and in this paper only the eigenvalue related parameters will be estimated.

### 3 Damage location by parameter subset selection

#### 3.1 Joint parameterization for damage location

Two types of subspaces are considered in this paper. The first is the joint subspace spanned by eigenvectors corresponding to the selected set of parameters, helping us to understand the effect of parameter modification as well as effective parameter selection for joint parameterization. The other is the subspace of the sensitivity matrix column space whose basis is the set of sensitivity matrix columns corresponding to the parameters of a particular joint (or other type of substructure for more general cases). This section deals with the former subspace type. The latter type of subspaces is dealt with in section 3.2.

The attempt to introduce the concept of a joint (or in general, substructure) subspace is driven by the need to have a unique entity or mathemetic construct representing the part of the structure in question, in terms of previously introduced families of generic elements and corresponding parameters. This uniqueness allows the location of damage in a structure. This is an alternative to the approach presented in [7] and [16], where subspaces of dimension one, i.e. single columns, were searched. The joint stiffness matrix is created by a fixed part and the part corresponding to eigenvalues, which are taken as joint parameters.

\[
K^J = K^J_F + K^J_P \left( \lambda_j \right) = K^J_F + \sum_{j \in P} \lambda_j \Phi_{0,j} \Phi_{0,j}^T
\]

(13)
According to Eq. (13), all modifications of the joint stiffness matrix are realized in the subspace spanned by vectors $\phi_{0,j}$, $j \in V$, where $V$ is an index set of selected eigenvectors/eigenvalues, and this provides a restriction for all possible cases covered by this joint parameterization. Any variation in stiffness is thus modeled as a combination of relative changes of selected eigenvector participation with respect to the reference state. This approach is suggested because any assumed damage scenario can result in more complex stiffness changes than those representable by single parameter. Therefore, here we are working with single (joint) subspaces instead of single parameters. Actually, an appropriately chosen joint subspace could cover more than a single damage scenario as is indicated by Eq. (14).

$$
\sum_{j \in V_1} \lambda_j \phi_{0,j}^T \phi_{0,j} + \sum_{k \in V_2} \lambda_k \phi_{0,k}^T \phi_{0,k} = \sum_{m \in V_3} \lambda_m \phi_{0,m}^T \phi_{0,m} + \sum_{l \in V_4} \lambda_l \phi_{0,l}^T \phi_{0,l}
$$

$$
V_1 \cap V_2 = \emptyset, V_1 \cup V_2 = V, \quad V_3 \cap V_4 = \emptyset, V_3 \cup V_4 = V, \quad V_2 \cap V_4 \neq \emptyset
$$

A negative aspect of the use of the joint subspace to represent damage scenarios is that with increasing dimension of the subspace (in an effort to describe complex damage scenarios), their capacity to cover other joint damage scenarios also increases, resulting into decreased resolution when using subspace angles for damage location. This problem is addressed in section 3.3.

### 3.2 Parameter subset selection – problem formulation

Parameter subset selection for damage location is described in [7] and [16]. The exposition here is restricted only to those details that are directly investigated in this paper. A characteristic feature of this approach is that it uses a parameterized mathematical model of the structure to produce modal sensitivity values. These are used, along with measured variations in corresponding (measured) modal values (called here response residuals), possibly due to the presence of damage, to find the most likely regions in the structure explaining these variations. The general formulation of the problem is shown in Eq. (15) along with the equation partitioning in Eq. (16). Partitioning is based on parameter groups corresponding to respective joints in the structure’s model using the concept of joint subspaces. Eq. (15) represents the first order approximation of the influence of parameter changes on the selected response quantities, based on a Taylor’s series expansion and adjusted to residual form. Response residuals are based solely on measured data. For more information about this concept see [7].

$$
S \delta p \approx \delta z
$$

$$
S \in \mathbb{R}^{n_z \times n_p}, \delta p \in \mathbb{R}^{n_p \times 1}, \delta z \in \mathbb{R}^{n_z \times 1}, \delta z = z_{m,\text{ref}} - z_{m,\text{cur}}
$$

Eq. (15) is partitioned according to following specification

$$
S = \begin{bmatrix} S_1^T, S_2^T, \ldots, S_{n_j}^T \end{bmatrix}, \quad S_k^T = \begin{bmatrix} \frac{\partial z}{\partial \lambda_1^j}, \frac{\partial z}{\partial \lambda_2^j}, \ldots, \frac{\partial z}{\partial \lambda_{n_{\text{gen},j}}^j} \end{bmatrix}^T
$$

$$
\delta p = \begin{bmatrix} \delta p_1^T, \delta p_2^T, \ldots, \delta p_{n_j}^T \end{bmatrix}^T, \quad \delta p_k = \begin{bmatrix} \lambda_1^j, \lambda_2^j, \ldots, \lambda_{n_{\text{gen},j}}^j \end{bmatrix}^T, \quad n_p = \sum_{j=1}^{n_j} n_{\text{gen},j}
$$

Matrix $S$ is the sensitivity matrix, i.e. derivatives of selected modal responses $z$ with respect to parameters of generic elements, $\delta p$ are parameter differences causing approximate differences in the response quantities $\delta z$ at a certain design point in parameter space, $z_{m,\text{ref}}$ are the measured reference and $z_{m,\text{cur}}$ are the measured current values of modal responses, $n_j$ is number of joints in structure, $S_k^T$ and $\delta p_k, k \in \{1, 2, \ldots, n_j\}$ are blocks of the sensitivity matrix and parameter vector corresponding to $k$-th
joint, respectively. Sensitivity matrix block $S^j_k$ contains modal response vector $z$ derivatives with respect to parameters of the $k$-th generic element, i.e. $\frac{\partial z}{\partial \lambda^j_i}$, $j \in \{1, 2, \ldots, n_{gen,k}\}$, parameter vector block $\delta p_k$ contains ordered parameters of the $k$-th generic element, $n_R$ and $n_P$ are the numbers of all response quantities and all parameters, respectively, and $n_{gen,k}$ is the number of parameters of $k$-th generic element.

Subscript of $\lambda^j_i$ in this and following parts of the paper represents the identification number of the eigenvalue.

If we introduce the $k$-th joint sensitivity subspace $\mathcal{J}_k$ as a vector space spanned by columns of matrix $S^j_k$,

$$\mathcal{J}_k = \text{Span} \left( \frac{\partial z}{\partial \lambda^j_1}, \frac{\partial z}{\partial \lambda^j_2}, \ldots, \frac{\partial z}{\partial \lambda^j_{n_{gen,k}}} \right)$$

then the principal subspace angle between the $k$-th joint sensitivity subspace and the measured response residual vector $\delta z$ is

$$\alpha_k = \angle (\mathcal{J}_k, \delta z), \quad \alpha_k \in \{0^\circ, 90^\circ\}$$

The value of the angle will be an indicator of how well any given joint with its parameterization can explain residual vector $\delta z$. High values of this angle, possibly close to $90^\circ$, would indicate that the region/joint corresponding to the $\mathcal{J}_k$ subspace has a low capacity to explain differences in the residual vector and conversely, low values, possibly close to $0^\circ$, would indicate that the region/joint corresponding to the $\mathcal{J}_k$ subspace has a high capacity to explain vector $\delta z$, i.e. that $\delta z$ is “almost” embedded in subspace $\mathcal{J}_k$. Thus, a subset of parameters is selected corresponding to a specific region of the structure.

The next step could be a detailed analysis of the selected subspace $\mathcal{J}_k$ in terms of the contribution of different parameter subspaces to the overall result, thus leading to the possibility to detect one out of a few possible damage scenarios.

The concept of principal subspace angle(s) is introduced in linear algebra as the generalization of an angle between two lines (i.e. one-dimensional subspaces) or an angle between a line and a plane (i.e. subspaces of dimension one and two), allowing the study of the “the amount of intersection” between two vector subspaces of arbitrary dimension with basis vectors of the same size. More about this subject, as well as about its computer implementation usually based on QR and SVD decompositions, can be found in any modern linear algebra textbooks, or in more specialist descriptions in [7] or [18].

### 3.3 Modal response vector for damage location

Section 3.2 provided a problem formulation for the application of parameter subset selection in damage location. However, there are a few problems that need to be addressed before presenting an example. Due to the nature of the algorithm, one has to fulfill condition (19) as a minimal requirement for being able to locate a region of potential damage, assuming that there is no linear dependency between any of the $n_P$ parameters.

$$\dim_{j \in U} \left( \mathcal{J}_j \right) < n_R, \quad U = \{1, 2, \ldots, n_j\}$$

$\dim(\cdot)$ is dimension of joint sensitivity subspace.

Condition (19) can sometimes be difficult to fulfill, especially if one wishes to use only natural frequencies, as is often preferred. Moreover, as natural frequencies are global dynamic properties, their sole use can cause damage resolution problems in cases where geometric symmetries are present in a structure, i.e. two or more locations cause an equal or very similar variation in natural frequencies. Hence,
the number of response quantities in vector $\delta z$ must be expanded and consequently the sensitivity matrix $S$ is enlarged. One such option, since Experimental Modal Analysis has already been performed, is to use modeshapes. Such a case is presented in Eq. (20) where weighted blocks of modeshape related data are added to the original formulation. Thus, if we take Eq. (15) and its matrices organized in row-wise blocks, the equation can be written in following form

$$
\begin{bmatrix}
S^\omega \\
\vdots \\
S^\phi
\end{bmatrix}
\delta p 
\cong
\begin{bmatrix}
\delta z^\omega \\
\vdots \\
\delta z^\phi
\end{bmatrix},
$$

where $w_j \in \mathbb{R}^+$, $j = \{1, 2, \ldots, n_\phi\}$.

$S^\omega, \delta z^\omega$ are blocks of the sensitivity matrix and the response residual vector corresponding to the natural frequencies and $S^\phi_j, \delta z^\phi_j, j = \{1, 2, \ldots, n_\phi\}$ are blocks of the sensitivity matrix and the response residual vector corresponding to the $j$-th modeshape, $n_\omega$ and $n_\phi$ are numbers of natural frequencies and modeshapes, respectively, and $w_j$ represents the relative weight applied to the supplementary modeshape data.

Assuming that we are using two different types of responses – natural frequencies and modeshapes, a scaling is desirable due to the wide numeric range of quantities present in the vector $\delta z$. This is implemented here as follows

$$
S^* \delta p \cong \delta z^*
$$

$$
S^* = C S, \delta z^* = C \delta z, \quad C = \text{diag} \left( [C^\omega, C^\phi, \ldots, C^\phi_{n_\phi}] \right)
$$

$C$ is block diagonal scaling matrix of total size $n_R$, separate scaling blocks are diagonal matrices by themselves and are based on the corresponding measured reference, i.e. undamaged, response values.

$$
C^\omega = \text{diag} \left( 1/\omega_{ref,1}, 1/\omega_{ref,2}, \ldots, 1/\omega_{ref,n_\omega} \right)
$$

$$
C^\phi_j = \left( 1/\max \left( [\phi_{ref,j}] \right) \right) I^{(n_{ref,j} \times n_{ref,j})}, j = \{1, 2, \ldots, n_\phi\}, n_R = n_\omega + \sum_{j=1}^{n_\phi} n_{\text{dof},j}
$$

$n_{\text{dof},j}, j = \{1, 2, \ldots, n_\phi\}$ is size of $j$-th selected modeshape used in response vector $\delta z$ and $I^{(n_{ref,j} \times n_{ref,j})}$ is a unit matrix of size $n_{\text{dof},j}$.

4  Example

4.1  Structure and simulated experiment description

The damage location concept presented in sections 2 and 3 was tested on the structure shown in Figure 1. The presentation will, at this stage, be performed on data from a simulated experiment, i.e. reference measured data as well as current, measured (potentially damaged) data will be simulated by the FE model. The FE model is a mathematical representation of a three-storey frame structure made of MEROFORM M12 components. A similar structure was tested and investigated from a model updating point of view in [11] and [19]. The two main construction components are standard aluminium nodes (0.08 kg) and 0.707 m long M12 aluminium tubes (outer diameter 0.022 m, wall thickness 0.001 m). The structure is assumed to be in free-free conditions. Figure 2 shows four selected Frequency Response Functions (FRF) in
frequency range from 0 to 200 Hz. The first 8 flexible modes are concentrated in the frequency range from 15 to 50 Hz and represent global dynamics of the structure. The frequency range starting at 100 Hz contains mostly local modes. Thus, there is a very distinctive separation between local and global dynamics of the structure. Due to this, only the first 8 modes will be used in the damage location study. Table 1 shows the “measured” reference (undamaged) values of natural frequencies.

A finite element model of the structure was created using 136 Euler-Bernoulli beams with 6 degrees of freedom at each node, i.e. four elements on each tube. Outer elements on each tube, i.e. 68 elements in total, were modeled by partially rigid elements with additional mass at each node representing the steel ends of the aluminium tubes. The geometry and material properties, lengths of rigid parts next to the MEROFORM nodes, as well as additional mass properties were based on the investigation presented in [19]. The structure contains 3, 4 and 5-element joints. Aluminium MEROFORM nodes were represented by mass elements with appropriate mass properties. This model was used to provide sensitivity information, as well as experimental data.

Structural damage was simulated by the reduction of Young’s modulus in all elements of the chosen joint, e.g. joint 6 in Figure 1. A parametric study showing the dependence of the first 8 non-zero natural frequencies on this parameter for joint 6 is shown in Figure 3. To better understand the influence of overall joint stiffness reduction via Young’s modulus reduction the modeshapes are also presented in Figure 4. Along with the first 8 global modeshapes, the 9th modeshape is also shown, providing illustration of the local character of the modeshapes corresponding to frequencies over 100 Hz. The algorithm was tested on the model with the extent of damage corresponding to a 6 % local reduction in Young’s modulus at joint no. 6. According to Figure 3, this corresponds to less than 0.5 % maximal reduction in the first 8 natural frequencies.

The following presentation consists of two parts. Firstly, the algorithm is tested in an environment without the influence of modeling uncertainty. Then, in an effort to include this influence the model that was used to provide experimental data has been modified by 1 % and 2 % reductions in the global Young’s modulus. Thus, the experimental structure has been slightly modified with respect to the nominal model of the structure.

A structural modification was used to reduce the negative influence of geometric symmetry on the localization of the damage. The modification was represented by a 1.5 kg mass added to both models used in the study. The location of this known mass is shown in Figure 1.

All computations were performed in Matlab by means of an in-house toolbox.

4.2 Study I – no uncertainty in experimental model

In this part, the study without the influence of modeling uncertainty is considered. The structure in Figure 1 was parameterized using 16 generic elements, each corresponding to one joint. As the damage was simulated by the reduction of Young’s modulus of joint elements, joint substructure stiffness matrices were chosen and parameterized by eigenvalue decomposition. It was assumed that the eigenvectors are retained as the damage progressed to the specified level, justified by the relatively low extent of the damage. Thus, only the eigenvalue related parameters were considered for actual parameterization and the joint substructures were parameterized according to Eq. (13) for all 16 joints. The full range of parameters, even after this considerable reduction, is 18 for 3-element joints, 24 for 4-element joints and 30 for 5-element joints. These numbers correspond to relatively high-dimensional joint subspaces requiring further reduction. The reduction is at this stage based on the study of the influence of damage increase on joint stiffness matrix eigenvalues, i.e. the sensitivities of matrix eigenvalues with respect to damage. The parameters used in following study are thus restricted to the eigenvalues from 14 to 18 (counting from the first zero eigenvalue and assuming 6 zero eigenvalues, or from the 8th to 12th non-zero eigenvalue) for each joint model. This is a subjective compromise between the sensitive part of a joint stiffness matrix spectra and the possible size of respective joint subspaces. Consequently, the dimension of all joint subspaces is 5. All angle computations were performed in scaled form as it is described in Eq. (21) and (22).
The results, namely the angles between joint sensitivity subspaces and residual vector $\delta z$, are presented in Figure 5 in the form of bar graphs with the height of each bar representing the size of corresponding angle. There are also error bounds present in these graphs centered on the top of the bars. These error bounds are computed as a standard deviation of the angles computed from cross-validation type studies using Eq. (20). The bars correspond to mean values of the angles. In these studies the angles are computed after removing pre-determined subset of equations, which amounts to rejection of a subset of experimental data from the angle computation. In our case, all instances of one-row removal are computed and the resulting angles and standard deviation are computed and presented in the Figure 5. This type of graph thus indicates the stability of the results with respect to the experimental data.

Figure 5a presents situation where only natural frequencies were used to compute the residual vector $\delta z$, as the number of experimental data values was bigger then the dimension of the joint subspaces. It is clear that due to geometric symmetry it is not possible to uniquely locate damage in the structure. Especially joints of the same type, i.e. eight 5-element joints, show a very similar ability to explain the residual vector $\delta z$ containing only natural frequency residuals. Thus, a modeshape is required to provide improved localization capacity. The third modeshape is selected for this purpose with its coordinates corresponding to measured locations in each MEROFORM node in X-Y plane. A weighting constant for the modeshape related rows of Eq. (20) is assumed to be $w_1 = 1$. The choice of this modeshape was based on the behavior of the MAC criterion for all 8 considered modeshapes while changing the local stiffness of the joint. Figure 5b presents these results. In contrast with Figure 5a, here the selection of the joints explaining the residual vector $\delta z$ (of size 40, i.e. 8 natural frequencies and 16x2 modeshape coordinates) can be narrowed to only two joints. This persisting, even though significantly reduced, ambiguity in damage location is due to the fact that the effect of the joint 7 is very similar to the changes introduced by joint 6, as can be also seen from Figure 1. Similar behavior can be observed also for the other joints, for instance for the case of the same amount of damage in joint 16 which is shown in Figure 5c. An attempt to overcome this problem by the use of an increased number of modeshapes resulted in no considerable improvement.

An alternative solution is suggested by the use of a known structural modification, in our case a known mass has been chosen as described above. The situation with added mass is presented in Figure 5d. Here, both natural frequencies and modeshape 3 were used, as in the case shown in Figure 5b. It is evident from this figure that the contrast between undamaged and damaged joints is significantly improved (note a change in the $y$ axis limits). The indication of the actual damaged region is correctly identified, however, the difference between angles corresponding to the joints 6 and 7 is very small.

### 4.3 Study II – simulated uncertainty in experimental model

This part presents a study where the influence of uncertainty has been included in the form of a modified experimental model as described in section 4.1. Figure 5e presents the angles computed for this case for the damage in joint 6 simulated by a 6 % reduction in Young’s modulus for the joint elements, for the structure without mass modification. The experimental model in this case has a reduced global Young’s modulus by 1 %. When this figure is compared to the equivalent without modeling error (Figure 5b), a deterioration of damage location capability is evident. The contrast between damaged and undamaged joints is smaller. The ambiguity between locating damage in joint 6 and 7 is still present. However, it is still possible to narrow the location of damage to these two joints. The same situation has been tested for the case of the structure with mass modification. The results of this case are shown in Figure 5f. The same behavior is observed as in the case of Figure 5d, however, now the contrast between the damaged joint and the rest of the structure is not as significant as in the case of Figure 5d.

The last figure shows the influence of a 2 % reduction in global Young’s modulus on damage indicating angles for the structure with mass modification and modeshape 3 used in the residual vector along with natural frequencies. In this case, Figure 5g, the damage location is not sufficiently indicated and the damage cannot be effectively located.
5 Conclusions

This paper presents two concepts that are tried to improve damage location. It introduces the concept of the joint subspace by considering the eigenvalue decomposition for the joint substructure stiffness matrix. Thus, a rich set of parameters is provided enabling the study of the quality of changes in stiffness matrix of damaged substructure on a more abstract level, retaining straightforward physical interpretation. Another concept considered in this paper is the concept of a joint sensitivity subspace resulting from the use of sensitivity type of equations and the parameter subset selection algorithm for damage location. An accompanying feature of this approach is a requirement for the quantity of experimental data in the damage indicating, modal residual vector. Here, the minimal number of experimental data required is related to the minimal size of joint sensitivity subspaces. Thus, modeshapes are often required with related considerations such as weighting and scaling.

These concepts were tested on a three-storey aluminium frame. The experimental data were simulated and the case with a relatively low damage extent was considered. The case with and without modeshape data was considered and the possibility of improved damage location with known structural modification was studied as well. The results suggest that these two concepts can provide a sensible indication of damage location even though at this stage the results seem to be somewhat sensitive to the model quality.

The use of parameters of generic elements in this paper is their first use in terms of joint subspaces. The implementation presented in the example part of this paper is the simplest possible and there is still considerable scope for further development, especially for parameters related to the eigenvector transformation, as the eigenvectors may be expected to change as a result of damage. Thus, a technique for parameter subspace identification with respect to certain damage scenarios could be the next step to making this approach more robust and usable in real cases.

Acknowledgements

Dr. Titurus gratefully acknowledges the support of the Royal Society, NATO and the Foreign and Commonwealth Office through the award of the Royal Society/NATO Postdoctoral Fellowship. The authors would like to express thanks to Dr. Mares for providing model data of the aluminium frame structure.

References


**Figure 1:** Aluminium MEROFORM M12 frame structure

**Figure 2:** Selected FRFs showing separation between global and local dynamics

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**Table 1:** Natural frequencies of reference FE model

**Figure 3:** Parameter variation study, Young’s modulus reduction for joint no.6
Figure 4: Modeshapes of the structure in undamaged state

Figure 5: Joint sensitivity subspace angles for different situations
Figure 5: Joint sensitivity subspace angles for different situations (continued from previous page)